

Limit Theorems for Height Fluctuations in a Class of Discrete Space and Time Growth Models

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We introduce a class of one-dimensional discrete space-discrete time stochastic growth models described by a height function $h_t(x)$ with corner initialization. We prove, with one exception, that the limiting distribution function of $h_t(x)$ (suitably centered and normalized) equals a Fredholm determinant previously encountered in random matrix theory. In particular, in the universal regime of large x and large t the limiting distribution is the Fredholm determinant with Airy kernel. In the exceptional case, called the critical regime, the limiting distribution seems not to have previously occurred. The proofs use the dual RSK algorithm, Gessel's theorem, the Borodin–Okounkov identity and a novel, rigorous saddle point analysis. In the fixed x , large t regime, we find a Brownian motion representation. This model is equivalent to the Seppäläinen–Johansson model. Hence some of our results are not new, but the proofs are.

KEY WORDS: Growth processes; shape fluctuations; limit theorems; digital boiling; random matrix theory; Airy kernel; Painlevé II; saddle point analysis; invariance principle.

1. INTRODUCTION

Growth processes have been extensively studied by mathematicians and physicists for many years (see, e.g., refs. 22, 32, 38 and references therein), but it was only recently that K. Johansson⁽²⁸⁾ proved that the *fluctuations* of the limiting shape in a class of growth models are described by certain distribution functions first appearing in random matrix theory (RMT).^(40, 41)

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Further work by Johansson,⁽²⁹⁾ Prähofer and Spohn^(35, 36) and Baik and Rains⁽⁸⁾ strongly suggests the universal nature of these RMT distribution functions. These developments are part of the recent activity relating Robinson–Schensted–Knuth (RSK) type problems of combinatorial probability to the distribution functions of RMT, see, e.g., refs. 1, 3, 5–7, 11, 26, 27, 31, 45, 34, 43, and 44.

In this paper we analyze a class of one-dimensional discrete space-discrete time stochastic growth models, called *oriented digital boiling*.^(20, 21) Digital boiling dynamics is a cellular automaton that models an excitable medium in the presence of persistent random spontaneous excitation. Alternatively, digital boiling models represent contour (constant height) lines for one of the simplest models for growing a connected interface. It is this latter point of view we adopt here; that is, we introduce a height function $h_t(x)$ that characterizes the state of the system. Figure 1 illustrates the height fluctuations in oriented digital boiling.

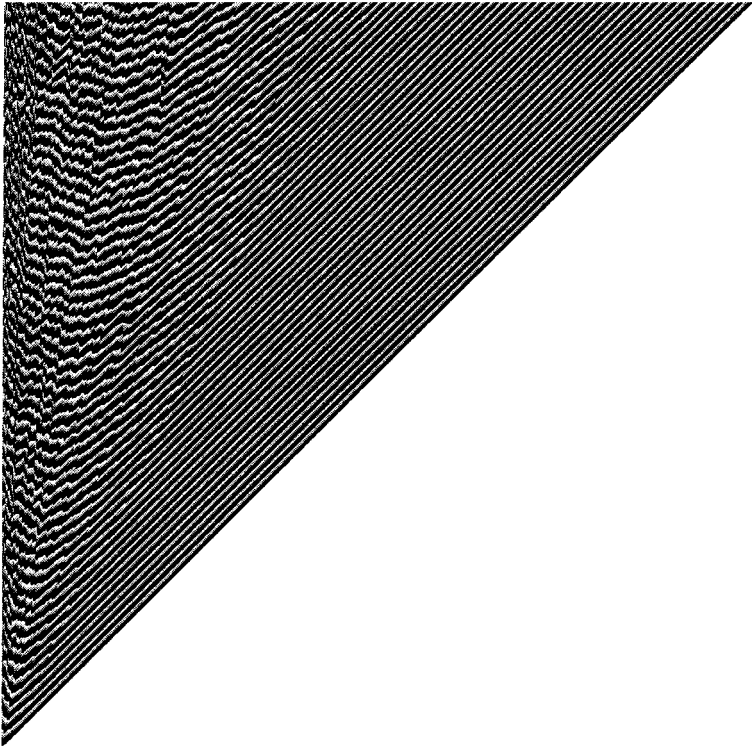


Fig. 1. ODB Simulations: The contour lines of $h_t(x)$ are drawn in space-time (x, t) . In these simulations $0 \leq t, x \leq 800$ and $p = 1/2$.

We shall derive various limit theorems for $h_t(x)$. We find four limiting regimes:

1. *GUE Universal Regime*: $x \rightarrow \infty$, $t \rightarrow \infty$ such that $p_c := 1 - x/t$ is fixed and $p < p_c$.
2. *Critical Regime*: $x \rightarrow \infty$, $t \rightarrow \infty$ such that $p_c := 1 - x/t$ is fixed and $p \sim p_c$.
3. *Deterministic Regime*: $x \rightarrow \infty$, $t \rightarrow \infty$ such that $p_c := 1 - x/t$ is fixed and $p > p_c$.
4. *Finite x GUE Regime*: Fixed x and $t \rightarrow \infty$.

The limit theorems are stated at the beginning of Section 3. Here is an outline of how they are obtained. First we show that $h_t(x)$ satisfies a last passage property, i.e., it equals the maximum over a certain class of paths in space-time. Then applying the dual RSK algorithm,^(30,39) we obtain a reformulation of the problem in terms of Young tableaux. This is followed by an application of a theorem of Gessel⁽¹⁸⁾ (see also ref. 44) which gives a Toeplitz determinant representation for the distribution function for $h_t(x)$. An identity of Borodin and Okounkov⁽¹⁰⁾ expresses the Toeplitz determinant in terms of the Fredholm determinant of an infinite matrix. Finally we use a saddle point analysis (steepest descent) to determine the limiting behavior of the entries, and therefore the Fredholm determinant, of the infinite matrix.

Along the way we identify⁴ ODB with a first-passage percolation model of Seppäläinen⁽³⁸⁾ whose limit law in the universal regime was determined by Johansson.⁽²⁹⁾ Thus we could have used the analysis in ref. 29 to establish our limit law in the universal regime, or alternatively used Riemann–Hilbert methods,^(5,15,16) to investigate the Toeplitz determinant asymptotics. But the method we present is in our opinion more straightforward and technically simpler than these, and it is very general. (The Fredholm determinant is easier to handle than the Toeplitz determinant, even though they are essentially equal.) Also, our analysis permits a nice conceptual understanding of the various limiting regimes. For example, the universal regime is characterized by the coalescence of two saddle points; and the emergence of the Airy kernel is related to the well-known appearance of Airy functions in such a saddle point analysis.⁽¹²⁾

Even in this simpler approach there are technical details to work out after the saddle point analysis gives us the answer. For example in the universal regime we need uniform estimates on the entries of the infinite

⁴ A referee points out that this identification can be made at the very beginning.

matrix in order to show that the matrix scales in trace norm to the Airy kernel. These details are given completely only for this regime.

In Regime 4 we give an independent proof that the suitably centered and normalized $h_t(x)$ has a limiting distribution. The proof proceeds through the introduction of a certain Brownian motion functional. This leads to some apparently new identities for n -dimensional Brownian motion; see (4.28) later.

The initial conditions are corner initialization. Due to the fact there is no known symmetry theorem for the *dual* RSK algorithm,^(30, 39) we are unable to prove limit theorems with different initial conditions, e.g., growth from a flat substrate. From work of Baik and Rains⁽⁷⁾ and Prähofer and Spohn,^(35, 36) it is natural to conjecture that the limiting distribution is now of GOE symmetry and hence given by the analogous distribution function in the GOE case.⁽⁴¹⁾

2. GROWTH MODELS AND INCREASING PATHS

In this section we introduce three classes of discrete space and discrete time stochastic growth models. Each of these models will have an equivalent path description, but only for one of these models are we able to prove limit theorems. Nevertheless, we believe it is useful to place this “solvable” case in a larger context.

We assume that the occupied set of our growth models can be described by a *height function* $h_t: \mathbf{Z}_+ \rightarrow \mathbf{Z}_+ \cup \{-\infty, \infty\}$, where \mathbf{Z}_+ is the set of nonnegative integers. Here, time $t=0, 1, 2, \dots$ proceeds in discrete steps. The *occupied set* at time t is thus given by

$$\eta_t = \{(x, y) \in \mathbf{Z}_+ \times \mathbf{Z}_+ : y \leq h_t(x)\}$$

In the models below we use the following one-dimensional neighborhood: $(x + \mathcal{N}) = \{x - 1, x\}$ (*the oriented case*) and assume *corner initialization*,

$$h_0(x) = \begin{cases} 0, & \text{if } x = 0 \\ -\infty, & \text{otherwise} \end{cases} \quad (2.1)$$

2.1. Oriented Digital Boiling

The first class of growth rules we call *oriented digital boiling* (ODB).^(20, 21)⁵ The rules for ODB are

⁵ For spatial dimensions greater than one, visual features of this dynamics resemble bubble formation, growth and annihilation in a boiling liquid, hence the process is called *digital boiling* and *oriented* refers to the choice of neighborhood \mathcal{N} , see Fig. 2 in ref. 20 or Feb. 12, 1996 Recipe of ref. 24.

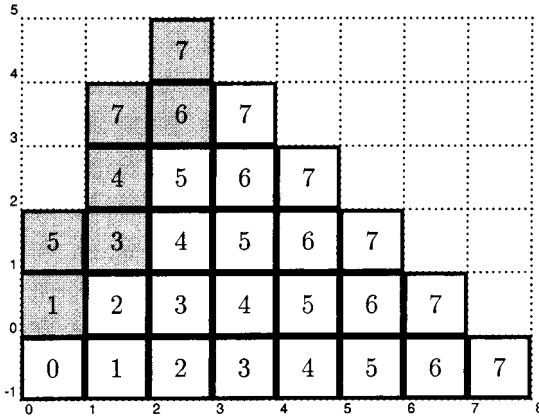


Fig. 2. Oriented Digital Boiling Process. The number in a box is the time this box was added and if the box is colored, then the box was added stochastically according to Rule 3.

- (1) $h_t(x) \leq h_{t+1}(x)$ for all x and t .
- (2) If $h_t(x - 1) > h_t(x)$, then $h_{t+1}(x) = h_t(x - 1)$.
- (3) Otherwise, then independently of the other sites and other times, $h_{t+1}(x) = h_t(x) + 1$ with probability p . (With probability $1 - p$, we have $h_{t+1}(x) = h_t(x)$.)

It follows from these rules that for every x and t , $h_t(x - 1) \leq h_t(x) + 1$.

This process can be readily visualized by imagining the growth proceeding by the addition of unit squares starting with the initial square centered at $(1/2, -1/2)$. We denote this initial time by placing a 0 in this box. At time $t = 1$ a box is added to the right (centered at $(3/2, -1/2)$) and with probability p a box is added to the top of the initial box (centered at $(1/2, 1/2)$). We place 1's in the boxes added at time $t = 1$. The boxes that are added stochastically (Rule (3)) are shaded. An example of this process run for seven time steps is shown in Fig. 2.

2.1.1. Path Description

As has been observed many times before (see, e.g., refs. 2, 13, 21, 23, 35, and 38), a most productive way to think about height processes is to introduce the (discrete) backwards lightcone of a point (x, t) . Precisely, if $\mathcal{S} = \mathbf{Z}_+ \times \mathbf{Z}_+$ denotes space-time, then

$$\mathcal{L}_B(x, t) = \{(x', t') \in \mathcal{S} : 0 \leq x' \leq x, x' \leq t' < x' + t - x\}$$

For those space-time points in \mathcal{L}_B at which a box was added stochastically (according to Rule (3)), we place a \times . We call such space-time points *marked*. We define the length of a sequence $\pi = \{(x_1, t_1), \dots, (x_k, t_k)\}$ of (distinct) space-time points in \mathcal{L}_B to be k . Such a sequence π is *increasing* if $0 \leq x_i - x_{i-1} \leq t_i - t_{i-1} - 1$ for $i = 2, \dots, k$. Let $L(x, t)$ equal the length of the longest increasing sequence of marked space-time points in $\mathcal{L}_B(x, t)$. (If $x \leq t$ and $\mathcal{L}_B(x, t)$ contains no increasing path, then $L(x, t) := 0$.) For the example in Fig. 2, the discrete backwards lightcone $\mathcal{L}_B(3, 7)$ and an increasing path are shown in Fig. 3. One observes that $h_7(3) = L(3, 7) = 4$. Indeed, this is a general fact. However, before proceeding with its proof, it is useful to change slightly the point of view of the process defined by Rules (1)–(3). Let $\Pi = \Pi(p)$ be a random subset of \mathcal{S} to which every point of \mathcal{S} belongs with probability p . We *mark* the points of \mathcal{S} that belong to Π . (Accordingly, we call these points *marked*.) $L(x, t)$ remains the same; namely, the length of the longest increasing sequence of marked space-time points in $\mathcal{L}_B(x, t)$. With regard to the process, we may intuitively think that all the “coins” used in Rule (3) are thrown in advance—of course, many of these are ignored as x at time t may become occupied deterministically by Rule (2). Precisely, Rule (3) is replaced with

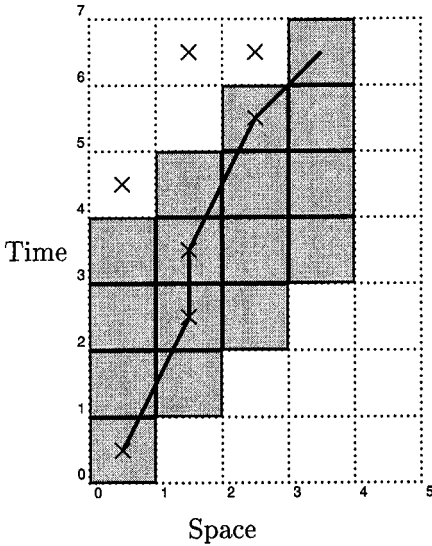


Fig. 3. The backwards lightcone of the point $(x, t) = (3, 7)$ for the process shown in Fig. 2. The \times 's denote the marked points and polygonal line gives a longest increasing path. The length of this path is equal to the number of \times 's in the path. This length equals $h_t(x)$.

