LIMIT THEOREMS FOR RANDOM TRANSFORMATIONS AND PROCESSES IN RANDOM ENVIRONMENTS

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ABSTRACT. I derive general relativized central limit theorems and laws of iterated logarithm for random transformations both via certain mixing assumptions and via the martingale differences approach. The results are applied to Markov chains in random environments, random subshifts of finite type, and random expanding in average transformations where I show that the conditions of the general theorems are satisfied and so the corresponding (fiberwise) central limit theorems and laws of iterated logarithm hold true in these cases. I consider also a continuous time version of such limit theorems for random suspensions which are continuous time random dynamical systems.

1. INTRODUCTION

Let $(\Omega, \mathcal{G}, P)$ be a probability space with an invertible $P-$preserving transformation $\theta : \Omega \to \Omega$, let $(\Xi, \mathcal{F})$ be another measurable space, and let Ξ be a measurable (with respect to the product $\mathcal{G} \times \mathcal{F}$) subset of $\Xi \times \Omega$ with the "fibers" $\Xi^\omega = \{\xi \in \Xi : (\xi, \omega) \in \Xi\} \in \mathcal{F}$. The set-up includes also a measurable map $\tau : \Xi \to \Xi$ such that $\tau(\xi, \omega) = (f_\omega \xi, \theta\omega)$ with $f_\omega : \Xi^\omega \to \Xi^{\theta\omega}$ being measurable "fiber" maps called random transformations (while τ is called the skew product transformation) with the composition rule $f_\omega^n = f_{\theta^{n-1}\omega} \circ \cdots \circ f_{\theta\omega} \circ f_\omega : \Xi^\omega \to \Xi^{\theta^n\omega}$.

Denote by $\mathcal{P}(\Xi)$ the space of probability measures μ on Ξ, and write $\mu \in \mathcal{P}_P(\Xi)$ if the marginal of μ on Ω is P. Such measures are determined by an essentially unique measurable family μ^ω such that μ^ω belongs to the space $\mathcal{P}(\Xi^\omega)$ of probability measures on Ξ^ω and

$$(1.1) \qquad \mu(d\xi, d\omega) = \mu^\omega(d\xi)P(d\omega).$$

It is easy to see that μ is $\tau-$invariant if and only if $f_\omega \mu^\omega = \mu^{\theta\omega}$.

There is already an extensive literature on the ergodic theory of random transformations, known also as the relativized ergodic theory, and, in particular, on the thermodynamic formalism (variational principle, equilibrium states, symbolic representations) for expanding random transformations and random subshifts of finite type (see [Ki1], [KK], [BG1], [BG2]) which yields also large deviations type results. In the deterministic case the study of the Ruelle-Perron-Frobenius (RPF) operator, which usually is the key point in the thermodynamic formalism approach, leads to estimates of decay of correlations which, in turn, via some probabilistic arguments yield central limit theorem (CLT) type results for corresponding dynamical systems

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(see [De]). Actually, the CLT requires much weaker conditions than one needs for the thermodynamic formalism set-up, but for many interesting classes of dynamical systems these conditions were verified only via RPF operator estimates.

In the random transformations framework the natural CLT has the following form. Let $\mu \in \mathcal{P}_P(\Xi)$ be $\tau-$invariant with the disintegration (1.1), and let $\varphi(\xi, \omega) = \varphi_\omega(\xi), \xi \in \Xi^\omega$, be a measurable function. The aim of this paper is to provide conditions (which can be verified for specific cases) which ensure that if $\psi_\omega = \varphi_\omega - \int \varphi_\omega d\mu^\omega$, then for $P-$almost all (a.a.)ω and any real a,

$$(1.2) \qquad \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \psi \circ \tau^i(\xi, \omega) \le a \right\} \underset{n \to \infty}{\longrightarrow} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{x^2}{2\sigma^2}} \, dx,$$

which may be called a relativized (or fiberwise) CLT or a CLT in random environments, depending on whether one takes a dynamical systems or a probabilistic point of view. I shall derive also a corresponding law of the iterated logarithm (LIL) saying that for $P-$a.a.ω and $\mu^\omega-$a.a.ξ the sum $\sum_{i=0}^{n-1} \psi \circ \tau^i(\xi, \omega)$ has the order $2\sigma^2 n \log \log(\sigma^2 n)$.

