Limit Theorems for Empirical Processes of Cluster Functionals

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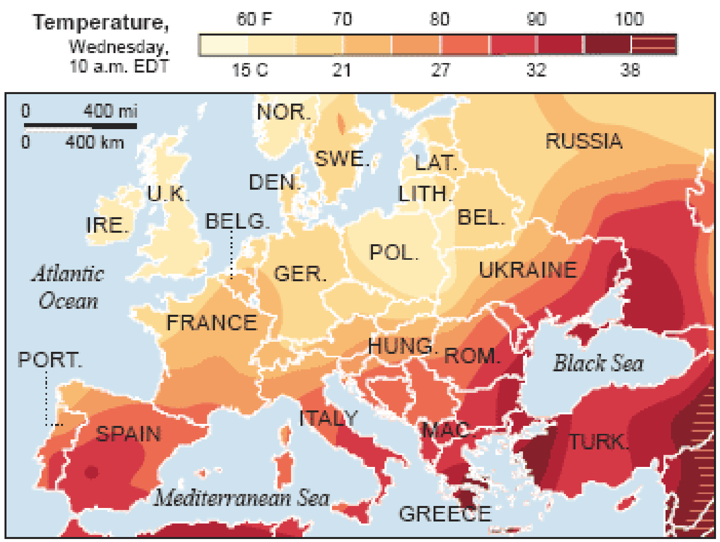
Why models for clusters of extremes?

- Heat waves may kill many 1000. Not just one high temperature which kills, but entire sequence of temperatures – cf 2003 heat wave in Europe.
- Floods may be caused by extreme precipitation when the ground is saturated by water from a previous rain – cf 2000 flood in northern Sweden



Why empirical process theory?

- Omnibus method to prove asymptotic normality of estimators and goodness of fit tests
- Proofs are important because
  - Sometimes heuristics are wrong
  - Sometimes exact limits of validity of heuristics are unclear
  - Sometimes they provide more detailed information
  - And, they lead to clearer and more general thinking which gives base for the next heuristic
- May stimulate development of new methods

Empirical limit theory – uniform central limit theorems

$X_1, X_2, \ldots$ i.i.d $\mathcal{F}$ class of functions, e.g. $\mathcal{F} = \{\mathbf{1}_{(-\infty,t]} \mid t > 0\}$

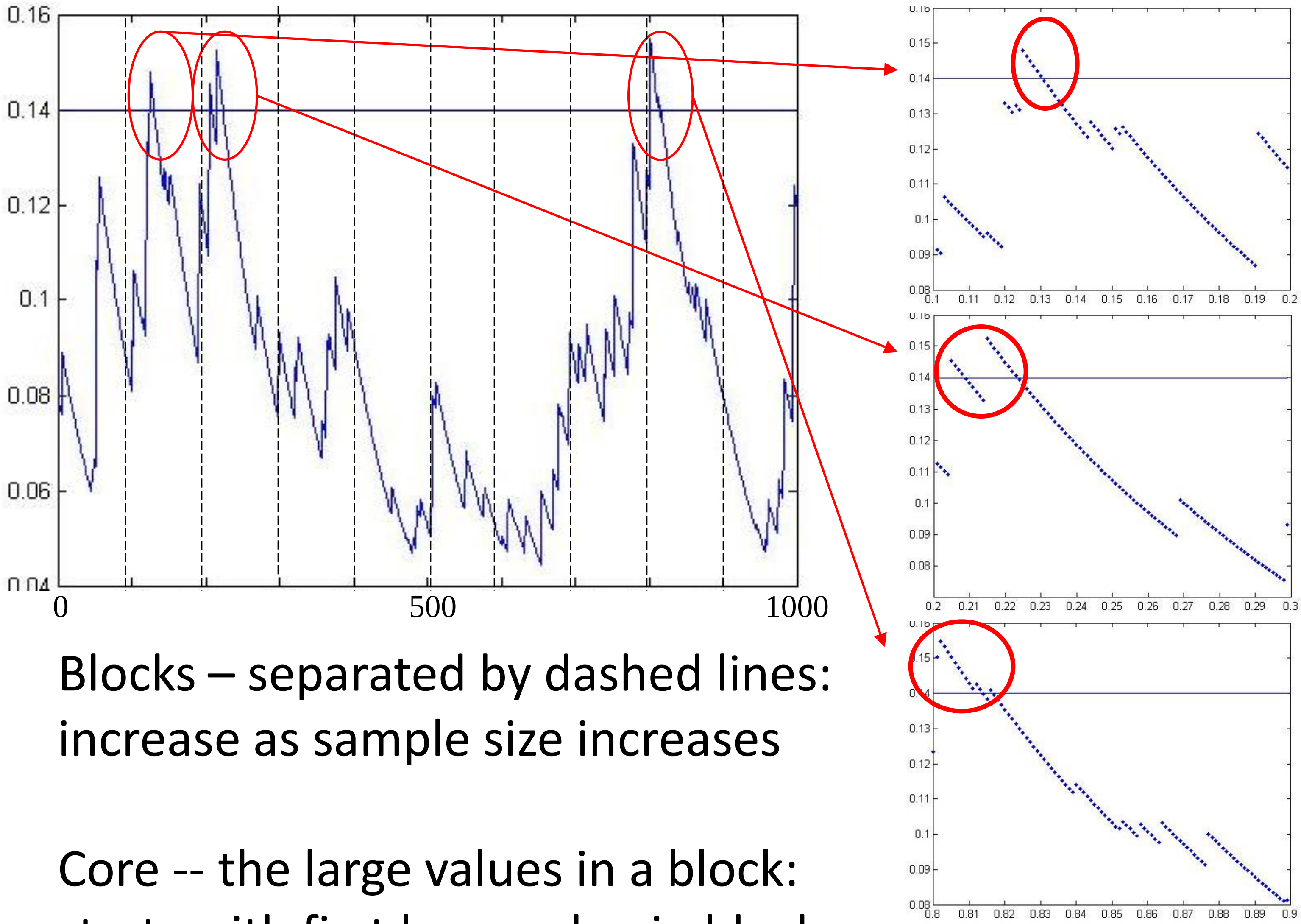
$Z_n : \mathcal{F} \rightarrow \mathbf{R}$ empirical function

