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# LIMIT THEOREMS FOR DIVISOR DISTRIBUTIONS

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ABSTRACT. For a positive integer $N$, let $X_N$ be a random variable uniformly distributed over the set $\{\log d\colon d \mid N\}$. Let $F_N$ be the normalized (to have expectation zero and variance one) distribution function for $X_N$. Necessary and sufficient conditions for the convergence of a sequence $F_{N_j}$ of distributions are given. The possible limit distributions are investigated, and the case where the limit distribution is normal is considered in detail.

**1. Introduction.** Let the positive integer $N$ have prime factorization $N = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Define $\mu_n(N)$, for positive integer $n$, by

$$\mu_n = \left(12^{-1} \sum_{1 \leqslant j \leqslant k} \left((\alpha_j + 1)^n - 1\right)(\log p_j)^n\right)^{1/n}.$$

The divisor distribution of $N$ refers to the function

$$F_N(x) = \tau^{-1} \sum_{d \mid N}{}' 1$$

where $\tau$ is the number of divisors of $N$, and the sum is restricted to those divisors satisfying $\log(d/\sqrt{N}) \leqslant x\mu_2$.

In this paper, we determine when a sequence of divisor distributions tends to a limit, and investigate the limit distributions that arise. Erdös and Nicolas **[2]** had previously shown the divisor distribution of $N_j = \prod_{p<j} p$ (we reserve the letters $p$ and $q$ for primes) to be asymptotically normal as $j \to \infty$. With regard to the normal distribution we prove

THEOREM 2. *The normal distribution*

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}t^2\right) dt$$

*is the only infinitely divisible distribution that can arise as the limit of a sequence $F_{N_j}$ of divisor distributions. A necessary and sufficient condition for convergence is that*

$$\lim_{j \to \infty} \left(\mu_2(N_j)\right)^{-1} \mu_\infty(N_j) = 0.$$

*Moreover*,

$$\sup_w \left|F_N(w) - \Phi(w)\right| \ll \frac{\mu_\infty}{\mu_2},$$

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*and*

$$\left|\frac{1 - F_N(x)}{1 - \Phi(x)} - 1\right| \ll \left(x + \frac{1}{x}\right)\frac{\mu_4}{\mu_2}$$

*for* $x \leqslant \mu_2 \mu_4^{-1}$.

Here $\mu_\infty(N)$ means $\lim_{n \to \infty} \mu_n(N)$.

We define the (Fourier) transform of a distribution $F$ as

$$\hat{F}(t) = \int_R e^{2\pi i t x}\, dF(x).$$

If $\hat{F}(t)$ is the restriction to $R$ of an entire function, we say that $\hat{F}$ is entire. In the general case we have

THEOREM 1. *A necessary and sufficient condition for a sequence* $F_{N_j}$ *of divisor distributions to converge to a distribution* $F$ *is that for each* $n$ *the limits* $a_n = \lim_{j \to \infty} \mu_{2n}(N_j)(\mu_2(N_j))^{-1}$ *exist. In this case* $\hat{F}$ *is entire and is represented in the disk* $|z| < 1/4$ *by*

$$\hat{F}(z) = \exp\left(-\sum_{n=1}^{\infty} (n(2n)!)^{-1} 6 B_n (2\pi a_n z)^{2n}\right),$$

*where*

$$B_n = 4n \int_0^\infty (e^{2\pi t} - 1)^{-1} t^{2n-1}\, dt$$

*are the Bernoulli numbers.*

We say that a sequence $F_j$ of distributions converges to $F$, if $F_j(\pm\infty) \to F(\pm\infty)$, and $F_j(x) \to F(x)$ at all continuity points $x$ of $F$.

A reasonable characterization of the possible limit distributions seems difficult. We do however have the following "factoring theorem":

THEOREM 3. *Suppose the sequence* $F_{N_j}$ *of divisor distributions converges to* $F$. *If* $F$ *is not a finite point mass distribution then, for some* $\phi \in [0, \pi/2)$,

$$F(x) = G(x \sec\phi) * H(x \csc\phi).$$

*The convolution factor* $G$ *is a normal, uniform, or singular distribution.* $H$ *is the limit of a sequence of divisor distributions when* $\phi > 0$, *and otherwise is to be interpreted as point mass at* 0. *Moreover, if* $\liminf_{j \to \infty} \omega(N_j) = K < \infty$, *then* $F$ *may be written as a convolution product involving no more than* $K$ *uniform or arithmetic distributions.*

Here $\omega(N)$ is the number of distinct prime divisors of $N$. A finite point mass distribution is a finite convolution of arithmetic distributions. An arithmetic distribution is a probability distribution with zero expectation, and, a step function, whose finitely many jump discontinuities are of equal height and occur along an arithmetic progression. A uniform distribution has density $(12)^{-1/2}\chi_{[-\sqrt{3},\sqrt{3}]}$ (where $\chi_I$ is the indicator of the interval $I$), and a singular distribution is continuous with zero derivative almost everywhere.

