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Limit theorems for one and two-dimensional random walks in random scenery[1]

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Abstract. Random walks in random scenery are processes defined by $Z_n := \sum_{k=1}^n \xi_{X_1+\cdots+X_k}$, where $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z}^d)$ are two independent sequences of i.i.d. random variables with values in $\mathbb{Z}^d$ and $\mathbb{R}$ respectively. We suppose that the distributions of X_1 and ξ_0 belong to the normal basin of attraction of stable distribution of index $\alpha \in (0, 2]$ and $\beta \in (0, 2]$. When $d = 1$ and $\alpha \neq 1$, a functional limit theorem has been established in (*Z. Wahrsch. Verw. Gebiete* **50** (1979) 5–25) and a local limit theorem in (*Ann. Probab.* To appear). In this paper, we establish the convergence in distribution and a local limit theorem when $\alpha = d$ (i.e. $\alpha = d = 1$ or $\alpha = d = 2$) and $\beta \in (0, 2]$. Let us mention that functional limit theorems have been established in (*Ann. Probab.* **17** (1989) 108–115) and recently in (An asymptotic variance of the self-intersections of random walks. Preprint) in the particular case when $\beta = 2$ (respectively for $\alpha = d = 2$ and $\alpha = d = 1$).

Résumé. Les promenades aléatoires en paysage aléatoire sont des processus définis par $Z_n := \sum_{k=1}^n \xi_{X_1+\cdots+X_k}$, où $(X_k, k \geq 1)$ et $(\xi_y, y \in \mathbb{Z}^d)$ sont deux suites indépendantes de variables aléatoires i.i.d. à valeurs dans $\mathbb{Z}^d$ et $\mathbb{R}$ respectivement. Nous supposons que les lois de X_1 et ξ_0 appartiennent au domaine d'attraction normal de lois stables d'indice $\alpha \in (0, 2]$ et $\beta \in (0, 2]$. Quand $d = 1$ et $\alpha \neq 1$, un théorème limite fonctionnel a été prouvé dans (*Z. Wahrsch. Verw. Gebiete* **50** (1979) 5–25) et un théorème limite local dans (*Ann. Probab.* To appear). Dans ce papier, nous prouvons la convergence en loi et un théorème limite local quand $\alpha = d$ (i.e. $\alpha = d = 1$ ou $\alpha = d = 2$) et $\beta \in (0, 2]$. Mentionnons que des théorèmes limites fonctionnels ont été établis dans (*Ann. Probab.* **17** (1989) 108–115) et récemment dans (An asymptotic variance of the self-intersections of random walks. Preprint) dans le cas particulier où $\beta = 2$ (respectivement pour $\alpha = d = 2$ et $\alpha = d = 1$).

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1. Introduction

Random walks in random scenery (RWRS) are simple models of processes in disordered media with long-range correlations. They have been used in a wide variety of models in physics to study anomalous dispersion in layered random flows [20], diffusion with random sources, or spin depolarization in random fields (we refer the reader to Le Doussal's review paper [16] for a discussion of these models).

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On the mathematical side, motivated by the construction of new self-similar processes with stationary increments, Kesten and Spitzer [15] and Borodin [4,5] introduced RWRS in dimension one and proved functional limit theorems. This study has been completed in many works, in particular in [3] and [10]. These processes are defined as follows. Let $\xi := (\xi_y, y \in \mathbb{Z}^d)$ and $X := (X_k, k \geq 1)$ be two independent sequences of independent identically distributed random variables taking values in $\mathbb{R}$ and $\mathbb{Z}^d$ respectively. The sequence ξ is called the *random scenery*. The sequence X is the sequence of increments of the *random walk* $(S_n, n \geq 0)$ defined by $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$, for $n \geq 1$. The *random walk in random scenery* Z is then defined by

$$Z_0 := 0 \quad \text{and} \quad \forall n \geq 1, \quad Z_n := \sum_{k=0}^{n-1} \xi_{S_k}.$$

Denoting by $N_n(y)$ the local time of the random walk S:

$$N_n(y) := \#\{k = 0, \dots, n-1\colon\ S_k = y\},$$

it is straightforward to see that Z_n can be rewritten as $Z_n = \sum_y \xi_y N_n(y)$.

As in [15], the distribution of ξ_0 is assumed to belong to the normal domain of attraction of a strictly stable distribution $\mathcal{S}_\beta$ of index $\beta \in (0, 2]$, with characteristic function ϕ given by

$$\phi(u) = \mathrm{e}^{-|u|^\beta (A_1 + \mathrm{i} A_2 \operatorname{sgn}(u))}, \quad u \in \mathbb{R},$$

where $0 < A_1 < \infty$ and $|A_1^{-1} A_2| \leq |\tan(\pi\beta/2)|$. We will denote by φ_ξ the characteristic function of the ξ_x's. When $\beta > 1$, this implies that $\mathbb{E}[\xi_0] = 0$. When $\beta = 1$, we will further assume the symmetry condition

$$\sup_{t>0} \bigl|\mathbb{E}[\xi_0 \mathbb{1}_{\{|\xi_0| \leq t\}}]\bigr| < +\infty. \tag{1}$$

Under these conditions (for $\beta \in (0; 2]$), there exists $C_\xi > 0$ such that we have

$$\forall t > 0, \quad \mathbb{P}\bigl(|\xi_0| \geq t\bigr) \leq C_\xi t^{-\beta}. \tag{2}$$

Concerning the random walk, the distribution of X_1 is assumed to belong to the normal basin of attraction of a stable distribution $\mathcal{S}'_\alpha$ with index $\alpha \in (0, 2]$.

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on $[0, \infty)$ and on $\mathbb{R}$ respectively, endowed with the Skorohod J_1-topology (see [2], Chapter 3):

$$\bigl(n^{-1/\alpha} S_{\lfloor nt \rfloor}\bigr)_{t \geq 0} \xrightarrow[n\to\infty]{\mathcal{L}} \bigl(U(t)\bigr)_{t \geq 0}$$

and

$$\left(n^{-1/\beta} \sum_{k=0}^{\lfloor nx \rfloor} \xi_{k e_1}\right)_{x \geq 0} \xrightarrow[n\to\infty]{\mathcal{L}} \bigl(Y(x)\bigr)_{x \geq 0}, \quad \text{with } e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d,$$

where U and Y are two independent Lévy processes such that $U(0) = 0$, $Y(0) = 0$, $U(1)$ has distribution $\mathcal{S}'_\alpha$, $Y(1)$ has distribution $\mathcal{S}_\beta$.

1.1. *Functional limit theorem*

Our first result is concerned with a limit theorem for $(Z_{[nt]})_{t \geq 0}$. Intuitively speaking,

- when $\alpha < d$, the random walk S_n is transient, its range is of order n, and Z_n has the same behaviour as a sum of about n independent random variables with the same distribution as the variables ξ_x. It was proved in [5] that for $\beta = 2$, $n^{-1/\beta}(Z_{[nt]})_{t \geq 0}$ converges in distribution in the space $D([0, \infty))$ of càdlàg functions endowed with the Skorohod J_1-topology, to a multiple of the process (Y_t). The case $\beta \in (0, 2]$ was also mentioned in [15] (see Remark 3). When $\beta < 1$ and the scenery is positive, a functional limit theorem in the space $D([0, \infty))$ endowed with the Skorohod M_1-topology, is proved in [1] or [13];

- when $\alpha > d$ (i.e. $d = 1$ and $1 < \alpha \le 2$), the random walk S_n is recurrent, its range is of order $n^{1/\alpha}$, its local times are of order $n^{1-1/\alpha}$, so that Z_n is of order $n^{1-1/\alpha+1/(\alpha\beta)}$. In this situation, [4] and [15] proved a functional limit theorem for $n^{-(1-1/\alpha+1/(\alpha\beta))}(Z_{[nt]}, t \ge 0)$ in the space $\mathbb{C}([0, \infty))$ of continuous functions endowed with the uniform topology, the limiting process being a self-similar process, but not a stable one;
- when $\alpha = d$ (i.e. $\alpha = d = 1$ or $\alpha = d = 2$), S_n is recurrent, its range is of order $n/\log(n)$, its local times are of order $\log(n)$ so that Z_n is of order $n^{1/\beta}\log(n)^{(\beta-1)/\beta}$. In this situation, a functional limit theorem in the space of continuous functions was proved in [3] for $d = \alpha = \beta = 2$, and in [10] for $d = \alpha = 1$ and $\beta = 2$.

Our first result gives a limit theorem for $\alpha = d$ and for any value of $\beta \in (0; 2)$. We establish the convergence in the sense of finite distributions, and prove that the convergence in distribution does not hold for the J_1-topology when $\beta \ne 2$ but that the convergence in distribution holds for the M_1-topology when $\beta \ne 1$ (for technical reasons, our proof does not apply when $\beta = 1$).

Theorem 1. *Let $\beta \in (0; 2)$. We assume that the random walk is strongly aperiodic and that*

(a) *either $d = 2$ and X_1 is centered, square integrable with invertible variance matrix Σ and then we define $A := 2\sqrt{\det \Sigma}$;*
(b) *or $d = 1$ and $(\frac{S_n}{n})_n$ converges in distribution to a random variable with characteristic function given by $t \mapsto \exp(-a|t|)$ with $a > 0$ and then we define $A := a$.*

Then, the sequence of random variables

$$\left(\left(\frac{Z_{[nt]}}{n^{1/\beta}\log(n)^{(\beta-1)/\beta}}\right)_{t\in[0,1]}\right)_{n\ge 2}$$

converges in the sense of finite distributions to the process

$$\left(\tilde{Y}_t := \left(\frac{\Gamma(\beta+1)}{(\pi A)^{\beta-1}}\right)^{1/\beta} Y(t)\right)_{t\in[0,1]}.$$

For $\beta < 2$, the convergence does not hold in $\mathcal{D}([0, 1])$ endowed with the J_1-topology, but when $\beta \ne 1$, the convergence holds in $\mathcal{D}([0, 1])$ endowed with the M_1-topology.

Remark 2. *For $d > \alpha$ and $\beta \ne 1$, the same proof as in Theorem 1 shows that the sequence $(n^{-1/\beta}Z_{[nt]}, t \in [0, 1])$ converges in $(\mathcal{D}([0, 1], M_1)$ to the process $(\mathbb{E}[N_\infty^{\beta-1}]^{1/\beta}Y(t), t \in [0, 1])$, where N_∞ is the total number of visits to 0 of a two-sided random walk $(S_n, n \in \mathbb{Z})$ such that $S_0 = 0$ and whose increments are distributed according to X_1 (see Remarks 6, 8, 9, 11 below).*

1.2. *Local limit theorem*

Our next results concern a local limit theorem for $(Z_n)_n$. The $d = 1$ case was treated in [7] for $\alpha \in (0; 2] \setminus \{1\}$ and all values of $\beta \in (0; 2]$. Here, we complete this study by proving a local limit theorem for $\alpha = d = 1$ (and $\beta \in (0; 2]$). By a direct adaptation of the proof of this result, we also establish a local limit theorem for $\alpha = d = 2$ (we just adapt the definition of "peaks," see Section 3.5). Let us notice that the same adaptation can be done from [7] (case $\alpha < 1$) to get local limit theorems for $d \ge 2$, $\alpha < d$ and $\beta \in (0; 2]$.

We give two results corresponding respectively to the case when ξ_0 is lattice and to the case when it is strongly nonlattice. We denote by φ_ξ the characteristic function of ξ_0.

Theorem 3. *Assume that ξ_0 takes its values in $\mathbb{Z}$. Let $d_0 \ge 1$ be the integer such that $\{u\colon |\varphi_\xi(u)| = 1\} = \frac{2\pi}{d_0}\mathbb{Z}$. Let $b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta}$. Under the previous assumptions on the random walk and on the scenery, for $\alpha = d \in \{1, 2\}$, for every $\beta \in (0, 2]$, and for every $x \in \mathbb{R}$,*

- *if* $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \notin d_0\mathbb{Z}) = 1$, *then* $\mathbb{P}(Z_n = \lfloor b_n x\rfloor) = 0$;
- *if* $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \in d_0\mathbb{Z}) = 1$, *then*

$$\mathbb{P}\big(Z_n = \lfloor b_n x\rfloor\big) = d_0\frac{C(x)}{b_n} + \mathrm{o}\big(b_n^{-1}\big)$$

uniformly in $x \in \mathbb{R}$, *where* $C(\cdot)$ *is the density function of* $\tilde{Y}_1$.