(3) Otherwise,

$$h_{t+1}(x) = \begin{cases} h_t(x) + 1, & \text{if } (x, t) \in \Pi \\ h_t(x), & \text{if } (x, t) \notin \Pi \end{cases}$$

We are now ready to prove the last passage property

Proposition [ref. 21]. $h_t(x) = L(x, t)$.

Proof. We first show that our process is *attractive*⁶ in the following sense: Let Π and Π' be two sets of marked points such that $\Pi \subset \Pi'$. Let h_t evolve using Π and h'_t using Π' , then $h_t \leq h'_t$ for all t . For if this were *not* true, then, for some t , $h_s \leq h'_s$, $s \leq t$, and $h_{t+1}(x) > h'_{t+1}(x)$ for some x . This, of course, implies that $h_t(x) = h'_t(x)$. But then $h_{t+1}(x) = h_t(x) + 1$ either because of Rule (2); in which case, $h'_t(x-1) \geq h_t(x-1) > h_t(x) = h'_t(x)$, so (by Rule (2)) $h'_{t+1}(x) = h'_t(x) + 1$; or, because $(x, t) \in \Pi \subset \Pi'$, so again $h'_{t+1}(x) = h'_t(x) + 1$. This is a contradiction. Thus we've established the attractiveness of our process.

The property of attractiveness immediately implies $h_t(x) \geq L(x, t)$, since any increasing path of length k will, without the addition of other marked points, cause $h_t(x) \geq k$.

We now show that $h_t(x) \leq L(x, t)$. We will show, by induction on k and t , that $h_t(x) = k$ implies there exists an increasing sequence of marked points of length k in $\mathcal{L}_B(x, t)$. This is obviously true for either $t = 0$ or $k = 0$. (Note that $h_t(x) \geq 0$ means that $x \leq t$.) Now assume the claim has been demonstrated for all $k' < k$ and $t' < t$. We can clearly assume that $h_{t-1}(x) = k - 1$, or else we can use the induction hypothesis right away. Therefore, we have two possibilities.

Case 1. $h_t(x) = h_{t-1}(x) + 1$ by application of Rule (2). This means (by Rule (2)) that $h_{t-1}(x-1) = k$. Thus by the induction hypothesis, there is an increasing sequence of length k in $\mathcal{L}_B(x-1, t-1) \subset \mathcal{L}_B(x, t)$.

Case 2. $h_t(x) = h_{t-1}(x) + 1$ by application of Rule (3'). This means that $h_{t-1}(x) = k - 1$ and $(x, t-1) \in \Pi$. By the induction hypothesis, $\mathcal{L}_B(x, t-1)$ contains an increasing path of length $k - 1$. Adjoin the marked point $(x, t-1)$ to the sequence. Observe that the increasing property is preserved. This completes the proof of the proposition.

⁶ For examples of the kind of exotic shapes that can occur from cellular automaton rules without this monotonicity property, see ref. 22.

We summarize this section by noting that h_t satisfies for all $t \geq 1$, $x \geq 0$,

$$h_t(x) = \max\{h_{t-1}(x-1), h_{t-1}(x) + \varepsilon_{x,t}\}$$

where $\varepsilon_{x,t} = 1$ if $(x, t) \in \Pi$ and 0 otherwise. The initial conditions are (2.1). (We take $h_t(-1) = -\infty$.) Formulated this way ODB is a “stochastic dynamic programming” problem.

2.1.2. The (0, 1)-Matrix Description of ODB

Without changing the increasing path property, the backwards light-cone \mathcal{L}_B of any space-time point (x, t) can be deformed into a rectangle of size $(t-x) \times (x+1)$. Thus the equivalent problem is to fix x and t and to set $m = t-x$, $n = x+1$, and to consider a (0, 1)-matrix A of size $m \times n$. We number the rows of A starting at the bottom of A and the columns of A starting at the left of A . A increasing path in \mathcal{L}_B becomes a sequence of 1's in A at, say, positions $\{(i_1, j_1), \dots, (i_k, j_k)\}$ such that the i_ℓ ($\ell = 1, \dots, k$) are increasing and the j_ℓ ($\ell = 1, \dots, k$) are weakly increasing. Any such (0, 1)-matrix A of size $m \times n$ corresponds (bijectively) to a two-line array⁷

$$w_A = \begin{pmatrix} j_1 & j_2 & \dots & j_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} \tag{2.2}$$

where $j_1 \leq j_2 \leq \dots \leq j_k$ and if $j_\ell = j_{\ell+1}$, then $i_\ell < i_{\ell+1}$ and the pair $\binom{j}{i}$ appears in w_A if and only if the (i, j) entry of A is 1. Note that the upper numbers belong to $\{1, 2, \dots, n\}$ and the lower numbers to $\{1, 2, \dots, m\}$. For example, the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

maps to the two-line array

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 7 \\ 3 & 4 & 5 & 2 & 4 & 5 & 2 & 3 & 5 & 3 & 5 & 6 & 1 & 3 & 4 & 5 & 1 & 3 & 5 & 6 \end{pmatrix}$$

⁷ We have chosen both a nonstandard labeling of A and a nonstandard bijection $A \leftrightarrow w_A$ so that our increasing path property remains (essentially) the same under the bijections.

(Recall the convention for row labels.) As an example, a longest increasing path (of length 5) is indicated in bold typeface. We remark that one can compute the length of an increasing path by *patience sorting*⁽³⁾ on the bottom row of w_A (from left to right) with the rule that a number is placed on the left most pile such that it is less than or equal to the number showing in the pile. Patience sorting on the above example results in the five piles

					3
	1	3	4		
	1	3	5		
	2	3	5	5	
	2	4	5	5	
	3	4	5	6	6

If N denotes the number of 1's in a random $m \times n$ $(0, 1)$ -matrix A , then the above mappings imply that for any nonnegative integer h ,

$$\begin{aligned} \text{Prob}(h_t(x) \leq h) &= \sum_{k \geq 0} \text{Prob}(h_t(x) \leq h \mid N = k) \text{Prob}(N = k) \\ &= \sum_{k=0}^{mn} \binom{mn}{k} p^k (1-p)^{mn-k} \text{Prob}(L_{m,n,k} \leq h) \end{aligned} \tag{2.3}$$

where $L_{m,n,k}$ is the length of the longest increasing path in a random $(0, 1)$ -matrix A with k 1's (or equivalently, in the associated w_A).

2.1.3. Tableaux Description of ODB

The dual RSK algorithm^(30, 39) is a bijection between $(0, 1)$ -matrices A of size $m \times n$ and pairs (P, Q) such that P^t (the transpose of P) and Q are semistandard Young tableaux (SSYT) with $\text{sh}(P) = \text{sh}(Q)$ where the elements of P are from $\{1, 2, \dots, m\}$ and the elements of Q are from $\{1, 2, \dots, n\}$. In terms of the associated w_A , (2.2), one forms P by successive row bumping of the second row of w_A starting with i_1 and with the rule an element i bumps the leftmost element $\geq i$. Thus each row of P is strictly increasing. A fundamental property of the dual RSK algorithm is that the length of the longest strictly increasing subsequence of the second row of w_A equals the number of boxes in the first row of P .

If $d_\lambda(M)$ denotes the number of SSTYs of shape λ with entries coming from $\{1, 2, \dots, M\}$, then the number of pairs (P, Q) of fixed shape λ in the above dual RSK algorithm is

$$d_{\lambda'}(m) d_\lambda(n)$$

where λ' is the conjugate partition. (Conjugate since P' is a SSYT.) Since there are $\binom{mn}{k}$ $(0, 1)$ -matrices with k 1's,

$$\text{Prob}(L_{m,n,k} \leq h) = \frac{1}{\binom{mn}{k}} \sum_{\substack{\lambda \vdash k \\ \lambda_1 \leq h}} d_{\lambda'}(m) d_{\lambda}(n) = \frac{1}{\binom{mn}{k}} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq h}} d_{\lambda}(m) d_{\lambda'}(n)$$

And hence from (2.3)

$$\text{Prob}(h_t(x) \leq h) = (1 - p)^{mn} \sum_{k=0}^{mn} r^k \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq h}} d_{\lambda}(m) d_{\lambda'}(n)$$

where $r = p/(1 - p)$. Observe that for $|\lambda| > mn$, $d_{\lambda}(m) d_{\lambda'}(n) = 0$. (A SSYT with entries from $\{1, 2, \dots, M\}$ can have at most M rows.) If \mathcal{P} denotes the set of all partitions (including the empty partition), then the above sum can be summed over all partitions without changing its value,

$$\text{Prob}(h_t(x) \leq h) = (1 - p)^{mn} \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq h}} r^{|\lambda|} d_{\lambda}(m) d_{\lambda'}(n) \tag{2.4}$$

Comparing (2.4) with Johansson's Krawtchouk ensemble results establishes the equivalence of ODB with the Seppäläinen–Johansson model.

2.1.4. Application of Gessel's Theorem and the Borodin–Okounkov Identity

Gessel's theorem^(18, 44) is

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq h}} r^{|\lambda|} s_{\lambda}(x) s_{\lambda}(y) = D_h(\varphi)$$

where s_{λ} are the Schur functions (see, e.g., ref. 39) and $D_h(\varphi)$ is the $h \times h$ Toeplitz determinant⁸ with symbol

$$\varphi(z) = \prod_{j=1}^{\infty} (1 - x_j z)^{-1} \prod_{j=1}^{\infty} (1 - y_j r z^{-1})^{-1}$$

⁸ If ϕ is a function on the unit circle with Fourier coefficients ϕ_k then $T_n(\phi)$ denotes the Toeplitz matrix $(\phi_{i-j})_{i,j=0,\dots,n-1}$ and $D_n(\phi)$ its determinant.

If we apply to both sides of this identity the automorphism ω (see Stanley, ⁽³⁹⁾ p. 332), $\omega(s_\lambda) = s_{\lambda'}$, to the symmetric functions in the x -variables we obtain

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq h}} r^{|\lambda|} s_{\lambda'}(x) s_\lambda(y) = D_h(\varphi) \tag{2.5}$$

where *now* the symbol is

$$\varphi(z) = \prod_{j=1}^{\infty} (1 + x_j z) \prod_{j=1}^{\infty} (1 - y_j r z^{-1})^{-1} \tag{2.6}$$

Recalling the specialization ps_n^1 (see Stanley, ⁽³⁹⁾ p. 303), we apply ps_n^1 to the x -variables and ps_m^1 to the y -variables in Gessel’s identity (2.5) and observe⁹ that the resulting LHS is precisely the RHS of (2.4). Since the specialization ps_n^1 is a ring homomorphism, we may apply it directly to the symbol (2.6). Doing so we obtain

$$\text{Prob}(h_t(x) \leq h) = (1 - p)^{mn} D_h(\varphi) \tag{2.7}$$

where

$$\varphi(z) = (1 + z)^n (1 - r/z)^{-m} \tag{2.8}$$

This derivation required $r < 1$. However, by (2.3) the left side is a rational function of r , and analytic continuation shows that (2.7) holds for all $r \geq 0$ if in the integral representing the Fourier coefficients of φ the contour has r on the inside.

The Borodin–Okounkov⁽¹⁰⁾ identity expresses a Toeplitz determinant in terms of a Fredholm determinant of an infinite matrix which in turn is a product of two Hankel matrices. Subsequent simplifications of the proof by Basor and Widom⁽⁹⁾ extended the identity to block Toeplitz determinants. We now apply this identity to the Toeplitz determinant (2.7). First we find the Wiener–Hopf factorization of $\varphi(z)$:

$$\varphi(z) = \varphi_+(z) \varphi_-(z)$$

where

$$\varphi_+(z) = (1 + z)^n, \quad \varphi_-(z) = (1 - r/z)^{-m}$$

⁹ Note that $\text{ps}_n^1 s_\lambda = d_\lambda(n)$ which follows from the combinatorial definition of the Schur function.

Define K_h acting on $\ell^2(\{0, 1, \dots\})$ by

$$K_h(j, k) = \sum_{\ell=0}^{\infty} (\varphi_- / \varphi_+)_{h+j+\ell+1} (\varphi_+ / \varphi_-)_{-h-k-\ell-1} \tag{2.9}$$

The Borodin–Okounkov identity is then

$$D_h(\varphi) = Z \det(I - K_h)$$

Since the determinant on the right tends to 1 as $h \rightarrow \infty$ as does $\text{Prob}(h_t(x) \leq h)$, we have $Z = (1 - p)^{-mm}$. Thus we have derived a representation of the distribution function of the random variable $h_t(x)$ in terms of a Fredholm determinant,

$$\text{Prob}(h_t(x) \leq h) = \det(I - K_h) \tag{2.10}$$

This derivation also required $r < 1$. As above, analytic continuation shows that (2.10) holds for all $r \geq 0$ if in the integral representing the Fourier coefficients of φ_- / φ_+ the contour has r on the inside and -1 on the outside. (In fact the contour must have -1 on the outside no matter what r is.)