## 2. Necessary and sufficient conditions.

LEMMA 1. *The transform $\hat{F}_N$ of the divisor distribution of $N$ is an entire function which is represented in the disk $|z| < \mu_2\mu_\infty^{-1}$ by*

$$\hat{F}_N(z) = \exp\left\{-\sum_{n=1}^{\infty}(n(2n)!)^{-1}6B_n\left(\mu_{2n}\mu_2^{-1}2\pi z\right)^{2n}\right\}.$$

PROOF. $\hat{F}_N$ is entire since $dF_N$ is compactly supported. For notational convenience, let $u = \pi\mu_2^{-1}t$. The transform of $F_n$ is

$$F(t) = \prod_{j=1}^{k}\left((\alpha_j+1)\sin(u\log p_j)\right)^{-1}\sin\left(u(\alpha_j+1)\log p_j\right). \tag{2.1}$$

Taking the logarithm of (2.1) yields

$$\hat{F}(t) = \exp\left\{\sum_{j=1}^{k}\int_{u\log p_j}^{u(\alpha_j+1)\log p_j}\cot x - x^{-1}\,dx\right\}. \tag{2.2}$$

Substituting the power series for $\cot x - x^{-1}$ in (2.2), integrating term by term, and interchanging the order of summation completes the proof. □

COROLLARY 1. *If $|y| < \lambda < 1/4$, then $\hat{F}_N(x+iy) \ll_\lambda 1$. Also, $\hat{F}$ has a zero at $\mu_2\mu_\infty^{-1}$.*

PROOF. (2.1) shows that $\hat{F}_N(\mu_2\mu_\infty^{-1}) = 0$. If $m$ is a positive integer, $m \leqslant 2n$, then

$$\mu_{2n}\mu_m^{-1} \leqslant (1-2^{-m})^{-1/m}12^{1/m-1/2n}. \tag{2.3}$$

Since $\hat{F}_N(x+iy) \ll \hat{F}_N(iy)$, using inequality (2.3) in Lemma 1 completes the proof.

PROOF OF THEOREM 1. By Corollary 1, the collection $\mathscr{F} = \{\hat{F}_{Nj}\}_1^\infty$ of analytic functions is uniformly bounded on compact subsets of $G_\lambda = \{x+iy \in \mathbf{C}\colon |y| < \lambda < 1/4\}$. Therefore, Montel's theorem **[1]** implies that *any* sequence of functions from $\mathscr{F}$ has a subsequence which converges uniformly on compact subsets of $G_\lambda$. This, together with the representation of $\hat{F}_N$ provided by Lemma 1, implies that the sequence $\hat{F}_{N_j}$ converges to a function $H$ if and only if the limits $a_n$ exist, and in such case,

$$H(z) = \exp\left\{-\sum_{n=1}^{\infty}(n(2n)!)^{-1}6B_n(a_n 2\pi z)^{2n}\right\}$$

in the disk $|z| < 1/4$. It follows from the continuity theorem for Fourier-Stieltjes transforms that the convergence of $\hat{F}_{N_j}$ to such a function $H$ is equivalent to the convergence of $F_{N_j}$ to some distribution $F$, and in such case, $\hat{F} = H$. It remains to show that $\hat{F}$ is entire.

The inequality (for $x \geqslant 0$)

$$1 - F(x) \leqslant F(-x) \leqslant \exp\{-x^2/6\} \tag{2.4}$$

implies that the sequence of entire functions

$$H_L(z) = \int_{-L}^{L} e^{2\pi izx}\,dF(x)$$

is uniformly Cauchy on compact subsets of **C**. It therefore suffices to prove (2.4). Note that for positive $\lambda$ and positive $x$,

$$(2.5)\qquad 1 - F_N(x) \leqslant F_N(-x) \leqslant \tau(N)^{-1} \sum_{d \mid N} \exp\left\{-\lambda x + \lambda \mu_2^{-1} \log \frac{\sqrt{N}}{d}\right\}.$$

With $\lambda_j = \lambda \mu_2^{-1} \log p_j$, the right-hand side of (2.5) is equal to

$$(2.6)\qquad e^{-\lambda x} \prod_{j=1}^{k} \left( \exp\left\{\frac{1}{2}\alpha_j \lambda_j\right\} (\alpha_j + 1)^{-1} \sum_{n=0}^{\alpha_j} \exp\{-n\lambda_j\} \right).$$

Using the convexity of $e^x$ and the inequality $\cosh(x) \leqslant \exp\{x^2/2\}$, we see that (2.6) is not greater than $\exp\{-\lambda x + \lambda^2/2\}$. Choosing $\lambda = 3^{-1}x$ finishes the proof.