When ξ_0 is strongly nonlattice, we establish the weak convergence of $b_n\mathbb{P}_{Z_n}$ to the Lebesgue measure on $\mathbb{R}$ (in the sense of compact supported function, see Definition 10.2 of [6]). More precisely we state the following result.

Theorem 4. *Assume now that* ξ_0 *is strongly nonlattice which means that*

$$\limsup_{|u|\to+\infty}\big|\varphi_\xi(u)\big| < 1.$$

We still assume that $\alpha = d \in \{1, 2\}$ *and* $\beta \in (0; 2]$. *Then, for every compactly supported continuous function* $g : \mathbb{R} \to \mathbb{C}$, *we have*

$$\lim_{n\to+\infty}\sup_{x\in\mathbb{R}}\left|b_n\mathbb{E}\big[g(Z_n - b_n x)\big] - C(x)\int_{\mathbb{R}} g(t)\,\mathrm{d}t\right| = 0,$$

with $b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta}$ *and where* $C(\cdot)$ *is the density function of* $\tilde{Y}_1$.

2. Proof of the functional limit theorem

Before proving the theorem, we prove some technical lemmas. For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \dots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \cdots < t_m$, we consider the sequences of random variables $(L_n(\gamma))_{n\geq 2}$ and $(L'_n(\gamma))_{n\geq 2}$ defined by

$$L_n(\gamma) := \frac{1}{n(\log n)^{\gamma-1}}\sum_{x\in\mathbb{Z}^d}\left|\sum_{i=1}^m \theta_i\big(N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)\big)\right|^\gamma$$

and

$$L'_n(\gamma) := \frac{1}{n(\log n)^{\gamma-1}}\sum_{x\in\mathbb{Z}^d}\left|\sum_{i=1}^m \theta_i\big(N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)\big)\right|^\gamma \operatorname{sgn}\left(\sum_{i=1}^m \theta_i\big(N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)\big)\right).$$

Lemma 5. *For any real number* $\gamma > 0$, *any integer* $m \geq 1$, *any* $\theta_1, \dots, \theta_m \in \mathbb{R}$, *any* $t_0 = 0 < t_1 < \cdots < t_m$, *the following convergences hold* $\mathbb{P}$*-almost surely*

$$\lim_{n\to+\infty} L_n(\gamma) = \frac{\Gamma(\gamma+1)}{(\pi A)^{\gamma-1}}\sum_{i=1}^m |\theta_i|^\gamma(t_i - t_{i-1}) \tag{3}$$

and

$$\lim_{n\to+\infty} L'_n(\gamma) = \frac{\Gamma(\gamma+1)}{(\pi A)^{\gamma-1}}\sum_{i=1}^m |\theta_i|^\gamma \operatorname{sgn}(\theta_i)(t_i - t_{i-1}). \tag{4}$$

Proof. We fix an integer $m \geq 1$ and $2m$ real numbers $\theta_1, \dots, \theta_m, t_1, \dots, t_m$ such that $0 < t_1 < \cdots < t_m$ and we set $t_0 := 0$. To simplify notations, we write $d_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)$. Following the techniques developed in [8], we first have to prove (3) and (4) for integer γ: for every integer $k \geq 1$, $\mathbb{P}$-almost surely, as n goes to infinity, we have

$$\frac{1}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=1}^m \theta_i d_{i,n}(x) \right)^k \longrightarrow \frac{\Gamma(k+1)}{(\pi A)^{k-1}} \sum_{i=1}^m \theta_i^k (t_i - t_{i-1}). \tag{5}$$

Let us assume (5) for a while, and let us end the proof of (3) and (4) for any positive real γ. Given the random walk $S := (S_n)_n$, let $(U_n)_{n\geq 1}$ be a sequence of random variables with values in $\mathbb{Z}^d$, such that for all n, U_n is a point chosen uniformly in the range of the random walk up to time $[nt_m]$, that is

$$\mathbb{P}(U_n = x | S) = R_{[nt_m]}^{-1} \mathbf{1}_{\{N_{[nt_m]}(x) \geq 1\}},$$

with $R_k := \#\{y\colon\ N_k(y) > 0\}$. Moreover, let U' be a random variable with values in $\{1, \dots, m\}$ and distribution

$$\mathbb{P}\big(U' = i\big) = (t_i - t_{i-1})/t_m$$

and let T be a random variable with exponential distribution with parameter one and independent of U'.

Then, for $\mathbb{P}$-almost every realization of the random walk S, the sequence of random variables

$$\left(W_n := \frac{\pi A}{\log(n)} \sum_{i=1}^m \theta_i d_{i,n}(U_n) \right)_n$$

converges in distribution to the random variable $W := \theta_{U'} T$. Indeed, the moment of order k of W_n given S is

$$\mathbb{E}\big(W_n^k | S\big) = \frac{(\pi A)^k}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=1}^m \theta_i d_{i,n}(x) \right)^k \frac{n}{\log(n) R([nt_m])}.$$

Using (5) and the fact that $((\log n) R_n / n)_n$ converges almost surely to πA (see [11,17]), the moments $\mathbb{E}(W_n^k | S)$ converges a.s. to $\mathbb{E}(W^k) = \Gamma(k+1) \sum_{i=1}^m \theta_i^k (t_i - t_{i-1})/t_m$. This proves the convergence of the conditional distribution of $(W_n)_n$ given S to W, since the distribution of W is identified by its moments (thanks to the Carleman condition). This ensures, in particular, the convergence in distribution of $(|W_n|^\gamma)_n$ and of $(|W_n|^\gamma \operatorname{sgn}(W_n))_n$ (given S) to $|W|^\gamma$ and $|W|^\gamma \operatorname{sgn}(W)$ respectively (for every real number $\gamma \geq 0$ and for $\mathbb{P}$-almost every realization of the random walk S). Since, conditional on S, any moment of $|W_n|$ can be bounded from above by an integer moment, we deduce that, for any $\gamma \geq 0$, we have $\mathbb{P}$-almost surely

$$\lim_{n \to +\infty} \mathbb{E}\big(|W_n|^\gamma \big| S\big) = \mathbb{E}\big(|W|^\gamma\big) \quad \text{and} \quad \lim_{n \to +\infty} \mathbb{E}\big(|W_n|^\gamma \operatorname{sgn}(W_n) \big| S\big) = \mathbb{E}\big(|W|^\gamma \operatorname{sgn}(W)\big),$$

which proves Lemma 5.

Let us prove (5). Let $k \geq 1$. According to Theorem 1 in [8] (proved for $\alpha = d = 2$, but also valid for $\alpha = d = 1$; see Appendix for additional comments on the proof of this theorem), we have

$$\forall i \in \{1, \dots, m\}, \quad \lim_{n \to +\infty} \frac{1}{n(\log n)^{k-1}} \sum_{x \in \mathbb{Z}^d} \big(d_{i,n}(x)\big)^k = \frac{\Gamma(k+1)}{(\pi A)^{k-1}} (t_i - t_{i-1}), \quad \mathbb{P}\text{-a.s.} \tag{6}$$

We define

$$\Sigma_n(\theta_1, \dots, \theta_m) := \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=1}^m \theta_i d_{i,n}(x) \right)^k - \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^m (\theta_i)^k \big(d_{i,n}(x)\big)^k. \tag{7}$$

According to (6), it is enough to prove that $\mathbb{P}$-a.s., $\Sigma_n(\theta_1,\dots,\theta_m)=\mathrm{o}(n(\log n)^{k-1})$. We observe that $\Sigma_n(\theta_1,\dots,\theta_m)$ is the sum of the following terms

$$\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{k}\bigl(\theta_{i_j}d_{i_j,n}(x)\bigr) \tag{8}$$

over all the k-tuple $(i_1,\dots,i_k)\in\{1,\dots,m\}^k$, with at least two distinct indices. We observe that

$$\bigl|\Sigma_n(\theta_1,\dots,\theta_m)\bigr|\le\max\bigl(|\theta_1|,\dots,|\theta_m|\bigr)^k\,\Sigma_n(1,\dots,1).$$

But, we have

$$\begin{aligned}\Sigma_n(1,\dots,1)&=\sum_{x\in\mathbb{Z}^d}\bigl(N_{[nt_m]}(x)\bigr)^k-\sum_{x\in\mathbb{Z}^d}\sum_{i=1}^{m}\bigl(d_{i,n}(x)\bigr)^k\\&=\sum_{x\in\mathbb{Z}^d}\bigl(N_{[nt_m]}(x)\bigr)^k-\sum_{i=1}^{m}\sum_{x\in\mathbb{Z}^d}\bigl(d_{i,n}(x)\bigr)^k=\mathrm{o}\bigl(n\log(n)^{k-1}\bigr),\end{aligned}$$

according to (6). □

Remark 6. *Case $d>\alpha$.*

In this case, R_n/n converges a.s. to $p=\mathbb{P}[S_k\neq 0$ for any $k\ge 1]$ (cf. [21]), and for all real number $k\ge 0$, $\frac{1}{n}\sum_{x\in\mathbb{Z}^d}N_n^k(x)$ converges a.s. to $\mathbb{E}[N_\infty^{k-1}]$ (see Remark 2 for a definition of N_∞ and the introduction of [15] for a proof of this fact). Setting $W_n=\sum_{j=1}^m\theta_j d_{j,n}(U_n)$, it follows that for all integer $k\ge 1$ $\mathbb{E}[W_n^k|S]$ tends to $\mathbb{E}_{\mathbb{Q}}[(\theta_{U'}N_\infty)^k]$, where $\mathbb{Q}$ is the probability on the random walk's paths space, whose density w.r.t. the random walk's law $\mathbb{P}$ is given by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}=1/(pN_\infty)$. This leads to the following two facts: for any real number $\gamma>0$, any integer $m\ge 1$, any $\theta_1,\dots,\theta_m\in\mathbb{R}$, any $t_0=0<t_1<\dots<t_m$, the following convergences hold $\mathbb{P}$-almost surely

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{x\in\mathbb{Z}^d}\left|\sum_{i=1}^m\theta_i(N_{[nt_i]}-N_{[nt_{i-1}]})\right|^\gamma=\mathbb{E}\bigl[N_\infty^{\gamma-1}\bigr]\sum_{i=1}^m|\theta_i|^\gamma(t_i-t_{i-1}) \tag{9}$$

and

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{n}\sum_{x\in\mathbb{Z}^d}\left|\sum_{i=1}^m\theta_i(N_{[nt_i]}-N_{[nt_{i-1}]})\right|^\gamma\operatorname{sgn}\left(\sum_{i=1}^m\theta_i(N_{[nt_i]}-N_{[nt_{j-1}]})\right)\\&\quad=\mathbb{E}\bigl[N_\infty^{\gamma-1}\bigr]\sum_{i=1}^m|\theta_i|^\gamma\operatorname{sgn}(\theta_i)(t_i-t_{i-1}).\end{aligned} \tag{10}$$

Lemma 7. *For any $\rho>0$,*

$$\sup_{x\in\mathbb{Z}^d}N_n(x)=\mathrm{o}\bigl(n^\rho\bigr)\quad a.s.$$

Proof. See Lemma 2.5 in [3]. □

Proof of Theorem 1. *Convergence of the finite-dimensional distributions.*

Let an integer $m\ge 1$ and $2m$ real numbers $\theta_1,\dots,\theta_m,t_1,\dots,t_m$ such that $0<t_1<\dots<t_m\le 1$. We set $t_0:=0$, Again, we use the notation $d_{i,n}(x):=N_{[nt_i]}(x)-N_{[nt_{i-1}]}(x)$, and set

$$b_n=n^{1/\beta}\bigl(\log(n)\bigr)^{(\beta-1)/\beta},\qquad \bar{Z}_n:=\frac{1}{b_n}\sum_{i=1}^m\theta_i(Z_{[nt_i]}-Z_{[nt_{i-1}]}).$$