A somewhat different direction (and one we do not follow here) is to apply isomonodromy and Riemann–Hilbert methods^(5, 14, 27) directly to the Toeplitz determinant $D_h(\varphi)$. This would result in the identification of $D_h(\varphi)$ as a τ -function of an integrable ODE.

2.2. Inhomogeneous ODB

In ODB the probability p appearing in Rule (3) is independent of the site x . *Inhomogeneous ODB* replaces Rule (3), for each site $x \in \mathbf{Z}_+$, with

(3_x) Otherwise, then independently of the other sites and other times, $h_{t+1}(x) = h_t(x) + 1$ with probability $0 < p_x < 1$ and $h_{t+1}(x) = h_t(x)$ with probability $q_x := 1 - p_x$.

Since the dual RSK algorithm is a bijection between $(0, 1)$ -matrices A and pairs (P, Q) such that P^t and Q are SSYTs with $\text{col}(A) = \text{type}(P)$ and $\text{row}(A) = \text{type}(Q)$,⁽³⁹⁾ we have

$$\text{Prob}(h_t(x) \leq h) = q_0^m \cdots q_x^m \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq h}} d_\lambda(m) s_\lambda(r) \tag{2.11}$$

where, as before, $m = t - x$, but now $r = (r_0, \dots, r_x, 0, \dots)$ with $r_j := p_j / q_j$. The proof of (2.11) is straightforward and similar to the proof of the analogous

result in ref. 26; therefore, we omit it. The right hand side of (2.11) clearly reduces to (2.4) in the homogeneous case.

We again apply Gessel’s theorem to obtain the Toeplitz determinant representation

$$\text{Prob}(h_t(x) \leq h) = q_1^m \cdots q_n^m D_h(\varphi)$$

where¹⁰

$$\varphi(z) = (1 - 1/z)^{-m} \prod_{j=0}^x (1 + r_j z) \tag{2.12}$$

Application of the Borodin–Okounkov identity results in a Fredholm determinant representation for this distribution function. Observe that from either (2.11) or (2.12) it follows that $\text{Prob}(h_t(x) \leq h)$ is a symmetric function of (p_0, p_1, \dots, p_x) . This property opens the possibility for an analysis of the spin glass version of ODB which we plan to address in future work.

2.3. Weak ODB and Strict ODB

Here are two natural variants of the ODB. We let the “spontaneous increase” in Rule (3) apply after Rule (2) has already taken effect to get *weak ODB*:

(1’) $h_t(x) \leq h_{t+1}(x)$ for all space-time points (x, t) .

(2’) If $h_t(x - 1) > h_t(x)$, then $\tilde{h}_t(x) = h_t(x - 1)$ else $\tilde{h}_t(x) = h_{t-1}(x)$. (Here \tilde{h}_t is an intermediate height function.)

(3’) Independently of the other sites and other times, $h_{t+1}(x) = \tilde{h}_t(x) + 1$ with probability p . (With probability $1 - p$, $h_{t+1}(x) = \tilde{h}_t(x)$.)

In *strict ODB* we require that the left neighbor is *rested*¹¹ for the spontaneous increase. (We take $h_t(x) = -\infty$ for $x < 0$ which in this model implies $h_t(0) = 0$ for every t .)

(1’’) $h_t(x) \leq h_{t+1}(x)$ for all space-time points (x, t) .

(2’’) If $h_t(x - 1) > h_t(x)$, then $h_{t+1}(x) = h_t(x - 1)$.

(3’’) Otherwise, if $x - 1$ is rested at time t , $h_t(x - 1) = h_t(x)$ then independently of other sites and times, $h_{t+1}(x) = h_t(x) + 1$ with probability p ($h_{t+1}(x) = h_t(x)$ with probability $1 - p$).

¹⁰ The homogeneous case of (2.12) does not directly reduce to (2.8). It does after $z \rightarrow z/r$ which corresponds to a similarity transformation of the Toeplitz matrix.

¹¹ The height at a site cannot increase at two consecutive times, i.e., it must rest for one time unit before it is allowed to increase.

In a similar way one shows

- In weak ODB, $h_t(x)$ equals, in distribution, the longest sequence (i_ℓ, j_ℓ) of positions in a random $(0, 1)$ -matrix of size $m \times n$ ($m = t - x + 1$, $n = x + 1$) which have entry 1 such that i_ℓ are j_ℓ are both weakly increasing. (The lower left corner of the matrix is fixed to be a 0.)
- In strict ODB, $h_t(x)$ equals, in distribution, the longest sequence (i_ℓ, j_ℓ) of positions in a random $(0, 1)$ -matrix of size $m \times n$ ($m = t - x$, $n = x$) which have entry 1 such that i_ℓ are j_ℓ are both strictly increasing.

3. LIMIT THEOREMS

In this section we derive limit theorems for the distribution function $\text{Prob}(h_t(x) \leq h)$ for ODB. Our starting point will be the Fredholm determinant representation (2.10). This distribution function is a function of four variables, x , t , h and p ; and accordingly, there are several asymptotic regimes:

(1) *GUE Universal Regime:* Let $x \rightarrow \infty$, $t \rightarrow \infty$ such that $p_c := 1 - x/t < 1$ is fixed. For fixed $p < p_c$ define

$$c_1 := 2p_c p - p + 2\sqrt{pp_c(1-p)(1-p_c)} \tag{3.1}$$

$$c_2 := (p_c(1-p_c))^{1/6} (p(1-p))^{1/2} \left[\left(1 + \sqrt{\frac{(1-p)(1-p_c)}{pp_c}} \right) \times \left(\sqrt{\frac{p_c}{1-p_c}} - \sqrt{\frac{p}{1-p}} \right) \right]^{2/3} \tag{3.2}$$

We will show that

$$\text{Prob} \left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} < s \right) \rightarrow F_2(s)$$

where⁽⁴⁰⁾

$$F_2(s) = \det(I - K_{\text{Airy}}) = \exp \left(- \int_s^\infty (x - s) q(x)^2 dx \right) \tag{3.3}$$

Here K_{Airy} is the operator with Airy kernel acting on $L^2((s, \infty))$ (see (3.5) below) and q is the (unique) solution of the Painlevé II equation

$$q'' = sq + 2q^3$$

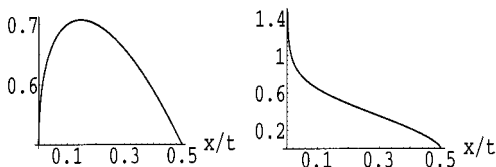


Fig. 4. In the GUE Universal Regime, the left figure displays the limiting shape, c_1 , as a function of x/t and the right figure displays the normalization constant, c_2 , as a function of x/t . In both cases, $p = 1/2$.

with boundary condition $q(s) \sim \text{Ai}(s)$ as $s \rightarrow \infty$. The limiting shape, c_1 , and the normalization constant, c_2 , as functions of x/t are shown in Fig. 4 for $p = 1/2$. The probability density, $f_2 = dF_2/ds$, is shown in Fig. 5.

(2) *Critical Regime:* Let $x \rightarrow \infty, t \rightarrow \infty$ such that

$$x = (1 - p)t + o(\sqrt{t})$$

For fixed $\Delta \in \mathbb{Z}_+$ we will show that

$$\text{Prob}(h_t(x) - (t - x) \leq -\Delta)$$

converges to a $\Delta \times \Delta$ determinant. One can think of this as

$$p = p_c + o\left(\frac{1}{\sqrt{t}}\right)$$

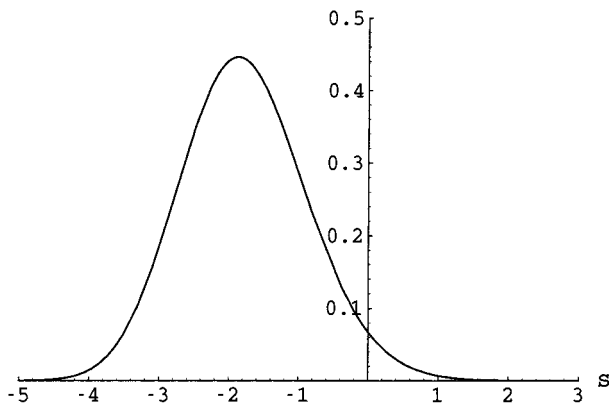


Fig. 5. The density $f_2(s) = dF_2/ds$ where F_2 is defined by (3.3) The distribution function F_2 has mean $\mu = -1.77109$, standard deviation $\sigma = 0.9018$, skewness $S = 0.2241$ and excess kurtosis $K = 0.0935$.

(3) *Deterministic Regime*: For $x \rightarrow \infty$, $t \rightarrow \infty$ and fixed $p > p_c$, we will show that

$$\text{Prob}(h_t(x) = p_c t) \rightarrow 1$$

(4) *Finite x GUE Regime*: Fix x and let $t \rightarrow \infty$, then we will show that

$$\text{Prob}\left(\frac{h_t(x) - pt}{(p(1-p)t)^{1/2}} < s\right)$$

converges to the distribution of the largest eigenvalue in the GUE of $(x + 1) \times (x + 1)$ hermitian matrices, denoted below by F_{x+1}^{GUE} .

3.1. GUE Universal Regime

It is convenient to use the variables $m = t - x$ and $n = x + 1$ rather than x and t and to translate back to the space-time variables at the end. We assume $p < p_c := m/(n + m)$. (This is asymptotically $1 - x/t$ as defined above.) Further, when there is no chance of confusion, we denote the random variable $h_t(x)$ by H . (We reserve lower case h to denote the values of H .) Set $h = cm + sm^{1/3}$, where c will be determined shortly, and $\alpha = n/m$. (In this notation the condition $p < p_c$ is $\alpha r < 1$.) For any v the matrix $((-v)^{k-j} K_h(j, k))$ has the same Fredholm determinant (the determinant of I minus the matrix) as $(K_h(j, k))$. We shall show that for a particular v and a certain constant $b > 0$ this matrix scales to a kernel with the same Fredholm determinant as

$$K_{\text{Airy}}(s/v(3b)^{1/3} + x, s/v(3b)^{1/3} + y) \tag{3.4}$$

on $(0, \infty)$, where

$$K_{\text{Airy}}(s + x, s + y) = \int_0^\infty \text{Ai}(t + s + x) \text{Ai}(t + s + y) dt \tag{3.5}$$

This gives

$$\lim_{m \rightarrow \infty} \text{Prob}\left(\frac{H - cm}{m^{1/3}} \leq s\right) = F_2(s/v(3b)^{1/3}) \tag{3.6}$$

Here is what we mean by scaling. Any matrix $(M(j, k))$ acting on $\ell^2(\mathbf{Z}_+)$ has the same Fredholm determinant as the kernel $M([x], [y])$ on $L^2(0, \infty)$ and this in turn has the same Fredholm determinant as

$M_m(x, y) = m^{1/3}M([m^{1/3}x], [m^{1/3}y])$. If this kernel has the limit $k(x, y)$ we say that the matrix $(M(j, k))$ has, after the scaling $j \rightarrow m^{1/3}x, k \rightarrow m^{1/3}y$, the limit $k(x, y)$. If $M_m(x, y)$ converges to $k(x, y)$ in trace norm then the Fredholm determinant of $(M(j, k))$ converges to that of $k(x, y)$. And if $(M(j, k))$ were the product of two matrices each having scaling limits in Hilbert–Schmidt norm (under the same scaling, of course), then the Fredholm determinant of the product converges to the Fredholm determinant of the product of the limits. This is what we shall show in our case.

There is a slightly awkward notational problem. Since h is always an integer and $h = cm + sm^{1/3}$, the quantity s as it appears here and the analysis which follows is not completely arbitrary. What we actually show is that if h and m tend to infinity, and s is defined in terms of them by the formula $h = cm + sm^{1/3}$, then

$$\text{Prob}(H \leq h) - F_2(s/v(3b)^{1/3}) \rightarrow 0 \tag{3.7}$$

uniformly for s lying in a bounded set. From this we easily deduce (3.6) for fixed s , which now has a different meaning. These observations are important when one tries to estimate errors. It can be shown that the difference in (3.7) is $O(m^{-2/3})$. But the difference between the right side of (3.6) and the probability on the left can only be expected to be $O(m^{-1/3})$. The reason is that if the quantity s' is defined by $cm + s'm^{1/3} = [cm + sm^{1/3}]$ then the probability is within $O(m^{-2/3})$ of $F_2(s'/v(3b)^{1/3})$, but $s - s'$ is very likely of the order $m^{-1/3}$.