Note that our method of proving Theorem 1 (via Montel's Theorem) shows that *any* sequence of $F_{N_j}$ (or $\hat{F}_{N_j}$) has a subsequence which converges to some $F$ (respectively $\hat{F}$).

Proof of Theorem 2. Suppose $F_{N_j}$ converges to $\Phi$. By Theorem 1, we have

$$\hat{\Phi}(t) = \exp\left\{ -\sum_{n=1}^{\infty} (n(2n)!)^{-1} 6 B_n (a_n 2\pi t)^{2n} \right\}.$$

On the other hand, $\hat{\Phi}(t) = \exp\{-2\pi^2 t^2\}$. It follows that $a_j = 0$ for $j > 1$. Conversely, if $a_j = 0$ for $j > 1$, then the sequence $F_{N_j}$ converges to some distribution $F$, where $\hat{F}(t) = \exp\{-2\pi^2 t^2\}$. Hence $F = \Phi$. It follows that $\mu_\infty \mu_2^{-1} \to 0$ is necessary and sufficient since

$$\mu_\infty \mu_2^{-1} \ll \mu_{2j} \mu_2^{-1} \ll \left(\mu_\infty \mu_2^{-1}\right)^{1/4}.$$

If $F_{N_j}$ does not converge to $\Phi$, then there is some compact interval of **R** containing infinitely many of the points $t_j = \mu_2(N_j)(\mu_\infty(N_j))^{-1}$. Let $t^*$ be a limit point. Each $t_j$ is a zero of $\hat{F}_{N_j}$ by Corollary 1, so if $F_{N_j}$ were to converge to $F$, then $F(t^*) = 0$. This precludes the possibility that $F$ is infinitely divisible, since such distributions have positive transforms.

The first inequality of Theorem 2 is a straightforward application of the following result, referred to as the Berry–Eseen inequality. Let $F$ and $G$ be probability distributions, and suppose $G$ has density $g$. Then for all $T > 0$

$$\sup_x |F(x) - G(x)| \ll T^{-1}\|g\|_\infty + \int_{-T}^{T} |t|^{-1} |\hat{F}(t) - \hat{G}(t)| \, dt$$

(Feller [3]).

To prove the second inequality of Theorem 2, define the measures $dV$ and $dG$ by

$$dV(x) = e^{-A+yx}\, dF_N(x), \qquad dG(x) = (2\pi)^{-1/2} e^{-(x-y)^2/2}\, dx.$$

Let $R(x) = \exp(x^2/2)(1 - \Phi(x))$. Then

$$(2.7)\qquad \frac{1 - F_N(y)}{1 - \Phi(y)} = e^{y^2/2} R(y)^{-1} \left( I + e^{A - y^2} R(y) \right),$$

where

$$I = e^A \int_y^\infty e^{-yx}\, d(V(x) - G(x)).$$

Now define $A - y^2/2$ to be the function

$$H(y) = -\sum_{n=2}^{\infty} 6B_n\left(n(2n)!\right)^{-1}\left(\mu_{2n}\mu_2^{-1}iy\right)^{2n}.$$

Assuming that $|V(x) - G(x)| \leqslant \Delta$ and $y > 0$, (2.7) becomes

$$\frac{1 - F_N(y)}{1 - \Phi(y)} = e^{H(y)} + O\left(\Delta R(y)^{-1} e^{H(y)}\right). \tag{2.8}$$

It is well known that $R(y)^{-1} \leqslant \sqrt{2\pi}(y + y^{-1})$ (see for example Mitrinovic [**4**]), so the proof is completed by establishing, for $0 < y \leqslant \mu_2/\mu_4$, the inequalities

$$|H(y)| \ll y^4(\mu_4/\mu_2)^4, \tag{2.9}$$

and

$$\Delta \ll \mu_4/\mu_2. \tag{2.10}$$

$H(m)$ is the sum of an alternating decreasing sequence, hence (2.9). The Berry–Eseen inequality, with $F = V$ and $G = G$, yields (2.10). □

**3. The factoring theorem.**

LEMMA 3. *If M and N are relatively prime positive integers, then*

$$\hat{F}_{MN}(t) = \hat{F}_M(t\cos\phi)\hat{F}_N(t\sin\phi)$$

*where*

$$\cos\phi = \frac{\mu_2(M)}{\sqrt{(\mu_2(M))^2 + (\mu_2(N))^2}}.$$

PROOF. The functions $(\mu_n(N))^n$ are additive. Therefore, Lemma 1 implies $\hat{F}_{MN}(t\mu_2(MN)) = \hat{F}_M(t\mu_2(M))\hat{F}_N(t\mu_2(N))$, and Lemma 3 follows.