The whole set-up can be reformulated in the language of processes in random environments. Namely, let $(\mathbf{X}, \mathcal{A})$ be another measurable space, X, a measurable subset of $\mathbf{X} \times \Omega$, and $Z : \Xi \to X$ a measurable "bundle" map, so that $Z(\xi, \omega) = (Z^\omega(\xi), \omega)$, where Z^ω maps Ξ^ω to $X^\omega \in \mathcal{A}$. I view Ω as an environments space and Ξ^ω and X^ω as the path and the phase (or state) spaces, respectively, in the environment ω for the process $Z_n^\omega = Z \circ \tau^n(\xi, \omega) = Z^{\theta^n \omega}(f_\omega^n \xi)$. If $\mu \in \mathcal{P}_P(\Xi)$ is $\tau-$invariant, then $Z_n^\omega(\xi)$ can be considered as a usual stationary process on the probability space $(\Xi, \mathcal{G} \times \mathcal{F}, \mu)$, but my goal is to study properties of the process $Z_n^\omega(\xi)$ with respect to the measure μ^ω for a fixed $P-$typical ω, which usually requires different arguments. In particular, if there exists a measurable family of probability measures $P^\omega(x, \cdot)$ on $X^{\theta\omega}, x \in X^\omega$, such that for any measurable $\Gamma_i \subset X^{\theta^i \omega}, i = 1, ..., n$,

$$(1.3) \qquad \mu^\omega \{ \xi \in \Xi^\omega : Z_1^\omega \in \Gamma_1, Z_2^\omega(\xi) \in \Gamma_2, ..., Z_n^\omega(\xi) \in \Gamma_n | Z_0^\omega(\xi) = x \}$$

$$= \int_{\Gamma_1} ... \int_{\Gamma_{n-1}} P^\omega(x, dy_1) \cdots P^{\theta^{n-2}\omega}(y_{n-2}, dy_{n-1}) P^{\theta^{n-1}\omega}(y_{n-1}, \Gamma_n),$$

then $Z_n^\omega(\xi)$ becomes for each ω a time-inhomogenious Markov chain and the whole object is called a Markov chain, in a random environment. A particular case of this set-up when all X^ω coincide with one countable set was introduced as a model for stochastic automata, and its ergodic theory was studied in a number of papers (see [Or] and the references there). An $\omega-$wise CLT for this situation under rather indirect assumptions was obtained in [Co], but even for this particular case I shall derive here more general and more comprehensive results. I shall obtain first a CLT and an LIL in rather general circumstances, both under some uniform mixing assumptions and using the martingale differences approach. Then I shall verify the conditions of these theorems in the case of Markov chains in random environments under the random Doeblin condition introduced in [Ki3], and in the cases of random subshifts of finite type under certain random topological mixing conditions and random expanding in average transformations. A corresponding law of the iterated logorithm holds true for these cases, as well. Note that the large deviations type results were obtained for Markov chains in random environments in [Se] and [Ki3], and for random expanding transformations in [Ki1].

In the deterministic dynamics an important class of continuous time dynamical systems (flows) can be modelled by suspensions constructed by a well-mixing transformation and a Hölder continuous function. I introduce also the notion of random suspensions and obtain under natural conditions a CLT for this class of random continuous time dynamical systems.

2. Preliminaries and main results

I shall start with the set-up which, includes a $\tau-$invariant probability measure μ on Ξ with the disintegration (1.1) and a measurable family of $\sigma-$algebras $\mathcal{F}_{m,n}^\omega, n \geq m, \omega \in \Omega$, of sets from Ξ^ω such that

$$(2.1) \qquad \mathcal{F}_{m,n}^\omega \subset \mathcal{F}_{m',n'}^\omega \text{ if } m' \leq m \text{ and } n' \geq n \text{ and } f_\omega^{-1}\mathcal{F}_{m,n}^{\theta\omega} = \mathcal{F}_{m+1,n+1}^\omega.$$