3.1.1. The Saddle Point Method

The matrix $(K_h(j, k))$ is the product of two matrices, the matrix on the right having j, k entry $(\varphi_+/\varphi_-)_{-h-j-k-1}$ and the one on the left having j, k entry $(\varphi_-/\varphi_+)_{h+j+k+1}$. Notice that the first vanishes if $h + j + k + 1 > m$ so we may assume that all our indices j and k satisfy $h + j + k < m$. We have

$$\begin{aligned} (\varphi_+/\varphi_-)_{-h-j-k-1} &= \frac{1}{2\pi i} \int (1+z)^n (z-r)^m z^{-m+h+j+k} dz \\ &= (-1)^{h+j+k} \frac{1}{2\pi i} \int (1+z)^n (r-z)^m (-z)^{-m+h+j+k} dz \end{aligned} \tag{3.8}$$

and a similar formula holds for $(\varphi_-/\varphi_+)_{h+j+k+1}$. If we set

$$\psi(z) = (1+z)^n (r-z)^m (-z)^{-(1-c)m}$$

then

$$(-1)^{h+j+k} (\varphi_+ / \varphi_-)_{-h-j-k-1} = \frac{1}{2\pi i} \int \psi(z) (-z)^{sm^{1/3}+j+k} dz$$

and

$$(-1)^{h+j+k} (\varphi_- / \varphi_+)_{h+j+k+1} = \frac{1}{2\pi i} \int \psi(z)^{-1} (-z)^{-sm^{1/3}-j-k-2} dz$$

The contours for the first integral surrounds 0 while the contour for the second integral has r on the inside and -1 on the outside. The restriction $h+j+k < m$ is the same as $sm^{1/3}+j+k < (1-c)m$. If we make the replacements $j \rightarrow m^{1/3}x$, $k \rightarrow m^{1/3}y$ these become

$$\frac{1}{2\pi i} \int \psi(z) (-z)^{m^{1/3}(s+x+y)} dz, \quad \frac{1}{2\pi i} \int \psi(z)^{-1} (-z)^{-m^{1/3}(s+x+y+2m^{-1/3})} dz$$

Our restrictions become $s+x+y < (1-c)m^{2/3}$. For convenience we replace $s+x+y$ by x , and we want to evaluate

$$\frac{1}{2\pi i} \int \psi(z) (-z)^{m^{1/3}x} dz, \quad \frac{1}{2\pi i} \int \psi(z)^{-1} (-z)^{-m^{1/3}x-2} dz \quad (3.9)$$

asymptotically. Our restriction is now $x < (1-c)m^{2/3}$.

To do a steepest descent we have to find the zeros of

$$\frac{d}{dz} \log \psi(z) = \frac{n}{1+z} + \frac{m}{z-r} - \frac{(1-c)m}{z}$$

or equivalently the zeros of

$$(c+\alpha)z^2 + (c+r-cr-\alpha)z + r(1-c)$$

(Recall that $\alpha = n/m$.) The discriminant of this quadratic equals zero when

$$c = \frac{1}{1+r} (2\sqrt{\alpha r} + (1-\alpha)r) \quad (3.10)$$

This is the value of c we take.¹² The critical probability is the condition $c = 1$, i.e., $p_c = m/(m + n)$. The single zero of the quadratic is then at $u = -v$ where

$$v = \frac{(1 - r)c + (1 - \alpha)r}{2(c + \alpha)} = \frac{1 - \sqrt{\alpha r}}{1 + \sqrt{\alpha/r}}$$

Note that $0 < v < 1$ since $0 < p < p_c$. (It is because $u < 0$ that we used powers of $-z$ rather than z in the definition of ψ .) We write

$$\begin{aligned} 6b := \frac{1}{m} \frac{d^3}{dz^3} \log \psi(z) \Big|_{z=u} &= \frac{2\alpha}{(1 + u)^3} + \frac{2}{(u - r)^3} - \frac{2(1 - c)}{u^3} \\ &= \frac{2(\sqrt{\alpha} + \sqrt{r})^5}{r \sqrt{\alpha}(1 + r)^3 (1 - \sqrt{\alpha r})} \end{aligned}$$

The quantity b is positive since $\alpha r < 1$. In the neighborhood of $z = u$,

$$\psi(z) \sim \psi(u) e^{mb(z-u)^3} \tag{3.11}$$

The steepest descent curves will come into u at angles $\pm \pi/3$ and $\pm 2\pi/3$. Call the former C^+ and the latter C^- . For the integral involving $\psi(z)$ we want $|\psi(z)|$ to have a maximum at that point of the curve and for the integral involving $\psi(z)^{-1}$ we want $|\psi(z)|$ to have a minimum there. Since $b > 0$ the curve for $\psi(z)$ must be C^+ and the curve for $\psi(z)^{-1}$ must be C^- . Both contours will be described downward near u . The curve C^+ will loop around the origin and close at r , the upper and lower parts making an angle there depending on c while C^- will loop around on both sides and go to infinity with slopes depending on c . (That C^\pm have these forms follows from the fact that the contours cannot cross and, since the only critical point is at $z = u$, the contours can end only where ψ , respectively ψ^{-1} , is zero.) The steepest descent curves are shown in Fig. 6.

Proceeding formally now, consider the $\psi(z)$ integral and make the substitution $z \rightarrow u + z = -v + z$. Then the old $-z$ becomes the new $v(1 - z/v) \sim ve^{-z/v}$, and recall (3.11). If we make these replacements in the integral we get

$$\psi(u) v^{m^{1/3}x} \frac{1}{2\pi i} \int_{\infty e^{i\pi/3}}^{\infty e^{-i\pi/3}} e^{mbz^3 - m^{1/3}xz/v} dz$$

¹² If there were two critical points, or if we took the negative square root in (3.10), the Fredholm determinant would tend exponentially to either zero or one. It is only for this value of c that we get a nontrivial limit.

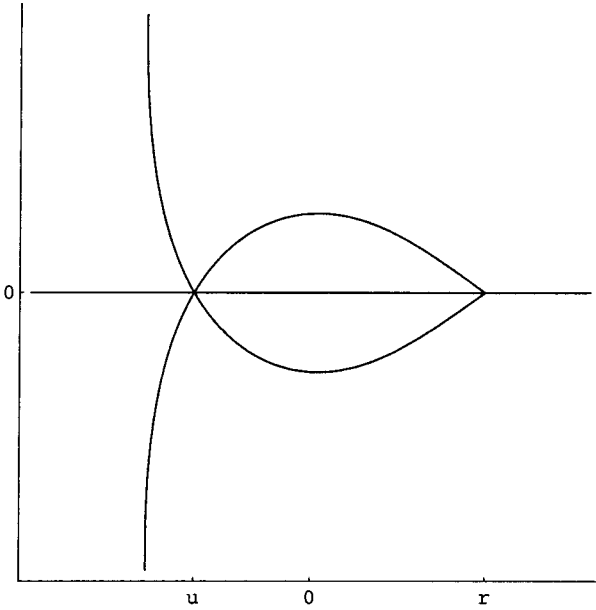


Fig. 6. The steepest descent curves C^\pm as described in the text.

The contour can be deformed to the imaginary axis since we only pass through regions where $\Re z^3$ is negative. If we then set $z = -i\zeta/m^{1/3}$ the above becomes

$$-\psi(u) m^{-1/3} v^{m^{1/3}x} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ib\zeta^3 + ix\zeta/v} d\zeta = -\frac{\psi(u) m^{-1/3} v^{m^{1/3}x}}{(3b)^{1/3}} \text{Ai}(x/v(3b)^{1/3})$$

If we recall that x was a replacement for $s + x + y$ we see that the matrix with j, k entry

$$(-1)^{h+j+k} \psi(u)^{-1} v^{-m^{1/3}s-j-k} (\varphi_+/\varphi_-)_{-h-j-k-1}$$

has the scaling limit

$$-\frac{1}{(3b)^{1/3}} \text{Ai}((s + x + y)/v(3b)^{1/3})$$

Similarly the matrix with j, k entry

$$(-1)^{h+j+k} \psi(u) v^{m^{1/3}s+j+k} (\varphi_-/\varphi_+)_{h+j+k+1}$$

has $1/v^2$ times exactly same scaling limit. Hence the scaling limit of $u^{k-j}K_h(j, k)$, which has the same Fredholm determinant as $(K_h(j, k))$, is the product of these scaling limits,

$$\begin{aligned} & \frac{1}{v^2(3b)^{2/3}} \int_0^\infty \text{Ai}((s+x+t)/v(3b)^{1/3}) \text{Ai}((s+t+y)/v(3b)^{1/3}) dt \\ &= \frac{1}{v(3b)^{1/3}} K_{\text{Airy}}((s+x)/v(3b)^{1/3}, (s+y)/v(3b)^{1/3}) \end{aligned}$$

And, as promised,¹³ this kernel has the same Fredholm determinant as

$$K_{\text{Airy}}(s/v(3b)^{1/3} + x, s/v(3b)^{1/3} + y)$$

3.1.2. Convergence Proof

Now for the justification. We have to obtain not only the pointwise limit, but uniform estimates to establish convergence of the operators in trace norm. We first obtain asymptotics under the assumption that x lies in a bounded set. (Notice that $x \geq s$ always.) We begin with

$$\frac{1}{2\pi i} \int_{C^+} \psi(z) z^{m^{1/3}x} dz \tag{3.12}$$

and denote by C_ϵ^+ the portion of C^+ which lies within ϵ of the critical point u_c .

Lemma 1. If in (3.12) we integrate only over C_ϵ^+ the error incurred is $O(|\psi(u)| e^{-\delta m})$ for some $\delta > 0$.

Proof. Define

$$\sigma(z) = \frac{1}{m} \log \psi(z) = \alpha \log(1+z) + \log(r-z) + (c-1) \log(-z)$$

Its maximum on C^+ (it is real-valued there) is $\sigma(u)$ and it is strictly less than this on the complement of C_ϵ in C^+ . Therefore $\psi(z)/\psi(u) = O(e^{-\delta m})$ for some $\delta > 0$ on the complement while $z^{m^{1/3}x} = e^{O(m^{1/3})}$. This gives the statement of the lemma.

¹³ The time constant $c_1 = p_c c$ and the normalization constant $c_2 = p_c^{1/3} v(3b)^{1/3}$. A computation then gives (3.1) and (3.2).

Lemma 2. We have as $m \rightarrow \infty$

$$\begin{aligned} m^{1/3} \psi(u)^{-1} v^{-m^{1/3}x} \frac{1}{2\pi i} \int \psi(z) z^{m^{1/3}x} dz \\ = -(3b)^{-1/3} \text{Ai}(x/v(3b)^{1/3}) + O(m^{-1/3}) \end{aligned}$$

uniformly for bounded x .

Proof. Near $z = u$

$$\begin{aligned} \sigma(z) &= \sigma(u) + b(z-u)^3 + O((z-u)^4) \\ \log(-z) &= \log v + \frac{1}{u}(z-u) + O((z-u)^2) \\ &= \log v - \frac{1}{v}(z-u) + O((z-u)^2) \end{aligned}$$

Hence, using Lemma 1, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{C^+} \psi(z) z^{m^{1/3}x} dz \\ = O(|\psi(u)| e^{-\delta m}) + \psi(u) v^{m^{1/3}x} \frac{1}{2\pi i} \\ \times \int_{C_\varepsilon^+} e^{mb(z-u)^3 - m^{1/3}x(z-u)/v + O(m(z-u)^4 + m^{1/3}x(z-u)^2)} dz \end{aligned}$$

We show that removing the O term in the exponential in the integrand leads to an error $O(m^{-2/3})$ in the integral. This error equals

$$\begin{aligned} \int_{C_\varepsilon^+} e^{mb(z-u)^3 - m^{1/3}x(z-u)/v} (e^{O(m(z-u)^4 + m^{1/3}x(z-u)^2)} - 1) dz \\ = \int_{C_\varepsilon^+} e^{mb(z-u)^3 - m^{1/3}x(z-u)/v + O(m(z-u)^4 + m^{1/3}x(z-u)^2)} \\ \times O(m(z-u)^4 + m^{1/3}x(z-u)^2) dz \end{aligned}$$

Now the exponential has the form

$$e^{mb(1+\eta_1)(z-u)^3 - m^{1/3}x(1+\eta_2)(z-u)/v}$$

where the η_i can be made arbitrarily small by taking ε small enough. If we make the substitution $z - u = \zeta/m^{1/3}$ the error becomes

$$m^{-2/3} \int e^{b(1+\eta_1)\zeta^3 - x(1+\eta_2)\zeta/v} O(\zeta^4 + x\zeta^2) d\zeta$$

The integral is now taken over a long contour lying in thin angles around the rays $|\arg \zeta| = \pi/3$, with ends having absolute value at least a constant times m . This integral is clearly bounded, uniformly in m for bounded x .

Therefore with the stated error we may remove the O terms from the exponential in the original integral. Then we make the same substitution. The integrand is exponentially small at the ends of the resulting contour. Therefore if we complete it so that it goes to infinity in the two directions $\pm \pi/3$ the error incurred will be exponentially small.

We have shown that

$$m^{1/3}\psi(u)^{-1} v^{-m^{1/3}x} \frac{1}{2\pi i} \int \psi(z) z^{m^{1/3}x} dz = \frac{1}{2\pi i} \int_{\infty e^{i\pi/3}}^{\infty e^{-i\pi/3}} e^{b\zeta^3 - x\zeta/v} d\zeta + O(m^{-1/3})$$

If we deform the contour to the imaginary axis and make the substitution $\zeta \rightarrow -i\zeta$ then the last integral, with its factor, becomes

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ib\zeta^3 + ix\zeta/v} d\zeta = -(3b)^{-1/3} \text{Ai}(x/v(3b)^{1/3})$$

This proves the lemma.

