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LIMIT SHAPES FOR GROWING EXTREME CHARACTERS OF $U(\infty)$

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We prove the existence of a limit shape and give its explicit description for certain probability distribution on signatures (or highest weights for unitary groups). The distributions have representation theoretic origin—they encode decomposition on irreducible characters of the restrictions of certain extreme characters of the infinite-dimensional unitary group $U(\infty)$ to growing finite-dimensional unitary subgroups $U(N)$. The characters of $U(\infty)$ are allowed to depend on N. In a special case, this describes the hydrodynamic behavior for a family of random growth models in $(2+1)$-dimensions with varied initial conditions.

1. Introduction. Decomposing the restriction of an irreducible representation of a group to its subgroup onto irreducible components is one of the basic problems of the representation theory. Under special circumstances, as the group and the subgroup become large, such decomposition may be subject to a Law of Large Numbers type concentration phenomenon—the bulk of the decomposition consists of representations that are in some sense close to each other. This paper is devoted to studying one of such situations.

Historically, the first example of this concentration phenomenon was discovered by Vershik–Kerov [26] and Logan–Shepp [16]. One way to phrase their result is to consider the infinite bisymmetric group $G = S(\infty) \times S(\infty)$, where $S(\infty)$ is the group of finite permutations of $\mathbb{N} := \{1, 2, \ldots\}$, and the growing subgroups being finite bisymmetric groups $G(n) = S(n) \times S(n)$, where $S(n)$ consists of permutations of a subset of $\mathbb{N}$ with $n \geq 1$ elements. Take the biregular representation of G in $\ell^2(S(\infty))$ with G acting by left and right shifts. It is well-known that it is irreducible (as for any countable group with infinite nontrivial conjugacy classes). Its restriction to $G(n)$ decomposes on isotypical components corresponding to irre-

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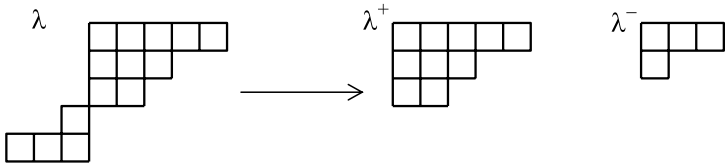

FIG. 1. *Signature* $\lambda = (5, 3, 2, -1, -3)$ *and Young diagrams* $\lambda^+ = (5, 3, 2)$ *and* $\lambda^- = (3, 1)$.

ducible representations of $S(n)$, or to partitions of n (equivalently, Young diagrams with n boxes). The corresponding spectral measure is the celebrated *Plancherel distribution* on Young diagrams with n boxes that assigns to λ the weight equal to the square of the number of standard Young tableaux of shape λ divided by $n!$.

The theorem of Vershik–Kerov–Logan–Shepp (see Kerov [13] and Ivanov–Olshanski [12] for a different proof that is closer to the present work) says that if we shrink the random Young diagram λ by the factor of $\sqrt{n}$ in both directions (so that its area is now 1), then as $n \to \infty$, the boundary of λ converges, in probability and in a suitable topology, to an explicit smooth curve usually referred to as the *limit shape*.

Vershik–Kerov in [27] also considered the case of other (unitary spherical) irreducible representations of G and their restrictions to $G(n)$, showing that while the law of large numbers is still there, it takes a drastically different form—one needs to normalize the row and column lengths of the random Young diagram λ by n to see the almost sure convergence to a point configuration (not a smooth curve) that essentially encodes the original representation of G.

In the present paper, we are dealing not with the symmetric groups $S(n)$ but with the compact unitary groups $U(N)$. The irreducible characters of $U(N)$ are parameterized by N-tuples $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$, which are called *signatures* of length N. Note that every such λ can be viewed as a couple (λ^+, λ^-) of Young diagrams (their row-lengths are, resp., the positive and the minus negative coordinates in λ; see an example in Figure 1). These two Young diagrams represent the *shape* of the signature.

Let us take $G(N) = U(N) \times U(N)$ and define G as the union of the growing groups $G(N)$. In other words, $G = U(\infty) \times U(\infty)$, where $U(\infty)$ is the group of unitary matrices of format $\mathbb{N} \times \mathbb{N}$ with finitely many entries U_{ij} distinct from δ_{ij}. The restriction of a (unitary spherical) irreducible representation of G to $G(N)$ decomposes on isotypical components parameterized by the signatures of length N. It is easiest to encode this decomposition via *characters*—central normalized positive-definite functions on $U(\infty)$ that are in one-to-one correspondence with the spherical unitary representations; see Olshanski [22, 23].

If $\chi : U(\infty) \to \mathbb{C}$ is a character of $U(\infty)$, then

$$\chi\big(\operatorname{diag}(z_1, \ldots, z_N, 1, 1, \ldots)\big) = \sum_{\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N} M_N^{\chi}(\lambda) \frac{s_\lambda(z_1, \ldots, z_N)}{s_\lambda(1, \ldots, 1)},$$

where s_λ's are the rational Schur functions [conventional irreducible characters for $U(N)$], and M_N^χ is the spectral measure of the decomposition, which is a probability distribution on the set of all signatures of length N.

Irreducible (spherical unitary) representations of G correspond to the extreme points of the convex set of characters of $U(\infty)$, often referred to as its *extreme characters*. The classification of the extreme characters is known as the Edrei–Voiculescu theorem (see Voiculescu [29], Edrei [11], Vershik–Kerov [28], Okounkov–Olshanski [20], Borodin–Olshanski [7]). They can be parameterized by the set

$$\Omega = (\alpha^+, \alpha^-, \beta^+, \beta^-, \delta^+, \delta^-) \in (\mathbb{R}_+^\infty)^4 \times (\mathbb{R}_+)^2,$$

where

$$\alpha^\pm = \alpha_1^\pm \ge \alpha_2^\pm \ge \cdots \ge 0, \qquad \beta^\pm = \beta_1^\pm \ge \beta_2^\pm \ge \cdots \ge 0,$$

$$\delta^\pm \ge 0, \qquad \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) \le \delta^\pm, \qquad \beta_1^+ + \beta_1^- \le 1.$$

Instead of $\delta^\pm$, we will use parameters $\gamma^\pm \ge 0$ defined by $\gamma^\pm := \delta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm)$. Each $\omega \in \Omega$ defines a function $\Phi^\omega : \{u \in \mathbb{C} : |u| = 1\} \to \mathbb{C}$ by

$$\begin{aligned}\Phi^\omega(u) &= \exp\bigl(\gamma^+(u-1) + \gamma^-(u^{-1}-1)\bigr) \\ &\quad \times \prod_{i=1}^\infty \frac{(1+\beta_i^+(u-1))}{(1-\alpha_i^+(u-1))} \frac{(1+\beta_i^-(u^{-1}-1))}{(1-\alpha_i^-(u^{-1}-1))},\end{aligned} \tag{1.1}$$

which we call the *Voiculescu function* with parameter ω. The corresponding extreme character has the form

$$\chi^\omega(U) := \prod_{u \in \mathrm{Spectrum}(U)} \Phi^\omega(u), \qquad U \in U(\infty), \tag{1.2}$$

where the product is over the eigenvalues of U [this product is essentially finite, because $\Phi^\omega(1) = 1$ and only finitely many of u's are distinct from 1].

We are thus interested in the limit shape phenomenon for the probability measures of the form $M_N^{\chi^\omega}$ as $N \to \infty$.

Let $\lambda = \lambda(N)$ be the random signature with distribution $M_N^{\chi^\omega}$, and let $\lambda^\pm$ be the corresponding Young diagrams. The row and column lengths of $\lambda^\pm$ (see Figure 1) divided by N almost surely converge, as $N \to \infty$, to the values of the $\alpha^\pm$ and $\beta^\pm$ coordinates (somewhat similarly to the case of $S(\infty)$, cf. Vershik–Kerov [27]). If all those coordinates are zero but $\gamma^\pm$ are not, then scaling by $\sqrt{N}$ leads to concentration of $M_N^{\chi^\omega}$ around two copies of the Vershik–Kerov–Logan–Shepp limit shape; see Borodin–Kuan [6].[4] The latter work also noted a hypothetical limit

[4]There is no proof of the measure concentration there, but there is substantial evidence that it holds.

shape formation as $\gamma^{\pm}$ *grow linearly in* N as $N \to \infty$ (as opposed to being independent of N), and suggested a formula for the limit shape. In the case when only γ^{+} is nonzero, the concentration around the limit shape was proved earlier by Biane [3].

In the present work, we prove that the limit shape phenomenon takes place in a much more general setting.

Let us state our main result.

Consider a sequence of points $\omega(N) \in \Omega$, $N \geq 1$, and assume that there exists an analytic function $P(z)$ defined in a neighborhood of the origin such that

$$\lim_{N\to\infty} \frac{1}{N}\bigl(\log \Phi^{\omega(N)}(z+1)\bigr) = P(z) \tag{1.3}$$

uniformly in a (possibly smaller) neighborhood of $z = 0$ (see the beginning of Section 3 below for simple sufficient conditions for the above convergence to hold).

THEOREM 1.1. *Let us fix an arbitrary sequence* $\{\omega(N)\}_{N\geq 1}$ *of elements in* Ω *satisfying the limit relation* (1.3). *For every* N, *let* $\lambda(N)$ *denote the random signature distributed according to* $M_N^{\chi^{\omega(N)}}$ *and let* $\lambda^{\pm}(N)$ *be the corresponding Young diagrams*.

Let us shrink the diagrams $\lambda^{\pm}(N)$ *by the factor of* N *in both directions. Then the resulting random shapes converge, as* $N \to \infty$, *to certain nonrandom shapes, which in principle can be obtained from the function* $P(z)$.

Here is another (and more precise) formulation of the result.

Denote by $\delta(x)$ the Dirac measure at a point $x \in \mathbb{R}$. To every signature $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$, we assign an atomic probability measure on $\mathbb{R}$:

$$\mu_\lambda := \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{\lambda_i - i + 1/2}{N}\right). \tag{1.4}$$

This measure encodes the (scaled) shape of λ (see Section 2.5 for more detail).

THEOREM 1.2. *Let* $\{\omega(N)\}_{N\geq 1}$*and* $\lambda(N)$ *be as above, and let* $\mu_{\lambda(N)}$ *be the random atomic measure on* $\mathbb{R}$ *corresponding to* $\lambda(N)$.

There exists a probability measure σ *with compact support on* $\mathbb{R}$ *such that*

$$\lim_{N\to\infty} \mu_{\lambda(N)} = \sigma \qquad (\textit{weak convergence in probability}).$$

The measure σ *is uniquely determined by its moments* $(1, m_1, m_2, \ldots)$, *which in turn are found from the fact that the two formal series in* z,

$$\exp(z + m_1 z + m_2 z^3 + \cdots) - 1 \quad \textit{and} \quad \frac{z}{1 + z(1+z)P'(z)}$$

are mutually inverse with respect to composition.

See Theorem 3.2 below. The fact that Theorem 1.2 implies Theorem 1.1 is explained in Proposition 2.2. That proposition also shows that the limit measure σ always has a density with respect to Lebesgue measure.

Concrete example of sequences $\{\omega(N)\}_{N\geq 1}$ and corresponding limit shapes can be found in the Appendix below.

The density of σ can be guessed using the determinantal structure of suitably defined correlation functions of measures M_N^ω found by Borodin–Kuan [6], and a steepest descent analysis of the double contour integral representation of the correlation kernel. We outline this route in Section 3.2 below. Note, however, that *proving* the concentration of measure phenomenon is a different task, and correlation functions are not well suited for it. In this work, we employ a different approach.

Our result also has a probabilistic interpretation. Measures of the form M_N^ω with ω having finitely many nonzero $\alpha^\pm$ and $\beta^\pm$ parameters can be obtained via a Markov growth process in $(2+1)$-dimensions; see Borodin–Ferrari [4]. Our main result then establishes the law of large numbers for a growing two-dimensional random interface. The growth process is *local*, and one can expect that the limit shape should be evolving in time according to a first-order PDE. Our result confirms that for a broad class of initial conditions; see Section 3.3 for details.[5]

If we have two sequences of extreme characters that lead to limit shapes, we can also consider the sequence whose members are products of those of the two original sequences (the set of extreme characters is closed under multiplication). The new sequence will also have a limit shape, and we thus obtain an operation on limiting measures σ. We call it "quantized free convolution"; it is a relative of the free convolution in free probability, and it degenerates to it; see Section 3.4 below. Bufetov–Gorin [9] show how this operation naturally arises through tensoring large irreducible representations of growing (but finite-dimensional) unitary groups and further decomposing them on irreducibles.

The particular examples of characters of $U(\infty)$ are the one-sided Plancherel character (the only nonzero parameter is γ^+) and the two-sided Plancherel chararcter (γ^+ and γ^- are nonzero). The probability measures arising from these characters were considered, for example, by Biane [3], Borodin–Bufetov [8], Borodin–Kuan [6]. However, we want to emphasize that the conditions of Theorem 1.2 are much more general because they allow to manipulate not only γ^+, γ^-, but all $4\cdot\infty+2$ parameters of extremal characters. Theorem 1.2 gives the same answer that was proved earlier by Biane [3] in the case of the one-sided Plancherel character and conjectured by Borodin–Kuan [6] in the case of the two-sided Plancherel character.

Having proved a law of large numbers, it is natural to ask about the central limit theorem. In the case of linearly growing parameter γ^+ and all other parameters being zero, it was shown in Borodin–Ferrari [4] and Borodin–Bufetov [8] that the

[5] In the case when the only nonzero paramater is γ^+, the corresponding PDE was found in [4].

fluctuations around the limit shape are described by the *two-dimensional Gaussian Free Field*. It is plausible that a similar description of fluctuations should exist under the (substantially more general) assumption of our theorem above.

Our proof is based on the method of moments. It bears a certain similarity with the work of Ivanov–Olshanski [12] for the Plancherel measures on symmetric groups and the work of Borodin–Bufetov [8] for the nonzero γ^+ case, but it is of course more involved because of the many parameters present. The key ingredients are provided by certain graph enumeration arguments, as we explain in Section 4.

2. Preliminaries.

2.1. *The infinite-dimensional unitary group and its characters.* Let $U(N) = \{[u_{ij}]_{i,j=1}^N\}$ be the group of $N \times N$ unitary matrices. Consider the tower of embedded unitary groups

$$U(1) \subset U(2) \subset \cdots \subset U(N) \subset U(N+1) \subset \cdots,$$

where the embedding $U(N) \subset U(N+1)$ is defined by $u_{i,N+1} = u_{N+1,i} = 0$, $1 \leq i \leq k$, $u_{N+1,N+1} = 1$. *The infinite-dimensional unitary group* is the union of these groups:

$$U(\infty) = \bigcup_{N=1}^{\infty} U(N).$$

Define a *character* of the group $U(\infty)$ as a function $\chi : U(\infty) \to \mathbb{C}$ that satisfies the following conditions:

(1) $\chi(e) = 1$, where e is the identity element of $U(\infty)$ (normalization);
(2) $\chi(ghg^{-1}) = \chi(h)$, where g, h are any elements of $U(\infty)$ (centrality);
(3) $[\chi(g_i g_j^{-1})]_{i,j=1}^n$ is an Hermitian and positive-definite matrix for any $n \geq 1$ and $g_1, \ldots, g_n \in U(\infty)$ (positive-definiteness);
(4) the restriction of χ to $U(N)$ is a continuous function for any $N \geq 1$ (continuity).