We have to prove that

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\bar{Z}_n}\big] \to \prod_{i=1}^{m} \phi\left(\theta_i (t_i - t_{i-1})^{1/\beta} \left(\frac{\Gamma(\beta+1)}{(\pi A)^{\beta-1}}\right)^{1/\beta}\right), \tag{11}$$

as n goes to infinity. We observe that $\bar{Z}_n = \frac{1}{b_n} \sum_{x\in\mathbb{Z}^d} \sum_{i=1}^m \theta_i d_{i,n}(x)\xi_x$. Hence we have

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\bar{Z}_n}|S\big] = \prod_{x\in\mathbb{Z}^d} \varphi_\xi\left(\frac{\sum_{i=1}^m \theta_i d_{i,n}(x)}{b_n}\right).$$

Observe next that

$$\big|\varphi_\xi(t) - \exp\big(-|t|^\beta\big(A_1 + \mathrm{i}A_2\,\mathrm{sgn}(t)\big)\big)\big| \le |t|^\beta h\big(|t|\big) \quad \text{for all } t \in \mathbb{R},$$

with h a continuous and monotone function on $[0,+\infty)$ vanishing in 0. According to Lemma 7, $\mathbb{P}$-almost surely, for every n large enough, we have

$$D_n := \sup_x \frac{|\sum_{i=1}^m \theta_i d_{i,n}(x)|}{b_n} \le m \max\big(|\theta_i|\big) \frac{\sup_x N_n(x)}{b_n} \le \varepsilon_0$$

and so

$$\left|\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\bar{Z}_n}|S\big] - \prod_{x\in\mathbb{Z}^d} \mathrm{e}^{-(|\sum_{i=1}^m \theta_i d_{i,n}(x)|^\beta / b_n^\beta)(A_1 + \mathrm{i}A_2\,\mathrm{sgn}(\sum_{i=1}^m \theta_i d_{i,n}(x)))}\right|$$

is less than $\sum_{x\in\mathbb{Z}^d} \frac{|\sum_{i=1}^m \theta_i b_{i,n}(x)|^\beta}{b_n^\beta} h(B_n)$. Hence, according to Lemmas 5 and 7, $\mathbb{P}$-almost surely, we have

$$\lim_{n\to+\infty} \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\bar{Z}_n}|S\big] = \mathrm{e}^{-(\Gamma(\beta+1)/(\pi A)^{\beta-1}) \sum_{i=1}^m |\theta_i|^\beta (t_i - t_{i-1})(A_1 + \mathrm{i}A_2\,\mathrm{sgn}(\theta_i))}$$

which gives (11) thanks to the Lebesgue dominated convergence theorem.

Remark 8. *Case $d > \alpha$.*
The proof is exactly the same with $b_n = n^{1/\beta}$.

Study of the tightness.
When $\beta = 2$, the sequence is known to be tight for the J_1 (so also M_1) topology (see [3]). For $\beta < 2$, we prove that the sequence $(\frac{Z_{[nt]}}{b_n})_{t\in[0;1]}$ is not tight in $(\mathcal{D}([0,1]), J_1)$. To this aim, let $(Z_n(t), t \in [0,1])$ denote the linear interpolation of $(Z_{[nt]}, t \in [0,1])$, i.e.

$$Z_n(t) = Z_{[nt]} + \big(nt - [nt]\big)\xi_{S_{[nt]}}.$$

Then, $\forall \epsilon > 0$,

$$\begin{aligned}
\mathbb{P}\Big[\sup_{t\in[0,1]} \big|Z_n(t) - Z_{[nt]}\big| \ge \epsilon b_n\Big] &= \mathbb{P}\Big[\max_{i=0}^{n-1} |\xi_{S_i}| \ge \epsilon b_n\Big] \\
&= \mathbb{P}\big[\exists x \in \{S_0, \ldots, S_{n-1}\} \text{ s.t. } |\xi_x| \ge \epsilon b_n\big] \\
&\le \mathbb{E}\big(\#\{S_0, \ldots, S_{n-1}\}\big)\mathbb{P}\big[|\xi_0| \ge \epsilon b_n\big] \\
&\le C\frac{n}{\log(n)} \epsilon^{-\beta} b_n^{-\beta} = C\epsilon^{-\beta} \log(n)^{-\beta},
\end{aligned}$$

where the last inequality comes from (2) and Theorem 6.9 of [17]. Therefore, if $((\frac{Z_{[nt]}}{b_n})_{t\in[0;1]})_{n\geq 2}$ converges in distribution in $(\mathcal{D}([0,1]), J_1)$ to $(\tilde{Y}_t)_{t\in[0,1]}$, the same is true for $((\frac{Z_n(t)}{b_n})_{t\in[0;1]})_{n\geq 2}$ which implies that $(\frac{Z_n(t)}{b_n})_{t\in[0;1]}$ converges in distribution in $\mathbb{C}([0,1])$, and that the limiting process $(\tilde{Y}_t)_{t\in[0,1]}$ is therefore continuous, which is false as soon as $\beta<2$.

M_1-tightness for $\beta>1$.

Set $\tilde{Z}_n(t)=\frac{Z_{[nt]}}{b_n}$, and let us prove the tightness of the sequence $(\tilde{Z}_n)_n$ in $\mathcal{D}([0,1])$ for the M_1-topology when $\beta>1$. For any y_1, y_2 and y_3 real, let us denote $\|y_2-[y_1,y_3]\|=\inf_{t\in[y_1,y_3]}|y_2-t|$. For any function $z=(z(t))_{t\in[0,1]}$ in $\mathcal{D}([0,1])$, we define

$$\omega(z,\delta)=\sup_{t\in[0,1]}\sup\bigl\{\bigl\|z(t_2)-\bigl[z(t_1),z(t_3)\bigr]\bigr\|:\ (t-\delta)\vee 0\leq t_1<t_2<t_3\leq(t+\delta)\wedge 1\bigr\}.$$

From Skorohod criteria (see [22] or [23], Chapter 12) it is enough to prove that for every $\varepsilon>0$,

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathbb{P}\biggl[\omega\biggl(\tilde{Z}_n,\frac{1}{k}\biggr)>\varepsilon\biggr]=0. \tag{12}$$

The proof is based on two distinct results: the first one by Louhichi and Rio in [19] where they prove that in the case of a sum of associated random variables, the above M_1-tightness criteria can be deduced from a maximal inequality for the sum; the second one by Louhichi in [18] where a maximal inequality for the sum of associated random variables without moment conditions (not necessarily stationary) is proved. Let us give the details. Since the sequence $(\xi_{S_k})_{k\geq 0}$ is stationary, we have for every $k\geq 3$,

$$\mathbb{P}\biggl[\omega\biggl(\tilde{Z}_n,\frac{1}{k}\biggr)>\varepsilon\biggr]\leq(k-2)\mathbb{P}\Bigl[\sup_{0\leq n_1<n_2<n_3\leq 1+\lfloor 3n/k\rfloor}\bigl\|Z_{n_2}-[Z_{n_1},Z_{n_3}]\bigr\|>\varepsilon b_n\Bigr].$$

Conditionally to the random walk $S=(S_n)_{n\geq 0}$, the sequence of random variables $(\xi_{S_k})_{k\geq 0}$ is associated, therefore by applying inequality (3) in [19], we have

$$\mathbb{P}\biggl[\omega\biggl(\tilde{Z}_n,\frac{1}{k}\biggr)>\varepsilon\biggr]\leq(k-2)\mathbb{E}\biggl[\mathbb{P}\biggl(\max_{0\leq j\leq 1+\lfloor 3n/k\rfloor}|Z_j|>\frac{\varepsilon b_n}{2}\Big|S\biggr)^2\biggr]. \tag{13}$$

Now let us apply Lemma 1 in [18] to the random variables $X=|\xi_0|$ and $X_i=\xi_{S_i}$, $i\geq 0$, conditionally to the random walk. For any sequence of positive reals $(\tilde{b}_n)_n$, there exist some constant $C>0$ depending on ε (the value of C may change from line to line in the following inequalities) s.t.

$$\begin{aligned}\mathbb{P}\biggl(\max_{0\leq j\leq 1+\lfloor 3n/k\rfloor}Z_j>\frac{\varepsilon b_n}{2}\Big|S\biggr)\leq C\biggl\{&\frac{(1+\lfloor 3n/k\rfloor)}{b_n^2}\mathbb{E}\bigl[\xi_0^2\mathbf{1}_{\{|\xi_0|\leq\tilde{b}_n\}}\bigr]+\frac{(1+\lfloor 3n/k\rfloor)}{b_n}\mathbb{E}\bigl[|\xi_0|\mathbf{1}_{\{|\xi_0|>\tilde{b}_n\}}\bigr]\\&+\biggl(1+\biggl\lfloor\frac{3n}{k}\biggr\rfloor\biggr)\biggl(\frac{\tilde{b}_n}{b_n}\biggr)^2\mathbb{P}\bigl[|\xi_0|>\tilde{b}_n\bigr]+\frac{1}{b_n^2}\sum_{0\leq i<j\leq 1+\lfloor 3n/k\rfloor}G_{ij}(\tilde{b}_n)\biggr\},\end{aligned}$$

where, in our setting, if we denote for $v\in\mathbb{R}_+$ by g_v the function $(u\wedge v)\vee(-v)$,

$$\begin{aligned}G_{ij}(v)&:=\mathbb{E}\bigl[g_v(\xi_{S_i})g_v(\xi_{S_j})|S\bigr]-\mathbb{E}\bigl[g_v(\xi_{S_i})|S\bigr]\mathbb{E}\bigl[g_v(\xi_{S_j})|S\bigr]\\&\leq\mathbb{E}\bigl[g_v(\xi_{S_i})^2|S\bigr]\mathbf{1}_{\{S_i=S_j\}}=\mathbb{E}\bigl[g_v(\xi_0)^2\bigr]\mathbf{1}_{\{S_i=S_j\}}.\end{aligned}$$

The same reasoning holds for the sequence $(-\xi_{S_i})_{i\geq 0}$, which is also associated, then since the function g_v is odd, we deduce, by denoting $I_n:=\sum_{i,j=0}^{n-1}\mathbf{1}_{\{S_i=S_j\}}$, the following maximal inequality

$$\begin{aligned}\mathbb{P}\biggl[\max_{0\leq j\leq 1+\lfloor 3n/k\rfloor}|Z_j|>\frac{\varepsilon b_n}{2}\Big|S\biggr]\leq C\biggl\{&\frac{(1+\lfloor 3n/k\rfloor)}{b_n^2}\mathbb{E}\bigl[\xi_0^2\mathbf{1}_{\{|\xi_0|\leq\tilde{b}_n\}}\bigr]+\frac{(1+\lfloor 3n/k\rfloor)}{b_n}\mathbb{E}\bigl[|\xi_0|\mathbf{1}_{\{|\xi_0|>\tilde{b}_n\}}\bigr]\\&+\biggl(1+\biggl\lfloor\frac{3n}{k}\biggr\rfloor\biggr)\biggl(\frac{\tilde{b}_n}{b_n}\biggr)^2\mathbb{P}\bigl[|\xi_0|>\tilde{b}_n\bigr]+\frac{I_{1+\lfloor 3n/k\rfloor}}{b_n^2}\mathbb{E}\bigl[g_{\tilde{b}_n}(\xi_0)^2\bigr]\biggr\}.\end{aligned}$$