The uniform mixing ($\phi-$mixing) coefficient is defined by

$$(2.2) \qquad \phi_{i,j}^\omega = \sup_{A \in \mathcal{F}_{0,i}^\omega, \mu^\omega(A) \neq 0, B \in \mathcal{F}_{j,\infty}^\omega} \left| \frac{\mu^\omega(A \cap B)}{\mu^\omega(A)} - \mu^\omega(B) \right|, j > i.$$

Let $\varphi = \varphi(\xi, \omega) = \varphi_\omega(\xi)$ be a measurable function on Ξ so that φ_ω as a function on Ξ^ω is $\mathcal{F}_{0,\infty}^\omega-$measurable, and so in view of (2.1), $\varphi_{\theta^i\omega} \circ f_\omega^i$ is $\mathcal{F}_{i,\infty}^\omega-$measurable as a function on Ξ^ω. The set-up includes also a measurable set $Q \subset \Omega$ with $P(Q) > 0$ and the corresponding sequence of hitting times

$$(2.3) \qquad k_{i+1}(\omega) = \min\{k > k_i(\omega) : \theta^k\omega \in Q\} \text{ with } k_0 \equiv 0.$$

Set

$$\psi_\omega = \varphi_\omega - E_{\mu^\omega}\varphi_\omega, \ \Psi(\xi, \omega) = \Psi_\omega(\xi) = \sum_{i=0}^{k_1(\omega)-1} \psi \circ \tau^i(\xi, \omega),$$

$$c(\omega) = (E_{\mu^\omega}|\psi_\omega|^2)^{1/2}, \ C(\omega) = (E_{\mu^\omega}|\Psi_\omega|^2)^{1/2},$$

$$d_n(\omega) = \left(E_{\mu^\omega}(\psi_\omega - E_{\mu^\omega}(\psi_\omega|\mathcal{F}_{0,n}^\omega))^2 \right)^{1/2},$$

$$D_n(\omega) = \left(E_{\mu^\omega}(\Psi_\omega - E_{\mu^\omega}(\Psi_\omega|\mathcal{F}_{0,n}^\omega))^2 \right)^{1/2},$$

where E_ν always denotes the expectation (i.e. the integral) with respect to a probability measure ν and $E_\nu(\cdot|\cdot)$ is the corresponding conditional expectation. Observe that

$$(2.4) \qquad C(\omega) \leq \sum_{i=0}^{k_1(\omega)-1} c(\theta^i\omega) \text{ and } D_n(\omega) \leq \sum_{i=0}^{k_1(\omega)-1} d_{n-i}(\theta^i\omega).$$

Set $\Theta = \theta^{k_1(\omega)}$, $F_\omega = f_\omega^{k_1(\omega)}$, and $T(\xi, \omega) = (F_\omega\xi, \Theta\omega)$. I always assume that $\theta : (\Omega, P) \longrightarrow (\Omega, P)$ is ergodic; then it is well known (see, for instance, [CFS]) that Θ is an ergodic measure-preserving transformation on the space (Q, P_Q), where $P_Q(A) = \frac{P(A \cap Q)}{P(Q)}$. I shall denote by Ξ_Q the restriction of Ξ to $\boldsymbol{\Xi} \times Q$ and by μ_Q the normalized restriction of μ to Ξ_Q, i.e. $d\mu_Q(\xi, \omega) = d\mu^\omega(\xi)dP_Q(\omega)$. It follows that μ_Q is invariant under the action of T.

2.1. Theorem. *Let* $\phi_j = \sup_{\omega, i \geq 1} \phi^{\omega}_{k_i(\omega), k_{i+j}(\omega)}$ *and* $\beta_j = (E_{P_Q} D^2_{k_j})^{1/2}$, *and suppose that*

$$(2.5) \qquad\qquad \sum_{j=1}^{\infty} \phi_j^{1/2} < \infty,$$

$$(2.6) \qquad\qquad \sum_{j=1}^{\infty} \beta_j < \infty,$$

and

$$(2.7) \qquad\qquad E_{P_Q}\left(\sum_{i=0}^{k_1-1} c \circ \theta^i \right)^2 < \infty.$$