The set of characters of $U(\infty)$ is obviously convex. The extreme points of this set are called the *extreme* characters; they replace irreducible characters in this setting. The classification of the extreme characters was described in the Introduction; see formulas (1.1) and (1.2) above.

2.2. *The Gelfand–Tsetlin graph and coherent systems of measures.* A *signature* (also called *highest weight*) of length N is a sequence of N weakly decreasing integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N), \qquad \lambda_i \in \mathbb{Z}, 1 \leq i \leq N.$$

It is well known that the irreducible (complex) representations of $U(N)$ can be parameterized by signatures of length N (see, e.g., [30, 31]). Let $\mathrm{Dim}_N(\lambda)$ be the dimension of the representation corresponding to λ. By χ^λ, we denote the *normalized* character of this representation, that is, the conventional character divided by $\mathrm{Dim}_N(\lambda)$.

Let $\mathbb{GT}_N$ denote the set of all signatures of length N. (Here, letters $\mathbb{GT}$ stand for "Gelfand–Tsetlin.") We say that $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N-1}$ *interlace*, notation $\mu \prec \lambda$, if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for any $1 \leq i \leq N-1$. We also define $\mathbb{GT}_0$ as a singleton consisting of an element that we denote as $\varnothing$. We assume that $\varnothing \prec \lambda$ for any $\lambda \in \mathbb{GT}_1$.

The *Gelfand–Tsetlin graph* $\mathbb{GT}$ is defined by specifying its set of vertices as $\bigcup_{N=0}^{\infty} \mathbb{GT}_N$ and putting an edge between any two signatures λ and μ such that either $\lambda \prec \mu$ or $\mu \prec \lambda$. A *path* between signatures $\kappa \in \mathbb{GT}_K$ and $\nu \in \mathbb{GT}_N$, $K < N$, is a sequence

$$\kappa = \lambda^{(K)} \prec \lambda^{(K+1)} \prec \cdots \prec \lambda^{(N)} = \nu, \qquad \lambda^{(i)} \in \mathbb{GT}_i,\ K \leq i \leq N.$$

It is well known that $\mathrm{Dim}_N(\nu)$ is equal to the number of paths between $\varnothing$ and $\nu \in \mathbb{GT}_N$. An *infinite path* is a sequence

$$\varnothing \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(k)} \prec \lambda^{(k+1)} \prec \cdots.$$

We denote by $\mathcal{P}$ the set of all infinite paths. It is a topological space with the topology induced from the product topology on the ambient product of discrete sets $\prod_{N \geq 0} \mathbb{GT}_N$. Let us equip $\mathcal{P}$ with the Borel σ-algebra.

For $N = 0, 1, 2, \ldots$, let M_N be a probability measure on $\mathbb{GT}_N$. We say that $\{M_N\}_{N=0}^{\infty}$ *is a coherent system of measures* if for any $N \geq 0$ and $\lambda \in \mathbb{GT}_N$,

$$M_N(\lambda) = \sum_{\nu\,:\,\lambda \prec \nu} M_{N+1}(\nu) \frac{\mathrm{Dim}_N(\lambda)}{\mathrm{Dim}_{N+1}(\nu)}.$$

Given a coherent system of measures $\{M_N\}_{N=1}^{\infty}$, define the weight of a cylindric set of $\mathcal{P}$ consisting of all paths with prescribed members up to $\mathbb{GT}_N$ by

$$P\bigl(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}\bigr) = \frac{M_N(\lambda^{(N)})}{\mathrm{Dim}_N(\lambda^{(N)})}. \tag{2.1}$$

Note that this weight depends on $\lambda^{(N)}$ only (and does not depend on $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N-1)}$). The coherency property implies that these weights are consistent, and they correctly define a Borel probability measure on $\mathcal{P}$.

Now let χ be an arbitrary character of $U(\infty)$ and χ_N denote its restriction to the subgroup $U(N)$. The function χ_N can be expanded into a series in χ^λ's,

$$\chi_N = \sum_{\lambda \in \mathbb{GT}_N} M_N(\lambda) \chi^\lambda. \tag{2.2}$$

It is readily seen that the coefficients $M_N(\lambda)$ form a coherent system of measures on $\mathbb{GT}$. Conversely, for any coherent system of measures on $\mathbb{GT}$ one can construct a character of $U(\infty)$ using the above formula.

Note also that if χ_N is smooth, then the coefficients of the expansion (2.2) rapidly decay as λ goes to infinity, so that any polynomial function in variables $\lambda_1, \dots, \lambda_N$ is summable on $\mathbb{GT}_N$ with respect to measure M_N.

2.3. *The algebra of shifted symmetric functions.* In this subsection, we review some facts about the algebra of shifted symmetric functions; see [12, 14, 21].

Let $\mathrm{Sym}^*(N)$ be the algebra of polynomials in N variables $x_1, \dots, x_N$, that are symmetric in shifted variables

$$y_i := x_i - i + \tfrac{1}{2}, \qquad i = 1, 2, \dots, N.$$

The standard filtration of $\mathrm{Sym}^*(N)$ is defined by the degree of a polynomial. Define a map $\mathrm{Sym}^*(N) \to \mathrm{Sym}^*(N-1)$ as specializing $x_N = 0$. The *algebra of shifted symmetric functions* Sym^* is the projective limit of the algebras $\mathrm{Sym}^*(N)$ with respect to these maps. Here, the limit is taken in the category of filtered algebras meaning that the degree does not grow.

The algebra Sym^* can be identified with the subalgebra in $\mathbb{R}[[x_1, x_2, \dots]]$ generated by the algebraically independent system $\{\mathbf{p}_k\}_{k=1}^{\infty}$, where

$$\mathbf{p}_k(x_1, x_2, \dots) := \sum_{i=1}^{\infty} \left(\left(x_i - i + \frac{1}{2} \right)^k - \left(-i + \frac{1}{2} \right)^k \right), \qquad k = 1, 2, \dots.$$

Let $\mathbb{Y}_n$ denote the set of partitions (or Young diagrams) $\nu = (\nu_1 \geq \nu_2 \geq \dots \geq 0)$ with $|\nu| := \sum_{i \geq 1} \nu_i = n$. Let $\rho, \nu \in \mathbb{Y} := \mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \mathbb{Y}_2 \cup \dots$, and let $r = |\rho|$, $n = |\nu|$. For $r = n$, denote by ψ_ρ^ν the value of the irreducible character of the symmetric group $S(n)$ corresponding to ν on the conjugacy class indexed by ρ (see, e.g., [17, 24] for details on symmetric groups). For $r < n$, denote by ψ_ρ^ν the value of the same character on the conjugacy class indexed by $\rho \cup 1^{n-r} = (\rho, 1, 1, \dots, 1) \in \mathbb{Y}_n$. Define $p_\rho^\# : \mathbb{Y} \to \mathbb{R}$ by

$$p_\rho^\#(\nu) = \begin{cases} n(n-1)\cdots(n-r+1)\dfrac{\psi_\rho^\nu}{\dim \nu}, & n \geq r; \\ 0, & n < r. \end{cases}$$

Note that elements of Sym^* are well-defined functions on the set of all infinite sequences with finitely many nonzero terms. It turns out that there is a unique element $\mathbf{p}_\rho^\# \in \mathrm{Sym}^*$ such that $\mathbf{p}_\rho^\#(\nu) = p_\rho^\#(\nu)$ for all $\nu \in \mathbb{Y}$. It is known that the set $\{\mathbf{p}_\rho^\#\}_{\rho \in \mathbb{Y}}$ is a linear basis in Sym^*. When ρ consists of a single row, $\rho = (k)$, we denote the element $\mathbf{p}_\rho^\#$ by $\mathbf{p}_k^\#$. It is also known that the set $\{\mathbf{p}_k^\#\}_{k=1}^{\infty}$ is an algebraically independent system of generators of Sym^*. See [12] for details.

The *weight* of $\mathbf{p}_\rho^\#$ is defined by

$$\mathrm{wt}(\mathbf{p}_\rho^\#) = |\rho| + l(\rho),$$

where $l(\rho)$ denotes the number of nonzero coordinates in ρ. We extend this definition to arbitrary elements $f \in \mathrm{Sym}^*$ in a natural way, namely, we expand f in the basis $\{\mathbf{p}_\rho^\#\}$ and define the *weight* $\mathrm{wt}(f)$ as the maximal weight of those basis elements that enter the expansion of f with nonzero coefficients. It turns out (see [12]) that $\mathrm{wt}(\cdot)$ is a filtration on Sym^*. It is called the *weight filtration*.

We will need the following formula (see [12], Proposition 3.7):

$$\mathbf{p}_k = \frac{1}{k+1}[u^{k+1}]\{(1+\mathbf{p}_1^\# u^2+\mathbf{p}_2^\# u^3+\cdots)^{k+1}\} + \text{lower weight terms}, \tag{2.3}$$

where "lower weight terms" denotes terms with weight $\le k$, and $[u^k]\{A(u)\}$ stands for the coefficient of u^k in a formal power series $A(u)$.

2.4. *An algebra of functions on* (*random*) *signatures*. In this section, we define an algebra of functions on the probability space $(\mathbb{GT}_N, M_N)$ and state some properties of these functions.

For any $N \ge 1$, define functions $p_k^{(N)} : \mathbb{GT}_N \to \mathbb{R}$ by

$$p_k^{(N)}(\lambda) = \sum_{i=1}^{N}\left(\left(\lambda_i - i + \frac{1}{2}\right)^k - \left(-i+\frac{1}{2}\right)^k\right), \qquad \lambda \in \mathbb{GT}_N,\ 1 \le k \le N. \tag{2.4}$$

Let $\mathbb{A}(N)$ be the algebra generated by $\{p_k^{(N)}\}_{k=1}^N$. It is easy to see that for a fixed $N \ge 1$, the functions $\{p_k^{(N)}\}_{k=1}^N$ are algebraically independent; therefore, they form a system of algebraically independent generators of $\mathbb{A}(N)$. Clearly, the algebras $\mathbb{A}(N)$ and $\mathrm{Sym}^*(N)$ are naturally isomorphic.

Consider the map $\mathrm{pr}_N : \mathrm{Sym}^* \to \mathbb{A}(N)$ such that $\mathrm{pr}_N(\mathbf{p}_k) = p_k^{(N)}$. Denote by $p_\rho^{\#(N)}$ the function $\mathrm{pr}_N(\mathbf{p}_\rho^\#)$.

Let χ be a character of $U(\infty)$, χ_N its restriction to $U(N)$, and M_N the corresponding probability measure on $\mathbb{GT}_N$, where $N = 1, 2, \ldots$. We consider the pair $(\mathbb{GT}_N, M_N)$ as a probability space. Then the functions from $\mathbb{A}(N)$ turn into random variables. Let $\mathbf{E}_N$ be the expectation on this probability space. Note that for any $f \in \mathrm{Sym}^*$ we can consider the random variable $\mathrm{pr}_N(f)$.

With some ambiguity that should not lead to any confusion, we omit the index N in the notation of $p_k^{(N)}$ and $p_\rho^{\#(N)}$.

The complexification of $U(N)$ is the group $\mathrm{GL}(N,\mathbb{C})$, which is an open subset of $\mathrm{Mat}(N,\mathbb{C})$, the space of $N \times N$ complex matrices. Let x_{ij} be the natural coordinates in $\mathrm{Mat}(N,\mathbb{C})$ (where $1 \le i, j \le N$) and ∂_{ij} be the abbreviation for the (holomorphic) partial derivative operator $\partial/\partial x_{ij}$. Note that any analytic function on the real manifold $U(N)$ can be extended to a holomorphic function in a neighborhood of the identity matrix in $\mathrm{Mat}(N,\mathbb{C})$.

PROPOSITION 2.1. *Assume that χ is such that for every $N = 1, 2, \dots$, the function χ_N is analytic and so admits a holomorphic extension to a neighborhood of 1 in* $\mathrm{Mat}(N, \mathbb{C})$. *Then the following formula holds*:

$$\mathbf{E}_N\bigl(p_\rho^{\#}\bigr) = \sum_{1 \le i_1, \dots, i_{|\rho|} \le N} \partial_{i_1 s(i_1)}\, \partial_{i_2 s(i_2)} \cdots \partial_{i_{|\rho|} i_{s(|\rho|)}}\, \chi_N(1 + X)\Big|_{X=0}, \tag{2.5}$$

where $s \in S(|\rho|)$ is an arbitrary permutation with cycle structure ρ, and $X = [x_{ij}]$ is a matrix from $\mathrm{Mat}(N, \mathbb{C})$ *close to* 0.

Before proceeding to the proof, let us note that we will apply this result only to the extreme characters, and all such characters satisfy the hypothesis of the proposition, because every Voiculescu function is analytic. However, there exist nonextreme characters χ for which the functions χ_N are not analytic and even not smooth.

PROOF OF PROPOSITION 2.1. Because χ_N is analytic, all functions from $\mathbb{A}(N)$ are summable with respect to M_N, so that the corresponding random variables have finite expectation. Thus, the left-hand side of (2.5) is well defined.

The key fact we need is Theorem 2 in Kerov–Olshanski [14] (see also Okounkov–Olshanski [21], Section 15). Here is its statement. Consider the differential operator

$$D_\rho = \sum_{\alpha_1, \dots, \alpha_k, i_1, \dots, i_k = 1}^{N} x_{\alpha_1 i_1}, \dots, x_{\alpha_k i_k}\, \partial_{\alpha_1 i_{s(1)}} \cdots \partial_{\alpha_k i_{s(k)}}$$

on $\mathrm{Mat}(N, \mathbb{C})$. Its restriction to the group $\mathrm{GL}(N, \mathbb{C})$ is invariant with respect to left and right shifts, and one has

$$D_\rho \chi^\lambda = p_\rho^{\#}(\lambda)\chi^\lambda$$

for any $\lambda \in \mathbb{GT}_N$. Let us recall that χ^λ denotes the normalized irreducible character of $U(N)$ indexed by λ, so that $\chi^\lambda(1) = 1$. Therefore, evaluating the both sides at 1 we get

$$p_\rho^{\#}(\lambda) = \bigl(D_\rho \chi^\lambda\bigr)(1).$$

Next, taking the expectation of the both sides with respect to M_N, we get

$$\mathbf{E}_N\bigl(p_\rho^{\#}\bigr) = (D_\rho \chi_N)(1).$$

Finally, under the specialization of the coefficients of the operator D_ρ at the point $1 \in \mathrm{Mat}(N, \mathbb{C})$ this operator simplifies and turns into the operator in (2.5). □

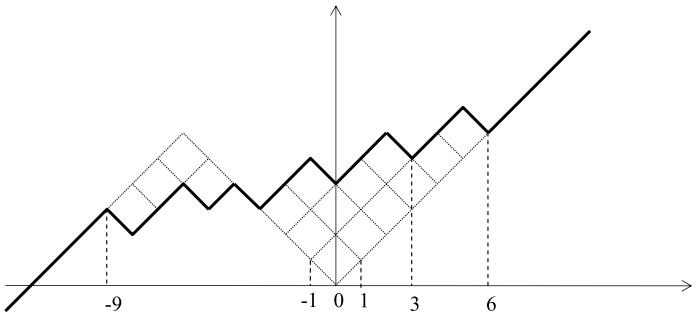

FIG. 2. *A piecewise linear function corresponding to the signature* $\lambda = (6, 4, 2, 0, -1, -3)$.