Since for every $x, y \in \mathbb{R}^+$, $(x+y)^2 \le 2(x^2+y^2)$, we get

$$\mathbb{E}\left[\mathbb{P}\left(\max_{0\le j\le 1+\lfloor 3n/k\rfloor}|Z_j| > \frac{\varepsilon b_n}{2}\Big|S\right)^2\right] \le C\sum_{i=1}^{4}\Sigma_i(n,k), \tag{14}$$

where

$$\Sigma_1(n,k) = \frac{(1+\lfloor 3n/k\rfloor)^2}{b_n^4}\mathbb{E}\big[\xi_0^2\mathbf{1}_{\{|\xi_0|\le\tilde{b}_n\}}\big]^2,$$

$$\Sigma_2(n,k) = \frac{(1+\lfloor 3n/k\rfloor)^2}{b_n^2}\mathbb{E}\big[|\xi_0|\mathbf{1}_{\{|\xi_0|>\tilde{b}_n\}}\big]^2,$$

$$\Sigma_3(n,k) = \left(1+\left\lfloor\frac{3n}{k}\right\rfloor\right)^2\left(\frac{\tilde{b}_n}{b_n}\right)^4\mathbb{P}\big[|\xi_0|>\tilde{b}_n\big]^2,$$

$$\Sigma_4(n,k) = \frac{\mathbb{E}(I^2_{1+\lfloor 3n/k\rfloor})}{b_n^4}\mathbb{E}\big[g_{\tilde{b}_n}(\xi_0)^2\big]^2.$$

Note that $\mathbb{E}[\xi_0^2\mathbf{1}_{\{|\xi_0|\le\tilde{b}_n\}}] \asymp \tilde{b}_n^{2-\beta}$, $\mathbb{E}[|\xi_0|\mathbf{1}_{\{|\xi_0|>\tilde{b}_n\}}] \asymp \tilde{b}_n^{1-\beta}$ for $\beta<1$, and $\mathbb{E}[g_{\tilde{b}_n}(\xi_0)^2] \asymp \tilde{b}_n^{2-\beta}$. Therefore, by choosing $\tilde{b}_n = (\frac{n}{\log(n)})^{1/\beta}$, we deduce that for $i=1,3$,

$$\limsup_{n\to+\infty}\Sigma_i(n,k) = 0, \tag{15}$$

and (recall that $\mathbb{E}(I_n^2) = \mathcal{O}((n\log(n))^2)$) for $i=2,4$, there exist two constants $C_i>0$ s.t.

$$\limsup_{n\to+\infty}\Sigma_i(n,k) \le \frac{C_i}{k^2}. \tag{16}$$

Therefore, by combining (13), (14), (15) and (16), there exists some constant $C>0$ s.t.

$$\limsup_{n\to+\infty}\mathbb{P}\left[\omega\left(\tilde{Z}_n,\frac{1}{k}\right)>\varepsilon\right] \le \frac{C}{k}$$

then (12) follows.

Remark 9. *Case $d>\alpha$ and $\beta>1$.*

It is easy to see that for $d>\alpha$, $\mathbb{E}(I_n^2) = \mathrm{O}(n^2)$. Taking $\tilde{b}_n = b_n = n^{1/\beta}$, the same proof leads to $\limsup_{n\to\infty}\Sigma_i(n,k) \le C_i/k^2$ for every $i\in\{1,\dots,4\}$, and to the tightness in M_1-topology.

M_1-tightness for $\beta<1$.

For $\beta<1$, to get a control of the oscillation, we write $\xi_x = \xi_x^+ - \xi_x^-$ to obtain the decomposition $\tilde{Z}_n = \tilde{Z}_n^+ - \tilde{Z}_n^-$, where $\tilde{Z}_n^+(t) := \frac{1}{b_n}Z^+_{[nt]}$, and Z_n^+ is the random walk in the random scenery $(\xi_x^+, x\in\mathbb{Z}^d)$:

$$Z_n^+ = \sum_{k=0}^{n-1}\xi^+_{S_k} = \sum_{x\in\mathbb{Z}^d}\xi_x^+ N_n(x).$$

$\tilde{Z}_n^-$ is defined in the same way as $\tilde{Z}_n^+$ using the negative part of the scenery. Since the processes $\tilde{Z}_n^-$, $\tilde{Z}_n^+$ are increasing, for any $\delta>0$, $\omega(\tilde{Z}_n^-,\delta) = \omega(\tilde{Z}_n^+,\delta) = 0$. Assume for a while that $\tilde{Z}_n^-(1)$ and $\tilde{Z}_n^+(1)$ both converge in distribution (this is false for $\beta\ge 1$ due to centering term). It follows that the processes $\tilde{Z}_n^-$ and $\tilde{Z}_n^+$ are tight in M_1-topology. To get the tightness of their difference $\tilde{Z}_n$, we have then to prove that the limiting processes of $\tilde{Z}_n^-$ and $\tilde{Z}_n^+$ do not have common discontinuity points (see Corollary 12.7.1 in [23]). This is the case if these two processes are independent. Therefore, all that remains to prove is the following lemma. □

Lemma 10. *Let an integer $m \geq 1$ and $3m$ real numbers $\theta_1, \dots, \theta_m, \gamma_1, \dots, \gamma_m, t_1, \dots, t_m$ such that $0 < t_1 < \cdots < t_m \leq 1$. We set $t_0 := 0$. Then*

$$\lim_{n\to\infty} \mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}\sum_{j=1}^{m}\big(\theta_j\big(\tilde{Z}_n^+(t_j) - \tilde{Z}_n^+(t_{j-1})\big) + \gamma_j\big(\tilde{Z}_n^-(t_j) - \tilde{Z}_n^-(t_{j-1})\big)\big)\Bigg)\Bigg]$$

$$= \prod_{j=1}^{m} \phi_1\big(\theta_j(t_j - t_{j-1})^{1/\beta}\big)\phi_2\big(\gamma_j(t_j - t_{j-1})^{1/\beta}\big),$$

where ϕ_1 and ϕ_2 are characteristic functions of positive β-stable laws.

Proof. We use the notation

$$d_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x), \qquad d_n(x) := \big(d_{1,n}(x), \dots, d_{m,n}(x)\big).$$

Observe that

$$\sum_{j=1}^{m}\big(\theta_j\big(\tilde{Z}_n^+(t_j) - \tilde{Z}_n^+(t_{j-1})\big) + \gamma_j\big(\tilde{Z}_n^-(t_j) - \tilde{Z}_n^-(t_{j-1})\big)\big) = \frac{1}{b_n}\sum_{x\in\mathbb{Z}^d} \xi_x^+\big\langle\theta; d_n(x)\big\rangle + \xi_x^-\big\langle\gamma; d_n(x)\big\rangle.$$

Therefore

$$\mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}\sum_{j=1}^{m}\big(\theta_j\big(\tilde{Z}_n^+(t_j) - \tilde{Z}_n^+(t_{j-1})\big) + \gamma_j\big(\tilde{Z}_n^-(t_j) - \tilde{Z}_n^-(t_{j-1})\big)\big)\Bigg)\Big|S\Bigg]$$

$$= \prod_{x\in\mathbb{Z}^d}\mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}\Bigg(\xi_x^+\frac{\langle\theta; d_n(x)\rangle}{b_n} + \xi_x^-\frac{\langle\gamma; d_n(x)\rangle}{b_n}\Bigg)\Bigg)\Big|S\Bigg].$$

Note that for any real s, t, $\mathbb{E}[\exp(\mathrm{i}(t\xi_0^+ + s\xi_0^-))] = \varphi_{\xi^+}(t) + \varphi_{\xi^-}(s) - 1$. Since ξ is in the domain of attraction of $\mathcal{S}_\beta$, the tails of the variables ξ^+ and ξ^- satisfy $\mathbb{P}[\xi^+ \geq t] \asymp p\mathbb{P}[|\xi| \geq t]$, $\mathbb{P}[\xi^- \geq t] \asymp (1-p)\mathbb{P}[|\xi| \geq t]$ for some $p \in [0, 1]$. Thus, ξ^+ and ξ^- belong to the domain of attraction of positive stable laws with index β whose characteristic functions are denoted by ϕ_+ and ϕ_-. Since $\beta < 1$, it follows (see Theorem 2, p. 448 in [12]) that $\frac{1}{n^\beta}\sum_{j=1}^{n}\xi_j^+$ converges to a β stable random variable with characteristic function ϕ_+. Therefore, we get $|\varphi_{\xi^+}(t) - \phi_+(t)| \leq |t|^\beta h_+(|t|)$ for some increasing continuous function h_+ such that $h_+(0) = 0$. The analogous statement is true for φ_{ξ^-}. Hence, for any real numbers s, t

$$\begin{aligned}
&\big|\varphi_{\xi^+}(t) + \varphi_{\xi^-}(s) - 1 - \phi_+(t)\phi_-(s)\big| \\
&\quad\leq \big|\varphi_{\xi^+}(t) - \phi_+(t)\big| + \big|\varphi_{\xi^-}(s) - \phi_-(s)\big| + \big|\big(\phi_+(t) - 1\big)\big(\phi_-(s) - 1\big)\big| \\
&\quad\leq |t|^\beta h_+\big(|t|\big) + |s|^\beta h_-\big(|s|\big) + C|s|^\beta|t|^\beta.
\end{aligned}$$

Note also that $|\langle\theta; d_n(x)\rangle| \leq m\max(|\theta_i|)N_n(x)$. It follows that

$$\begin{aligned}
&\bigg|\mathbb{E}\big[\mathrm{e}^{\mathrm{i}(\sum_{j=1}^{m}(\theta_j(\tilde{Z}_n^+(t_j) - \tilde{Z}_n^+(t_{j-1})) + \gamma_j(\tilde{Z}_n^-(t_j) - \tilde{Z}_n^-(t_{j-1}))))}|S\big] - \prod_x \phi_+\bigg(\frac{\langle\theta, d_n(x)\rangle}{b_n}\bigg)\phi_-\bigg(\frac{\langle\gamma, d_n(x)\rangle}{b_n}\bigg)\bigg| \\
&\quad\leq \sum_x\bigg|\varphi_{\xi^+}\bigg(\frac{\langle\theta, d_n(x)\rangle}{b_n}\bigg) + \varphi_{\xi^-}\bigg(\frac{\langle\gamma, d_n(x)\rangle}{b_n}\bigg) - 1 - \phi_+\bigg(\frac{\langle\theta, d_n(x)\rangle}{b_n}\bigg)\phi_-\bigg(\frac{\langle\gamma, d_n(x)\rangle}{b_n}\bigg)\bigg| \\
&\quad\leq C_{\beta,\gamma,\theta}\frac{\sum_x N_n^\beta(x)}{b_n^\beta}\bigg[h_+\bigg(\frac{m\|\theta\|N_n^*}{b_n}\bigg) + h_-\bigg(\frac{m\|\gamma\|N_n^*}{b_n}\bigg) + \bigg(\frac{N_n^*}{b_n}\bigg)^\beta\bigg],
\end{aligned}$$

where $N_n^* = \sup_x N_n(x)$ and $\|\theta\| = \max(|\theta_i|)$. Using Lemmas 5 and 7, the above quantity tends to 0 almost surely. Now, ϕ_+ and ϕ_- get the same form as ϕ (with $A_2/A_1 = -\tan(\pi\beta/2)$). And as in the proof of the convergence o the finite-dimensional distributions, we get that almost surely

$$\lim_{n\to+\infty} \prod_x \phi_+\left(\frac{\langle\theta; d_n(x)\rangle}{b_n}\right) = \prod_{j=1}^m \phi_+\left(\frac{\theta_j(t_j - t_{j-1})^{1/\beta}\Gamma(\beta+1)^{1/\beta}}{(\pi A)^{(\beta-1)/\beta}}\right).$$

The same is true for $\prod_x \phi_-(\frac{\langle\gamma; d_n(x)\rangle}{b_n})$. □

Remark 11. *Case $d > \alpha$ and $\beta < 1$.*
The proof is exactly the same using (9) *and* (10).