The second integral in (3.9) is similar.

Lemma 3. We have as $m \rightarrow \infty$

$$\begin{aligned} m^{1/3}\psi(u) v^{m^{1/3}x} \frac{1}{2\pi i} \int_{C^-} \psi(z)^{-1} z^{-m^{1/3}x} dz \\ = -(3b)^{-1/3} \text{Ai}(x/v(3b)^{1/3}) + O(m^{-1/3}) \end{aligned}$$

uniformly for bounded x .

Proof. The derivation is essentially the same. The exponentials are replaced by their negatives and the directions $\pm \pi/3$ are replaced by $\pm 2\pi/3$. The fact that C^- is unbounded causes no difficulty since the integrand now behaves at infinity like a large negative power of z . We get the same Airy function in the end, as we have already seen.

Now for the tricky part. We need estimates that are uniform for all x and where the error term contains a factor which is very small for large x . In fact we shall show that the statements of Lemmas 2 and 3 hold, uniformly for all x , when the error terms are replaced by $m^{-1/3}e^{-x}$. (The x in the exponential can be improved to a constant times $x^{3/2}$ but that makes no difference.) To do this we have to be more careful and use the steepest descent curves for the full integrands in (3.9), not just for the factors $\psi^{\pm 1}$. We consider in detail only the first integral in (3.9); as before, the second is treated analogously.

Set

$$\psi(z, c') = (1+z)^n (r-z)^m (-z)^{-(1-c')m} = \psi(z)(-z)^{(c'-c)m}$$

We are interested in the asymptotics of

$$I(c') = \frac{1}{2\pi i} \int \psi(z, c') dz \quad (3.13)$$

when $c' - c = m^{-2/3}x$. Our condition on x says that $c' < 1$ and, in view of what we already know, we may assume x is positive and bounded away from zero, so $c' > c$.

We let C be the steepest descent curve for $\psi(z, c')$. This curve now passes vertically through one of the critical points of $\psi(z, c')$. For $c' > c$ there are two critical points

$$u_{c'}^{\pm} = \frac{-(1-r)c' - (1-\alpha)r \pm \sqrt{((1+r)c' + (\alpha-1)r)^2 - 4\alpha r}}{2(\alpha + c')}$$

which are real and satisfy

$$-1 < u_{c'}^- < -v < u_{c'}^+ < 0$$

To determine which critical point our curve passes through we consider the function

$$\sigma(z, c') = \frac{1}{m} \log \psi(z, c') = \sigma(z) + (c' - c) \log(-z)$$

The critical points $u_{c'}^{\pm}$ are the zeros of $\sigma_z(z, c')$. (Subscripts here and below denote derivatives in the usual way.) We use the fact that $u_{c'}^{\pm}$ are smooth

functions of $\gamma = \sqrt{c' - c}$ and compute, recalling that $\sigma_z(u, c) = \sigma_{zz}(u, c) = 0$ and observing that $dc'/d\gamma = 0$ when $\gamma = 0$,

$$\frac{d}{d\gamma} \sigma_{zz}(u_{c'}^\pm, c') \Big|_{\gamma=0} = \sigma_{zzz}(u, c) \frac{du_{c'}^\pm}{d\gamma} \Big|_{\gamma=0} \tag{3.14}$$

The first factor on the right is positive (we denoted it by $6b$), while

$$\frac{du_{c'}^\pm}{d\gamma} \Big|_{\gamma=0} = \pm \beta \tag{3.15}$$

where

$$\beta = \frac{(\alpha r)^{1/4} (1+r)^{3/2}}{(\sqrt{\alpha} + \sqrt{r})^2}$$

Since $\sigma_{zz}(u_{c'}^\pm, c') = 0$ when $\gamma = 0$ we deduce that for c' close to, but greater than, c we have

$$\sigma_{zz}(u_{c'}^+, c') > 0, \quad \sigma_{zz}(u_{c'}^-, c') < 0$$

These inequalities hold for all c' since the second derivative can be zero only when $c' = c$. This shows that the steepest descent curve C for $\psi(z, c')$ passes through $u_{c'}^+$, because on the curve $|\psi(z, c')|$ has a maximum at the critical point. (Similarly the steepest descent curve for $\psi(z, c')^{-1}$ passes through $u_{c'}^-$.) To make the notation less awkward we write $u_{c'}$ instead of $u_{c'}^+$. First, we have the analogues of Lemmas 2 and 3.

Lemma 4. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $I(c') = O(|\psi(u, c')| e^{-\delta m})$ if $c' - c > \varepsilon$.

Proof. The function $\sigma(z, c')$ is decreasing for $u < z < u_{c'}$ since it decreases near and to the left of $u_{c'}$ and has no critical point in this interval. Hence $\sigma(u_{c'}, c') < \sigma(u, c')$ so $\sigma(u_{c'}, c') - \sigma(u, c')$ is negative and bounded away from zero for $c' > c + \varepsilon$. Since the maximum of $|\psi(z, c')|$ on C is at $z = u_{c'}$ the statement follows.

In view of Lemma 4 we may assume in what follows that $c' - c$ is as small as we please. We denote by C_ε the portion of C which lies within ε of the critical point $u_{c'}$.

Lemma 5. If in the integral (3.13), in which we integrate over C , we integrate only over C_ε the error incurred is $O(|\psi(u, c')| e^{-\delta m})$ for some $\delta > 0$.

Proof. The maximum of $|\psi(z, c')|^{1/m}$ on C occurs at $u_{c'}$ and it is strictly smaller on the complement of C_ε in C than it is at $u_{c'}$. Therefore the integral in question is $O(|\psi(u_{c'}, c')| e^{-\delta m})$ for some $\delta > 0$. Since $\sigma(u_{c'}, c') < \sigma(u, c')$, as we saw in the proof of the last lemma, this one is established.

Because of Lemmas 4 and 5 we need only compute the behavior of $\psi(z, c')$, or equivalently $\sigma(z, c')$, for z near $u_{c'}$. Recall that $u = u_c = -v$.

Lemma 6. We have

- (i) $\sigma(u_{c'}, c') = \sigma(u) + (c' - c) \log v - \frac{2}{3} \frac{\beta}{v} (c' - c)^{3/2} + O((c' - c)^2)$;
- (ii) $\sigma_z(u_{c'}, c') = 0$;
- (iii) $\sigma_{zz}(u_{c'}, c') = 6b\beta \sqrt{c' - c} + O(c' - c)$;
- (iv) $\sigma_{zzz}(u_{c'}, c') = 6b + O(\sqrt{c' - c})$.

Proof. From (3.15) and the fact that $u_{c'}$ is a smooth function of γ (or directly) we see that

$$u_{c'} - u = \beta \sqrt{c' - c} + O(c' - c) \tag{3.16}$$

Consequently, since $u = -v$,

$$\frac{u_{c'}}{u} = 1 - \frac{\beta}{v} \sqrt{c' - c} + O(c' - c) \tag{3.17}$$

Now since $\sigma_z(u_{c'}, c') = 0$ we have

$$\frac{d}{dc'} \sigma(u_{c'}, c') = \frac{\partial}{\partial c'} \sigma(z, c') \Big|_{z=u_{c'}} = \log(-u_{c'}) = \log v + \log \frac{u_{c'}}{u}$$

Integrating with respect to c' from c to c' and using (3.17) we obtain (i). Of course (ii) is immediate. As for (iii) and (iv), these follow from (3.14) and (3.15) and the fact that $u_{c'}$ is a smooth functions of γ .

Lemma 7. The conclusions of Lemmas 2 and 3 hold uniformly for all x when the error terms are replaced by $O(e^{-\delta m}) + O(m^{-1/3}e^{-x})$ for some $\delta > 0$.

Proof. We consider (3.12), which is $I(c')$ with $c' - c = m^{-2/3}x$. Putting together Lemmas 5 and 6 we deduce that

$$I(c') = O(|\psi(u) v^{(c'-c)m}| e^{-\delta m}) + \psi(u) v^{(c'-c)m} e^{-(2\beta/3v)(c'-c)^{3/2}m} \\ \times \frac{1}{2\pi i} \int_{C_\varepsilon} e^{mb(z-u_{c'})^3 + 3mb\beta \sqrt{c'-c} (z-u_{c'})^2 + O(m[(c'-c)^2 + (c'-c)|z-u_{c'}|^2 + \sqrt{c'-c}|z-u_{c'}|^3 + |z-u_{c'}|^4])} dz$$

If $c' - c = m^{-2/3}x$ the exponential factor equals $e^{-(2\beta/3v)x^{3/2}}$ while the integral equals

$$\frac{1}{2\pi i} \int_{C_\varepsilon} e^{mb(z-u_{c'})^3 + 3m^{2/3}bx^{1/2}\beta(z-u_{c'})^2 + O(m^{-1/3}x^2 + m^{1/3}x|z-u_{c'}|^2 + m^{2/3}x^{1/2}|z-u_{c'}|^3 + m|z-u_{c'}|^4)} dz$$

Now C_ε , rather than looking like two rays near the critical point, looks like one branch of a hyperbola.

Note that by Lemma 4 we may assume that $c' - c = m^{-2/3}x$ is as small as desired. It follows that the exponent, without the $O(m^{-1/3}x^2)$ term, can be written

$$m(b + \eta_1)(z - u_{c'})^3 + 3m^{2/3}(b + \eta_2)x^{1/2}\beta(z - u_{c'})^2$$

where, if ε is chosen small enough, the η_i can be made as small as desired. Upon making the variable change $z - u_{c'} = \zeta/m^{1/3}$ the integral becomes

$$\frac{m^{-1/3}}{2\pi i} \int e^{(b + \eta_1)\zeta^3 + 3(b + \eta_2)x^{1/2}\beta\zeta^2} d\zeta$$

taken over a long contour in the right half-plane on which $|\arg \zeta| > \pi/3 - \eta$, with another small η . The integral here is uniformly bounded.

To take care of the term $O(m^{-1/3}x^2)$ in the exponential in the original integral, observe that if $m^{-2/3}x$ is small enough then $m^{-1/3}x^2$ will be at most a small constant times $x^{3/2}$ and so

$$e^{-(2\beta/3v)x^{3/2}}(e^{O(m^{-1/3}x^2)} - 1) = O(m^{-1/2}x^2e^{-(\beta/2v)x^{3/2}}) = O(m^{-1/3}e^{-x})$$

Thus removing the term from the exponential leads to an eventual error $O(m^{-2/3}e^{-x})$. That removing the other O terms from the exponential leads to the same error is seen as it was in the proof of Lemma 2—the substitution in the integral representing the error results in an extra factor $m^{-1/3}$ and there is the exponential factor $e^{-(2\beta/3v)x^{3/2}}$ outside the integral.

After removing all the O terms and making the variable change $z - u_{c'} = \zeta/m^{1/3}$ the integral becomes

$$\frac{m^{-1/3}}{2\pi i} \int e^{b\zeta^3 + 3bx^{1/2}\beta\zeta^2} d\zeta$$

taken over a long contour in the right half-plane on which $|\arg \zeta| > \pi/3 - \eta$. Completing the contour so that it goes to infinity in the directions $\arg \zeta = \pm \pi/3$ leads to an exponentially small error. It follows that (the first

part of) the lemma holds with the negative of the Airy function in the statement replaced by

$$e^{-(2\beta/3v)x^{3/2}} \frac{1}{2\pi i} \int_{\infty e^{i\pi/3}}^{\infty e^{-i\pi/3}} e^{b\zeta^3 + 3bx^{1/2}\beta\zeta^2} d\zeta$$

If we complete the cube and make the substitution $\zeta \rightarrow \zeta - \beta x^{1/2}$ this becomes, upon noting that $3b\beta^2 = 1/v$,

$$\int_{\infty e^{i\pi/3}}^{\infty e^{-i\pi/3}} e^{b\zeta^3 - x\zeta/v} d\zeta = -(3b)^{-1/3} \text{Ai}(x/(v(3b)^{1/3}))$$

The second part of the lemma is analogous, just as the proof of Lemma 3 was analogous to the proof of Lemma 2.

We have now shown that if we set $j = m^{1/3}x$, $k = m^{1/3}y$ then

$$\begin{aligned} & (-1)^{h+j+k} m^{1/3} \psi(u) v^{m^{1/3}s+j+k} (\varphi_+/\varphi_-)_{-h-j-k-1} \\ & \rightarrow -(3b)^{-1/3} \text{Ai}((s+x+y)/v(3b)^{1/3}) \end{aligned}$$

and the difference between the two is $O(m^{-1/3}e^{-(x+y)}) + O(e^{-\delta m})$. It follows easily from this that if we denote the matrix on the left, without the factor $m^{1/3}$, by $(M(j, k))$ and the kernel on the right by $A(x, y)$ then the kernel $m^{1/3}M([m^{1/3}x], [m^{1/3}y])$ converges in Hilbert–Schmidt norm to the kernel $A(x, y)$ on $(0, \infty)$. (Recall that j and k are at most $O(m)$. Therefore the error term $O(e^{-\delta m})$ can only contribute an exponentially small error to the norm and so can be ignored. Similarly we can let our indices j and k run to infinity.) Thus, under the scaling $j \rightarrow m^{1/3}x$, $k \rightarrow m^{1/3}y$ the matrices with j, k entry

$$(-1)^{h+j+k} \psi(u) v^{m^{1/3}s+j+k} (\varphi_+/\varphi_-)_{-h-j-k-1}$$

scale in Hilbert–Schmidt norm to the kernel $A(x, y)$. Similarly so do the matrices with j, k entry

$$(-1)^{h+j+k} \psi(u)^{-1} v^{-(m^{1/3}s+j+k)} (\varphi_-/\varphi_+)_{h+j+k+1}$$

Therefore the product of the matrices scale in trace norm to the (operator) square of the kernel, which is the Airy kernel (3.4). This or completeness the justification.