LEMMA 4. *Let p and q be primes, and $\alpha$ a positive integer. Then $F_{p^\alpha} = F_{q^\alpha}$ is an arithmetic distribution with discontinuities at the points*

$$\left\{(2\alpha^{-1}k - 1)\left((\alpha + 2)^{-1}3\alpha\right)^{1/2}\right\}_{k=0}^{\alpha}.$$

*As $\alpha \to \infty$, $F_{p^\alpha}$ converges to the uniform distribution U having density* $(12)^{-1/2}\chi_{[-\sqrt{3},\sqrt{3}]}$.

Lemma 4 follows immediately from Lemma 1 and Theorem 1.

LEMMA 5. *Let $F_j$ be a sequence of arithmetic distributions with $d_j > 1$ discontinuities such that $dF_j$ is supported in* $[-1, 1]$. *Let $s_j$ be the distance between discontinuities of $F_j$, and assume $v_j$ is a sequence of positive numbers such that*

(1) $\sum_{j>J} v_j < \frac{1}{4}s_J v_J$ *for* $J = 1, 2, \ldots,$

(2) $(\prod_{j=1}^{J} d_j)\sum_{j>J} v_j \to 0$ *as* $J \to \infty$.

*Then the convolution $H_k(x) = F(x/v_1) * \cdots * F_k(x/v_k)$ converges to a singular distribution as $k \to \infty$.*

The proof is easy, and will be omitted.

We now prove Theorem 3. Let $N_j$ have prime factorization

$$N_j = p_j(1)^{\alpha_j(1)} \cdots p_j(k_j)^{\alpha_j(k_j)}.$$

We abbreviate $p_j(i)^{\alpha_j(i)}$ as $(j, i)$, and use $\langle x \rangle$ to mean $2^x$.

First consider the case $\liminf_{j\to\infty} \omega(N_j) = K$. By passing to a subsequence and reindexing, we may assume $\omega(N_j) = k$, and $(j, k) > (j, l)$ for $k < l$. Let $v_j(k) = \mu_2((j, k))(\mu_2(N_j))^{-1}$, and $M_j(k) = (j, k)^{-1}N_j$. Assume that $F$ is not a finite point mass distribution.

Repeated use of Lemma 3 gives

$$\hat{F}_{N_j}(t) = \prod_{k=1}^{K} \hat{F}_{(j,k)}\big(v_j(k)t\big). \tag{3.1}$$

Since each $v_j(k) \in [0, 1]$, we may pass to a subsequence and assume $v_j(k) \to v_k$ as $j \to \infty$. If any $v_k = 0$, then $\hat{F}_{(j,k)}(v_j(k)t) \to 1$ for all $t$. Hence such a factor can be ignored when considering $\lim_{j\to\infty} \hat{F}_{N_j}$, and so we may assume $v_k > 0$. We may also pass to a subsequence and assume each $F_{(j,k)}$ in (3.1) converges. Therefore, Theorem 1 implies that either $\alpha_j(k) \to \infty$ or the sequence $\alpha_j(k)$ becomes constant, say $\alpha_j(k) = \alpha_k$, for large $j$. If for all $k$, $\alpha_j(k) \to \alpha_k$, then (3.1) and Lemma 4 give

$$\hat{F}_{N_j}(t) \to \hat{F}_{\langle \alpha_1 \rangle}(v_1 t) \cdots \hat{F}_{\langle \alpha_k \rangle}(v_K t),$$

so that $F$ would be a finite point mass distribution, contrary to hypothesis. Therefore, let $k$ be such that $\alpha_j(k) \to \infty$, and let $M_j = M_j(k)$. By Lemma 3 we have

$$\hat{F}_{N_j}(t) = \hat{F}_{M_j}\big(t \sin\phi_j\big)\hat{F}_{(j,k)}\big(t\cos\phi_j\big),$$

where $\cos\phi_j = v_j(k)$. As $j \to \infty$, we have $\cos\phi_j \to \cos\phi = v_k > 0$, and by Lemma 4, $\hat{F}_{(j,k)}(t) \to \hat{U}(t)$. Passing to a subsequence, we have also $\hat{F}_{M_j} \to \hat{H}$ as $j \to \infty$. Therefore, $F(x) = U(x \sec\phi) * H(x \csc\phi)$.