3. Proof of the local limit theorem in the lattice case

3.1. *The event Ω_n*

Set

$$N_n^* := \sup_y N_n(y) \quad \text{and} \quad R_n := \#\{y:\ N_n(y) > 0\}.$$

We also define, for every $n \geq 1$,

$$V_n := \sum_{i,j=0}^{n-1} N_n(x)^\beta.$$

Lemma 12. *For every $n \geq 1$ and $1 > \gamma > 0$, set*

$$\Omega_n = \Omega_n(\gamma) := \left\{R_n \leq \frac{n}{(\log\log(n))^{1/4}} \text{ and } N_n^* \leq n^\gamma\right\}.$$

Then, $\mathbb{P}(\Omega_n) = 1 - \mathrm{o}(b_n^{-1})$. Moreover, the following also holds on Ω_n:

$$\big(\log\log(n)\big)^{1/4} \leq N_n^* \quad \text{and} \quad V_n \geq n^{1-\gamma(1-\beta)_+}. \tag{17}$$

Proof. We first prove that

$$\mathbb{P}\big(R_n \geq n\big(\log\log(n)\big)^{-1/4}\big) = \mathrm{o}\big(b_n^{-1}\big). \tag{18}$$

Let us recall that for every $a, b \in \mathbb{N}$, we have

$$\mathbb{P}(R_n \geq a+b) \leq \mathbb{P}(R_n \geq a)\mathbb{P}(R_n \geq b). \tag{19}$$

The proof is given for instance in [9]. We will moreover use the fact that $\mathbb{E}[R_n] \sim cn(\log(n))^{-1}$ and $\mathrm{Var}(R_n) = \mathrm{O}(n^2\log^{-4}(n))$ (see [17]). Hence, for n large enough, there exists $C > 0$ such that we have

$$\begin{aligned}\mathbb{P}\left(R_n \geq \frac{n}{(\log\log(n))^{1/4}}\right) &\leq \mathbb{P}\left(R_n \geq \left\lfloor\frac{n(\log\log(n))^{1/4}}{\log(n)}\right\rfloor\right)^{\lfloor\log(n)(\log\log(n))^{-1/2}\rfloor} \\ &\leq \mathbb{P}\left(\left|R_n - \mathbb{E}[R_n]\right| \geq \frac{1}{2}\left\lfloor\frac{n(\log\log(n))^{1/4}}{\log(n)}\right\rfloor\right)^{\lfloor\log(n)(\log\log(n))^{-1/2}\rfloor}\end{aligned}$$

$$\le \left(\frac{5\,\mathrm{Var}(R_n)\log^2(n)}{n^2(\log\log(n))^{1/2}}\right)^{\lfloor \log(n)(\log\log(n))^{-1/2}\rfloor}$$
$$\le \left(\frac{Cn^2\log^2(n)/\log^4(n)}{n^2\sqrt{\log\log(n)}}\right)^{\lfloor \log(n)(\log\log(n))^{-1/2}\rfloor}$$
$$\le \left(\frac{C}{(\log(n))^2}\right)^{\lfloor \log(n)(\log\log(n))^{-1/2}\rfloor}$$
$$= \exp\left(-\log(n)\sqrt{\log\log(n)}\left(1-\frac{\log(C)}{2\log\log(n)}\right)\right).$$

This ends the proof of (18).

Let us now prove that

$$\mathbb{P}\big[N_n^* \ge n^\gamma\big] = \mathrm{o}\big(b_n^{-1}\big). \tag{20}$$

We have

$$\begin{aligned}
\mathbb{P}\big(N_n^* \ge n^\gamma\big) &\le \sum_x \mathbb{P}\big(N_n(x) \ge n^\gamma\big)\\
&= \sum_x \mathbb{P}\big(T_x \le n;\, N_n(x) \ge n^\gamma\big), \quad \text{where } T_x := \inf\{n > 1,\ \text{s.t. } S_n = x\},\\
&\le \sum_x \mathbb{P}(T_x \le n)\mathbb{P}\big(N_n(0) \ge n^\gamma\big)\\
&\le \mathbb{E}[R_n]\mathbb{P}(T_0 \le n)^{n^\gamma}.
\end{aligned}$$

Hence, (20) follows now from $\mathbb{E}[R_n] \sim cn(\log(n))^{-1}$, and from $\mathbb{P}(T_0 > n) \sim C/\log(n)$.

Since $n = \sum_y N_n(y) \le R_n N_n^*$, we get that $N_n^* \ge \frac{n}{R_n} \ge (\log\log(n))^{1/4}$ on Ω_n.

To prove the lower bound for V_n, note that, for $\beta \ge 1$, $V_n = \sum_y N_n(y)^\beta \ge \sum_y N_n(y) = n$. For $\beta < 1$, on Ω_n, we have

$$n = \sum_y N_n(y) = \sum_y N_n(y)^\beta N_n(y)^{1-\beta} \le V_n\big(N_n^*\big)^{1-\beta} \le V_n n^{\gamma(1-\beta)}.$$

□

3.2. *Scheme of the proof*

It is easy to see (cf. the proof of Lemma 5 in [7]) that $\mathbb{P}(Z_n = \lfloor b_n x\rfloor) = 0$ if $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \notin d_0\mathbb{Z}) = 1$, and that if $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \in d_0\mathbb{Z}) = 1$,

$$\mathbb{P}\big(Z_n = \lfloor b_n x\rfloor\big) = \frac{d_0}{2\pi}\int_{-\pi/d_0}^{\pi/d_0} \mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor}\mathbb{E}\left[\prod_y \varphi_\xi\big(tN_n(y)\big)\right]\mathrm{d}t.$$

In view of Lemma 12, we have to estimate

$$\frac{d_0}{2\pi}\int_{-\pi/d_0}^{\pi/d_0} \mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor}\mathbb{E}\left[\prod_y \varphi_\xi\big(tN_n(y)\big)\mathbf{1}_{\Omega_n}\right]\mathrm{d}t.$$

This is done in several steps presented in the following propositions.

Proposition 13. *Let* $\gamma \in (0, 1/(\beta+1))$ *and* $\delta \in (0, 1/(2\beta))$ *s.t.* $\gamma \frac{(1-\beta)_+}{\beta} < \delta < 1/\beta - \gamma$. *Then, we have*

$$\frac{d_0}{2\pi} \int_{\{|t| \le n^\delta / b_n\}} \mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor} \mathbb{E}\Big[\prod_y \varphi_\xi\big(tN_n(y)\big)\mathbf{1}_{\Omega_n}\Big]\,\mathrm{d}t = d_0 \frac{C(x)}{b_n} + \mathrm{o}\big(b_n^{-1}\big),$$

uniformly in $x \in \mathbb{R}$.

Recall next that the characteristic function ϕ of the limit distribution of $(n^{-1/\beta}\sum_{k=1}^n \xi_{ke_1})_n$ has the following form:

$$\phi(u) = \mathrm{e}^{-|u|^\beta (A_1 + \mathrm{i}A_2 \operatorname{sgn}(u))},$$

with $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \le |\tan(\pi\beta/2)|$. It follows that the characteristic function φ_ξ of ξ_0 satisfies:

$$1 - \varphi_\xi(u) \sim |u|^\beta \big(A_1 + \mathrm{i}A_2 \operatorname{sgn}(u)\big) \quad \text{when } u \to 0. \tag{21}$$

Therefore there exist constants $\varepsilon_0 > 0$ and $\sigma > 0$ such that

$$\max\big(\big|\phi(u)\big|, \big|\varphi_\xi(u)\big|\big) \le \exp\big(-\sigma|u|^\beta\big) \quad \text{for all } u \in [-\varepsilon_0, \varepsilon_0]. \tag{22}$$

Since $\overline{\varphi_\xi(t)} = \varphi_\xi(-t)$ for every $t \ge 0$, the following propositions achieve the proof of Theorem 3:

Proposition 14. *Let* δ *and* γ *be as in Proposition* 13. *Then there exists* $c > 0$ *such that*

$$\int_{n^\delta/b_n}^{\varepsilon_0 n^{-\gamma}} \mathbb{E}\Big[\prod_y \big|\varphi_\xi\big(tN_n(y)\big)\big|\mathbf{1}_{\Omega_n}\Big]\,\mathrm{d}t = \mathrm{o}\big(\mathrm{e}^{-n^c}\big).$$

Proposition 15. *There exists* $c > 0$ *such that*

$$\int_{\varepsilon_0 n^{-\gamma}}^{\pi/d_0} \mathbb{E}\Big[\prod_y \big|\varphi_\xi\big(tN_n(y)\big)\big|\mathbf{1}_{\Omega_n}\Big]\,\mathrm{d}t = \mathrm{o}\big(\mathrm{e}^{-n^c}\big).$$

3.3. *Proof of Proposition* 13

Remember that $V_n = \sum_{z\in\mathbb{Z}^d} N_n^\beta(z)$. We start by a preliminary lemma.

Lemma 16. (1) *If* $\beta > 1$, $\sup_n \mathbb{E}[(\frac{n\log(n)^{\beta-1}}{V_n})^{1/(\beta-1)}] < +\infty$.

(2) *If* $\beta \le 1$, $\forall p \in \mathbb{N}$, $\sup_n \mathbb{E}[(\frac{n\log(n)^{\beta-1}}{V_n})^p] < +\infty$.

Proof. For $\beta > 1$, using Hölder's inequality with $p = \beta$, we get

$$n = \sum_x N_n(x) \le V_n^{1/\beta} R_n^{(\beta-1)/\beta}$$

which means that

$$\left(\frac{n\log(n)^{\beta-1}}{V_n}\right)^{1/(\beta-1)} \le \frac{\log(n) R_n}{n}.$$

But it is proved in [17], Eq. (7.a), that $\mathbb{E}[R_n] = \mathcal{O}(n/\log(n))$. The result follows.

The result is obvious for $\beta = 1$. For $\beta < 1$, Hölder's inequality with $p = 2 - \beta$ yields

$$n = \sum_x N_n^{\beta/(2-\beta)}(x) N_n^{2(1-\beta)/(2-\beta)}(x) \le V_n^{1/(2-\beta)} \left(\sum_x N_n^2(x)\right)^{(1-\beta)/(2-\beta)}$$

and so

$$\frac{n\log(n)^{\beta-1}}{V_n}\le\left(\frac{\sum_x N_n^2(x)}{n\log(n)}\right)^{1-\beta}.$$

It is therefore enough to prove that there exists $c>0$ such that

$$\sup_n \mathbb{E}\left[\exp\left(c\frac{\sum_x N_n^2(x)}{n\log(n)}\right)\right]<\infty. \tag{23}$$

Note that $\sum_x N_n^2(x)=\sum_{k=0}^{n-1}N_n(S_k)$. By Jensen's inequality, we get thus

$$\mathbb{E}\left[\exp\left(c\frac{\sum_x N_n^2(x)}{n\log(n)}\right)\right]\le\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}\left[\exp\left(c\frac{N_n(S_k)}{\log(n)}\right)\right].$$

Observe now that $N_n(S_k)=\sum_{j=0}^{k}\mathbf{1}_{\{S_k-S_j=0\}}+\sum_{j=k+1}^{n-1}\mathbf{1}_{\{S_j-S_k=0\}}\overset{(d)}{=}N_{k+1}(0)+N'_{n-k}(0)-1$, where $(N'_n(x),n\in\mathbb{N},x\in\mathbb{Z}^d)$ is an independent copy of $(N_n(x),n\in\mathbb{N},x\in\mathbb{Z}^d)$. Hence,

$$\mathbb{E}\left[\exp\left(c\frac{\sum_x N_n^2(x)}{n\log(n)}\right)\right]\le\mathbb{E}\left[\exp\left(c\frac{N_n(0)}{\log(n)}\right)\right]^2.$$

But, $\forall t>0$,

$$\mathbb{P}\big(N_n(0)\ge t\log(n)\big)\le\mathbb{P}(T_0\le n)^{\lceil t\log(n)\rceil}$$

and

$$\mathbb{E}\left[\exp\left(c\frac{N_n(0)}{\log(n)}\right)\right]\le 1+\int_0^\infty c\exp(ct)\exp\big(-\lceil t\log(n)\rceil\mathbb{P}(T_0>n)\big)\,\mathrm{d}t.$$

Now (23) follows then from the fact that $\exists C>0$ such that $\mathbb{P}(T_0>n)\sim C/\log(n)$ for any integer $n\ge 1$. □

The next step is

Lemma 17. *Under the hypotheses of Proposition 13, we have*

$$\int_{\{|t|\le n^\delta/b_n\}}\mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor}\mathbb{E}\left[\left\{\prod_y\varphi_\xi\big(tN_n(y)\big)-\mathrm{e}^{-|t|^\beta(A_1+\mathrm{i}A_2\operatorname{sgn}(t))V_n}\right\}\mathbf{1}_{\Omega_n}\right]\mathrm{d}t=\mathrm{o}\big(b_n^{-1}\big),$$

uniformly in $x\in\mathbb{R}$.