3.2. Critical Regime: $p \sim p_c$

When $p = p_c$ ($\alpha r = 1$),¹⁴ the analysis of the previous section must be modified. We set $h = m - \Delta h$ ($\Delta h = 0, 1, 2, \dots$) and introduce the new ψ

$$\psi = (1 + z)^n (z - r)^m$$

and the corresponding new σ

$$\sigma(z) = \frac{1}{m} \log \psi = \alpha \log(1 + z) + \log(z - r)$$

The saddle point now occurs at $z = 0$ with $\sigma''(0) = -\alpha(1 + \alpha)$. Thus in the neighborhood of $z = 0$

$$\psi(z) \sim (-1)^m r^m e^{-m\alpha(1 + \alpha) z^2/2} \tag{3.18}$$

Since $(\varphi_+/\varphi_-)_{-h-k-j-1}$ vanishes for $h + j + k + 1 > m$, we can again assume $h + j + k < m$ which becomes the condition $j + k < \Delta h$. As before our starting point is the integral expression

$$(\varphi_+/\varphi_-)_{-h-j-k-1} = \frac{1}{2\pi i} \int \psi(z) z^{-m+h+j+k} dz$$

where the contour is a circle centered at 0 with radius $\rho < 1$. Taking this ρ sufficiently small so that we may use the approximation (3.18) on the integrand, we obtain after making the change of variables

$$\zeta = \left(\frac{m\alpha(1 + \alpha)}{2} \right)^{1/2} z = z/S$$

$$\begin{aligned} (\varphi_+/\varphi_-)_{-h-j-k-1} &\sim (-1)^m r^m S^{j+k-\Delta h+1} \frac{1}{2\pi i} \int e^{-\zeta^2} \zeta^{j+k-\Delta h} d\zeta \\ &= \begin{cases} (-1)^m r^m S^{j+k-\Delta h+1} [(-1)^{L/L}] & \text{if } \Delta h - j - k - 1 = 2L = 0, 2, 4, \dots \\ 0 & \text{if } \Delta h - j - k - 1 = \text{odd integer} \end{cases} \end{aligned}$$

Our second integral is

$$(\varphi_-/\varphi_+)_{h+j+k+1} = \frac{1}{2\pi i} \int \psi(z)^{-1} z^{m-h-j-k-2} dz$$

¹⁴ See the remark at end of this section.

where the contour has -1 on the outside and r on the inside. We deform the contour to the imaginary axis going from $i\infty$ to $-i\infty$ with an infinitesimal indentation going around 0 to the left. The part of the contour lying in the right half plane is exponentially small because of the factor $(1+z)^{-n}$ and can therefore be neglected. For the integral along the imaginary axis we can replace ψ by (3.18) with an error that is exponentially small. Thus the above integral is asymptotically equal to

$$(-1)^m \frac{r^{-m}}{2\pi i} \int_{i\infty}^{-i\infty} e^{z^2/S^2} z^{\Delta h - j - k - 2} dz$$

which in turn equals

$$(-1)^m i^{\Delta h - j - k - 1} r^{-m} S^{\Delta h - j - k - 1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\zeta^2} \zeta^{\Delta h - j - k - 2} d\zeta$$

where there is an indentation above $\zeta = 0$. If we now substitute $\zeta = \sqrt{t}$, the above integral becomes

$$(-1)^m i^{\Delta h - j - k - 1} r^{-m} S^{\Delta h - j - k - 1} \frac{1}{4\pi i} \int_{\infty}^{0^+} e^{-t} t^{(\Delta h - j - k - 1)/2 - 1} dt$$

The contour starts at $+\infty$, loops around 0 in the positive direction and then returns to $+\infty$. This last integral is Hankel's integral representation of the Γ function. Thus

$$\begin{aligned} (\varphi_- / \varphi_+)_{h+j+k+1} &\sim \frac{(-1)^{h+j+k+1}}{2\pi} r^{-m} S^{\Delta h - j - k - 1} \\ &\quad \times \sin\left(\frac{\pi}{2}(\Delta h - j - k - 1)\right) \Gamma\left(\frac{\Delta h - j - k - 1}{2}\right) \end{aligned}$$

We now use these two asymptotic expressions along with the condition $j+k < \Delta h$ in (2.9) to obtain (after a short calculation)

$$K_h(j, k) \sim \frac{(-S)^{k-j}}{2\pi} \sum_{\ell=0}^{[(\Delta h - k - 1)/2]} \frac{1}{\ell!} \sin \frac{\pi}{2} (k-j) \Gamma\left(\ell + \frac{k-j}{2}\right) \quad (3.19)$$

When $\ell + (k-j)/2$ is a nonpositive integer, the product of the sine and gamma functions is replaced by

$$\frac{(-1)^\ell \pi}{\left(\frac{j-k}{2} - \ell\right)!}$$

Table I. Limiting Distribution when $p \sim p_c$

Δh	$\lim_{m \rightarrow \infty} \text{Prob}(H - m \leq -\Delta h)$	Numerical value
0	1	1.0
1	$\frac{1}{2}$	0.5
2	$\frac{1}{4} - \frac{1}{2\pi}$	9.08451×10^{-2}
3	$\frac{1}{8} - \frac{3}{8\pi}$	5.63379×10^{-3}
4	$\frac{1}{16} + \frac{1}{3\pi^2} - \frac{29}{96\pi}$	1.17616×10^{-4}
5	$\frac{1}{32} + \frac{41}{144\pi^2} - \frac{145}{768\pi}$	8.22908×10^{-7}
6	$\frac{1}{64} - \frac{32}{135\pi^3} + \frac{1169}{3840\pi^2} - \frac{1249}{10240\pi}$	1.92570×10^{-9}
7	$\frac{1}{128} - \frac{49}{225\pi^3} + \frac{198827}{921600\pi^2} - \frac{8743}{122880\pi}$	1.50565×10^{-12}
8	$\frac{1}{256} + \frac{4096}{23625\pi^4} - \frac{10289}{36000\pi^3} + \frac{5773487}{34406400\pi^2} - \frac{145603}{3440640\pi}$	3.92048×10^{-16}
9	$\frac{1}{512} + \frac{15376}{91875\pi^4} - \frac{5528469}{25088000\pi^3} + \frac{279234531}{2569011200\pi^2} - \frac{436809}{18350080\pi}$	3.42524×10^{-20}

The factor $(-S)^{k-j}$ may be dropped when computing the determinant $\det(I - K_h)$ since it does not change its value. We evaluate this determinant and display the results for $\Delta h \leq 9$ in Table I.

Remark. Since m and n are integers it is extremely unlikely that $p = p_c = m/(m+n)$. If p is irrational this never occurs. However the preceding analysis shows that if $\alpha r = 1 + o(m^{-1/2})$ rather than 1 then in the integrals one gets extra factors $(1+z)^{o(m^{1/2})}$. Then after the substitution $z = S\zeta$ this drops out since $S = O(m^{-1/2})$. The upshot is that the asymptotics hold for any p when m and n go to infinity in such a way that $m/(m+n) = p + o(m^{-1/2})$.

3.3. Deterministic Regime: $p > p_c$

3.3.1. Large Deviations Approach

Assume that $p > p_c$. Then there exists an $\varepsilon > 0$ so that n/m approaches $(1 + \varepsilon)(1/p - 1)$. To simplify the statements, we will just assume that $n = (1 + \varepsilon)(1/p - 1)m$.

Imagine the random $m \times n$ matrix A from Section 2.1 as the lower left corner of an infinite matrix of 0's and 1's, created by the independent coin flips. Fix a position (i, j) ($i, j \geq 1$) in this infinite random matrix. Define J as the column index of the first entry, from left to right, with a 1 on the row above (i, j) and in the columns larger or equal j . Then define $\xi_{(i, j)} = J - j$. In the example given, $\xi_{(3, 1)} = 0$ and $\xi_{(5, 1)} = 3$.

Now create a sequence of i.i.d. random variables ξ_1, ξ_2, \dots , as follows. Let ξ_1 equal the column index minus one of the first 1 on the first row. Then let $\xi_2 = \xi_{(1, 1 + \xi_1)}$, $\xi_3 = \xi_{(2, 1 + \xi_1 + \xi_2)}$, ... The basic observation is that, since we are always taking the best positioned 1 on the next line, we have equality of the two events

$$\{ \text{there is an increasing path of length } m \text{ in } A \} = \{ \xi_1 + \dots + \xi_m < n \}$$

Therefore, we need to show that

$$\text{Prob}(\xi_1 + \dots + \xi_m \geq n)$$

goes to 0 exponentially as $m \rightarrow \infty$. However, $\text{Prob}(\xi_1 = i) = p(1 - p)^i$, $i = 0, 1, \dots$ and so $E(\xi_1) = 1/p - 1$. By elementary large deviations (e.g., Section 1.9 in ref. 17),

$$-m^{-1} \log P(\xi_1 + \dots + \xi_m \geq n) \rightarrow \gamma(\varepsilon)$$

where an elementary calculation shows

$$\gamma(\varepsilon) = (1/p - 1)(1 + \varepsilon) \log(1 + \varepsilon) - p^{-1}(1 + \varepsilon - \varepsilon p) \log(1 + \varepsilon - \varepsilon p)$$

which is positive whenever $\varepsilon > 0$.

3.3.2. Saddle Point Approach

For completeness, we show how the saddle point method gives the same result. Thus we show that when $p > p_c$ (or $\alpha r > 1$)

$$\det(I - K_h) \rightarrow 0$$

exponentially as $m \rightarrow \infty$ even when $h = m - 1$, thus establishing assertion 1(c) in Section 3 with exponential approach to the limit.

As we saw at the beginning in the last section we need only consider the entries $K_h(j, k)$ when $h + j + k < m$, which in the present situation means $j = k = 0$. Our claim is therefore that $K_{m-1}(0, 0) \rightarrow 1$ exponentially as $m \rightarrow \infty$. The first integral to consider is

$$(\varphi_+ / \varphi_-)_{-m} = \frac{1}{2\pi i} \int \psi(z) z^{-1} dz = \psi(0) = (-r)^m$$

The second integral is

$$(\varphi_-/\varphi_+)_m = \frac{1}{2\pi i} \int \psi(z)^{-1} z^{-1} dz$$

Recall that the contour here surrounds 0 and has -1 on the outside, r on the inside. The critical point for steepest descent is at $z = u$ where

$$\frac{\alpha}{1+u} + \frac{1}{u-r} = 0, \quad u = \frac{\alpha r - 1}{\alpha + 1}$$

The steepest descent curve will pass vertically through this point and go to ∞ in two directions. But notice that since u is positive, in order to deform our original contour to this one we have to pass through $z = 0$. The residue of the integrand there equals $(-r)^{-m}$ and so

$$(\varphi_-/\varphi_+)_m = (-r)^{-m} + \frac{1}{2\pi i} \int \psi(z)^{-1} z^{-1} dz$$

where now the integral is taken over the steepest descent curve. This integral is asymptotically a constant times $m^{-1/2}$ times the value of the integrand at $z = u$, and this value equals $(-1)^m$ times

$$\left(\frac{\alpha(r+1)}{\alpha+1}\right)^{-\alpha m} \left(\frac{r+1}{\alpha+1}\right)^{-m}$$

Our claim is therefore equivalent to the statement that this is exponentially smaller than r^{-m} , which in turn is equivalent to the inequality

$$(r+1)^{\alpha+1} \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} > r$$

It is an elementary exercise that this is true for all $r \geq 0$ except for $r = 1/\alpha$, when equality holds. But in our case $r > 1/\alpha$ so the inequality holds.