Note that $\omega(M_j) < k = \omega(N_j)$; so, by redefining $N_j$ as $M_j$, the above argument can be repeated at most $k - 1$ times.

Now consider the case $\omega(N_j) \to \infty$. Assume that $F$ has no uniform or normal convolution factors, and is not a finite point mass distribution. We will show that $F$ either has a singular convolution factor, or is the limit of a sequence $F_{L_j}$ with $\omega(L_j) = O(1)$.

Since $F$ is not normal, Theorem 2 gives the existence of a $\delta > 0$ such that, for infinitely many $j$, $\mu_\infty(N_j)(\mu_2(N_j))^{-1} > \delta$. Passing to a subsequence we may assume this for all $j$. Lemma 3 gives

$$\hat{F}_{N_j}(t) = \hat{F}_{M_j(1)}\big(t\sin\phi_j\big)\hat{F}_{(j,1)}\big(t\cos\phi_j\big), \tag{3.2}$$

where $\cos\phi_j = v_j(1)$. Inequality (2.3) implies that $v_j(1) > \delta/4$, so we may pass to a subsequence and assume $\cos\phi_j \to \cos\phi = v_1 \geqslant \delta/4$ as $j \to \infty$. If $\sin\phi = 0$, then $\hat{F}_{M_j(1)}(t\sin\phi_j) \to 1$ for all $t$. Hence, this factor could be ignored when considering $\lim_{j\to\infty} F_{N_j}$, and $F$ would be the limit of a sequence $F_{L_j}$ with $\omega(L_j) = O(1)$ (take $L_j = (j, 1)$). By passing to a subsequence, we may assume that each factor in (3.2) converges. Since $F$ has no uniform convolution factor, this implies that the sequence $\alpha_j(1)$ becomes constant, say $\alpha_j(1) = \alpha_1$, for large $j$.

If we assume that $F$ is not the limit of a sequence $F_{L_j}$ with $\omega(L_j) = O(1)$, then by redefining $N_j$ as $M_j(1)$, the above argument can be repeated indefinitely. The $k$th application of the argument produces a subsequence $\hat{F}_{N_{k1}}, \hat{F}_{N_{k2}}, \ldots$ of the sequence generated at the $k-1$st stage along which

$$\hat{F}_{(j,k)}(v_j(k)t) \to \hat{F}_{\langle \alpha_k \rangle}(v_k t).$$

Let $N_j = N_{jj}$ be the diagonal sequence. It follows that

$$\hat{F}_{N_j}(t) = \prod_{k=1}^{k_j} \hat{F}_{\langle \alpha_j(k) \rangle}(v_j(k)t), \tag{3.3}$$

where for any $k$, $v_j(k) \to v_k$ and $\alpha_j(k) \to \alpha_k$ as $j \to \infty$.

Fatou's Lemma gives

$$\sum_{k=1}^{\infty} v_k^2 \leqslant \liminf_{j\to\infty} \sum_{k=1}^{k_j} v_j^2(k) = 1.$$

Hence there exists a subset $\{v_{k*}\}_{k=1}^{\infty}$ of the set $\{v_k\}_{k=1}^{\infty}$ satisfying the conditions of Lemma 5 with respect to the distributions $F_{\langle \alpha_{k^*} \rangle}(x)$. Let $f\colon Z^+ \to Z^+$ be a nondecreasing function satisfying the following conditions:

(A) $f(j) \leqslant k_j$ and $\lim_{j\to\infty} f(j) = \infty$,

(B) $k < f(j)$ implies $\alpha_j(k^*) = \alpha_{k^*}$ and $|v_j(k^*) - v_{k^*}| < 2^{-f(j)} v_{k^*}$.

Let $M_j = \prod_{k<f(j)}(j, k^*)$, and $\cos\phi_j = (\mu_2(N_j))^{-1}\mu_2(M_j)$. Applying Lemma 3, we reorganize (3.3) as

$$\hat{F}_{N_j}(t) = \hat{F}_{M_j^{-1}N_j}(t\sin\phi_j) \prod_{k<f(j)} F_{\langle \alpha_{k^*} \rangle}(v_j(k^*)t). \tag{3.4}$$

By passing to a subsequence, the first factor on the right-hand side of (3.4) converges to $\hat{H}(t\sin\phi)$ for some distribution $H$. The proof is completed by noting that the second factor converges to

$$\prod_{k=1}^{\infty} F_{\langle \alpha_{k^*} \rangle}(v_{k^*}t),$$

which by Lemma 5 is the transform of a singular distribution. □

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