Then $P-a.s.(almost\ surely),$

(2.8)

$$\sigma^2 \stackrel{def}{=} \lim_{n \to \infty} \frac{1}{n} E_{\mu^{\omega}} \left(\sum_{j=0}^{n-1} \psi_{\theta^j \omega} \circ f^j_{\omega} \right)^2 = P(Q)\left(E_{\mu_Q} \Psi^2 + \sum_{l=1}^{\infty} E_{\mu_Q}(\Psi(\Psi \circ T^l)) \right)$$

and the series in the right hand side of (2.8) converges. Furthermore, $P-a.s.$ for any number a,

$$(2.9) \qquad \lim_{n \to \infty} \mu^{\omega}\left\{ \xi \in \Xi^{\omega} : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \psi \circ \tau^i(\xi, \omega) \leq a \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2\sigma^2}} dx,$$

i.e. for $P-a.a.\omega$ the $\mu^{\omega}-distribution$ of $n^{-1/2} \sum_{i=0}^{n-1} (\psi \circ \tau^i)_{\omega}$ converges to the normal distribution with zero mean and variance σ^2, which in case $\sigma = 0$ is understood as the unit mass at 0. Finally, $\sigma = 0$ if and only if there exists a function η on Ξ_Q from $L^2(\Xi_Q, \mu_Q)$ such that $\mu_Q-a.s.$,

$$(2.10) \qquad\qquad \Psi \circ T = \eta \circ T - \eta.$$

Moreover, assuming that $\sigma > 0$, the following invariance principle for the law of the iterated logarithm (LIL) holds true. If $\zeta(t) = (2t \log \log t)^{1/2}$ and

$$\eta_n(t) = (\zeta(\sigma^2 n))^{-1} \sum_{j=0}^{k-1} \left(\psi \circ \tau^j + (nt - k)\psi \circ \tau^k \right)$$

$$for\ t \in [\frac{k}{n}, \frac{k+1}{n}),\ k = 0, 1, ..., n-1,$$

then $\mu_Q-a.s.$ the sequence of functions $\{\eta_n(\cdot),\ n \geq 3/\sigma^2\}$ is relatively compact in the space $C[0,1]$ (of continuous functions on $[0,1]$ considered with the supremum norm), and the set of its limit points as $n \to \infty$ coincides with the set K of absolutely continuous $x \in C[0,1]$ with $\int_0^1 (\dot{x}(t))^2 dt \leq 1$.

I note that though there are some results on CLT for nonstationary processes (see, for instance, [VR]) which at first sight could be used in Theorem 2.1, it turns out that the assumptions needed for these results are not satisfied for the interesting models I have in mind. Namely, [VR] requires that

$$(n - m)^{-1} E_{\mu^{\omega}} \sum_{m \leq j \leq n-1} \Psi^2 \circ T^i$$

is uniformly in n and m bounded away from zero and infinity, which may be satisfied only when k_1 is uniformly bounded—a very restrictive precondition. Thus in order to derive Theorem 2.1 one has to take into account in an essential way the stationary evolution on the space (Ω, P). My proof of Theorem 2.1 borrows few elements from [Co], where a partial case of Markov measures μ^ω (with a countable phase space of the corresponding Markov chain) and functions φ_ω measurable with respect to $\mathcal{F}^\omega_{0,0}$ was considered. The extension to a class of functions φ_ω more general than in [Co] follows by a slight modification of arguments from Section 20 of [Bi1]. The LIL part of Theorem 2.1 follows via a random martingale differences representation, which provides also another proof of the CLT part of Theorem 2.1. After this paper was accepted B.-Z. Rubshtein showed me [Ru] where CLT was proved in a similar setup using martingale differences under rather general conditions.