$$f \rightarrow Z_n(f) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(X_i) - E(f(X_i)))$$

often $\sqrt{n}\left(\hat{\theta}_n - \theta\right) \approx h(Z_n)$ and one can hope that $Z_n \xrightarrow{d} Z$ implies that $\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} h(Z)$

"True" if $h : \ell^\infty(\mathcal{F}) \rightarrow R$ is continuous (and this in fact is the "definition" of $\xrightarrow{d}$)

Cores --- clusters of extremes



Blocks – separated by dashed lines: increase as sample size increases

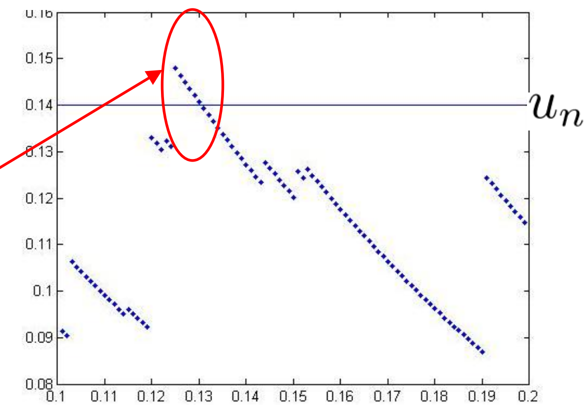
Core -- the large values in a block: starts with first large value in block, ends with the last

$(X_{n,i})$ triangular array of real variables, stationary in rows

typical example: $X_{n,i} = \frac{X_i - u_n}{a_n}$, with X_i stationary

$Y_{n,i} = (X_{n,j})_{(i-1)r_n < j \le ir_n}$ the *i*-th block:

blocklength



core: shortest consecutive vector containing all positive values in a block

$Y^c_{n,i} = (Y_{n,i,j})_{M_i \le j \le N_i}$ (or 0 if no pos value)

$Y_{n,i},\ Y^c_{n,i} \in \mathbf{R}_\cup := \cup_{d=1}^{\infty} \mathbf{R}^d$

goal: ”limit distribution of cores” in **general setting**

$f : \mathbf{R}_\cup \to \mathbf{R}$ cluster functional if f only depends on the core, i.e. if

$$f(Y) = f(Y^c), \;\; Y \in \mathbf{R}_\cup, \quad \text{and} \quad f(0) = 0$$

Yun (2000)
Segers (2003)