Proof. Let

$$E_n(t):=\prod_y\varphi_\xi\big(tN_n(y)\big)-\prod_y\exp\big(-|t|^\beta N_n^\beta(y)\big(A_1+\mathrm{i}A_2\operatorname{sgn}(t)\big)\big).$$

Since $\gamma+\delta<\beta^{-1}$, we get, on Ω_n and if $|t|\le n^\delta b_n^{-1}$

$$\big|E_n(t)\big|\le\sum_y\big|\varphi_\xi\big(tN_n(y)\big)-\exp\big(-|t|^\beta N_n^\beta(y)\big(A_1+\mathrm{i}A_2\operatorname{sgn}(t)\big)\big)\big|\exp\left(-\sigma|t|^\beta\sum_{z\ne y}N_n^\beta(z)\right)$$

for n large enough. Observe next that (21) implies

$$\big|\varphi_\xi(u)-\exp\big(-|u|^\beta\big(A_1+\mathrm{i}A_2\operatorname{sgn}(u)\big)\big)\big|\le|u|^\beta h\big(|u|\big)\quad\text{for all }u\in\mathbb{R},$$

with h a continuous and monotone function on $[0,+\infty)$ vanishing at 0. Therefore we get

$$\left|E_n(t)\right| \le |t|^\beta h\left(n^{\gamma+\delta} b_n^{-1}\right) \sum_y N_n^\beta(y) \exp\left(-\sigma |t|^\beta \sum_{z\neq y} N_n^\beta(z)\right).$$

Now, according to (17) and since $\gamma < \frac{1}{\beta+1} \le \frac{1}{\beta+(1-\beta)_+}$, if n is large enough, we have on Ω_n

$$\sum_{z\neq y} N_n^\beta(z) \ge V_n/2 \quad \text{for all } y \in \mathbb{Z}.$$

By using this and the change of variables $v = t V_n^{1/\beta}$, we get

$$\int_{\{|t|\le n^\delta b_n^{-1}\}} \mathbb{E}\left[\left|E_n(t)\right| \mathbf{1}_{\Omega_n}\right] \mathrm{d}t \le h\left(n^{\gamma+\delta} b_n^{-1}\right) \mathbb{E}\left[V_n^{-1/\beta}\right] \int_{\mathbb{R}} |v|^\beta \exp\left(-\sigma |v|^\beta/2\right) \mathrm{d}v = \mathrm{o}\left(\mathbb{E}\left[V_n^{-1/\beta}\right]\right),$$

which proves the result according to Lemma 16. □

Finally Proposition 13 follows from the

Lemma 18. *Under the hypotheses of Proposition 13, we have*

$$\frac{d_0}{2\pi} \int_{\{|t|\le n^\delta b_n^{-1}\}} \mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor} \mathbb{E}\left[\mathrm{e}^{-|t|^\beta V_n (A_1 + \mathrm{i} A_2 \operatorname{sgn}(t))} \mathbf{1}_{\Omega_n}\right] \mathrm{d}t = d_0 \frac{C(x)}{b_n} + \mathrm{o}\left(b_n^{-1}\right),$$

uniformly in $x \in \mathbb{R}$.

Proof. Set

$$I_{n,x} := \int_{\{|t|\le n^\delta b_n^{-1}\}} \mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor} \mathrm{e}^{-|t|^\beta V_n (A_1 + \mathrm{i} A_2 \operatorname{sgn}(t))} \mathrm{d}t = \int_{\{|t|\le n^\delta b_n^{-1}\}} \mathrm{e}^{-\mathrm{i}t\lfloor b_n x\rfloor} \phi\left(t V_n^{1/\beta}\right) \mathrm{d}t.$$

Since $|\lfloor b_n x\rfloor - b_n x| \le 1$ and $\delta < (2\beta)^{-1}$, we have

$$I_{n,x} = \int_{\{|t|\le n^\delta b_n^{-1}\}} \mathrm{e}^{-\mathrm{i}t b_n x} \phi\left(t V_n^{1/\beta}\right) \mathrm{d}t + \mathrm{o}\left(b_n^{-1}\right).$$

Next, with the change of variable $v = t b_n$, we get:

$$\int_{\{|t|\le n^\delta b_n^{-1}\}} \mathrm{e}^{-\mathrm{i}t b_n x} \phi\left(t V_n^{1/\beta}\right) \mathrm{d}t = b_n^{-1} \left\{ V_n^{-1/\beta} b_n f\left(x V_n^{-1/\beta} b_n\right) - J_{n,x} \right\}, \tag{24}$$

where f is the density function of the distribution with characteristic function ϕ and where

$$J_{n,x} := \int_{\{|v|\ge n^\delta\}} \mathrm{e}^{-\mathrm{i}vx} \phi\left(v b_n^{-1} V_n^{1/\beta}\right) \mathrm{d}v.$$

By Lemma 5 (applied with $m = 1$, $t_1 = \theta_1 = 1$, $\gamma = \beta$), $(W_n := b_n V_n^{-1/\beta})_n$ converges almost surely, as $n \to \infty$, to the constant $\Gamma(\beta+1)^{-1/\beta}(\pi A)^{1-1/\beta}$. Moreover, Lemma 16 ensures that the sequence $(W_n, n \ge 1)$ is uniformly integrable, so actually the convergence holds in $\mathbb{L}^1$. From which we conclude that

$$\mathbb{E}\left[W_n f(x W_n)\right] = \mathbb{E}\left[W f(x W)\right] + \mathrm{o}(1) = C(x) + \mathrm{o}(1),$$

uniformly in x.

In view of (24), it only remains to prove that $\mathbb{E}[J_{n,x}\mathbf{1}_{\Omega_n}]=\mathrm{o}(1)$ uniformly in x. But this follows from the basic inequality

$$\mathbb{E}\big[|J_{n,x}\mathbf{1}_{\Omega_n}|\big]\le\int_{\{|v|\ge n^\delta\}}\mathbb{E}\big[\mathrm{e}^{-A_1|v|^\beta V_n/b_n^\beta}\mathbf{1}_{\Omega_n}\big]\,\mathrm{d}v,$$

and from the lower bound for V_n given in (17) and from the choice $\delta>\gamma(1-\beta)_+/\beta$. □

3.4. *Proof of Proposition* 14

Recall that on Ω_n, $N_n(y)\le n^\gamma$, for all $y\in\mathbb{Z}^d$. Hence by (22),

$$K_n:=\int_{n^\delta/b_n}^{\varepsilon_0 n^{-\gamma}}\mathbb{E}\Big[\prod_y\big|\varphi_\xi\big(tN_n(y)\big)\big|\mathbf{1}_{\Omega_n}\Big]\,\mathrm{d}t\le\int_{n^\delta/b_n}^{\varepsilon_0 n^{-\gamma}}\mathbb{E}\big[\exp\big(-\sigma t^\beta V_n\big)\mathbf{1}_{\Omega_n}\big]\,\mathrm{d}t.$$

With the change of variable $s=tV_n^{1/\beta}$, we get

$$\begin{aligned}K_n&\le\mathbb{E}\Big[V_n^{-1/\beta}\int_{n^\delta V_n^{1/\beta}b_n^{-1}}^{\varepsilon_0 n^{-\gamma}V_n^{1/\beta}}\exp\big(-\sigma s^\beta\big)\,\mathrm{d}s\,\mathbf{1}_{\Omega_n}\Big]\\&\le\frac{1}{n^{1/\beta-\gamma(1-\beta)_+/\beta}}\int_{n^{\delta-\gamma(1-\beta)_+/\beta}\log(n)^{(1-\beta)/\beta}}^{+\infty}\exp\big(-\sigma s^\beta\big)\,\mathrm{d}s,\end{aligned}$$

which proves the proposition since $\delta>\gamma(1-\beta)_+/\beta$.

3.5. *Proof of Proposition* 15

We adapt the proof of [7], Proposition 10. We will see that the argument of "peaks" still works here. We endow $\mathbb{Z}^d$ with the ordered structure given by the relation $<$ defined by

$$(\alpha_1,\dots,\alpha_d)<(\beta_1,\dots,\beta_d)\leftrightarrow\exists i\in\{1,\dots,d\},\alpha_i<\beta_i,\forall j<i,\alpha_j=\beta_j.$$

We consider $\mathcal{C}^+=(x_1,\dots,x_T)\in(\mathbb{Z}^d\setminus\{0\})^T$ for some positive integer T such that:

- $x_1+\cdots+x_T=0$;
- for every $i=1,\dots,T$, $\mathbb{P}(X_1=x_i)>0$;
- there exists $I_1\in\{1,\dots,T\}$ such that
  - for every $i=1,\dots,I_1$, $x_i>0$,
  - for every $i=I_1+1,\dots,T$, $x_i<0$.

Let us write $\mathcal{C}^-:=(x_{T-i+1})_{i=1,\dots,T}$. We define $B:=\sum_{i=1}^{I_1}x_i$. We observe that

$$p:=\mathbb{P}\big((X_1,\dots,X_T)=\mathcal{C}^+\big)=\mathbb{P}\big((X_1,\dots,X_T)=\mathcal{C}^-\big)>0.$$

We notice that $(X_1,\dots,X_T)=\mathcal{C}^+$ corresponds to a trajectory visiting B only once before going back to the origin at time T (and without visiting $-B$). Analogously, $(X_1,\dots,X_T)=\mathcal{C}^-$ corresponds to a trajectory that goes down to $-B$ and comes back up to 0 (and without visiting B), and staying at a distance smaller than $\tilde{d}/2$ of the origin with $\tilde{d}:=\sum_{i=1}^T|x_i|$ (where $|\cdot|$ is the absolute value if $d=1$ and $|(a,b)|=\max(|a|,|b|)$ if $d=2$). We introduce now the event

$$\mathcal{D}_n:=\Big\{C_n>\frac{np}{2T}\Big\},$$

where

$$C_n:=\#\Big\{k=0,\dots,\Big\lfloor\frac{n}{T}\Big\rfloor-1:\ (X_{kT+1},\dots,X_{(k+1)T})=\mathcal{C}^\pm\Big\}.$$

Since the sequences $(X_{kT+1},\dots,X_{(k+1)T})$, for $k\ge 0$, are independent of each other, Chernoff's inequality implies that there exists $c>0$ such that

$$\mathbb{P}(\mathcal{D}_n)=1-\mathrm{o}\bigl(\mathrm{e}^{-cn}\bigr).$$

We introduce now the notion of "loop." We say that there is a loop based on y at time n if $S_n=y$ and $(X_{n+1},\dots,X_{n+T})=\mathcal{C}^{\pm}$. We will see (in Lemma 19 below) that, on $\Omega_n\cap\mathcal{D}_n$, there is a large number of $y\in\mathbb{Z}^d$ on which are based a large number of loops. For any $y\in\mathbb{Z}^d$, let

$$C_n(y):=\#\biggl\{k=0,\dots,\biggl\lfloor\frac{n}{T}\biggr\rfloor-1:\ S_{kT}=y\text{ and }(X_{kT+1},\dots,X_{(k+1)T})=\mathcal{C}^{\pm}\biggr\},$$

be the number of loops based on y before time n (and at times which are multiple of T), and let

$$p_n:=\#\biggl\{y\in\mathbb{Z}:\ C_n(y)\ge\frac{\log\log(n)^{1/4}p}{4T}\biggr\},$$

be the number of sites $y\in\mathbb{Z}$ on which at least $a_n:=\lfloor\frac{\log\log(n)^{1/4}p}{4T}\rfloor$ loops are based.