2.5. *Geometric interpretation of signatures.* Let us depict signatures $\lambda \in \mathbb{GT}_N$ in the way shown in Figure 2. This figure explains how to assign to λ a continuous piecewise linear function $w_\lambda(x)$ (bold line in the figure).

Formally, $w_\lambda(x)$ is uniquely determined by the following properties:

- $w'_\lambda(x)$ may have jump discontinuities only at points $n \in \mathbb{Z}$ of the x-axis;
- $w'_\lambda(x) = \pm 1$ for $x \notin \mathbb{Z}$;
- $w_\lambda(x) = x$ for $x \geq \lambda_1$ and $w_\lambda(x) = x + 2N$ for $x \leq \lambda_N - N$, so that $w'_\lambda(x) = 1$ outside $[\lambda_N - N, \lambda_1]$;
- inside $(\lambda_N - N, \lambda_1)$, there are exactly N unit intervals $(n, n+1)$ where $w'_\lambda(x) = -1$, and these are those with the midpoints $\lambda_i - i + \frac{1}{2}$, $i = 1, \dots, N$.

In particular, the function $w_0(x)$ corresponding to the signature $(0, \dots, 0) \in \mathbb{GT}_N$ has exactly two derivative jumps, at the points $x = -N$ and $x = 0$.

We regard w_λ as the *shape* of λ. Note that the part of the graph of w_λ above (resp., below) the broken line w_0 visualizes the diagram λ^+ (resp., λ^-); see the Introduction for the definition of $\lambda^\pm$.

We also need the function $\frac{1}{N} w_\lambda(Nx)$, which describes the *scaled shape* of λ.

Next, recall definition (1.4) of the probability measure on $\mathbb{R}$ associated with λ:

$$\mu_\lambda := \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i - i + 1/2}{N} \right),$$

where $\delta(x)$ denotes the Dirac measure at x. Clearly, λ is uniquely determined by μ_λ.

We are going to show that the concentration of random measures μ_λ implies the concentration of the scaled shapes.

PROPOSITION 2.2. *Assume that for every* $N = 1, 2, \dots$ *we are given an ensemble of random signatures* $\lambda = \lambda(N)$ *distributed according to a probability measure on* $\mathbb{GT}_N$. *Next, let us assume that, as* $N \to \infty$, *the corresponding random*

measures μ_λ weakly converge, in probability, to a nonrandom probability measure σ with support in a bounded interval $[a,b]\subset\mathbb{R}$.

(i) *The limit measure σ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ and so has a density $p(x)$ vanishing outside $[a,b]$.*

(ii) *The random functions $\frac{1}{N}w_\lambda(Nx)$ uniformly converge in probability to a nonrandom function $w(x)$, uniquely determined by the following three properties: $w(x)=x$ for $x>b$, $w(x)=x+2$ for $x<a$ and $w'(x)=1-2p(x)$ almost everywhere on $[a,b]$.*

PROOF. (i) The assumption of the proposition means that for any bounded continuous function $f(x)$ on $\mathbb{R}$,

$$\lim_{N\to\infty}\langle f,\mu_\lambda\rangle=\langle f,\sigma\rangle \qquad \text{in probability,} \tag{2.6}$$

where the angular brackets denote the pairing between functions and measures.

Let us assume that f is compactly supported and nonnegative. By the very definition of μ_λ,

$$\langle f,\mu_\lambda\rangle=\frac{1}{N}\sum_{i=1}^{N}f\left(\lambda_i-i+\frac{1}{2}\right)\le\frac{1}{N}\sum_{n\in\mathbb{Z}}f\left(n+\frac{1}{2}\right).$$

Since the last expression is the Riemann sum for the integral of f against Lebesgue measure, passing to a limit as $N\to\infty$, we see that $\langle f,\sigma\rangle$ is bounded from above by that integral. If follows that σ has a density $p(x)$ with respect to Lebesgue measure and, moreover, $p(x)\le 1$ almost everywhere.

(ii) Let us define an auxiliary piecewise linear function $\tilde{w}_\lambda(x)$ by

$$\tilde{w}_\lambda(x):=x+2\big(1-\mu_\lambda\big((-\infty;x]\big)\big)=x+2\mu_\lambda\big((x;+\infty)\big).$$

It readily follows that for x such that $w'_\lambda(Nx)=1$ we have $\frac{1}{N}w_\lambda(Nx)=\tilde{w}_\lambda(x)$, and for all x,

$$\left|\tilde{w}_\lambda(x)-\frac{1}{N}w_\lambda(Nx)\right|\le\frac{1}{N}. \tag{2.7}$$

Let us define $w(x)$ as the primitive function of $1-2p(x)$ such that $w(x)=x$ for $x\gg 0$. By virtue of (2.6) and claim (i), $\mu_\lambda(\mathbb{R}\setminus[a,b])$ converges in probability to 0 as $N\to\infty$. The uniform convergence of $\tilde{w}_\lambda(x)$ to $w(x)$ outside of $[a,b]$ directly follows from this fact. Equation (2.7) implies that the functions $\frac{1}{N}w_\lambda(Nx)$ also uniformly converge to $w(x)$ outside of $[a,b]$.

The definition of $\tilde{w}_\lambda(x)$ and the convergence of $\mu_\lambda(\mathbb{R}\setminus[a,b])$ to 0 implies that for any bounded continuous function f we have

$$\lim_{N\to\infty}\int_a^b f(x)\tilde{w}_\lambda(x)\,dx=\int_a^b f(x)w(x)\,dx \qquad \text{in probability.}$$

Using (2.7), we obtain

$$\lim_{N\to\infty}\int_a^b f(x)\frac{1}{N}w_\lambda(Nx)\,dx=\int_a^b f(x)w(x)\,dx \qquad \text{in probability.} \tag{2.8}$$

Note that the functions $\frac{1}{N}w_\lambda(Nx)$ and $w(x)$ are Lipschitz functions with Lipschitz constant 1, and for such functions, the convergence of the integrals (2.8) with an arbitrary continuous test function f on $[a,b]$ is equivalent to the uniform convergence on $[a,b]$ (see, e.g., [12], Lemma 5.7). This completes the proof. □

2.6. *Convergence of random measures*. In this subsection, we prove a technical lemma about convergence of random measures.

Let $\{X_{i,j}\}_{i=1,2,\dots;\,j=1,2,\dots,i}$ be a set of random variables. Let

$$\nu_N:=\frac{1}{N}\sum_{i=1}^{N}\delta(X_{i,N})$$

be a (random) measure on $\mathbb{R}$. Assume that the following conditions hold:

$$\lim_{N\to\infty}\mathbf{E}\int x^k\nu_N(dx)=\mathbf{a}_k,\qquad k=1,2,3,\dots, \tag{2.9}$$

$$\lim_{N\to\infty}\mathbf{E}\Big(\int x^k\nu_N(dx)\Big)^2=\mathbf{a}_k^2,\qquad k=1,2,3,\dots. \tag{2.10}$$

Also assume that there exists a constant $C>0$ such that

$$\mathbf{a}_k<C^k,\qquad k=1,2,3,\dots. \tag{2.11}$$

LEMMA 2.3. *Let $\{X_{i,j}\}_{i=1,2,\dots;\,j=1,2,\dots,i}$ be a set of random variables such that conditions (2.9)–(2.11) hold. Then there exists a measure ν such that*

$$\int x^k\nu(dx)=\mathbf{a}_k,\qquad k=1,2,3,\dots,$$

and we have

$$\lim_{N\to\infty}\nu_N=\nu \qquad \textit{weakly; in probability.}$$

In greater detail, for any bounded continuous f we have

$$\lim_{N\to\infty}\int f\,d\nu_N=\int f\,d\nu, \qquad \textit{in probability.}$$

PROOF. We follow [1], Section 2.1.2.

Define a (deterministic) measure $\bar\nu_N$ on $\mathbb{R}$ by its values on test functions via

$$\int f\,d\bar\nu_N:=\mathbf{E}\int f\,d\nu_N \qquad \text{for any bounded continuous } f.$$

It follows from the Chebyshev inequality that for any $B > 1$ we have

$$\mathbf{P}\left(\int x^k \mathbf{1}_{|x|>B}\nu_N(dx) > \varepsilon\right) \le \frac{1}{\varepsilon}\mathbf{E}\int x^k \mathbf{1}_{|x|>B}\nu_N(dx) \le \frac{\mathbf{E}\int x^{2k}\nu_N(dx)}{\varepsilon B^k}.$$

Conditions (2.9) and (2.11) imply

$$\limsup_{N\to\infty}\mathbf{P}\left(\int x^k \mathbf{1}_{|x|>B}\nu_N(dx) > \varepsilon\right) \le \frac{\mathbf{a}_{2k}}{\varepsilon B^k} \le \frac{(C^2)^k}{\varepsilon B^k}.$$

Note that for any $K > k$ we have

$$\begin{aligned}\limsup_{N\to\infty}\mathbf{P}\left(\int x^k \mathbf{1}_{|x|>B}\nu_N(dx) > \varepsilon\right) &\le \limsup_{N\to\infty}\mathbf{P}\left(\int x^{2K} \mathbf{1}_{|x|>B}\nu_N(dx) > \varepsilon\right)\\ &\le \frac{C^{4K}}{\varepsilon B^{2K}}.\end{aligned}$$

Choosing $B = C^2 + 1$ and letting K to infinity, we have

$$\limsup_{N\to\infty}\mathbf{P}\left(\int x^k \mathbf{1}_{|x|>B}\nu_N(dx) > \varepsilon\right) = 0. \tag{2.12}$$

Therefore, we obtain

$$\lim_{N\to\infty}\mathbf{E}\int x^k \mathbf{1}_{[-B;B]}\nu_N(dx) = \mathbf{a}_k, \qquad k = 1, 2, 3, \ldots.$$

Since the unit ball in $(C[-B; B])^*$ is weakly compact, the sequence $\bar{\nu}_N$ converges (weakly) to a probability measure ν with support in $[-B; B]$, and we have

$$\int x^k \nu(dx) = \mathbf{a}_k.$$

Note that (2.9) and (2.10) imply that the sequence $\int x^k\, d\nu_N$ converges to $\mathbf{a}_k$ in probability.

Let $f(x)$ be a continuous bounded function. The Weierstrass theorem implies that for any $\delta > 0$ there exists a polynomial $Q_\delta(x)$ such that

$$\sup_{x\in[-B;B]}\left|Q_\delta(x) - f(x)\right| < \delta/10.$$

Then

$$\begin{aligned}&\mathbf{P}\left(\left|\int f(x)\nu_N(dx) - \int f(x)\nu(dx)\right| > \delta\right)\\ &\quad\le \mathbf{P}\left(\left|\int Q_\delta(x)\nu_N(dx) - \int Q_\delta(x)\nu(dx)\right| > \delta/4\right)\\ &\qquad + \mathbf{P}\left(\left|\int Q_\delta(x)\mathbf{1}_{|x|>B}\nu_N(dx)\right| > \delta/4\right).\end{aligned}$$

The first term converges to zero due to the convergence in probability of $\int x^k \nu_N(dx)$ to $\mathbf{a}_k$, and the second term converges to zero due to (2.12). This completes the proof of the lemma. □

3. Main result and discussion.

3.1. *The main result.* In this section, we state the main result of this paper.

Recall that elements $\omega \in \Omega$ parameterize extreme characters of the group $U(\infty)$. We consider a sequence $\omega(N) \in \Omega$ depending on (growing) integer N. Let $\chi^{\omega(N)}$ be the extreme character of $U(\infty)$ corresponding to $\omega(N)$, and let M_N be the probability measure on $\mathbb{GT}_N$ determined by this character (see Section 2.2). Let $\lambda^{(N)} \in \mathbb{GT}_N$ be a random signature distributed according to M_N, and let

$$\mu^{(N)} := \mu_{\lambda^{(N)}} = \frac{1}{N}\sum_{i=1}^{N} \delta\Big(\frac{\lambda_i^{(N)} - i + 1/2}{N}\Big)$$

be the random measure on $\mathbb{R}$ associated with $\lambda^{(N)}$ (see Section 2.5). We are interested in the limit behavior of this random measure.

Let $\Phi^{\omega(N)}(z)$ be the Voiculescu function depending on parameters $\gamma^{\pm}(N)$, $\{\alpha_i^{\pm}(N)\}$, $\{\beta_j^{\pm}(N)\}$, see (1.1). Consider the following condition on these sequences of parameters.

Main condition. Assume that for some $\varepsilon > 0$ the analytic function $\log \Phi^{\omega(N)}(z+1)$ uniformly converges to an analytic function $P(z)$ on $\{z \in \mathbb{C} \mid |z| \le \varepsilon\}$:

$$\lim_{N\to\infty} \frac{1}{N}\big(\log \Phi^{\omega(N)}(z+1)\big) = P(z). \tag{3.1}$$

By t_i, $i \in \mathbb{N}$, we denote the coefficients of the Taylor series for $P'(z)$:

$$P'(z) =: t_1 + t_2 z + t_3 z^2 + \cdots. \tag{3.2}$$

It is convenient for us to also formulate a stronger condition that describes more explicitly how the parameters $\gamma^{\pm}(N)$, $\{\alpha_i^{\pm}(N)\}$, $\{\beta_j^{\pm}(N)\}$ can change.

Sufficient condition. Let

$$\mathcal{A}_N^{\pm} := \frac{1}{N}\sum_{i=1}^{\infty} \delta\big(\alpha_i^{\pm}(N)\big), \qquad \mathcal{B}_N^{\pm} := \frac{1}{N}\sum_{i=1}^{\infty} \delta\big(\beta_i^{\pm}(N)\big),$$

be measures on $\mathbb{R}$.

We say that a sequence $\omega(N)$ satisfies the sufficient condition if there exist limits

$$\lim_{N\to\infty} \frac{\gamma^{\pm}(N)}{N} \tag{3.3}$$

and

$$\lim_{N\to\infty} \mathcal{A}_N^{\pm} = \mathcal{A}^{\pm}, \qquad \lim_{N\to\infty} \mathcal{B}_N^{\pm} = \mathcal{B}^{\pm} \qquad \text{weak convergence}, \tag{3.4}$$

for some finite measures $\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}$ on $\mathbb{R}$ with compact support. Moreover, we require that there exist positive constants C_1, C_2 such that

$$\big|\alpha_i^{\pm}(N)\big| < C_1, \qquad \big|\beta_i^{\pm}(N)\big| < C_1 \qquad \text{for all } i \ge 1, \tag{3.5}$$

and the number of nonzero parameters $\alpha_i^{\pm}(N)$ and $\beta_i^{\pm}(N)$ is less than $C_2 N$.

For example, let $\alpha_1 = \cdots = \alpha_N = \alpha$, where $\alpha > 0$ is a fixed constant, and all other Voiculescu's parameters are equal to 0. It is clear that this sequence of parameters satisfies the sufficient condition (3.3)–(3.5). Another example is given by $\gamma^+ = \gamma N$, where $\gamma > 0$ is a fixed constant, and all other Voiculescu's parameters are equal to 0. More examples can be found in the Appendix.

PROPOSITION 3.1. *Let $\{\omega(N)\}_{N\geq 1}$ be a sequence of points in Ω. Assume it satisfies the sufficient condition* (3.3)–(3.5). *Then it also satisfies the main condition* (3.1).