It is much more convenient to formulate the assumptions of Theorem 2.1 and of other results of this section in terms of the function Ψ and not in terms of the original function φ and the set Q, but it is useful to have also some sufficient conditions on φ and on k_1 which yield these assumptions. I shall now formulate a sufficient condition of this type (which will be proved at the end of Section 3) in terms of mixing and approximations with respect to a family of $\sigma-$algebras $\mathcal{G}_{m,n} \subset \mathcal{G}, -\infty \le m \le n \le \infty$, on Ω such that $\mathcal{G}_{m,n} \subset \mathcal{G}_{m',n'}$ if $m' \le m, n' \ge n$ and $\theta\mathcal{G}_{m,n} = \mathcal{G}_{m-1,n-1}$. The strong mixing ($\alpha-$mixing) coefficient with respect to this family of $\sigma-$algebras is defined by

$$(2.11) \qquad \alpha_n = \sup_{k, A \in \mathcal{G}_{-\infty,k}, B \in \mathcal{G}_{k+n,\infty}} \left| P(A \cap B) - P(A)P(B) \right|.$$

Set also

$$(2.12) \qquad \gamma_n = E_P \left| \mathbb{I}_Q - E_P(\mathbb{I}_Q|\mathcal{G}_{-n,n}) \right|,$$

where $\mathbb{I}_Q(\omega) = 1$ if $\omega \in Q$ and $= 0$, otherwise.

2.2. Proposition. *Suppose that*

(2.13)

$$\sup_\omega c(\omega) < \infty \text{ and } \sum_{l=1}^\infty l^2(\alpha_l + \gamma_l) < \infty \text{ or } E_{P_Q} c^4 < \infty \text{ and } \sum_{l=1}^\infty l^{5/4}(\alpha_l + \gamma_l)^{1/4} < \infty.$$

Then (2.7) holds true. If $Q = Q_L = \{\omega : L_\omega \le L\}$ for some random variable L_ω on (Ω, P) and a constant L, then $\gamma_n \le 2\tilde\gamma_n$ with $\tilde\gamma_n = E_P|L_\omega - E_P(L_\omega|\mathcal{G}_{-n,n})|$, and so one can write $\tilde\gamma_l$ in place of γ_l in (2.13), which provides a sufficient condition directly in terms of L_ω, and the latter has an explicit representation in the specific cases of Theorems 2.4-2.6 below.

Next, I shall describe another approach which sometimes is more convenient since it does not require an explicit uniform mixing condition with respect to a certain family of $\sigma-$algebras as in Theorem 2.1. The set-up includes the same objects $f_\omega, \theta, \tau, \varphi, \psi, \mu, Q, k_i, \ \Psi, F_\omega, \Theta, T, P_Q, \Xi_Q,$ and μ_Q as above, but now in place of a family of $\sigma-$fields $\mathcal{F}^\omega_{m,n}$ I consider another family of $\sigma-$algebras $\mathcal{T}^\omega_l = (f^l_\omega)^{-1}\mathcal{T}^{\theta^l \omega}_0$, where $\mathcal{T}^\omega_0$ for each ω is the restriction of the $\sigma-$algebra $\mathcal{F}$ to $\Xi^\omega \subset \Xi$. Then $\mathcal{T}^\omega_l, l = 0, 1, ...,$ is a nonincreasing sequence of $\sigma-$algebras on Ξ^ω. Let

$u_\omega : L^2(\Xi^{\theta\omega}, \mu^{\theta\omega}) \longrightarrow L^2(\Xi^\omega, \mu^\omega)$ be an isometry operator acting by the formula $u_\omega \varphi(\xi) = \varphi(f_\omega \xi)$ and let $u_\omega^* : L^2(\Xi^\omega, \mu^\omega) \longrightarrow L^2(\Xi^{\theta\omega}, \mu^{\theta\omega})$ be its conjugate defined by $\int \varphi(u_\omega^* \tilde\varphi) d\mu^{\theta\omega} = \int (u_\omega \varphi) \tilde\varphi d\mu^\omega$ for any $\varphi \in L^2(\Xi^{\theta\omega}, \mu^{\theta\omega})$ and $\tilde\varphi \in L^2(\Xi^\omega, \mu^\omega)$. It is easy to see that $u_\omega u_\omega^* : L^2(\Xi^\omega, \mu^\omega) \longrightarrow L^2(\Xi^\omega, \mu^\omega)$ is the orthogonal projection to $u_\omega L^2(\Xi^{\theta\omega}, \mu^{\theta\omega})$, and the last set is exactly the set of $\mathcal{T}_1^\omega$−measurable functions in $L^2(\Xi^\omega, \mu^\omega)$. Introduce also the operator $U_\omega = u_\omega^{k_1(\omega)}$, where $u_\omega^n = u_{\theta^{n-1}\omega} \circ \cdots \circ u_{\theta\omega} \circ u_\omega$, and let U_ω^* be its conjugate. The proof of the following result emplois nonstationary backwards martingale differences apporoximations and proceeds along the scheme similar to [Li], which, in turn, is a modification of [Go], though both papers dealt with the stationary case only.