Lemma 19. *On $\Omega_n\cap\mathcal{D}_n$, we have, $p_n\ge c'n^{1-\gamma}$ with $c'=p/(4T)$.*

Proof. Note that $C_n(y)\le N_n^*$ for all $y\in\mathbb{Z}^d$. Thus on $\Omega_n\cap\mathcal{D}_n$, we have

$$\begin{aligned}\frac{np}{2T}&\le\sum_{y\in\mathbb{Z}^d:C_n(y)<a_n}C_n(y)+\sum_{y\in\mathbb{Z}^d:C_n(y)\ge a_n}C_n(y)\\&\le R_na_n+N_n^*p_n\le\frac{np}{4T}+p_nn^{\gamma},\end{aligned}$$

according to Lemma 12. This proves the lemma. □

We have proved that, if n is large enough, the event $\Omega_n\cap\mathcal{D}_n$ is contained in the event

$$\mathcal{E}_n:=\bigl\{p_n\ge c'n^{1-\gamma}\bigr\}.$$

Now, on $\mathcal{E}_n$, we consider $(Y_i)_{i=1,\dots,\lfloor c''n^{1-\gamma}\rfloor}$ (with $c'':=c'/(2\tilde{d})$ if $d=1$ and with $c'':=c'/2\tilde{d}^2$ if $d=2$) such that

- on each Y_i, at least a_n loops are based;
- for every i,j such that $i\ne j$, we have $|Y_i-Y_j|>\tilde{d}/2$.

For every $i=1,\dots,\lfloor c''n^{1-\gamma}\rfloor$, let $t_i^{(1)},\dots,t_i^{(a_n)}$ be the a_n first times (which are multiples of T) when a loop is based on the site Y_i. We also define $N_n^0(Y_i+B)$ as the number of visits of S before time n to Y_i+B, which do not occur during the time intervals $[t_i^{(j)},t_i^{(j)}+T]$, for $j\le a_n$.

Since our construction is basically the same as in [7], Section 2.8, the proof of the following lemma is exactly the same as the proof of [7], Lemma 16, and we do not prove it again.

Lemma 20. *Conditionally to the event $\mathcal{E}_n$, $(N_n(Y_i+B)-N_n^0(Y_i+B))_{i\ge 1}$ is a sequence of independent identically distributed random variables with binomial distribution $\mathcal{B}(a_n;\frac{1}{2})$. Moreover this sequence is independent of $(N_n^0(Y_i+B))_{i\ge 1}$.*

Let η be a real number such that $\gamma<\eta<(1-\gamma)/\beta$ (this is possible since $\gamma<1/(\beta+1)$). We define

$$\forall n\ge 1,\quad d_n:=n^{-\eta}.$$

Let now $\rho := \sup\{|\varphi_\xi(u)|:\ d(u, \frac{2\pi}{d_0}\mathbb{Z}) \ge \varepsilon_0\}$. According to Formula (22) and since $\lim_{n\to\infty} d_n = 0$, for n large enough, we have

$$\begin{aligned}\left|\varphi_\xi(u)\right| &\le \rho \mathbf{1}_{\{d(u,(2\pi/d_0)\mathbb{Z})\ge\epsilon_0\}} + \exp\left(-\sigma d\left(u, \frac{2\pi}{d_0}\mathbb{Z}\right)^\beta\right)\mathbf{1}_{\{d(u,(2\pi/d_0)\mathbb{Z})<\epsilon_0\}}\\ &\le \exp\left(-\sigma d_n^\beta\right),\end{aligned}$$

as soon as $d(u, \frac{2\pi}{d_0}\mathbb{Z}) \ge d_n$. Therefore, for n large enough,

$$\prod_z \left|\varphi_\xi\left(tN_n(z)\right)\right| \le \exp\left(-\sigma d_n^\beta \#\left\{z:\ d\left(tN_n(z), \frac{2\pi}{d_0}\mathbb{Z}\right) \ge d_n\right\}\right). \tag{25}$$

Then notice that

$$d\left(tN_n(z), \frac{2\pi\mathbb{Z}}{d_0}\right) \ge d_n \quad\Longleftrightarrow\quad N_n(z) \in \mathcal{I} := \bigcup_{k\in\mathbb{Z}} I_k, \tag{26}$$

where for all $k \in \mathbb{Z}$,

$$I_k := \left[\frac{2k\pi}{d_0 t} + \frac{d_n}{t}, \frac{2(k+1)\pi}{d_0 t} - \frac{d_n}{t}\right].$$

In particular $\mathbb{R}\setminus\mathcal{I} = \bigcup_{k\in\mathbb{Z}} J_k$, where for all $k \in \mathbb{Z}$,

$$J_k := \left(\frac{2k\pi}{d_0 t} - \frac{d_n}{t}, \frac{2k\pi}{d_0 t} + \frac{d_n}{t}\right).$$

Lemma 21. *Under the hypotheses of Proposition 15, for every $i \le \lfloor c'' n^{1-\gamma}\rfloor$, $t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0)$ and n large enough,*

$$\mathbb{P}\left(N_n(Y_i + B) \in \mathcal{I} | \mathcal{E}_n, N_n^0(Y_i + B)\right) \ge \frac{1}{3} \quad \textit{almost surely}.$$

Assume for a moment that this lemma holds true and let us finish the proof of Proposition 15. Lemmas 20 and 21 ensure that conditionally to $\mathcal{E}_n$ and $((N_n^0(Y_i + B), i \ge 1)$, the events $\{N_n(Y_i + B) \in \mathcal{I}\}$, $i \ge 1$, are independent of each other, and all happen with probability at least $1/3$. Therefore, since $\Omega_n \cap \mathcal{D}_n \subseteq \mathcal{E}_n$, there exists $c > 0$, such that

$$\mathbb{P}\left(\Omega_n \cap \mathcal{D}_n, \#\left\{i:\ N_n(Y_i + B) \in \mathcal{I}\right\} \le \frac{c'' n^{1-\gamma}}{4}\right) \le \mathbb{P}\left(B_n \le \frac{c'' n^{1-\gamma}}{4}\right) = \mathrm{o}\left(\exp\left(-cn^{1-\gamma}\right)\right),$$

where for all $n \ge 1$, B_n has binomial distribution $\mathcal{B}(\lfloor c'' n^{1-\gamma}\rfloor; \frac{1}{3})$.

But if $\#\{z:\ N_n(z) \in \mathcal{I}\} \ge \frac{c'' n^{1-\gamma}}{4}$, then by (25) and (26), there exists a constant $c > 0$, such that

$$\prod_z \left|\varphi_\xi\left(tN_n(z)\right)\right| \le \exp\left(-cn^{1-\gamma} d_n^\beta\right),$$

which proves Proposition 15 since $1 - \gamma - \beta\eta > 0$.

Proof of Lemma 21. First notice that by Lemma 20, for any $H \ge 0$,

$$\mathbb{P}\left(N_n(Y_i + B) \in \mathcal{I} | \mathcal{E}_n, N_n^0(Y_i + B) = H\right) = \mathbb{P}(H + \beta_n \in \mathcal{I}), \tag{27}$$

where β_n is a random variable with binomial distribution $\mathcal{B}(a_n; \frac{1}{2})$. We will use the following result whose proof is postponed.

Lemma 22. *Under the hypotheses of Proposition 15, for every $t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0)$ and for n large enough, the following holds*:

(i) *For any integer k such that all the elements of $I_k - H$ are smaller than $\frac{a_n}{2}$*,

$$\mathbb{P}\big(\beta_n \in (I_k - H)\big) \ge \mathbb{P}\big(\beta_n \in (J_k - H)\big).$$

(ii) *For any integer k such that all the elements of $I_k - H$ are larger than $\frac{a_n}{2}$*,

$$\mathbb{P}\big(\beta_n \in (I_k - H)\big) \ge \mathbb{P}\big(\beta_n \in (J_{k+1} - H)\big).$$

Now call k_0 the largest integer satisfying the condition appearing in (i) and k_1 the smallest integer satisfying the condition appearing in (ii). We have $k_1 = k_0 + 1$ or $k_1 = k_0 + 2$. According to Lemma 22, we have

$$\begin{aligned}\mathbb{P}(H + \beta_n \in \mathcal{I}) &\ge \sum_{k \le k_0} \mathbb{P}(H + \beta_n \in I_k) + \sum_{k \ge k_1} \mathbb{P}(H + \beta_n \in I_k) \\ &\ge \sum_{k \le k_0} \mathbb{P}(H + \beta_n \in J_k) + \sum_{k \ge k_1} \mathbb{P}(H + \beta_n \in J_{k+1}) \\ &= \mathbb{P}(H + \beta_n \notin \mathcal{I}) - \mathbb{P}(H + \beta_n \in J_{k_0+1} \cup J_{k_1}).\end{aligned}$$

Hence,

$$\mathbb{P}(H + \beta_n \in \mathcal{I}) \ge \frac{1}{2}\big[1 - \mathbb{P}(H + \beta_n \in J_{k_0+1} \cup J_{k_1})\big].$$

The interval J_{k_1} being of length $2d_n/t$, according to the uniform version of the local limit theorem for β_n, for every $t \ge \varepsilon_0 n^{-\gamma}$, we have

$$\mathbb{P}(H + \beta_n \in J_{k_1}) \le \left(\frac{2d_n}{\varepsilon_0 n^{-\gamma}} + 1\right) a_n^{-1/2}.$$

We conclude that $\mathbb{P}(H + \beta_n \in J_{k_1}) = \mathrm{o}(1)$. The same holds for $\mathbb{P}(H + \beta_n \in J_{k_0+1})$, so that for n large enough,

$$\mathbb{P}(H + \beta_n \in \mathcal{I}) \ge \frac{1}{2}\big[1 - \mathrm{o}(1)\big] \ge \frac{1}{3}.$$

Together with (27), this concludes the proof of Lemma 21. □

Proof of Lemma 22. We only prove (i), since (ii) is similar. So let k be an integer such that all the elements of $I_k - H$ are smaller than $\frac{a_n}{2}$. Assume that $(J_k - H) \cap \mathbb{Z}$ contains at least one nonnegative integer (otherwise $\mathbb{P}(\beta_n \in (J_k - H)) = 0$ and there is nothing to prove). Let z_k denote the greatest integer in $J_k - H$, so that by our assumption $\mathbb{P}(\beta_n = z_k) > 0$ (remind that $0 \le z_k < \frac{a_n}{2}$). By monotonicity of the function $z \mapsto \mathbb{P}(\beta_n = z)$, for $z \le \frac{a_n}{2}$, we get

$$\mathbb{P}(\beta_n \in J_k - H) \le \mathbb{P}(\beta_n = z_k)\#\big((J_k - H) \cap \mathbb{Z}\big) \le \mathbb{P}(\beta_n = z_k)\left\lceil \frac{2d_n}{t} \right\rceil.$$

In the same way,

$$\mathbb{P}(\beta_n \in I_k - H) \ge \mathbb{P}(\beta_n = z_k)\#\big((I_k - H) \cap \mathbb{Z}\big) \ge \mathbb{P}(\beta_n = z_k)\left\lfloor \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \right\rfloor.$$

Hence

$$\mathbb{P}(\beta_n \in I_k - H) \ge \frac{\lfloor 2\pi/(d_0 t) - 2d_n/t \rfloor}{\lceil 2d_n/t \rceil}\mathbb{P}(\beta_n \in J_k - H).$$

But $\pi/(d_0 t) \geq 1$ and $\lim_{n\to+\infty} d_n = 0$ by hypothesis. It follows immediately that for n large enough, we have $2d_n < \pi/(2d_0)$, and so

$$\left\lfloor \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \right\rfloor \geq \left\lfloor \frac{3\pi}{2d_0 t} \right\rfloor \geq 1 + \left\lfloor \frac{\pi}{2d_0 t} \right\rfloor \geq \left\lceil \frac{\pi}{2d_0 t} \right\rceil \geq \left\lceil \frac{2d_n}{t} \right\rceil.$$

This concludes the proof of the lemma. □

4. Proof of the local limit theorem in the strongly nonlattice case

As in [7], the proof in the strongly nonlattice case is closely related to the proof in the lattice case. We assume here that ξ is strongly nonlattice. In that case, there exist $\varepsilon_0 > 0$, $\sigma > 0$ and $\rho < 1$ such that $|\varphi_\xi(u)| \leq \rho$ if $|u| \geq \varepsilon_0$ and $|\varphi_\xi(u)| \leq \exp(-\sigma|u|^\beta)$ if $|u| < \varepsilon_0$.