PROOF. We have (omitting the dependence on N in notation)

$$\begin{aligned}
&\frac{1}{N}\log \Phi^{\omega(N)}(z+1)\\
&\quad = \frac{\gamma^+}{N} + \frac{\gamma^-}{N}\left(\frac{1}{z+1} - 1\right)\\
&\quad\quad + \frac{1}{N}\Bigg(\sum_{i\geq 1}\log(1+\beta_i^+ z) - \sum_{i\geq 1}\log(1-\alpha_i^+ z)\\
&\quad\quad\quad + \sum_{i\geq 1}\log\left(1 - \frac{\beta_i^- z}{1+z}\right) - \sum_{i\geq 1}\log\left(1 + \frac{\alpha_i^- z}{1+z}\right)\Bigg).
\end{aligned} \tag{3.6}$$

Conditions (3.4) and (3.5) imply that there exist limits

$$\frac{1}{N}\sum_i \mathbf{s}_i^k \qquad \text{for all } k \geq 0,$$

where $\mathbf{s}_i$ is equal to $\alpha_i^{\pm}$ or $\beta_i^{\pm}$.

This fact and condition (3.3) imply that the Taylor coefficients of (3.6) converge to some limiting coefficients t_i, and the power series determined by these t_i converges in a neighborhood of 0 (because the supports of $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ are compact). Condition (3.5) implies that this convergence is uniform. □

From now on we assume that $\omega = \omega(N)$ satisfies the main condition (3.1).

Let

$$Q(z) = 1 + z(1+z)\left(t_1 + t_2 z + t_3 z^2 + \cdots\right)$$

be a formal power series depending on coefficients $t_1, t_2, \ldots$. Define a formal power series $v_0(z)$ via

$$v_0(z) := \left(\frac{z}{Q(z)}\right)^{(-1)},$$

where in the right-hand side the formal inversion of power series is used.[6]

Let

$$S(z) := \log\bigl(1 + v_0(z)\bigr) = z + m_1 z^2 + m_2 z^3 + \cdots. \tag{3.7}$$

Later on (see Section 4.1) we will prove that there exists a unique probability measure on $\mathbb{R}$ with moments $\{1, m_1, m_2, \ldots\}$. Denote this measure by σ.

The main result of this paper is the following.

THEOREM 3.2. *Let $\{\omega(N)\}_{N\geq 1}$ be a sequence of points in Ω satisfying condition (3.1). Then*

$$\lim_{N\to\infty} \mu^{(N)} = \sigma \qquad \textit{weak convergence in probability.}$$

Equivalently, for any bounded continuous function f we have

$$\lim_{N\to\infty} \int f\, d\mu^{(N)} = \int f\, d\sigma \qquad \textit{in probability.}$$

The proof of this theorem is given in Section 4.

The density of σ (which is well defined by virtue of Proposition 2.2) can be obtained from the known Stieltjes transform $S(1/z)$ [see (3.7)] with the use of standard methods of complex analysis; see also the end of the next subsection.

3.2. *A heuristic derivation of the limit shape.* In this section, we sketch an argument which shows how one can compute the measure σ (see Theorem 3.2) via determinantal point processes. This yields the correct formula but not a complete proof, because the very existence of the concentration remains unclear. Our proof of Theorem 3.2 is obtained in a very different way (see Section 4).

In [6], it was shown that the correlation functions of the random point configuration $(\lambda_1 - 1, \lambda_2 - 2, \ldots)$ corresponding to the restriction of the extreme character of $U(\infty)$ with Voiculescu function $\Phi^\omega(z)$ to $U(N)$ have determinantal structure (necessary definitions can be found, e.g., in [6]). The correlation kernel of this process has the following form:

$$K(x, y) = \frac{-1}{4\pi^2} \oint\oint \frac{\Phi^\omega(u^{-1})}{\Phi^\omega(w^{-1})} \frac{u^x (1-u)^N}{w^{1+y}(1-w)^N} \frac{du\, dw}{u - w},$$

where the u-contour is a counterclockwise oriented circle with center 0 and radius $\varepsilon \ll 1$, and the w-contour is a counterclockwise oriented circle with center 1 and radius $\delta \ll 1$.

[6]Here and below we consider formal power series of the form

$$z + a_2 z^2 + a_3 z^3 + \cdots, \qquad a_i \in \mathbb{R}.$$

It is well known that such a series has a unique inverse (with respect to composition) of this form. For example, if $A(z) = \sum_{i=1}^{\infty} z^i$ then $A^{(-1)}(z) = \sum_{i=1}^{\infty} (-1)^{i-1} z^i$.

If one already knows that the random point process $(\lambda_1 - 1, \lambda_2 - 2, \ldots)$ satisfies the law of large numbers type theorem, then it is natural to assume that the density of the limit measure is equal to the limit of the diagonal values of the kernel $N^{-1}K(xN, xN)$ (this is the so-called density function) as $N \to \infty$.

Let us find (informally) the limit of $N^{-1}K(xN, xN)$ as $N \to \infty$. A useful general approach to asymptotic analysis of such integrals is the steepest decent method. In order to apply this method, we write the integrand in the form

$$\frac{\exp(N(\mathcal{S}(z) - \mathcal{S}(w)))}{z - w},$$

where

$$\mathcal{S}(u) := \frac{\log f(u^{-1}) + x \log u + N \log(1 - u)}{N}.$$

Following the logic of [19] (see also [5]), we need to deform the contours of integration in such a way that they pass through the critical points of $\mathcal{S}(z)$ which are the roots of

$$\frac{1}{N}\left(\left(\log \Phi^{\omega}(z^{-1})\right)' + \frac{x}{z} - \frac{N}{1 - z}\right) = 0. \tag{3.8}$$

We are interested in the root $z_+ = z_+(x)$ which has the positive imaginary part.

Then the steepest decent method gives the following asymptotics for the one-dimensional correlation function (cf. [5, 19]):

$$\frac{1}{N}K(xN, xN) \approx \frac{1}{\pi}\arg(z_+), \qquad N \to \infty. \tag{3.9}$$

Let us apply a change of variable $z = 1/w$; with the use of (3.1), equation (3.8) can be written in the form

$$P'(w - 1) - \frac{x + 1}{w} + \frac{1}{w - 1} = 0. \tag{3.10}$$

Let $w_0 = w_0(x)$ be the complex root of (3.10) in the complex upper half-plane.

Recall that the Stieltjes transform of a probability measure $\hat{\mu}$ with compact support is given by

$$\mathrm{Stil}_{\hat{\mu}}(z) := \int_{\mathbb{R}} \frac{\hat{\mu}(dt)}{z - t}, \qquad z \in \mathbb{C} \setminus \mathrm{supp}(\hat{\mu}).$$

Observe that if one denotes the moments of $\hat{\mu}$ by $1, m_1, m_2, \ldots$, then $\mathrm{Stil}_{\hat{\mu}}(z)$ is obtained from the right-hand side of (3.7) by the change of variable $z \mapsto z^{-1}$.

The Stieltjes transform can be inverted. For a measure $\hat{\mu}$ with density $\hat{p}(x)$ with respect to the Lebesgue measure, we have

$$\hat{p}(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi}\Im\left(\mathrm{Stil}_{\hat{\mu}}(x + i\varepsilon)\right).$$

Assume that $w_0 = w_0(x)$ is a real-analytic function. Using the analytic continuation, one can view $w_0 = w_0(x)$ as a complex analytic function. It is natural to think that for the principal branch of the function $\log(x)$ we have

$$\lim_{\varepsilon \to 0} \Im \log\bigl(w_0(x + i\varepsilon)\bigr) = \arg w_0(x), \qquad x \in \mathbb{R}.$$

Note also that $\arg(z_+(x)) = \arg(w_0(x))$. Therefore, it is natural to assume that the Stieltjes transform of the limit measure is equal to $\log(w_0(x))$.

Let $v_0(z)$ be the formal power series defined in Section 3.1. Note that the series $y_0 = v_0(1/z)$ solves the following equation:

$$z = \frac{1}{y} + (1 + y)P'(y). \tag{3.11}$$

Equations (3.10) and (3.11) imply that the formal power series $v_0(1/z)$ satisfies the same equation as $w_0 - 1$. Thus, the result stated in Theorem 3.2 coincides with the heuristic answer coming from the determinantal processes.

3.3. *Markov dynamics on two-dimensional arrays.* This subsection details the relation of the present work to random growth of surfaces in $(2 + 1)$-dimensions. This connection served as our original motivation, but it is not necessary for understanding the rest of the paper, and thus the reader should feel free to omit it.

Consider a two-dimensional triangular array of particles

$$\mathcal{W} = \bigl\{\{x_k^m\}_{m=1,\dots,\infty;k=1,\dots,m} \subset \mathbb{Z}^{n(n+1)/2} | x_k^{m+1} \geq x_k^m > x_{k+1}^{m+1}\bigr\};$$

we interpret the number x_k^m as the position of the particle with label (k, m).

For any N, the extreme character $\omega(N)$ is determined by a set of Voiculescu's parameters $\alpha^{\pm}(N)$, $\beta^{\pm}(N)$, and $\gamma^{\pm}(N)$ (below we omit the dependence on N in notation). Suppose that the number of parameters of types $\alpha^{\pm}$, $\beta^{\pm}$ is finite and equals $T = T(N)$. Let us enumerate these parameters by the numbers $1, \dots, T$ in an arbitrary way. We interpret this enumeration dynamically as follows: At time 1, we take only the first parameter; at time 2, the second parameter is added, etc. Let χ_a be the extreme character of $U(\infty)$ determined by the first a parameters in our ordering, $1 \leq a \leq T$. This character gives rise to a probability measure on $\mathbb{GT}_N$; denote it by $\mu_a^{(N)}$.

It turns out that the measure $\mu_a^{(N)}$ can be obtained in the following way. One can define (see [4], Section 2.6) a discrete time Markov dynamics on the triangular arrays as above with the following property: For any $N \geq 1$, at each time a the distribution of the vector $\{x_k^N - N\}_{k=1,\dots,N}$ coincides with the distribution of $\{\lambda_i - i\}_{i \geq 1}$, where $\{\lambda_i\}$ are the coordinates of the random signature distributed according to the measure $\mu_a^{(N)}$. In particular, for $a = T$ the distribution of the Nth level of the array coincides with the measure $\mu^{(N)}$. Parameters $\gamma^{\pm}$ can also be realized under a similar Markov dynamics with continuous time (see [4], Section 1).

An important feature of these Markov processes is the locality of interactions between the particles—the behavior of each individual particle is only influenced by particles whose coordinates differ at most by 1 from those of the chosen one.

The evolution of the whole array of particles can be fully encoded by the *height function* $h : \mathbb{R} \times \mathbb{R}_{\geq 1} \times \{1, 2, \dots, T\} \to \mathbb{Z}_{\geq 0}$ defined by

$$h(\mathbf{x}, \mathbf{y}, a) = \#\{k | x_k^{[\mathbf{y}]}(a) > \mathbf{x}\},$$

where $x_k^m(a)$ stands for the position of the particle x_k^m at the time a.

Suppose now that a sequence of characters $\omega(N)$ satisfies the general condition (3.1) with a function $P_0(z)$. Let us fix a large N; at this stage, we have a certain set of parameters $\{\alpha^{\pm}, \beta^{\pm}\}$. We want to add to these parameters another set of parameters satisfying condition (3.1). For simplicity, we consider six special cases: adding tN parameters of one of the possible types $\alpha^{\pm}$, $\beta^{\pm}$, or increasing $\gamma^{\pm}$ by tN. Then the function $P(z)$ describing such a model can be written in the form

$$P(z) = P_0(z) + tF(z),$$

where $F(z)$ is determined by the choice of one of the special cases mentioned above. Let

$$h(x, y, T + t) = \lim_{N \to \infty} \frac{\mathbf{E}h([xN], [yN], (T + t)N)}{N}$$

be the limiting height function.

The plot of the height function with a fixed 3rd coordinate can be viewed as a random two-dimensional surface in $\mathbb{R}^3$. As was mentioned above, the growth of the height function can be realized as a result of Markov dynamics with local interactions. The theory of hydrodynamic limits of random growth models allows one to predict the type of the modification of the limit shape when we add parameters with the use of local Markov dynamics. Namely, one can expect that the limit height function obeys an evolution equation of the form

$$\frac{\partial h(x, y, T + t)}{\partial t} = \mathcal{F}\left(\frac{\partial h(x, y, T + t)}{\partial x}, \frac{\partial h(x, y, T + t)}{\partial y}\right),$$

where $\mathcal{F}$ is a function of two variables uniquely determined by the function F (or, equivalently, by the type of parameters that we add).

Let us verify that the limit measure coming from Theorem 3.2 satisfies such an equation. We use the answer in the form given in Section 3.2, namely, let

$$\begin{aligned} S(z) &= P\left(\frac{1}{z} - 1\right) + x \log z + y \log(1 - z) \\ &= P_0\left(\frac{1}{z} - 1\right) + tF\left(\frac{1}{z} - 1\right) + x \log z + y \log(1 - z). \end{aligned}$$

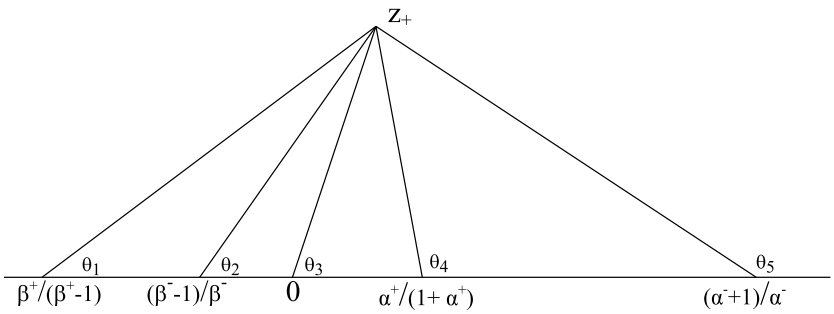

FIG. 3. *Angles which determine the growth of the limit shape.*

Then the density of the limit measure is equal to $\frac{1}{\pi}\arg(z_+(x,y,t))$, where z_+ is the root of the equation $S'(z)=0$ lying in the upper half-plane. Hence,

$$h(x,y,T+t) = -\frac{1}{\pi}\Im\big(S\big(z_+(x,y,t)\big)\big).$$

Differentiating this equality and taking into account that $S'(z_+)=0$, we obtain

$$\frac{\partial h(x,y,T+t)}{\partial x} = -\frac{1}{\pi}\Im\big(\log(z_+)\big) = -\frac{1}{\pi}\arg(z_+),$$

$$\frac{\partial h(x,y,T+t)}{\partial y} = -\frac{1}{\pi}\Im\big(\log(1-z_+)\big) = -\frac{1}{\pi}\arg(1-z_+),$$

$$\frac{\partial h(x,y,T+t)}{\partial t} = -\frac{1}{\pi}\Im\left(F\left(\frac{1}{z_+}-1\right)\right).$$

Note that the arguments of z_+ and $1-z_+$ uniquely determine the complex number z_+ with a positive imaginary part. Therefore, the function F and the derivatives of $h(x,y,T+t)$ with respect to x and y uniquely determine the derivative of $h(x,y,T+t)$ with respect to t.

When we add equal α^+ parameters we have $F(z) = -\log(1-\alpha^+ z)$. After computations, we obtain

$$\frac{\partial h(x,y,T+t)}{\partial t} = \frac{1}{\pi}\left(\arg\left(z_+ - \frac{\alpha^+}{1+\alpha^+}\right) - \arg(z_+)\right) = \frac{\theta_4-\theta_3}{\pi},$$

where the angles θ_i are shown in Figure 3.