2.3. Theorem. *Assume that (2.7) holds true and that the following two conditions are satisfied:*

(i)
$$E_{P_Q} \sum_{n=0}^\infty \left| E_{\mu^\omega}(\Psi_\omega (\Psi \circ T^n)_\omega) \right| < \infty$$

and

(ii)
$$E_{P_Q} E_{\mu^\omega} \left(\sum_{n=0}^\infty |U_{\Theta^{-n}\omega}^{*n} \Psi_{\Theta^{-n}\omega}| \right)^2 < \infty,$$

where $U_\omega^{*n} = U_{\Theta^{n-1}\omega}^* \circ \cdots \circ U_{\Theta\omega}^* U_\omega^*$. Then P−a.s. (2.8) and (2.9) hold true, and the criterion (2.10) for $\sigma = 0$ remains valid, as well. Moreover, assuming that $\sigma > 0$, the invariance principle for the law of iterated logarithm (LIL) stated in Theorem 2.1 holds true.

Actually, by an approximation argument similar to [Li] one can prove Theorem 2.3 assuming the weaker condition

(ii′)
$$E_{P_Q} \left(\sum_{n=0}^\infty E_{\mu^\omega} |U_{\Theta^{-n}\omega}^{*n} \Psi_{\Theta^{-n}\omega}| \right)^2 < \infty$$

in place of (ii), but since for the models I have in mind the verification of (ii′) is not easier than that of (ii) and since, on the other hand, the proof of Theorem 2.3 under (ii′) is more complicated, I restrict myself to the case when (ii) holds true.

Next, I shall discuss specific examples of processes in random environments and random transformations for which either the mixing conditions of Theorem 2.1 or the convergence conditions of Theorem 2.3 needed for martingale differences approximations hold true. First, I shall consider the case when the μ^ω's are Markov measures, and using Theorem 2.1 I shall obtain a result which both generalizes and specifies the CLT for Markov chains in random environments from [Co], deriving it for a more general class of functions and showing that the uniform mixing condition required in Theorem 2.1 follows if one assumes the random Doeblin condition for transition probabilities from [Ki3]. Thus, assume that μ^ω is given by (1.3), but in place of the conditional probabilities there I write

(2.14)

$$\mu^\omega \left\{ \xi \in \Xi^\omega : Z_1^\omega(\xi) \in \Gamma_1, Z_2^\omega(\xi) \in \Gamma_2, ..., Z_n^\omega(\xi) \in \Gamma_n \right\}$$
$$= \int_{X^\omega} \eta^\omega(dx) \int_{\Gamma_1} ... \int_{\Gamma_{n-1}} P^\omega(x, dy_1) \cdots P^{\theta^{n-2}\omega}(y_{n-2}, dy_{n-1}) P^{\theta^{n-1}\omega}(y_{n-1}, \Gamma_n),$$

where $X^\omega, P^\omega(x, \cdot)$, and Γ_i are the same as in (1.3) and $\eta^\omega, \omega \in \Omega$, is a measurable family of probability measures on X^ω such that for any measurable $\Gamma \subset X^\omega$,

$$(2.15) \qquad \int_{X^\omega} \eta^\omega(dx) P^\omega(x, \Gamma) = \eta^{\theta\omega}(\Gamma).$$