We use here the notations of Section 3 with the hypotheses on γ, and δ of Proposition 13. According to Lemma IV-5 of [14], it is enough to prove that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| b_n \mathbb{E}\big[h(Z_n - b_n x)\big] - C(x)\hat{h}(0)\right| = 0 \tag{28}$$

for any positive, Lebesgue-integrable and continuous real function h with continuously differentiable and compactly supported Fourier transform (let us notice that such functions exist, take for example $h_0(u) := \int_{u-\pi/2}^{u+\pi/2} (\frac{\sin t}{t})^4 \,\mathrm{d}t$). Let h be such a function. By Fourier inverse transform, we have

$$b_n \mathbb{E}\big[h(Z_n - b_n x)\big] = \frac{b_n}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}u b_n x} \mathbb{E}\Big[\prod_{x\in\mathbb{Z}^d} \varphi_\xi\big(u N_n(x)\big)\Big] \hat{h}(u)\,\mathrm{d}u.$$

Since $\hat{h}$ is L^1, we can restrict our study to the event Ω_n of Lemma 12. The part of the integral corresponding to $|u| \leq n^\delta b_n^{-1}$ is treated exactly as in Proposition 13. The only change is that we have to check that

$$\lim_{n\to\infty} b_n \int_{\{|u|\leq n^\delta b_n^{-1}\}} \mathbb{E}\big[\mathrm{e}^{-A_1|u|^\beta V_n} \mathbf{1}_{\Omega_n}\big] \sup_{|u|\leq n^\delta b_n^{-1}} \big|\hat{h}(u) - \hat{h}(0)\big|\,\mathrm{d}u = 0,$$

which is obviously true since $\hat{h}$ is a Lipschitz function.

Now, since $\hat{h}$ is bounded, the part corresponding to $n^\delta b_n^{-1} \leq |u| \leq \varepsilon_0 n^{-\gamma}$ is treated as in the proof of Proposition 14 (since it only uses the behavior of φ_ξ around 0, which is the same).

Finally, it remains to prove that

$$\lim_{n\to\infty} b_n \int_{\{|u|\geq\varepsilon_0 n^{-\gamma}\}} \left|\mathbb{E}\Big[\prod_x \varphi_\xi\big(u N_n(x)\big)\mathbf{1}_{\Omega_n}\Big]\right| \big|\hat{h}(u)\big|\,\mathrm{d}u = 0. \tag{29}$$

We note that, if $|u| \geq \varepsilon_0 n^{-\gamma}$ and $x \in \mathbb{Z}^d$, we have

$$\begin{aligned}
\big|\varphi_\xi\big(u N_n(x)\big)\big| &\leq \exp\big(-\sigma|u|^\beta N_n^\beta(x)\big)\mathbf{1}_{\{|uN_n(x)|\leq\varepsilon_0\}} + \rho\mathbf{1}_{\{|uN_n(x)|\geq\varepsilon_0\}} \\
&\leq \exp\big(-\sigma\varepsilon_0^\beta n^{-\gamma\beta} N_n^\beta(x)\big)\mathbf{1}_{\{|uN_n(x)|\leq\varepsilon_0\}} + \rho\mathbf{1}_{\{|uN_n(x)|\geq\varepsilon_0\}}.
\end{aligned}$$

For n large enough, $\rho \leq \exp(-\sigma\varepsilon_0^\beta n^{-\gamma\beta})$. Therefore, if n is large enough, then for all x and u such that $N_n(x) \geq 1$ and $|u| \geq \varepsilon_0 n^{-\gamma}$, we have

$$\big|\varphi_\xi\big(u N_n(x)\big)\big| \leq \exp\big(-\sigma\varepsilon_0^\beta n^{-\gamma\beta}\big).$$

Hence,

$$\left|\mathbb{E}\Big[\prod_x \varphi_\xi\big(u N_n(x)\big)\mathbf{1}_{\Omega_n}\Big]\right| \leq \mathbb{E}\big[\exp\big(-\sigma\varepsilon_0^\beta n^{-\gamma\beta} R_n\big)\mathbf{1}_{\Omega_n}\big] \leq \exp\big(-\sigma\varepsilon_0^\beta n^{1-\gamma(1+\beta)}\big).$$

Therefore, since $\gamma(1+\beta)<1$ and $\hat{h}$ is compactly supported, we have

$$\lim_{n\to\infty} b_n \int_{\{|u|\geq\varepsilon_0 n^{-\gamma}\}} \left|\mathbb{E}\left[\prod_x \varphi_\xi\big(uN_n(x)\big)\mathbf{1}_{\Omega_n}\right]\right| \left|\hat{h}(u)\right| \mathrm{d}u = 0.$$

This concludes the proof of Theorem 4.

Appendix: Complement to Cerny's paper

There is a missing argument in the proof of (6) in [8]. It concerns the control of the term

$$A_n := \sum_{(m_0,\dots,m_{2k-1})\in M_n} \big(\mathbb{P}(S_{m_u+m_v}=0)-\mathbb{P}(S_{m_u+\dots+m_v}=0)\big) \prod_{i\in\{1,\dots,2k-1\}\setminus\{u,v\}} \mathbb{P}(S_{m_i}=0),$$

where $k\geq 2$, $1\leq u\leq k-1$ and $v=u+k$ are fixed integers and $M_n:=\{(m_0,\dots,m_{2k-1})\in\mathbb{N}^{2k}\colon\ m_0+\dots+m_{2k-1}\leq n;\forall i\notin\{u,v\}, m_i\geq 1\}$. In order to obtain (6), it is necessary to prove that

$$A_n = \mathrm{O}\big(n^2(\ln n)^{2k-4}\big).$$

In [8], this estimate is proved using Karamata's Tauberian theorem. However, it is not clear that the sequence A_n is monotone.

To be complete, let us explain how this can be solved thanks to the argument used in [10] by Deligiannidis and Utev to prove their Theorem 2.2.

Summing over $m_0,\dots,m_{u-1},m_{v+1},\dots,m_{2k-1}$, and using the fact that $\mathbb{P}(S_n=0)=\mathrm{O}(n^{-1})$, we have

$$|A_n| \leq \mathrm{O}\big(n(\ln n)^{k-2}\big)B_n$$

with

$$B_n := \sum_{(m_u,\dots,m_v)\in M'_n} \left|\mathbb{P}(S_{m_u+m_v}=0)-\mathbb{P}(S_{m_u+\dots+m_v}=0)\right| \prod_{i=u+1}^{v-1}\mathbb{P}(S_{m_i}=0),$$

and $M'_n:=\{(m_u,\dots,m_v)\in\mathbb{N}^{k+1}\colon\ m_u+\dots+m_v\leq n;\forall i=u+1,\dots,v-1,m_i\geq 1\}$. Summing over m_u,m_v, we get

$$B_n = \sum_{(m_1,\dots,m_{k-1})\in\tilde{M}_{k-1,n}} \sum_{N=0}^{n-\sum_{i=1}^{k-1}m_i} (N+1)\left|\mathbb{P}(S_N=0)-\mathbb{P}(S_{N+\sum_{i=1}^{k-1}m_i}=0)\right| \prod_{i=1}^{k-1}\mathbb{P}(S_{m_i}=0)$$

with $\tilde{M}_{k-1,n}:=\{(m_1,\dots,m_{k-1})\in(\mathbb{N}\setminus\{0\})^{k-1}\colon\ m_1+\dots+m_{k-1}\leq n\}$. Now from the assumptions on the random walk, there exists $\sigma>0$ such that, for every $t\in[-\pi,\pi]^d$ ($d=1,2$) and every $j\in\mathbb{N}$, we have $|\varphi_{X_1}(t)|\leq \mathrm{e}^{-\sigma|t|^d}$ and $|1-(\varphi_{X_1}(t))^j|\leq(2+\sigma)\min(j|t|^d,1)$. Therefore, we have

$$B_n \leq \mathrm{O}(1)\sum_{(m_1,\dots,m_{k-1})\in\tilde{M}_{k-1,n}}\left(\prod_{i=1}^{k-1}\frac{1}{m_i}\right)\sum_{N=0}^{n-\sum_{i=1}^{k-1}m_i}(N+1)\int_{[-\pi,\pi]^d}\left|\varphi_{X_1}(t)\right|^N\left|1-\big(\varphi_{X_1}(t)\big)^{\sum_{i=1}^{k-1}m_i}\right|\mathrm{d}t$$

$$\leq \mathrm{O}(1)\sum_{(m_1,\dots,m_{k-1})\in\tilde{M}_{k-1,n}}\left(\prod_{i\in 1,\dots,k-1}\frac{1}{m_i}\right)\sum_{N=0}^{n-\sum_{i=1}^{k-1}m_i}(N+1)J_N\left(\sum_{i=1}^{k-1}m_i\right)$$

with

$$J_N(x) := \int_0^{\pi\sqrt{d}} \mathrm{e}^{-N\sigma t} \min(tx, 1)\,\mathrm{d}t.$$

We observe that $J_0 \le \pi\sqrt{d}$ and that, for every $N \ge 1$ and every $x \in \mathbb{N}$, we have

$$J_N(x) \le \frac{x}{(N\sigma)^2}\bigl(1 - \mathrm{e}^{-N\sigma/x}\bigr) + \frac{\mathrm{e}^{-N\sigma/x}}{N\sigma} = \frac{1}{N\sigma} f\left(\frac{N\sigma}{x}\right), \tag{30}$$

where $f(y) = \frac{1}{y}(1 - \mathrm{e}^{-y}) + \mathrm{e}^{-y}$. Since $f(y) \asymp 1$ for $y \ll 1$, and $f(y) \asymp \frac{1}{y}$ for $y \gg 1$, there exists a constant C such that $f(y) \le Cg(y)$, where $g(y) := \mathbb{1}_{[0,1]}(y) + \frac{1}{y}\mathbb{1}_{[1,+\infty[}(y)$. Hence, we have for $1 \le x \le n$,

$$\begin{aligned}
\sum_{N=0}^{n-x} (N+1) J_N(x) &\le \mathrm{O}(1)\left(1 + \sum_{N=1}^{n-x} g\left(\frac{N\sigma}{x}\right)\right) \\
&\le \mathrm{O}(1)\left(1 + \frac{x}{\sigma}\int_0^{n\sigma/x} g(y)\,\mathrm{d}y\right) \\
&\le \mathrm{O}(1)\left(x + x\log\left(\frac{n}{x}\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
B_n &\le \mathrm{O}(1) \sum_{(m_1,\ldots,m_{k-1}) \in \tilde{M}_{k-1,n}} \left(\prod_{i=1}^{k-1} \frac{1}{m_i}\right)\left(\sum_{i=1}^{k-1} m_i\right)\left[1 + \ln\left(\frac{n}{\sum_{i=1}^{k-1} m_i}\right)\right] \\
&= \mathrm{O}(1) \sum_{i=1}^{k-1} \sum_{(m_1,\ldots,m_{k-1}) \in \tilde{M}_{k-1,n}} \left(\prod_{j=1, j\neq i}^{k-1} \frac{1}{m_j}\right)\left[1 + \ln\left(\frac{n}{\sum_{i=1}^{k-1} m_i}\right)\right] \\
&\le \mathrm{O}(1) I_n,
\end{aligned}$$

with

$$\begin{aligned}
I_n &:= \sum_{(m_1,\ldots,m_{k-1}) \in \tilde{M}_{k-1,n}} \left(\prod_{i=1}^{k-2} \frac{1}{m_i}\right)\left[1 + \ln\left(\frac{n}{\sum_{i=1}^{k-1} m_i}\right)\right] \\
&= \sum_{(m_1,\ldots,m_{k-2}) \in \tilde{M}_{k-2,n}} \left(\prod_{i=1}^{k-2} \frac{1}{m_i}\right) \sum_{l=\sum_{i=1}^{k-2} m_i + 1}^{n} \left[1 + \ln\left(\frac{n}{l}\right)\right] \\
&\le \sum_{(m_1,\ldots,m_{k-2}) \in \tilde{M}_{k-2,n}} \left(\prod_{i=1}^{k-2} \frac{1}{m_i}\right) n \int_0^1 (-\ln x + 1)\,\mathrm{d}x = \mathrm{O}\bigl(n(\ln n)^{k-2}\bigr).
\end{aligned}$$

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