Analogous computations for five other cases (equal β^+'s, α^-'s, β^-'s and the growth of γ^+ or γ^-) show that (similar computations were performed in [10]), respectively,

$$\frac{\partial h(x,y,T+t)}{\partial t} = \frac{1}{\pi}\left(-\arg\left(z_+ + \frac{\beta^+}{1-\beta^+}\right) - \arg(z_+)\right) = \frac{-\theta_1-\theta_3}{\pi},$$

$$\frac{\partial h(x,y,T+t)}{\partial t} = \frac{1}{\pi}\left(\arg\left(z_+ - \frac{1+\alpha^-}{\alpha^-}\right) - \pi\right) = \frac{\theta_5-\pi}{\pi},$$

$$\frac{\partial h(x,y,T+t)}{\partial t} = -\frac{1}{\pi}\left(\arg\left(z_+ + \frac{1-\beta^-}{\beta^-}\right)\right) = \frac{\theta_2}{\pi},$$
$$\frac{\partial h(x,y,T+t)}{\partial t} = -\frac{\gamma^+}{\pi}\Im\left(\frac{1}{z_+}\right),$$
$$\frac{\partial h(x,y,T+t)}{\partial t} = -\frac{\gamma^-}{\pi}\Im(z_+).$$

3.4. *A convolution of measures.* Let $\mathcal{M}$ be the space of probability measures that can be obtained as limit measures σ from Theorem 3.2. For $\nu \in \mathcal{M}$, let P'_ν be the function defined in (3.1), and let S_ν be the generating function of moments defined in (3.7). These two functions uniquely determine each other by

$$S_\nu(z) = \log\left(1 + \left(\frac{z}{1+z(1+z)P'_\nu(z)}\right)^{(-1)}\right) \tag{3.12}$$

and

$$P'_\nu(z) = \frac{1}{(1+z)(\exp(S_\nu(z))-1)^{(-1)}} - \frac{1}{z(1+z)}. \tag{3.13}$$

Let χ^1 and χ^2 be two extreme characters of $U(\infty)$. Consider the product of these characters, which is also an extreme character of $U(\infty)$:

$$\chi^{1,2}(U) := \chi^1(U)\chi^2(U), \qquad U \in U(\infty).$$

It is natural to think that this operation corresponds to a tensor product of representations of $U(\infty)$ determined by the characters χ^1 and χ^2 (although these are infinite-dimensional objects and one needs to explain what that means).

Assume that χ^1_N and χ^2_N are sequences of extreme characters of $U(\infty)$ satisfying condition (3.1). By Theorem 3.2, there are limit measures σ_1 and σ_2 corresponding to these sequences. Then the sequence $\chi^1_N\chi^2_N$ also satisfies (3.1); let $\sigma_{1,2}$ be the limit measure for this sequence. Note that

$$P'_{\sigma_{1,2}}(z) = P'_{\sigma_1}(z) + P'_{\sigma_2}(z). \tag{3.14}$$

Thus, these formulas allow to define a natural operation of "quantized free convolution" for measures $\sigma_1, \sigma_2 \in \mathcal{M}$; the result of convolution is $\sigma_{1,2} \in \mathcal{M}$. The measure $\sigma_{1,2}$ is completely determined by equations (3.13), (3.14) and (3.12).

Special cases considered in the Appendix can serve as examples of this convolution. In particular, the limit measures for one-sided Plancherel characters with parameter γN or the characters corresponding to aN parameters $\alpha^+_j \equiv 1$, form one-parameter subgroups with respect to this convolution.

This operation of convolution can be defined by the same formulas for a more general class of measures; the setting for such a generalization is as follows. Let T_{λ_1} and T_{λ_2}, $\lambda_1, \lambda_2 \in \mathbb{GT}_N$, be two irreducible representations of $U(N)$. Let us

consider the Kronecker tensor product $T_{\lambda_1} \otimes T_{\lambda_2}$ and decompose it onto irreducible representations. As $N \to \infty$, under appropriate scaling regime one can prove a law of large numbers type theorem for this decomposition; see [9].

For the first time a similar problem was considered by Biane [2]; the resulting operation on measures was the free convolution. However, we consider a different scaling, and in our situation the resulting operation is not the free convolution (see [9], Section 1, for more details). In fact, for a certain degeneration turning the branching of signatures in the Gelfand–Tsetlin graph into the branching of eigenvalues (describing the eigenvalues of *corners* of Hermitian matrices), which corresponds to the degeneration of the Gelfand–Tsetlin graph to the "graph" of spectra of Hermitian matrices, our convolution turns into the free convolution. Let us show how this happens.

Let $R_\nu(x)$ be Voiculescu's R-function of the measure ν (see, e.g., [18]). Then it is easy to see that

$$P'_\nu(z) = \frac{1}{1+z} R_\nu\bigl(\log(1+z)\bigr) + \frac{1}{(1+z)\log(1+z)} - \frac{1}{z(1+z)}.$$

For the degeneration to the "graph" of spectra, we need to consider measures with homothetically growing supports and for values of variables that are close to 1. Let L be a large parameter, and let us change the variable $z = y/L$. The new R-function satisfies

$$R_\nu\bigl(\log(1+z)\bigr) = L R_{\tilde\nu}\left(L \log\left(1 + \frac{y}{L}\right)\right),$$

where $\tilde\nu$ is the measure arising after the degeneration.

Thus, we have

$$\frac{P'_\nu(z)}{L} = R_{\tilde\nu}(y) + O\bigl(L^{-1}\bigr) \underset{L\to\infty}{\to} R_{\tilde\nu}(y).$$

Therefore, in this limit the linearizing function P'_ν becomes the R-function of a measure, and the tensor product of representations gives rise to the free convolution.

4. Proof of Theorem 3.2. In this section, we prove our main result, Theorem 3.2. Because the proof is rather long, let us describe first its main ideas.

To establish the existence of a limit shape, we use the method of moments. Recall that we interpret signatures $\lambda \in \mathbb{GT}_N$ as certain measures on $\mathbb{Z}$, so random signatures become random measures. The moments of the random measures, as well as products of moments, are thus random functions. We have to examine the limit of their expectations as $N \to \infty$.

Our key technical tool is the algebra Sym^* of shifted symmetric functions. As explained in Section 2, elements of Sym^* can be converted, via the maps pr_N, into functions on signatures. We are dealing with two bases in Sym^*, $\{\mathbf{p}_\rho\}$ and

$\{\mathbf{p}^{\#}_{\rho}\}$. The products of moments that we need to control are given by the elements of the first basis, whereas the expectations are initially expressed in terms of the second basis. This is the source of the problem, because the transition coefficients between both basis have a very complicated structure and hardly can be written down explicitly.

Fortunately, we do not need to know the transition coefficients exactly, because for our purpose it suffices to compute their large-N asymptotics, so that we may drop many asymptotically negligible terms. This allows us to solve the problem by reducing it to combinatorial analysis of certain special graphs (Sections 4.2 and 4.3). In the process we recover the noncrossing partitions which make the connection with free probability (see Section 3.4) less surprising.

Note that a similar difficulty of transition between two bases in Sym^* arose in Kerov's proof of his central limit theorem for the Plancherel measure (see Ivanov–Olshanski [12]). However, in our case the limit regime is different, the emerging technical problems are more serious, and the required combinatorial machinery is substantially more sophisticated.

4.1. *Plan of the proof.* Let us modify the measure $\mu^{(N)}$ by adding atoms of weight $-\frac{1}{N}$ at locations $-\frac{i}{N}$, $i = 1, 2, \dots, N$. Let $\tilde{\mu}^{(N)}$ denote the resulting signed measure. Note that its total weight equals 0. As $N \to \infty$, the negative part of $\tilde{\mu}^{(N)}$ converges to the measure with density -1 on the interval $[-1; 0]$.

Recall that the functions from $\mathbb{A}(N)$ (see Sections 2.3 and 2.4) are defined on $\mathbb{GT}_N$. By (2.4) the functions p_k are the moments of the measure $\tilde{\mu}^{(N)}$. We know the limit of the negative part of $\tilde{\mu}^{(N)}$; therefore, the information about the limit of $\{p_k\}_{k\geq 1}$ is sufficient for describing the limit measure σ (see Section 3.1).

Let $\tilde{\sigma}$ be the sum of the measure σ and the negative Lebesgue measure on $[-1; 0]$. Let $\tilde{m}_k$ be the moments of $\tilde{\sigma}$. Define a formal power series $\tilde{S}(z)$ by

$$\tilde{S}(z) = \tilde{m}_1 z^2 + \tilde{m}_2 z^3 + \cdots.$$

It is easy to see that

$$S(z) = \tilde{S}(z) + \log(1 + z). \tag{4.1}$$

We recall that by $[u^k]A(u)$, where $A(u)$ is a formal power series of the form $a_1 u + a_2 u^2 + \cdots$, we denote the coefficient of u^k in $A(u)$.

Recall that the functions $p^{\#}_k$ are defined on $\mathbb{GT}_N$; see Section 2.4.

Let $\mathbf{E}_N$ denote the expectation with respect to M_N.

PROPOSITION 4.1. *For any $k \geq 1$, we have*

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(p^{\#}_k)}{N^{k+1}} = \frac{1}{k+1}\big[u^k\big](1 + t_1 u + t_2 u^2 + \cdots)^{k+1} =: c_k.$$

Note that the weight of $p^{\#}_k$ equals $k + 1$.

The proof of this proposition is given in Section 4.2.

PROPOSITION 4.2. *For any partition* $\rho = (k_1, k_2, \dots, k_{l(\rho)})$, *we have*

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(p_\rho^{\#})}{N^{k_1+k_2+\cdots+k_{l(\rho)}+l(\rho)}} = c_{k_1} c_{k_2} \cdots c_{k_{l(\rho)}}.$$

The proof of this proposition is given in Section 4.3.

Let us recall that $\{p_\rho^{\#}\}_{\rho\in\mathbb{Y}}$ is a linear basis in Sym^*; therefore, these two propositions give us complete information about expectations of functions from $\mathbb{A}(N)$. In particular, these propositions imply

$$\lim_{N\to\infty} \mathbf{E}_N(f) = O\bigl(N^{\mathrm{wt}(f)}\bigr),$$

where $\mathrm{wt}(f)$ is the weight filtration.

PROPOSITION 4.3. *For any* $k \geq 1$, *we have*

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(p_k)}{N^{k+1}} = \tilde{m}_k, \qquad \lim_{N\to\infty} \frac{\mathbf{E}_N(p_k^2)}{N^{2(k+1)}} = \tilde{m}_k^2.$$

The proof of this proposition is given in Section 4.4.

LEMMA 4.4. *There exists a constant* $C_1 > 0$ *such that* $m_k < C_1^k$ *for all* $k \geq 1$.

PROOF. The general condition (3.1) implies that there exists a constant $C > 0$ such that

$$t_k < C^k \qquad \text{for any } k \geq 1.$$

It is easy to see that if the coefficients of a formal power series are majorated by a geometric progression, then the coefficients of the inverse power series are also majorated by some geometric progression. Therefore, the coefficients of the series

$$v_0(z) = \left(\frac{z}{1 + z(1+z)(t_1 + t_2 z + \cdots)} \right)^{(-1)}$$

are majorated by a geometric progression. By definition, m_k is the coefficient of z^{k+1} in $\log(1 + v_0(z))$; this implies the statement of the lemma. □

PROOF OF THEOREM 3.2. Let $\lambda_i^{(N)}$, $i = 1, 2, \dots, N$, be the coordinates of a random signature distributed according to M_N. Note that Proposition 4.3 implies

$$\lim_{N\to\infty} \mathbf{E}_N \left[\frac{1}{N} \sum_{i=1}^{N} \bigl(\lambda_i^{(N)} - i + 1/2\bigr)^k \right] = m_k, \tag{4.2}$$

$$\lim_{N\to\infty} \mathbf{E}_N \left[\frac{1}{N} \left(\sum_{i=1}^{N} \bigl(\lambda_i^{(N)} - i + 1/2\bigr)^k \right) \right]^2 = m_k^2. \tag{4.3}$$

It remains to apply Lemma 2.3 (note that the existence of the limit measure with moments $\{m_k\}$ follows from this lemma); the conditions of the lemma hold due to (4.2) and Lemma 4.4. □

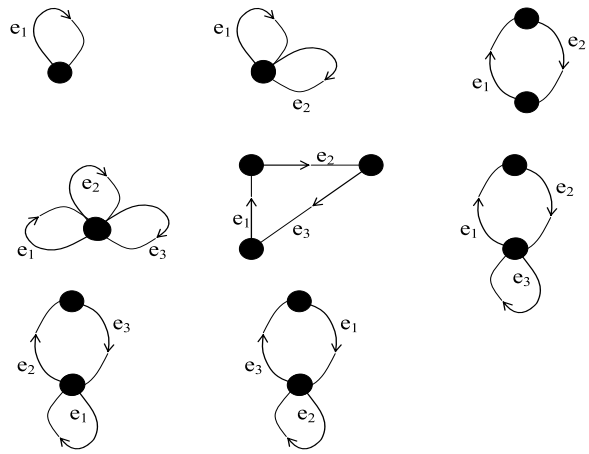

FIG. 4. *All Eulerian graphs with* $k = 1, 2, 3$.

4.2. *Proof of Proposition* 4.1. Equations (2.5) and (3.1) imply the following formula for $\mathbf{E}_N(p_k^{\#})$:

$$\mathbf{E}_N(p_k^{\#}) = \sum_{1\le i_1,i_2,\dots,i_k\le N} \partial_{i_1 i_2}\,\partial_{i_2 i_3}\cdots\partial_{i_k i_1} \times \exp\Bigg(N\Bigg(\mathbf{t}_1(N)\,\mathrm{Tr}(X) + \frac{\mathbf{t}_2(N)}{2}\,\mathrm{Tr}(X^2) + \cdots + \frac{\mathbf{t}_r(N)}{r}\,\mathrm{Tr}(X^r) + \cdots\Bigg)\Bigg)\Bigg|_{X=0}, \tag{4.4}$$

where the coefficients $\mathbf{t}_i(N)$ satisfy $\lim_{N\to\infty}\mathbf{t}_i(N) = t_i$ [the coefficients t_i are given by (3.2)].

To deal with this formula, we need to introduce a bit of combinatorial formalism. Below we use the term *graph* to denote a finite connected oriented graph, possibly with loops and multiple edges. A *cycle* in such a graph is a closed oriented path without repeated edges. A cycle is *simple* if it does not contain repeated vertices. A cycle is said to be *Eulerian* if it contains all the edges of the graph. By an *Eulerian graph*, we mean a graph together with a distinguished enumeration of the edges such that it forms an Eulerian cycle (note that this slightly differs from the conventional terminology).

Let $\mathcal{G}_k$ denote the set of (equivalence classes of) Eulerian graphs with k edges. For $G \in \mathcal{G}_k$ we denote by $\mathbf{e} = (e_1, \dots, e_k)$ the distinguished Eulerian cycle of G.

All Eulerian graphs with $k = 1, 2, 3$ are shown in Figure 4.