$\mathcal{F}$ class of cluster functionals

the number of blocks

$$Z_n(f) := \frac{1}{\sqrt{nv_n}} \sum_{i=1}^{m_n} \left(f(Y_{n,i}) - E(f(Y_{n,i})) \right), \;\; f \in \mathcal{F}$$ empirical process

$$= P(X_{n,1} > 0) \to 0 \;\; \text{and} \;\; nv_n \to \infty$$

Prove uniform central limit theorems for Z_n

for interesting classes $\mathcal{F}$ under suitable conditions

”big blocks – small blocks” proofs

- beta-mixing:

$$\beta_{n,\ell_n} \frac{n}{r_n} \to 0 \quad \text{for} \quad \beta_{n,\ell} = \sup_{k \geq 1} E_n \sup_{A \in \mathcal{B}^n_{n,\ell+k+1}} |P_n(A|\mathcal{B}^k_{n,1}) - P_n(A)|$$

length of short blocks $= o(r_n)$

- sums over small blocks tend to zero (i.i.d. calculation)

makes it possible to assume blocks are independent in proofs (assumed without further comment in sequel)

Finite-dimensional convergence:

$$(Z_n(f_1), \ldots Z_n(f_d)) \xrightarrow{d} (Z(f_1), \ldots Z(f_d)),\ \forall d, f_1, \ldots f_d \in \mathcal{F}$$

- Lindeberg (typically consequence of tightness condition)
- covariances converge:

Covariance function for limit *Z*

$$\frac{1}{r_n v_n} Cov\left(f(Y_{n,1}), g(Y_{n,1})\right) \to r(f,g), \quad \text{all} \quad f, g \in \mathcal{F}$$

(can use results of Segers (2003) to prove convergence of covariances)

Asymptotic equicontinuity / tightness

Ensures:

- Distribution of entire process can be approximated by finite-dimensional distributions
- Continuity of limiting Gaussian process

Tail empirical function (uniformly distributed variables)

If $$f_t : \mathbf{R}_\cup \to \mathbf{R}, \qquad f_t(y_1, \ldots y_k) = \sum_{i=0}^{k} \mathbf{1}_{(t,\infty)}(y_i)$$

then $$Z_n(f_t) \approx \frac{1}{\sqrt{nv_n}} \sum_{i=1}^{n} \left(\mathbf{1}_{\{(X_i)-u_n)/a_n>t\}} - \bar{F}(a_n t + u_n) \right)$$

Uniform central limit theorem if

$$\frac{1}{r_n v_n} E\left((\sum_{i=1}^{r_n} \mathbf{1}_{\{x<X_{n,i}\le y\}})^2 \right) \le |\log(y-x)|^{-(1+\epsilon)}$$

$$\frac{1}{r_n v_n} Cov\,(f_s(Y_{n,1}), f_t(Y_{n,1})) \to r(s,t), \quad \text{all} \quad f, g \in \mathcal{F}$$

Special case of tail array sums (Leadbetter)

Φ class of functions $\phi : \mathbf{R}^k \to \mathbf{R}$ with $\phi(\mathbf{0}) = 0$
Define $\mathcal{F} = \{f_\phi \mid \phi \in \Phi\}$ by

$$f_\phi : \mathbf{R}_\cup \to \mathbf{R}, \qquad f_\phi(y_1, \dots y_k) = \sum_{i=0}^{k-d} \phi(y_{i+1}^+, \dots y_{i+d}^+)$$

- $\phi_t(y) = \mathbf{1}_{(t,\infty)}(y)$ tail empirical process
- $\phi_t(y) = (y - t)^+$ total mass above t
- $\phi_{s,t}(y_1, y_2) = \mathbf{1}_{(-\infty,s]\times(t,\infty)}(y_1, y_2)$ number of upcrossings of (s, t]

Joint distribution function of cluster values

$\ell(y)$ = length of core of y
$\mathcal{F}$ = indicator functions of sets of the form

$$\{y_1^c \in [0,t_1], \ldots, y_j^c \in [0,t_j], \ell(x) = j\}, \quad j = 1, 2, \ldots$$

Uniform central limit theorem if

finite-dimensional marginal distributions belong to the domain of attraction of multivariate extreme value distribution

$$E\left(\ell(Y_{n,1})^{1/2+\epsilon} \middle| \, Y_{n,1} \neq 0\right) = O(1), \quad \text{some} \quad \epsilon > 0$$

Extensions/problems

- Other cluster definitions than the blocks definition, e.g. clusters start when there is an "r_n -upcrossing", i.e a positive $X_{n,i}$ preceeded by r_n values which are negative or zero.

- Random level, say the c_n-*th* largest order statistic instead of the non-random u_n: gives different limit, and new difficulties of proof