3.4. Finite GUE Regime: Fixed x and $t \rightarrow \infty$

3.4.1. Saddle Point Calculation

We return to (3.8) and this time set

$$h = \frac{r}{1+r} m + sm^{1/2} = pm + sm^{1/2}$$

and make the substitutions $j \rightarrow xm^{1/2}$, $k \rightarrow ym^{1/2}$ to write the integral (3.8) as

$$\frac{1}{2\pi i} \int (1+z)^n (r-z)^m (-z)^{-m/(1+r)} (-z)^{(s+x+y)m^{1/2}} dz \quad (3.20)$$

Now we set

$$\psi(z) = (r-z)^m (-z)^{-m/(1+r)}$$

which is the main part of the integrand. There is a single critical point, $z = -1$, and at this point $d^2/dz^2 \log \psi(z)$ is equal to

$$m \frac{r}{(1+r)^2} = mp(1-p)$$

This is positive and so the steepest descent curve is vertical at the critical point; it goes around the origin and closes at $z = r$. The main contribution to the integral comes from the immediate neighborhood of the critical point. If we make the variable change

$$z = -1 + \frac{\zeta}{\sqrt{m}}$$

and take into account the other factors in the integrand we see the integral is asymptotically

$$-\frac{(r+1)^m}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{\zeta}{\sqrt{m}}\right)^n e^{(1/2)p(1-p)\zeta^2 - (s+x+y)\zeta} \frac{d\zeta}{\sqrt{m}} \quad (3.21)$$

Now

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{a\zeta^2 - b\zeta} d\zeta = \frac{e^{-b^2/4a}}{2\sqrt{a\pi}}$$

and so

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \zeta^n e^{a\zeta^2 - b\zeta} d\zeta &= \frac{(-1)^n}{2\sqrt{a\pi}} \frac{d^n}{db^n} e^{-b^2/4a} \\ &= \frac{1}{\sqrt{\pi}(2\sqrt{a})^{n+1}} e^{-b^2/4a} H_n \left(\frac{b}{2\sqrt{a}} \right) \end{aligned}$$

(H_n are the Hermite polynomials.) Hence our first integral (3.20) is asymptotically equal to $-(r+1)^m/\sqrt{m}^{n+1}$ times this expression with

$$a = \frac{1}{2}p(1-p), \quad b = s+x+y$$

Thus we have shown that the matrix with j, k entry

$$(-1)^{-h-j-k} (r+1)^{-m} m^{n/2} (\varphi_+/\varphi_-)_{-h-j-k-1}$$

scales to the operator on $(0, \infty)$ with kernel

$$-\frac{1}{\sqrt{\pi}(2\sqrt{a})^{n+1}} e^{-(s+x+y)^2/4a} H_n\left(\frac{s+x+y}{2\sqrt{a}}\right)$$

with a as given above.

Next, with the same substitutions in the integral,

$$(\varphi_-/\varphi_+)_{h+j+k+1}$$

$$= \frac{1}{2\pi i} \int (1+z)^{-n} (z-r)^{-m} z^{m-h-j-k-2} dz$$

$$= (-1)^{-h-j-k} \frac{1}{2\pi i} \int (1+z)^{-n} (r-z)^{-m} (-z)^{m/(1+r)} (-z)^{-(s+x+y)m^{1/2}-2} dz$$

The contour here encloses 0 and r and has -1 on the outside. The steepest descent curve should go through the critical point -1 horizontally. We deform the given contour to a curve starting at $-\infty + 0i$, going above the the real axis, looping around $z = -1$ clockwise, then back below the real axis to $-\infty - 0i$. The original contour can be deformed to this because the integrand is small at ∞ . The main contribution is again in the neighborhood of $z = -1$. Making the same variable change as before leads to an integral which is asymptotically

$$-\frac{(r+1)^{-m}}{2\pi i} \int \left(\frac{\zeta}{\sqrt{m}}\right)^{-n} e^{-(1/2)p(1-p)\zeta^2+(s+x+y)\zeta} \frac{d\zeta}{\sqrt{m}} \tag{3.22}$$

where now the contour is a circle going around $\zeta = 0$ counterclockwise. Using now the fact

$$\begin{aligned} \frac{1}{2\pi i} \int \zeta^{-n} e^{-a\zeta^2+b\zeta} d\zeta &= \frac{a^{(n-1)/2}}{(n-1)!} e^{-b^2/4a} \frac{d^{n-1}}{d\zeta^{n-1}} e^{-(\zeta-b/2\sqrt{a})^2} \Big|_{\zeta=0} \\ &= \frac{a^{(n-1)/2}}{(n-1)!} H_{n-1}\left(\frac{b}{2\sqrt{a}}\right) \end{aligned}$$

we find that the matrix with j, k entry

$$(-1)^{h+j+k} (r+1)^m m^{-n/2} (\varphi_- / \varphi_+)_{h+j+k+1}$$

scales to the operator on $(0, \infty)$ with kernel

$$-\frac{a^{(n-1)/2}}{(n-1)!} H_{n-1} \left(\frac{s+x+y}{2\sqrt{a}} \right)$$

Combining, we see that the product of the two matrices (aside from a factor $(-1)^{j-k}$, which does not affect the determinant) has scaling limit the operator with kernel

$$\frac{1}{\sqrt{\pi} 2^{n+1} a (n-1)!} \int_0^\infty e^{-(s+x+z)^2/4a} H_n \left(\frac{s+x+z}{2\sqrt{a}} \right) H_{n-1} \left(\frac{s+z+y}{2\sqrt{a}} \right) dz$$

Instead of a direct evaluation of this last integral, we will not evaluate our ζ integrals (3.21) and (3.22), but rather consider them as integrals with variables ζ_1 and ζ_2 , combine and integrate with respect to z . We see that the scaled kernel for the product is

$$-\frac{1}{4\pi^2} \int_0^\infty \iint \left(\frac{\zeta_1}{\zeta_2} \right)^n e^{a(\zeta_1^2 - \zeta_2^2) - (s+x+z)\zeta_1 + (s+z+y)\zeta_2} dz d\zeta_1 d\zeta_2$$

where the ζ_1 contour is a vertical line described upward and the ζ_2 contour goes around 0 counterclockwise. If the vertical line is to the right of the circle we can integrate first with respect to z , yielding

$$-\frac{1}{4\pi^2} \iint \left(\frac{\zeta_1}{\zeta_2} \right)^n e^{a(\zeta_1^2 - \zeta_2^2) - (s+x)\zeta_1 + (s+y)\zeta_2} \frac{d\zeta_1 d\zeta_2}{\zeta_1 - \zeta_2}$$

Let's call this $L_n(x, y)$. This is 0 when $n = 0$, and

$$L_k(x, y) - L_{k-1}(x, y) = -\frac{1}{4\pi^2} \iint \frac{\zeta_1^{k-1}}{\zeta_2^k} e^{a(\zeta_1^2 - \zeta_2^2) - (s+x)\zeta_1 + (s+y)\zeta_2} d\zeta_1 d\zeta_2$$

This integral is a product and we can use the computations we did above to see that it equals

$$\frac{1}{2^k \sqrt{\pi a} (k-1)!} e^{-(s+x)^2/4a} H_{k-1} \left(\frac{s+x}{2\sqrt{a}} \right) H_{k-1} \left(\frac{s+y}{2\sqrt{a}} \right)$$

If φ_k are the oscillator wave functions¹⁵ then this equals

$$\frac{1}{2\sqrt{a}} \varphi_{k-1}\left(\frac{s+x}{2\sqrt{a}}\right) \varphi_{k-1}\left(\frac{s+y}{2\sqrt{a}}\right)$$

times the factor

$$e^{-(s+x)^2/8a} e^{(s+y)^2/8a}$$

It follows that if $K_{H,n}$, is the Hermite kernel then

$$L_n(x, y) = e^{-(s+x)^2/8a} \frac{1}{2\sqrt{a}} K_{H,n}\left(\frac{s+x}{2\sqrt{a}}, \frac{s+y}{2\sqrt{a}}\right) e^{(s+y)^2/8a} \quad (3.23)$$

We deduce that

$$\lim_{m \rightarrow \infty} \text{Prob}(H \leq pm + sm^{1/2})$$

is equal to the Fredholm determinant of

$$\frac{1}{2\sqrt{a}} K_{H,n}\left(\frac{s+x}{2\sqrt{a}}, \frac{s+y}{2\sqrt{a}}\right)$$

over $(0, \infty)$, or equivalently the Fredholm determinant of $K_{H,n}(x, y)$ over $(s/2\sqrt{a}, \infty)$. It is notationally convenient to introduce

$$\sigma^2 := 2a = p(1-p)$$

and to define

$$F_n^{\text{GUE}}(s) := \lim_{m \rightarrow \infty} \text{Prob}\left(\frac{H - pm}{\sigma\sqrt{m}} \leq s\right)$$

This equals the Fredholm determinant of $K_{H,n}$ over $(s/\sqrt{2}, \infty)$ and is equal to the distribution of the largest eigenvalue in the finite n GUE.¹⁶

¹⁵ The oscillator wave functions are $\varphi_k(x) := e^{-x^2/2} H_k(x) / \sqrt{2^k k! \pi^{1/2}}$ and form an orthonormal basis for $L^2((0, \infty))$. The Hermite kernel is $K_{H,n}(x, y) := \sum_{k=0}^{n-1} \varphi_k(x) \varphi_k(y)$.

¹⁶ Our normalization of F_n^{GUE} differs from the usual one^(33, 42) by a factor of $\sqrt{2}$, i.e., the usual normalization is the Fredholm determinant of the Hermite kernel over (s, ∞) .

3.4.2. Moments of F_n^{GUE}

From the theory of random matrices, e.g., refs. 33 and 42, we know that the distribution function $\det(I - K_{H,n})$ has an alternative representation as an $n \times n$ determinant. Explicitly,

$$F_n^{\text{GUE}}(s) = \det \left(\delta_{i,j} - \int_{s/\sqrt{2}}^{\infty} \varphi_i(x) \varphi_j(x) dx \right)_{0 \leq i, j \leq n-1}$$

where φ_j are the oscillator functions previously introduced. This last representation implies that the F_n^{GUE} are expressible in terms of elementary functions and the error function with increasing complexity for increasing values of n . In the simplest case, $n=1$, F_1^{GUE} is the standard normal; a result easily anticipated from the original formulation of the growth model. The next simplest case is $n=2$,

$$F_2^{\text{GUE}}(s) = \frac{1}{4} - \frac{1}{2\pi} e^{-s^2} - \frac{1}{2^{3/2} \sqrt{\pi}} s e^{-s^2/2} + \frac{1}{2} \left(1 - \frac{1}{\sqrt{2\pi}} s e^{-s^2/2} \right) \text{erf}(s/\sqrt{2}) + \frac{1}{4} \text{erf}(s/\sqrt{2})^2$$

The moments of F_n^{GUE} are, of course,

$$\mu_j(n) := \int_{-\infty}^{\infty} s^j f_n^{\text{GUE}}(s) ds, \quad j = 1, 2, \dots$$

where $f_n^{\text{GUE}} = dF_n^{\text{GUE}}/ds$. For $1 \leq n \leq 5$ we have,

First Moments:

$$\mu_1(1) = 0$$

$$\mu_1(2) = \frac{2}{\sqrt{\pi}} \approx 1.128379$$

$$\mu_1(3) = \frac{27}{8\sqrt{\pi}} \approx 1.904140$$

$$\mu_1(4) = -\frac{7}{48\sqrt{2}\pi^{3/2}} + \frac{475}{128\sqrt{\pi}} + \frac{475}{64\pi^{3/2}} \arcsin(1/3) \approx 2.528113$$

$$\mu_1(5) = \frac{13715}{4096\sqrt{\pi}} - \frac{16975}{41472\sqrt{2}\pi^{3/2}} + \frac{41145}{2048\pi^{3/2}} \arcsin(1/3) \approx 3.063268$$

Second Moments:

$$\mu_2(1) = 1$$

$$\mu_2(2) = 2$$

$$\mu_2(3) = 3 + \frac{9\sqrt{3}}{4\pi} \approx 4.240490$$

$$\mu_2(4) = 4 + \frac{16}{\sqrt{3}\pi} \approx 6.940420$$

$$\mu_2(5) = 5 - \frac{155\sqrt{5}}{864\pi^2} + \frac{2495}{108\sqrt{3}\pi} + \frac{499}{54\sqrt{3}\pi^2} \arcsin(1/4) \approx 1.977575$$

Third Moments:

$$\mu_3(1) = 0$$

$$\mu_3(2) = \frac{7}{\sqrt{\pi}} \approx 3.949327$$

$$\mu_3(3) = \frac{297}{16\sqrt{\pi}} \approx 10.472769$$

$$\mu_3(4) = \frac{333}{32\sqrt{2}\pi^{3/2}} + \frac{7109}{256\pi^{1/2}} + \frac{7109}{128\pi^{3/2}} \arcsin(1/3) \approx 20.378309$$

$$\mu_3(5) = \frac{2595475}{82944\sqrt{2}\pi^{3/2}} + \frac{259385}{8192\sqrt{\pi}} + \frac{778155}{4096\pi^{3/2}} \approx 33.432221$$

Fourth Moments:

$$\mu_4(1) = 3$$

$$\mu_4(2) = 9$$

$$\mu_4(3) = 19 + \frac{33\sqrt{3}}{2\pi} \approx 28.096927$$

$$\mu_4(4) = 33 + \frac{496}{3\sqrt{3}\pi} \approx 63.384348$$

$$\mu_4(5) = 51 + \frac{7475\sqrt{5}}{1296\pi^2} + \frac{99575}{324\sqrt{3}\pi} + \frac{99575}{162\sqrt{3}\pi^2} \arcsin(1/4) \approx 117.872208$$

Let H_n^∞ denote the weak limit $m \rightarrow \infty$, n fixed, of

$$\frac{H - pm}{\sigma \sqrt{m}}$$

and H^∞ the weak limit $m \rightarrow \infty$, $n \rightarrow \infty$, $\alpha = n/m$ fixed, of

$$\frac{1}{v(3b)^{1/3} m^{1/3}} (H - cm)$$

(Thus the distribution functions of H_n^∞ and H^∞ are F_n^{GUE} and F_2 , respectively.) For $\alpha \rightarrow 0$, $c = p + 2\sigma \sqrt{\alpha} + O(\alpha)$, $(3b)^{1/3} \sim \sigma/\alpha^{1/6}$, and $v \sim 1$. Proceeding heuristically,

$$\begin{aligned} H &\sim cm + m^{1/3}v(3b)^{1/3} H^\infty \\ &\sim pm + 2\sigma \sqrt{\alpha} m + m^{1/3}\sigma\alpha^{-1/6} H^\infty \\ &\sim pm + \sigma m^{1/2} \left\{ 2 \sqrt{n} + \frac{H^\infty}{n^{1/6}} \right\} \end{aligned}$$

Thus we expect

$$H_n^\infty \sim 2 \sqrt{n} + \frac{H^\infty}{n^{1/6}}$$

and hence

$$E(H_n^\infty) \approx 2 \sqrt{n} + \frac{E(H^\infty)}{n^{1/6}}, \quad E(H^\infty) = -1.77109... \tag{3.24}$$

$$\text{Var}(H_n^\infty) \approx \frac{\text{Var}(H^\infty)}{n^{1/3}}, \quad \text{Var}(H^\infty) = 0.8132... \tag{3.25}$$

For $2 \leq n \leq 9$, these approximations are compared with the exact moments in Table II. We also compute the skewness and the excess kurtosis¹⁷ of F_n^{GUE} .