REMARK 4.5. There exists a one-to-one correspondence $G \leftrightarrow \pi$ between the graphs $G \in \mathcal{G}_k$ and the set partitions of $[k] := \{1, \dots, k\}$. Indeed, let us consider

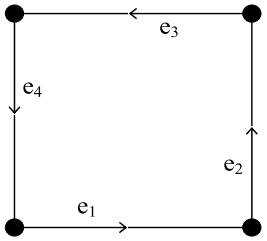

FIG. 5. *A simple Eulerian graph with* 4 *vertices and* 4 *edges.*

first the finest partition,

$$\pi_0 := \{1\} \cup \{2\} \cup \cdots \cup \{k\}.$$

By definition, the corresponding graph $G_0 \leftrightarrow \pi_0$ is the (unique) Eulerian graph with k edges and k vertices (see an example in Figure 5). Let us enumerate the vertices of G_0 in such a way that

$$e_1 = (1 \to 2), \qquad \ldots, \qquad e_{k-1} = (k-1 \to k), \qquad e_k = (k \to 1).$$

Then, given an arbitrary set partition π of $[k]$, we glue together the vertices of G_0 corresponding to every block of π; the result is the graph $G \leftrightarrow \pi$.

Equivalently, the vertices of G are identified with the blocks of π, and the ith edge e_i is directed from the block containing i to that containing $i+1$ (with the understanding that $k+1$ is identified with 1).

By $v(G)$, we will denote the number of vertices of G; this is the same as the number of blocks in the corresponding set partition π.

By a *cycle structure* on G, we mean a partition $C = (C_1, \ldots, C_p)$ of the edge set $\{e_1, \ldots, e_k\}$ such that each block C_j is a cycle; we also assume that the blocks are enumerated in the ascending order of their minimal elements. Below we write the number of blocks by $p(C)$ and denote by $|C_j|$ the size of the jth block. The set of all cycle structures on G is denoted by $\mathcal{C}(G)$.

Note that cycle structures exist for every Eulerian graph G. For instance, the Eulerian cycle $\mathbf{e}$ is itself a cycle structure with a single block. Another example is obtained when one cuts $\mathbf{e}$ into simple cycles, which is always possible, but sometimes can be made in different ways.

Examples of cycle structures are shown in Figure 6.

To shorten the notation, let us abbreviate

$$\mathbf{t}_1 := \mathbf{t}_1(N), \qquad \mathbf{t}_2 := \mathbf{t}_2(N), \ldots.$$

LEMMA 4.6. *The right-hand side of* (4.4) *can be written in the form*

$$\sum_{G \in \mathcal{G}_k} N(N-1)\cdots\big(N-v(G)+1\big) \sum_{C \in \mathcal{C}(G)} \alpha(C) N^{p(C)} \prod_{j=1}^{p(C)} \mathbf{t}_{|C_j|}, \tag{4.5}$$

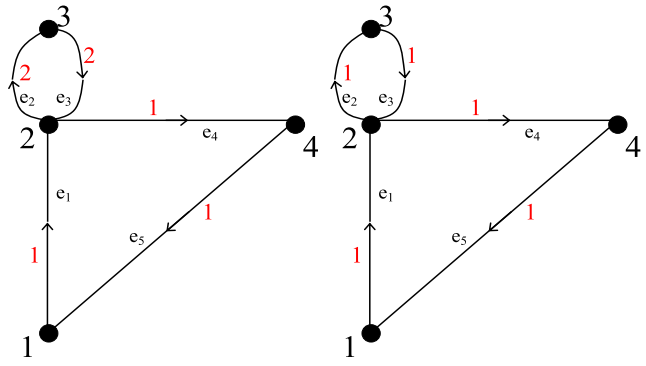

FIG. 6. *All cycle structures on the Eulerian graph* $1 \to 2 \to 3 \to 2 \to 4 \to 1$.

$\alpha(C)$ is a coefficient depending on C only. If all the blocks of C are simple cycles, then $\alpha(C) = 1$.

PROOF. *Step* 1. Let us fix a sequence $\mathbf{i} = (i_1, i_2, \dots, i_k) \in [N]^k$. The corresponding term in (4.4) can be written as

$$\partial_{i_1 i_2} \cdots \partial_{i_k i_1} \exp\Bigg(N \Bigg(\mathbf{t}_1 \sum_{1 \le j_1 \le N} x_{j_1 j_1} + \frac{\mathbf{t}_2}{2} \sum_{1 \le j_1, j_2 \le N} x_{j_1 j_2} x_{j_2 j_1} + \cdots + \frac{\mathbf{t}_r}{r} \sum_{1 \le j_1, j_2, \dots, j_r \le N} x_{j_1 j_2} x_{j_2 j_3} \cdots x_{j_r j_1} + \cdots \Bigg) \Bigg) \Bigg|_{x=0}, \tag{4.6}$$

where "$x = 0$" means that finally all the x-variables are set to be equal to 0.

The order of partial derivatives is not important; let us assume that one applies $\partial_{i_1 i_2}$ first, then $\partial_{i_2 i_3}$, etc. Since the sum inside the exponential converges uniformly, we can differentiate this expression term by term. Namely, each differentiation operator ∂ can be applied to one of the terms inside the exponential (as a result, a pre-exponential polynomial appears), or it can be applied to a pre-exponential factor which was brought down by previous differentiations. However, due to the final substitution $x = 0$, a nonzero contribution can only come from those terms for which the pre-exponential factors do not contain the x-variables.

Step 2. We will encode such terms by means of cycle structures.

First, we assign to $\mathbf{i}$ an Eulerian graph $G = G_{\mathbf{i}}$ with k edges—the vertex set of G is the subset of $[N]$ consisting of the numbers entering the sequence $\mathbf{i}$, and the edges are

$$e_1 = (i_1 \to i_2), \qquad e_2 = (i_2 \to i_3), \qquad \dots, \qquad e_{k-1} = (i_{k-1} \to i_k),$$
$$e_k = (i_k \to i_1).$$

In other words, we associate the edges with the ∂-operators in (4.6).

Next, given a term whose preexponential factor does not contain the x-variables, we assign to it a partition $C = (C_1, \dots, C_p)$ of the edge set $\{e_1, \dots, e_k\}$ in the following way. The first block C_1 starts with the edge $e_1 \leftrightarrow \partial_{i_1 i_2}$, and the remaining edges correspond to the ∂-operators killing the x-variables from the preexponential factor that arises after application of $\partial_{i_1 i_2}$ to the exponential. The second block starts with the edge labeling the next ∂-operator that is being applied to the exponential, etc.

We claim that C is a cycle structure, that is, all blocks are cycles. Indeed, a preexponential factor that may result from the application of $\partial_{i_1 i_2}$ to the exponential always has the form

$$\frac{N}{r}\mathbf{t}_r x_{j_1 j_2} \cdots \widehat{x_{j_m j_{m+1}}} \cdots x_{j_r j_1}, \qquad j_m = i_1,\ j_{m+1} = i_2 \tag{4.7}$$

(with the understanding that $m + 1 = 1$ if $m = r$; the hat over $x_{i_m i_{m+1}}$ means that this variable has to be omitted). Our assumption is that $r = |C_1|$ and the ∂-operators corresponding to the edges from C_1 different from e_1 kill all the x-variables from the above monomial. But this just means that the edges of C_1 form a cycle.

For the blocks C_2, C_3, etc. the argument is the same.

Step 3. The reasoning of step 2 shows that the quantity (4.6) can be represented as the sum of contributions coming from various cycle structures $C \in \mathcal{C}(G)$. Let us fix $C = (C_1, \dots, C_p)$ and analyze its contribution in more detail. Assume first that all the cycles are simple. Let us focus on the first cycle and keep the notation of step 2. The fact that C_1 is simple just means that the indices $j_1, \dots, j_r$ must be pairwise distinct. Therefore, given a simple cycle C_1, there are exactly $r = |C_1|$ eligible r-tuples $(j_1, \dots, j_r)$ that correspond to values $m = 1, \dots, r$. Then the summation over these r variants results in the cancellation of the factor r in the denominator of (4.7). The same argument applies to all the cycles, and we finally obtain that the whole contribution of C is equal to

$$N^p \mathbf{t}_{|C_1|} \cdots \mathbf{t}_{|C_p|},$$

as desired.

In the general case, when the cycles are not necessarily simple, we argue as above, and the only difference is that the contribution of C may involve a constant numeric factor $\alpha(C)$. For instance, if the graph G has a single vertex and k loops, then there is a single one-component cycle structure whose contribution equals $(k-1)!N\mathbf{t}_k$, so that in this case $\alpha(C) = (k-1)!$.

Step 4. We have explained the origin of the interior sum in (4.5). It remains to explain the exterior sum, and this is easy. Namely, we observe that the whole contribution of a given k-tuple $\mathbf{i} \in [N]^k$ depends solely on the equivalence class of the corresponding Eulerian graph $G_{\mathbf{i}}$. Indeed, two k-tuples producing equivalent graphs can be transformed to each over by a permutation of $[N]$, which does not

affect the quantity (4.6). Finally, given $G \in \mathcal{G}_k$, the number of k-tuples $\mathbf{i} \in [N]^k$ such that $G_{\mathbf{i}}$ is equivalent to G is equal to

$$N(N-1)\cdots\big(N-v(G)+1\big)$$

(to see this one may use Remark 4.5). This completes the proof. $\square$

Let us rewrite (4.5) as

$$\sum_{(G,C)} \alpha(C) N(N-1)\cdots\big(N-v(G)+1\big) N^{p(C)} \prod_{j=1}^{p(C)} \mathbf{t}_{|C_j|},$$

where the summation is taken over all pairs (G, C) such that $G \in \mathcal{G}_k$ and $C \in \mathcal{C}(G)$. For N large, the contribution of a fixed pair (G, C) grows as $N^{v(G)+p(C)}$, so that the leading part in the asymptotics comes from the pairs with the maximal possible value of the quantity $v(G) + p(C)$. Our goal now is to describe such pairs.

Assume A is an ordered set and $A_1 \subset A$ and $A_2 \subset A$ are two nonempty disjoint subsets. Then A_1 and A_2 are said to be *crossing* if there exists a quadruple $a < b < c < d$ of elements such that a and c are in one of these subsets while b and d are in another subset; otherwise A_1 and A_2 are said to be *noncrossing*. Next, a *noncrossing partition* of A is a set partition of A whose blocks are pairwise noncrossing.

By the very definition, every cycle structure $C = (C_1, \dots, C_p)$ on a graph $G \in \mathcal{G}_k$ is a partition of the set $\{e_1, \dots, e_k\}$. We introduce the natural order $e_1 < \cdots < e_k$ on the Eulerian cycle, so that $\{e_1, \dots, e_k\}$ becomes an ordered set isomorphic to $[k]$.

LEMMA 4.7. *Let us fix $k = 1, 2, \dots$ and let (G, C) range over the set of pairs such that $G \in \mathcal{G}_k$ and $C \in \mathcal{C}(G)$.*

Then the maximal possible value of the quantity $v(G) + p(C)$ is equal to $k + 1$. It is attained exactly for those pairs (G, C) for which all the cycles of C are simple and the set partition $\sigma(C)$ is noncrossing.

Moreover, under the identification $\{e_1, \dots, e_k\} \leftrightarrow [k]$ of ordered sets, for every noncrossing partition σ of the set $[k]$, there exists exactly one pair (G, C) such that $v(G) + p(C) = k + 1$ and $C \leftrightarrow \sigma$.

PROOF. *Step* 1. Let us fix a pair (G, C) with $C = (C_1, \dots, C_p)$, and estimate $v(G) + p(C)$.

Let us observe that C_2 always has a common vertex with C_1, C_3 has a common vertex with $C_1 \cup C_2$, and so on. Indeed, this follows from the very definition of a cycle structure (in particular, we use the fact the cycles in C are enumerated in the ascending order of their minimal elements).

Let $v(\cdot)$ stand for the number of vertices in a given cycle or a union of cycles. We have

$$v(C_1) \le |C_1|, \qquad \dots, \qquad v(C_p) \le |C_p|$$

and, by virtue of the above observation,

$$\begin{aligned} v(C_1 \cup \cdots \cup C_m) &\le v(C_1 \cup \cdots \cup C_{m-1}) + v(C_m) - 1 \\ &\le v(C_1 \cup \cdots \cup C_{m-1}) + |C_m| - 1 \end{aligned}$$

for $m = 2, \ldots, p$. Since $|C_1| + \cdots + |C_p| = k$, it follows that

$$v(G) = v(C_1 \cup \cdots \cup C_p) \le k - (p-1),$$

so that $v(G) + p \le k + 1$.

Moreover, the equality $v(G) + p = k + 1$ is attained if and only if the following two conditions are satisfied:

(1) $v(C_m) = |C_m|$ for every $m = 1, \ldots, p$, which is equivalent to saying that all cycles are simple.

(2) For every $m = 2, \ldots, p$, the cycle C_m has a single common vertex with the union $C_1 \cup \cdots \cup C_{m-1}$.

Step 2. Let us assume that (G, C) is such that C satisfies condition (1) above; we are going to show that C satisfies condition (2) if and only if C is noncrossing.

The key observation is that if (G, C) is such that C satisfies both (1) and (2), then removing the last cycle C_p we still get a pair (G', C') with the same properties. Likewise, if σ is a noncrossing set partition, then removing its last block we still get a noncrossing partition σ' (we always assume that the blocks are ordered according to the order of their minimal elements).

This suggests the idea to prove the desired claim by induction on p, the number of blocks. The base of induction is obvious: if $p = 1$, then there is nothing to prove. To justify the induction step, we observe that the possible transitions $(G', C') \to (G, C)$ preserving property (2) are directed by exactly the same mechanism as the possible transitions $\sigma' \to \sigma$ preserving the noncrossing property.

Indeed, in the first case, we may insert a simple cycle of length $|C_p|$ at any place of the Eulerian cycle of G' which is after the minimal edge of C_{p-1} (which is the last cycle of C'). Likewise, in the second case, we may insert a block of the same size after the minimal element of the last block of σ'. (Let us emphasize that in both cases, we have to insert a new cycle/block as a whole.)

Step 3. The argument of step 3 shows that both the pairs (G, C) satisfying conditions (1) and (2), and the noncrossing set partitions σ can be obtained by one and the same recursive procedure. This completes the proof. □

REMARK 4.8. The recursive procedure described above assigns a pair (G, C) to every noncrossing partition σ of the set $[k]$. On the other hand, according to Remark 4.5, the graph G is completely determined by a set partition π of $[k]$. One can show that the correspondence $\sigma \mapsto \pi$ that arises in this way is just the *complementation* operation first discovered by Kreweras [15]: it is a nontrivial involution on the set of noncrossing partitions of $[k]$ (see an example in Figure 7).

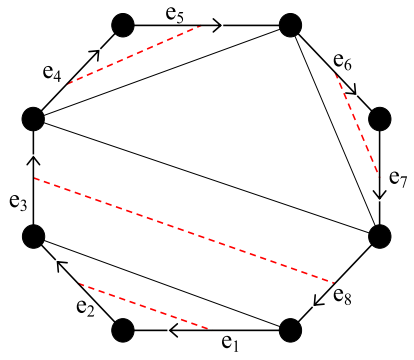

FIG. 7. *A noncrossing partition of edges* (*dashed lines*) *gives rise to a noncrossing partition of vertices* (*solid lines*).