The densities f_n^{GUE} , $1 \leq n \leq 7$, are graphed in Fig. 7.

¹⁷ The skewness of a random variable X is $E((\frac{X-\mu}{\sigma})^3)$ and the excess kurtosis is $E((\frac{X-\mu}{\sigma})^4) - 3$. Here $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$.

Table II. The Mean (μ) and the Variance (σ^2) of H_n^∞ , $2 \leq n \leq 9$, (H_n^∞ Has Distribution Function F_n^{GUE}) Are Compared with the Approximations (3.24) and (3.25), Respectively. Also Displayed Are the Skewness (S) and Excess Kurtosis (K) of H_n^∞ . H^∞ Has $S \approx 0.2241$ and $K \approx 0.0935$

n	μ	Approx	σ^2	Approx	S	K
2	1.12838	1.251	0.72676	0.645	0.08465	0.01053
3	1.90414	1.989	0.61474	0.564	0.11862	0.02192
4	2.52811	2.594	0.54907	0.512	0.13749	0.03042
5	3.06327	3.118	0.50426	0.476	0.14972	0.03683
6	3.53861	3.585	0.47101	0.448	0.15838	0.04184
7	3.97026	4.011	0.44497	0.425	0.16490	0.04586
8	4.36822	4.405	0.42379	0.407	0.17001	0.04917
9	4.73920	4.772	0.40609	0.391	0.17414	0.05195

4. BROWNIAN MOTION REPRESENTATION IN THE FINITE x GUE REGIME

Let $B(t) = (B_0(t), \dots, B_x(t))$ be the $(x + 1)$ -dimensional Brownian motion. Let F be the following functional on continuous functions $f = (f_0, \dots, f_x)$ from $[0, 1]$ to \mathbf{R}^{x+1} , which satisfy $f(0) = 0$,

$$F(f) := \max \{ f_0(t_0) + f_1(t_1) - f_1(t_0) + \dots + f_x(t_x) - f_x(t_{x-1}) : 0 \leq t_0 \leq t_1 \leq \dots \leq t_x = 1 \}$$

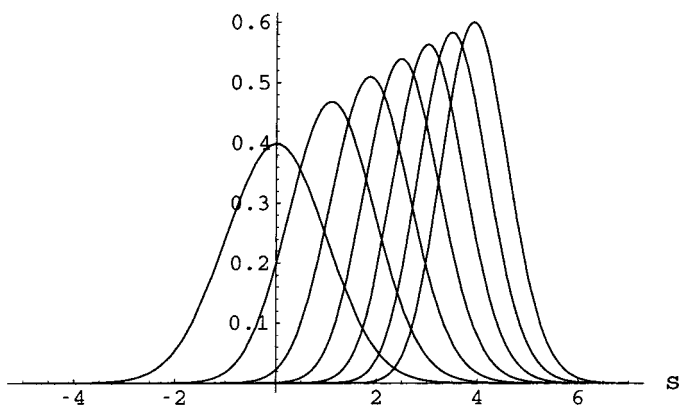


Fig. 7. The densities f_n^{GUE} , $1 \leq n \leq 7$. For increasing values of n the maximum of f_n^{GUE} moves to the right.

Note that F is continuous in the L^∞ metric. Finally, let

$$M_x := F(B)$$

Theorem. For $x \in \mathbf{Z}_+$, and $t \rightarrow \infty$, we have

$$\frac{h_t(x) - pt}{\sigma \sqrt{t}} \xrightarrow{d} M_x$$

where $\sigma^2 = p(1 - p)$.

Proof. First, the path representation tells us that we change $h_t(x)$ by at most a constant if we only obey the increasing property within the same column. That is, we change $L(x, t)$ to $L'(x, t)$, where $L'(x, t)$ is the longest path (x_i, t_i) , $i = 1, \dots, k$, of marked points such that $0 \leq t_i - t_{i-1} - 1$ if $x_i = x_{i-1}$, while $0 \leq x_i - x_{i-1} \leq t_i - t_{i-1} - 1$ if $x_i \neq x_{i-1}$. Thus, the first observation is

$$|L(x, t) - L'(x, t)| \leq x$$

Let S_k^i equal the length of the longest increasing sequence of points (t, i) , $0 \leq t \leq k$. Then

$$L'(x, t) = \max \{ S_{k_1}^0 + S_{k_2}^1 - S_{k_1}^1 + \dots + S_{k_x}^x - S_{k_{x-1}}^x : 0 \leq k_1 \leq k_2 \leq \dots \leq k_x = t - x \} \quad (4.26)$$

Now for every fixed i , S_k^i is independent of S_k^j for $j \neq i$. Let X_k^i equal the indicator of the event that (i, k) is a marked point. Of course, $S_k^i = \sum_{\ell=1}^k X_\ell^i$.

Let $S^i(\tau)$, $\tau \in \mathbf{R}_+$, equal S_k^i when $\tau = k$ and be obtained by linear interpolation off the integers. Moreover, let \tilde{S}^i be the centered versions $\tilde{S}^i(\tau) = S^i(\tau) - p\tau$, and $\tilde{S}(\tau) = (\tilde{S}_0(\tau), \dots, \tilde{S}_x(\tau))$. For $0 \leq \tau \leq 1$, define

$$X_t(\tau) := \frac{\tilde{S}(t\tau)}{\sigma \sqrt{t}}$$

Then the standard invariance principle (see, e.g., ref. 17, Chapt. 7) implies that X_t converges as $t \rightarrow \infty$ in distribution to the $(x + 1)$ -dimensional Brownian motion B .

Now define

$$L''(x, t) = \max \{ \tilde{S}^0(t_0) + \tilde{S}^1(t_1) - \tilde{S}^1(t_0) + \dots + \tilde{S}^x(t_x) - \tilde{S}^x(t_{x-1}) : 0 \leq t_0 \leq t_1 \leq \dots \leq t_x = t \} \quad (4.27)$$

Then

$$|L'(x, t) - pt - L''(x, t)| \leq 5x$$

(The linear interpolation gives an error of at most four at each t_i and we incur an additional x by replacing $t - x$ by t .) Note now that (by making a substitution $t'_i = t_i/t$)

$$\frac{L''(x, t)}{\sigma \sqrt{t}} = F(X_t)$$

It follows (by continuity of F) that the theorem holds with $L''(x, t)$ in place of $h_t(x)$, but this is clearly enough.

Remark 1. We should note that this theorem clearly holds in more general circumstances. For example, we could make every \times count an independent random number of jumps. We would get the same theorem, with the only assumption that the said random number has finite variance.

Remark 2. The theorem also holds for random words over an alphabet with n letters,⁽⁴⁴⁾ except that the $x + 1$ Brownian motions are not independent, but they have to sum to 0, so the covariances Γ_{ij} equals $(n - 1)/n^2$ when $i = j$ and $-1/n^2$ otherwise. In the case of two equiprobable letters, the limiting distribution of the centered and normalized length of the longest weakly increasing subsequence in a random word is equal to the distribution of the random variable

$$\begin{aligned} X &= \max_{0 \leq t \leq 1} (B_0(t) + B_1(1) - B_1(t)) \\ &= 2 \max_{0 \leq t \leq 1} (B_0(t)) - B_0(1) \\ &= 2M - N \end{aligned}$$

(M denotes the random variable $\max_{0 \leq t \leq 1} B_0(t)$ and N denotes the random variable $B_0(1)$.) From the reflection principle it follows (see, e.g., p. 395 in ref. 17) that the joint density of (M, N) is

$$f_{M, N}(m, n) = \sqrt{\frac{2}{\pi}} (2m - n) e^{-(2m - n)^2/2}, \quad \text{for } m \geq 0, \quad m \geq n$$

Thus the density of X equals¹⁸

$$f_X(x) = \int_0^x f_{M,N}(m, 2m - x) dm = \sqrt{\frac{2}{\pi}} x^2 e^{-x^2/2}$$

Remark 3. Limiting distribution of the centered and normalized $h_t(1)$: Here we have

$$\begin{aligned} M_1 &= \max_{0 \leq t \leq 1} (B_1(t) + (B_2(1) - B_2(t))) \\ &= \max_{0 \leq t \leq 1} (B_1(t) - B_2(t)) + B_2(1) = M + N \end{aligned}$$

(The random variables M and N are defined by the last equality; and therefore, are not to be confused with the random variables of the previous remark.) Note that N is standard normal. Since $(B_1 - B_2)/\sqrt{2}$ is the standard Brownian motion, M equals, in distribution, $\sqrt{2} |N|$ again by the reflection principle. Even though M and N are not independent, $E(M_1) = \sqrt{2} E(|N|) = 2/\sqrt{\pi}$. Moreover, the conditional distribution of N given the entire path of $W := B_1(t) - B_2(t)$, $0 \leq t \leq 1$, depends only on its final point $W(1)$. Given this final point S equals s , the distribution is normal with mean $-s/2$ and variance $1/2$. That is, if \mathcal{F}_t is the Brownian filtration for W , $S = W(1)$, then

$$\begin{aligned} \text{Prob}(N \in dn \mid \mathcal{F}_1) &= \text{Prob}(N \in dn \mid S = s) \\ &= \frac{1}{\pi} e^{-(x+s/2)^2} dn \end{aligned}$$

This makes it immediately possible to compute the second moment of M_1 , since

$$\begin{aligned} E(MN) &= E(E(MN \mid \mathcal{F}_1)) = E(ME(N \mid \mathcal{F}_1)) \\ &= -E(MS)/2 = -E((M/\sqrt{s})(S/\sqrt{2})) = -1/2 \end{aligned}$$

by a straightforward computation with the joint density above. Therefore $E(M_1^2) = E(M^2) + E(N^2) + 2E(MN) = 2$.

¹⁸C. Grinstead, in unpublished notes, also found a random walk interpretation of the two-letter random word problem and used this to determine the limiting distribution in this case.

In this way, the density of M_1 is

$$\begin{aligned} f_{M_1}(x) &= E(\text{Prob}(M_1 = x \mid M, S)) \\ &= \int f_{M,S}(m, s) \text{Prob}(M_1 = x \mid M = m, S = s) \, dm \, ds \\ &= \int_0^\infty dm \int_{-\infty}^m \frac{1}{2} f_{M,N}(m/\sqrt{2}, n/\sqrt{2}) \frac{1}{\sqrt{\pi}} e^{-(x-m+n/2)^2} \, dn \end{aligned}$$

An explicit evaluation shows this last integral equals, as it must, $f_2^{\text{GUE}}(x)$.

Remark 4. Since the distribution function of M_x equals F_{x+1}^{GUE} , it follows from RMT⁽³³⁾ that we have the alternative representation

$$\text{Prob}(M_x \leq s) = c_n \int_{-\infty}^s \cdots \int_{-\infty}^s \Delta(x)^2 e^{-(1/2)\sum x_j^2} \, dx_1 \cdots dx_n \tag{4.28}$$

where

$$\Delta(x) = \Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

is the Vandermonde determinant, $c_n^{-1} = 1! 2! \cdots n! (2\pi)^{n/2}$, and $n = x + 1 \geq 2$. In the context of Brownian motion, can one *directly* prove (4.28)?

Remark 5. For connections between Brownian motion exit times and random matrices, see Grabiner.⁽¹⁹⁾

Remark 6. The Brownian motion functional M_x has appeared previously in Glynn and Whitt,⁽²⁵⁾ and consequently (4.28) provides an exact formula for the limiting distribution of the departure time of the first $(x + 1)$ customers from n single server queues. Glynn and Whitt also consider the case when $x = t^a$, $0 < a < 1$, and prove what would, in our setting, be the following limit theorem

$$\lim_{t \rightarrow \infty} \frac{h_t(x) - pt}{\sigma \sqrt{tx}} = \alpha := \lim_{x \rightarrow \infty} \frac{M_x}{\sqrt{x}}$$

with both limits in probability. They conjectured that $\alpha = 2$, and this was later proved by Seppäläinen⁽³⁷⁾ via a hydrodynamic limit for simple exclusion. We note that our paper proves that $\alpha = 2$ as well, by a completely

different route. Namely, one only needs to apply the result (see, e.g., ref. 4) that the largest eigenvalue in the finite n GUE scales as $2\sqrt{n}$.¹⁹

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