Denote by NC_k the set of noncrossing partitions of $[k]$. Define the *weight* of a partition $\sigma = (\sigma_1 \cup \cdots \cup \sigma_p) \in \mathrm{NC}_k$ as the monomial

$$\mathrm{wt}(\sigma) := t_{|\sigma_1|} \cdots t_{|\sigma_p|}.$$

Lemmas 4.6 and 4.7 show that the leading term of the large-N asymptotics can be written as

$$N^{k+1} \sum_{\sigma \in \mathrm{NC}_k} \mathrm{wt}(\sigma). \tag{4.8}$$

LEMMA 4.9. *For any $k \geq 1$, we have*

$$\sum_{\sigma \in \mathrm{NC}_k} \mathrm{wt}(\sigma) = \frac{1}{k+1} [u^k] \{(1 + t_1 u + t_2 u^2 + \cdots)^{k+1}\}.$$

PROOF. Given $\sigma \in \mathrm{NC}_k$, let $(1^{s_1} 2^{s_2} \cdots)$ denote the corresponding ordinary partition of the *number* k, written in the multiplicative notation; this means that σ has exactly s_i blocks of size i, where $i = 1, 2, \ldots$. We say that $(1^{s_1} 2^{s_2} \cdots k^{s_k})$ is the *type* of σ. Obviously,

$$\mathrm{wt}(\sigma) = t_1^{s_1} t_2^{s_2} \cdots t_k^{s_k}.$$

Therefore, we have to prove that

$$\sum_{\sigma \in \mathrm{NC}_k} t_1^{s_1} t_2^{s_2} \cdots t_k^{s_k} = \frac{1}{k+1} [u^k] \{(1 + t_1 u + t_2 u^2 + \cdots)^{k+1}\}.$$

Now we apply Exercise 5.35a in Stanley [25], which says that the number of partitions $\sigma \in \mathrm{NC}_k$ of a given type $(1^{s_1} 2^{s_2} \cdots k^{s_k})$ is equal to

$$\frac{k(k-1)\cdots(k-\ell+2)}{s_1! s_2! \cdots s_k!} = \frac{1}{k+1} \frac{(k+1)k(k-1)\cdots(k-\ell+2)}{s_1! s_2! \cdots},$$

$$\ell := s_1 + s_2 + \cdots + s_k.$$

This is equivalent to the desired formula. $\square$

4.3. *Proof of Proposition* 4.2. Let us abbreviate $l := l(\rho)$. Relations (2.5) and (3.1) imply

$$\begin{aligned}\lim_{N\to\infty}\frac{\mathbf{E}_N(p_\rho^{\#})}{N^{\mathrm{wt}(\rho)}} &= \lim_{N\to\infty}\frac{1}{N^{k_1+k_2+\cdots+k_l+l}}\\ &\quad\times\sum_{\mathbf{i}\in[N]^k}\partial_{i_1i_2}\,\partial_{i_2i_3}\cdots\partial_{i_{k_1}i_1}\,\partial_{i_{k_1+1}i_{k_1+2}}\cdots\\ &\quad\times\exp\Biggl(\mathbf{t}_1(N)\Bigl(\sum_j x_j\Bigr)+\frac{\mathbf{t}_2(N)}{2}\Bigl(\sum_{j_1,j_2}x_{j_1j_2}x_{j_2j_1}\Bigr)+\cdots\\ &\qquad+\frac{\mathbf{t}_r(N)}{r}\Bigl(\sum_{j_1,j_2,\ldots,j_r}x_{j_1j_2}x_{j_2j_3}\cdots x_{j_rj_1}\Bigr)+\cdots\Biggr)\Bigr|_{x_{ij}\equiv0},\end{aligned}\tag{4.9}$$

where the coefficients $\mathbf{t}_i(N)$ satisfy $\lim_{N\to\infty}\mathbf{t}_i(N)=t_i$ [the numbers t_i were defined in (3.2)].

We shall deal with this formula in the same way as in Section 4.2. To every sequence $\mathbf{i}=(i_1,\ldots,i_k)\in[N]^k$ we assign an oriented graph $G_{\mathbf{i}}$ whose edges correspond to the ∂-operators from (4.9). This graph is composed from l Eulerian graphs, which may be glued together or disjoint, depending on whether the subsequences

$$(i_1,\ldots,i_{k_1}),\qquad (i_{k_1+1},\ldots,i_{k_1+k_2}),\qquad\ldots,\qquad (i_{k_1+\cdots+k_{l-1}+1},\ldots,i_k)\tag{4.10}$$

have common indices or not.

First, let us consider the case when there are no common indices, so that the corresponding Eulerian graphs are pairwise disjoint. Then the differential operators from different graphs are applied to nonintersecting sets of x-variables, and the arguments of Section 4.2 show that the total contribution from such $\mathbf{i}$'s equals

$$c_{k_1}c_{k_2}\cdots c_{k_l}N^{(k_1+1)+(k_2+1)+\cdots+(k_l+1)}+O\bigl(N^{k+l-1}\bigr).$$

It remains to show that the contribution from the remaining sequences $\mathbf{i}$ [those for which the subsequences in (4.10) have common indices] has lower degree in N.

To simplify the argument, let us assume that $l=2$, so that $k=k_1+k_2=r+(k-r)$. Thus, there are two subsequences in (4.10), which we denote as

$$(i_1,\ldots,i_r),\qquad (i_{r+1},\ldots,i_k),$$

and these two subsequences share a common index, say $i_a=i_{r+b}$ for some $a\in\{1,\ldots,r\}$ and $b\in\{1,\ldots,k-r\}$.

Then it is readily seen that the term corresponding to the differential operator

$$\partial_{i_1i_2}\,\partial_{i_2i_3}\cdots\partial_{i_ri_1}\,\partial_{i_{r+1}i_{r+2}}\,\partial_{i_{r+2}i_{r+3}}\cdots\partial_{i_ki_{r+1}}$$

is equal to the contribution of a *single* Eulerian graph with k edges, corresponding to the sequence

$$i_1, \dots, i_a, i_{r+b+1}, i_{r+b+2}, \dots, i_k, i_{r+1}, i_{r+2}, \dots, i_{r+b}, i_{a+1}, i_{a+2}, \dots, i_r, i_1.$$

Therefore, this contribution has order at most N^{k+1}, which is less than $N^{k+l} = N^{k+2}$.

The same argument holds when $l > 2$ as well.

4.4. *Proof of Proposition* 4.3. Recall that we consider the functions p_k as random variables on the probability space $(\mathbb{GT}_N, M_N)$. First, let us prove that after scaling the functions p_k converge to constants in L^2.

LEMMA 4.10. *There exist constants $\bar{m}_k$, $k = 1, 2, \dots$, such that for any $k \geq 1$*

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(p_k)}{N^{k+1}} = \bar{m}_k, \qquad \lim_{N\to\infty} \frac{\mathbf{E}_N(p_k^2)}{N^{2(k+1)}} = \bar{m}_k^2.$$

PROOF. Let $f \in \mathrm{Sym}^*$ be arbitrary. Since f is a linear combination of $p_\rho^\#$'s with $\mathrm{wt}(\rho) \leq \mathrm{wt}(f)$, and there exist limits of $\mathbf{E}_N(p_\rho^\#)/N^{k+1}$, we obtain

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(f)}{N^{k+1}} = a_f,$$

for some constants a_f.

It is known (see [12]) that

$$p_{\rho_1}^\# p_{\rho_2}^\# = p_{\rho_1\cup\rho_2}^\# + \text{lower weight terms},$$

where $\rho_1 \cup \rho_2$ stands for the union of the partitions ρ_1 and ρ_2, and "lower weight terms" denotes terms with weight $\leq \mathrm{wt}(p_{\rho_1}^\#) + \mathrm{wt}(p_{\rho_2}^\#) - 1$. Hence,

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(p_{\rho_1}^\# p_{\rho_2}^\#)}{N^{\mathrm{wt}(p_{\rho_1}^\#)+\mathrm{wt}(p_{\rho_2}^\#)}} = \lim_{N\to\infty} \frac{\mathbf{E}_N(p_{\rho_1\cup\rho_2}^\#)}{N^{\mathrm{wt}(p_{\rho_1}^\#)+\mathrm{wt}(p_{\rho_2}^\#)}}.$$

This equality and Proposition 4.2 imply that

$$\lim_{N\to\infty} \frac{\mathbf{E}_N(f^2)}{N^{2\,\mathrm{wt}(f)}} = a_f^2.$$

Therefore, the functions f converge to a_f in L^2.

Choosing the function p_k as f we obtain the statement of the lemma. By $\bar{m}_k$ we denote the limit constant. □

It remains to prove that $\bar{m}_k = \tilde{m}_k$ for all $k = 1, 2, \dots$ (recall that the constants $\tilde{m}_k$ were defined in Section 3.1).

Consider formal power series of the form

$$a(z) = a_1 z + a_2 z^2 + \cdots, \qquad a_i \in \mathbb{R}.$$

Recall that a series of this form is invertible if and only if $a_1 \neq 0$. Let $a^{(-1)}(z)$ denote the inverse of $a(z)$, that is, $a^{(-1)}(a(z)) = z$. Set

$$\bar{A}(z) = \bar{m}_1 z^2 + \bar{m}_2 z^3 + \bar{m}_3 z^4 + \cdots$$

and

$$C(z) = 1 + c_1 z^2 + c_2 z^3 + c_3 z^4 + \cdots.$$

LEMMA 4.11. *The formal power series $z\exp(\bar{A}(z))$ and $z/C(z)$ are inverse to each other.*

PROOF. Recall that [see (2.3)]

$$\mathbf{p}_k = \frac{1}{k+1}[u^{k+1}]\{(1 + \mathbf{p}_1^{\#} u^2 + \mathbf{p}_2^{\#} u^3 + \cdots)^{k+1}\} + \text{lower weight terms},$$

where "lower weight terms" denotes terms with weight $\leq k$. Since

$$\mathbf{E}_N(f) = O\bigl(N^{\mathrm{wt}(f)}\bigr), \qquad f \in \mathbb{A}(N),$$

the "lower weight terms" do not affect the asymptotics of $\mathbf{E}_N(p_k)$, and we have

$$\bar{m}_k = \frac{1}{k+1}[u^{k+1}]\{(1 + c_1 u^2 + c_2 u^3 + \cdots)^{k+1}\}.$$

The lemma follows from this formula; cf. [12], Propositions 3.6, 3.7. □

Let us find an expression for $C(z)$ using the formula for c_k's given by Proposition 4.1.

LEMMA 4.12. *We have*

$$C(z) = 1 - z + \left(\frac{z}{1 + t_1 z + t_2 z^2 + \cdots}\right)^{(-1)}.$$

PROOF. This is a direct consequence of the Lagrange inversion formula (see, e.g., [25], Theorem 5.4.2). □

Two previous lemmas imply that

$$\bar{m}_1 z^2 + \bar{m}_2 z^3 + \cdots = \log\left(\frac{1}{z}\left(\frac{z}{1 - z + (z/(1 + t_1 z + t_2 z^2 + \cdots))^{(-1)}}\right)^{(-1)}\right).$$

In order to show that $\bar{m}_k = \tilde{m}_k$, $k = 1, 2, \ldots$, we prove the equality of their generating functions. Formulas (3.7) and (4.1) imply

$$\tilde{m}_1 z^2 + \tilde{m}_2 z^3 + \cdots = \log\left(\frac{z}{1 + z(1+z)(t_1 + t_2 z + \cdots)}\right)^{(-1)} - \log(1+z).$$

Therefore, the following lemma completes the proof.

LEMMA 4.13. *We have*

$$\begin{aligned}&\left(\frac{z}{1+z(1+z)(t_1+t_2z+\cdots)}\right)^{(-1)}+1\\&\quad=\frac{z+1}{z}\left(\left(\frac{z}{1-z+(z/(1+t_1z+t_2z^2+\cdots))^{(-1)}}\right)^{(-1)}\right).\end{aligned}\tag{4.11}$$

PROOF. It is easy to see that both series have the form

$$a_0+a_1z+a_2z^2+\cdots,$$

and the coefficients a_0 and a_1 of both of the series are equal to 1. Let us prove the equality of the coefficients of z^n, $n\geq 2$.

Recall that the Lagrange inversion formula has the following form (see, e.g., [25], Theorem 5.4.2)

$$n[z^n]F^{(-1)}(z)^k=k[z^{n-k}]\left(\frac{z}{F(z)}\right)^n.$$

Let $s(z)$ denote the series $1+t_1z+t_2z^2+\cdots$. We have

$$1+z(1+z)(t_1+t_2z+\cdots)=s(z)+zs(z)-z.$$

Hence, the coefficient of z^n in the left-hand side of (4.11) can be written in the following form:

$$\begin{aligned}&\frac{1}{n}[z^{n-1}](s(z)+zs(z)-z)^n\\&\quad=\frac{1}{n}\sum_{i_1+i_2+i_3=n;i_1\geq 1}\frac{n!}{i_1!i_2!i_3!}[z^{n-1}]s(z)^{i_1}z^{i_2}s(z)^{i_2}(-1)^{i_3}z^{i_3}\\&\quad=\sum_{i_1+i_2+i_3=n;i_1\geq 1}\frac{(n-1)!}{i_1!i_2!i_3!}[z^{i_1-1}]s(z)^{i_1+i_2}(-1)^{i_3}.\end{aligned}\tag{4.12}$$

Now let us consider the expression in the right-hand side of (4.11). The Lagrange inversion formula implies that the coefficient of z^n in the right-hand side can be written as

$$\frac{1}{n}[z^{n-1}]C(z)^n+\frac{1}{n+1}[z^n]C(z)^{n+1}.$$

Since

$$C(z)=1-z+\left(\frac{z}{s(z)}\right)^{(-1)},$$

the coefficient of z^l (for any $l\geq 1$) in an arbitrary power of $C(z)$ can be found by the Lagrange inversion formula again. We obtain

$$[z^l]\left(\left(\frac{z}{s(z)}\right)^{(-1)}(z)\right)^k=\frac{k}{l}[z^{k-l}]s(z)^l.$$

Therefore,

$$\frac{1}{n}[z^{n-1}]C(z)^n = \frac{1}{n}\Bigg(\sum_{i_1+i_2+i_3=n;\, i_1\ge 1,\, i_3\ge 1} (-1)^{i_2}\frac{i_3}{n-1-i_2}[z^{i_1-1}]s(z)^{n-1-i_2}\frac{n!}{i_1!i_2!i_3!} + (-1)^{n-1}n\Bigg).$$

This formula with n replaced by $n+1$ reads

$$\frac{1}{n+1}[z^{n}]C(z)^{n+1} = \frac{1}{n}\Bigg(\sum_{j_1+j_2+j_3=n+1;\, j_1\ge 1,\, j_3\ge 1} (-1)^{j_2}\frac{j_3}{n-j_2}[z^{j_1-1}]s(z)^{n-j_2}\frac{(n+1)!}{j_1!j_2!j_3!} + (-1)^{n}(n+1)\Bigg).$$

Now add the two equalities above and combine the coefficients of $[z^a]s(z)^b$ for all a and b. We obtain the sum

$$\sum_{j_1+j_2+j_3=n;\, j_1\ge 0} (-1)^{j_2}\frac{(n-1)!}{j_1!j_2!j_3!}[z^{j_1-1}]s(z)^{j_1+j_3}.$$

Changing the notation of indices in the summation, we see that this expression coincides with (4.12). This completes the proof. □

APPENDIX: EXAMPLES OF LIMIT SHAPES

In this section, we consider several examples of sequences $\omega = \omega(N)$ which satisfy the main condition (3.1).

(a) *One-sided Plancherel character*. Let $\gamma^+ = \gamma N$, where γ is a fixed constant, and all other Voiculescu's parameters are equal to 0. In this case, the main condition holds with $t_1 = \gamma$ and $t_k = 0$, for $k \ge 2$. Then we have

$$Q(z) = 1 + \gamma z(1+z).$$

It follows that

$$v_0(z) = \frac{1 - \gamma z - \sqrt{y^2(z^2-4\gamma) - 2\gamma z + 1}}{2\gamma z}.$$

Using (3.7), one can derive the expression for the Stieltjes transform:

$$S\left(\frac{1}{z}\right) = \mathrm{Stil}^{\mathrm{Planch}}(z) = \log\frac{z+\gamma-\sqrt{(z-\gamma)^2-4\gamma}}{2\gamma}.$$

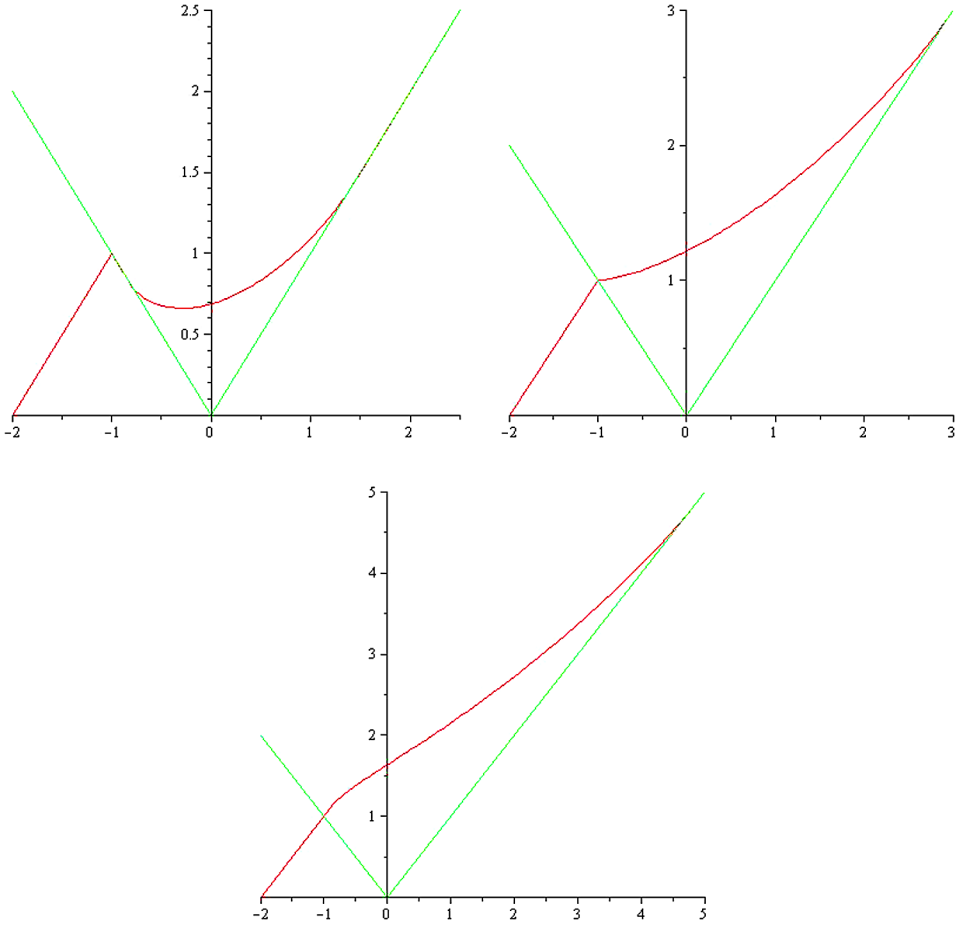

FIG. 8. *Limit shapes for the one-sided Plancherel character with* $\gamma = 0.3, 1, 2$, *respectively.*

Given the Stieltjes transform of a measure, there is a standard way to compute the density of this measure; see, for example, [1], Section 2.4, and the end of Section 3.2 above. After computations, we obtain that for $\gamma > 1$ we have

$$d^{\text{Planch}}(x) = \frac{1}{\pi} \arccos \frac{x+\gamma}{2\sqrt{\gamma(x+1)}} \qquad \text{for } x \in [\gamma - 2\sqrt{\gamma}; \gamma + 2\sqrt{\gamma}] \,,$$

and for $\gamma < 1$ we have

$$d^{\text{Planch}}(x) = \begin{cases} \dfrac{1}{\pi} \arccos \dfrac{x+\gamma}{2\sqrt{\gamma(x+1)}}, & \text{for } x \in [\gamma - 2\sqrt{\gamma}; \gamma + 2\sqrt{\gamma}], \\ 1, & \text{for } x \in [-1; \gamma - 2\sqrt{\gamma}]. \end{cases}$$

Examples of these limit shapes are shown in Figure 8.

After rescaling, these limit shapes coincide with Biane's limit shapes (see [3]).

(b) *One multiple α^+-parameter.* Assume that $\alpha_1^+ = \alpha_2^+ = \cdots = \alpha_{[aN]}^+ = \alpha$, and all other Voiculescu's parameters are equal to 0. Note that we fix two different real numbers a and α. Then $t_1 = a\alpha$, $t_2 = a\alpha^2, \ldots, t_k = a\alpha^k, \ldots$.

In this case, we have

$$Q(z) = 1 + \frac{z(1+z)a\alpha}{1-\alpha z}.$$

After computations, we obtain

$$\mathrm{Stil}^{\text{multi-}\alpha}(z) = \log \frac{\alpha(a+1) + (2\alpha+1)z - \sqrt{(z-\alpha(a+1))^2 - 4a\alpha(\alpha+1)}}{2\alpha(a+z)}.$$

The limiting density is given by the following formulas.

For $a \geq (\alpha+1)/\alpha$, we have

$$d^{\text{multi-}\alpha}(x) = \frac{1}{\pi} \arccos \frac{\alpha(a+1) + (2\alpha+1)x}{2\sqrt{\alpha(\alpha+1)(x+1)(x+a)}},$$
$$x \in \left[\alpha(a+1) - 2\sqrt{a\alpha(\alpha+1)}; \alpha(a+1) + 2\sqrt{a\alpha(\alpha+1)}\right].$$

For $\alpha/(\alpha+1) \leq a \leq (\alpha+1)/\alpha$, we have

$$d^{\text{multi-}\alpha}(x) = \begin{cases} 1, & x \in \left[-1; \alpha(a+1) - 2\sqrt{a\alpha(\alpha+1)}\right], \\ \dfrac{1}{\pi} \arccos \dfrac{\alpha(a+1) + (2\alpha+1)x}{2\sqrt{\alpha(\alpha+1)(x+1)(x+a)}}, & \\ & x \in \left[\alpha(a+1) - 2\sqrt{a\alpha(\alpha+1)}; \alpha(a+1) + 2\sqrt{a\alpha(\alpha+1)}\right]. \end{cases}$$

Finally, for $a \leq \alpha/(\alpha+1)$ we have

$$d^{\text{multi-}\alpha}(x) = \begin{cases} 1, & x \in [-1; -a], \\ \dfrac{1}{\pi} \arccos \dfrac{\alpha(a+1) + (2\alpha+1)x}{2\sqrt{\alpha(\alpha+1)(x+1)(x+a)}}, & \\ & x \in \left[\alpha(a+1) - 2\sqrt{a\alpha(\alpha+1)}; \alpha(a+1) + 2\sqrt{a\alpha(\alpha+1)}\right]. \end{cases}$$

Limit shapes for $\alpha = 1$ and various a are shown in Figure 9.

(c) *One multiple β^+-parameter.* Let us fix two positive real numbers b and $\beta \leq 1$. Assume that $\beta_1^+ = \beta_2^+ = \cdots = \beta_{[bN]}^+ = \beta$.

The computations in this case are equivalent to the previous one:

$$Q(z) = 1 + z(1+z)\frac{b\beta}{1+\beta z},$$

$$\mathrm{Stil}^{\text{multi-}\beta}(z) = \log \frac{z(1-2\beta) + \beta(b-1) - \sqrt{(z-\beta(b-1))^2 + 4b\beta^2 - 4b\beta}}{2\beta b - 2\beta z}.$$

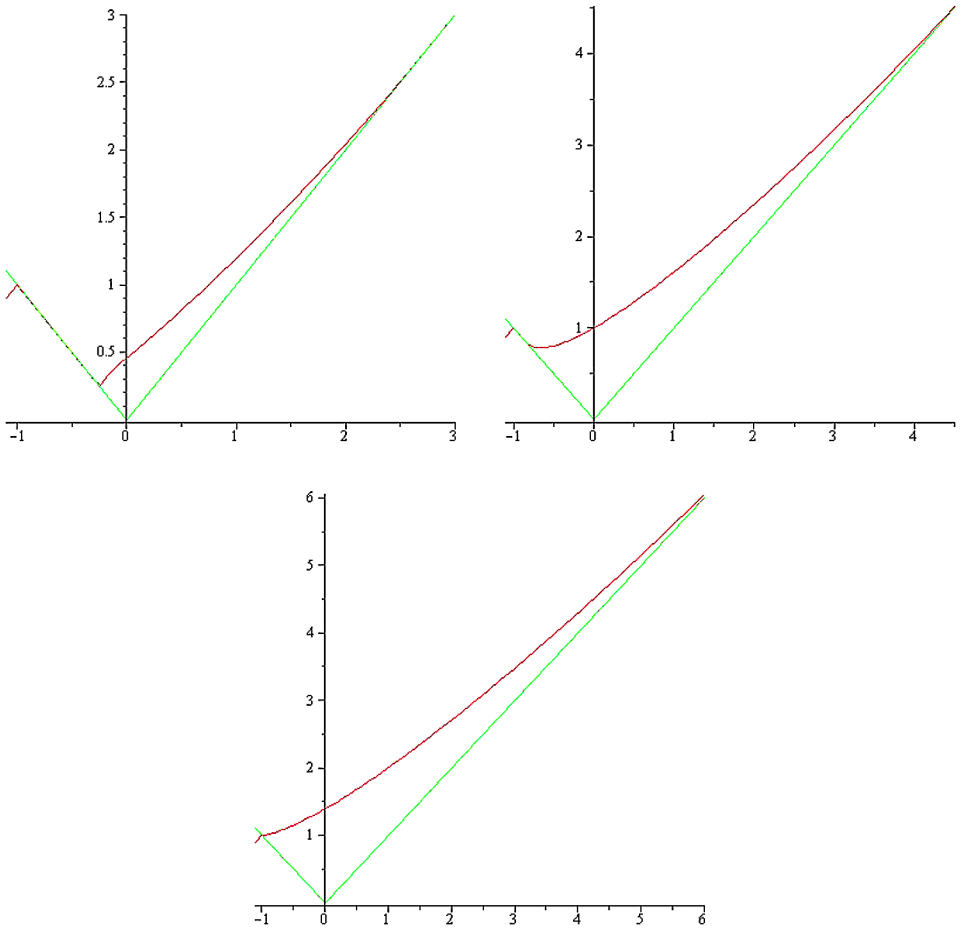

FIG. 9. *The limit shapes for one folded α^+-parameter with $\alpha = 1$ and $a = 0.25, 1, 2$, respectively.*

Inside the interval $[\beta(b-1) - 2\sqrt{b\beta(1-\beta)}; \beta(b-1) + 2\sqrt{b\beta(1-\beta)}]$, the density has the following form:

$$d^{\text{multi-}\beta}(x) = \frac{1}{\pi} \arccos \frac{(1-2\beta)x + \beta(b-1)}{2\sqrt{\beta(1-\beta)(1+z)(b-z)}}.$$

Furthermore, as in the previous case, for some parameters β and b there exist intervals with constant density which is equal to 1.

(d) *Two-sided Plancherel character.* Assume that $\gamma^+ = \gamma_1 N$, $\gamma^- = \gamma_2 N$ for fixed γ_1 and γ_2, and all other parameters are equal to 0. Then

$$Q(z) = 1 + z(1+z)\left(\gamma_1 - \frac{\gamma_2}{(z+1)^2}\right).$$

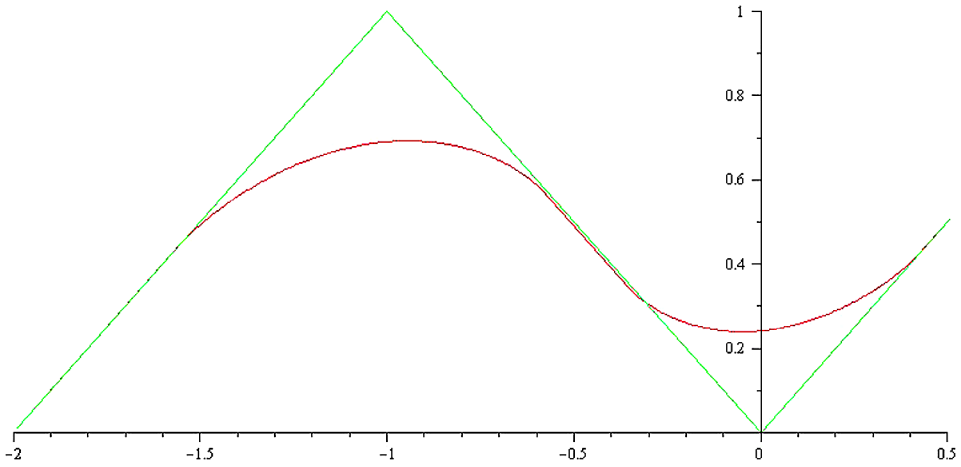

FIG. 10. *The limit shape for two-sided Plancherel character with parameters* $\gamma_1 = 0.04$, $\gamma_2 = 0.06$.

Hence, to obtain an explicit formula for the answer one need to solve the cubic equation

$$\frac{z(z+1)}{(z+1)+\gamma_1 z(z+1)^2-\gamma_2 z} = y.$$

If $z = v_0(y)$ is the formal power series that satisfies this equation, then the Stieltjes transform is equal to

$$\mathrm{Stil}^{2sP}(z) = \log\left(1 + v_0\left(\frac{1}{z}\right)\right).$$

An example of such a shape is shown in Figure 10.

In greater detail, this case was studied in [6].

(e) *The case of one multiple* α^+*-parameter and one multiple* α^-*-parameter.* Assume that $\alpha_1^+ = \alpha_2^+ = \cdots = \alpha_{[aN]}^+ = \alpha^+$ and $\alpha_1^- = \alpha_2^- = \cdots = \alpha_{[aN]}^- = \alpha^-$. Then

$$P'(z) = \frac{a\alpha}{1-\alpha_1 z} - \frac{\tilde{a}\tilde{\alpha}}{(1+z)(1+(\tilde{\alpha}+1)z)}.$$

In this case, the generating function of moments is determined by the solution of the cubic equation

$$\frac{z}{1+z(z+1)(a\alpha/(1-\alpha_1 z)-(\tilde{a}\tilde{\alpha}/((1+z)(1+(\tilde{\alpha}+1)z))))} = y.$$

(g) *The case of the continuous limit measure.* Assume that $\alpha_i^+ = i/N$ for $i = 1, \ldots, N$. It is easy to see that

$$P'(z) = \frac{-z-\log(1-z)}{z^2}.$$

Then the generating function of moments is determined by the solution of the equation

$$\frac{z^2}{-z^2-(1+z)\log(1-z)}=y.$$

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