Let

$$P^\omega(n, x, \Gamma) = \int_{X^{\theta\omega}} ... \int_{X^{\theta^{n-1}\omega}} P^\omega(x, dy_1) P^{\theta\omega}(y_1, dy_2) \cdots P^{\theta^{n-1}\omega}(y_{n-1}, \Gamma)$$

and assume that there exist random variables $N = N_\omega \in \mathbb{Z}_+ = \{1, 2, ...\}$ and $\gamma_\omega > 0$ and a measurable family of measures $m^\omega \in \mathcal{P}(X^\omega)$ such that for $P-$a.a.$\omega \in \Omega$, any $x \in X^{\theta^{-N}\omega}$, and each measurable $\Gamma \subset X^\omega$,

$$(2.16) \qquad P^{\theta^{-N}\omega}(N, x, \Gamma) \geq \gamma_\omega m^\omega(\Gamma),$$

which is a randomized version of the Doeblin condition introduced in [Ki3]. It follows from [Ki3] that (2.16) implies the existence and uniqueness of a measure η^ω satisfying (2.15). Let $\sigma-$algebras $\mathcal{F}^\omega_{m,n}$ be generated by all sets of the form $\{\xi : Z^\omega_l(\xi) \in \Gamma\}$, $l = m, m+1, ..., n$, for measurable $\Gamma \subset X^{\theta^l\omega}$.

2.4. Theorem. *Let $Q = Q_L = \{\omega : \max(N_\omega, \gamma_\omega^{-1}) \leq L\}$ for a sufficiently large L, so that $P(Q) > 0$. Suppose that (2.16) is satisfied, μ^ω is defined by (2.14), and the $k_i(\omega)$'s are defined by (2.3). Then the condition (2.5) holds true. Thus if φ is a measurable function on Ξ as in Theorem 2.1 satisfying the conditions (2.6) and (2.7), then (2.8) and (2.9) hold true, as well. If $\varphi_\omega(\xi) = \varphi_\omega(Z^\omega(\xi))$ is, in fact, a function on X (and, as before, $\psi_\omega = \varphi_\omega - E_{\mu^\omega}\varphi_\omega$), then (2.9) can be written in the form*

$$(2.17) \quad \lim_{n\to\infty} \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi_{\theta^k\omega}(Z^\omega_k(\xi)) \leq a \Big| Z^\omega_0(\xi) = x \right\}$$

$$= \lim_{n\to\infty} \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi_{\theta^k\omega}(Z^\omega_k(\xi)) \leq a \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{u^2}{2\sigma^2}} \, du,$$

which is satisfied for any initial point $x \in X^\omega$. A corresponding LIL described in Theorem 2.1 holds true for this case, as well.

Next, I shall deal with random subshifts of finite type (see [BG2] and [KK]). In this case $\Xi^\omega = \{\xi = (\xi_0, \xi_1, ...) : \xi_i \in \{1, ..., \ell(\theta^i\omega)\}$ and $a_{\xi_i\xi_{i+1}}(\theta^i\omega) = 1$ for all $i = 0, 1, ...\}$, where $\ell = \ell(\omega)$, is a random variable on (Ω, P) satisfying

$$(2.18) \qquad \int \log \ell \, dP < \infty$$

and $A(\omega) = (a_{ij}(\omega))$, $\omega \in \Omega$ is a measurable family of $\ell(\omega) \times \ell(\theta\omega)-$matrices with 0 and 1 entries such that $P-$a.s. $A(\omega)$ has no zero row. A random subshift of finite type is called topologically mixing if there exists a $\mathbb{Z}^+ = \{1, 2, ...\}-$ valued random variable $\tilde{N} = \tilde{N}_\omega < \infty$ on $(\Omega, \mathcal{G}, P)$ so that $A(\theta^{\tilde{N}}\omega) \cdots A(\theta^{-2}\omega) A(\theta^{-1}\omega)$ is a matrix with positive entries for $P-$a.a.ω. This property is equivalent to existence of random variables $N = N_\omega < \infty$ and $\tilde{N} = \tilde{N}_\omega < \infty$ such that for any $n \geq N_\omega$ and $k \geq \tilde{N}_\omega$ the matrices $A(\omega) A(\theta\omega) \cdots A(\theta^n\omega)$ and $A(\theta^{-k}\omega) A(\theta^{-k+1}\omega) \cdots A(\theta^{-1}\omega)$ have only positive entries. The maps f_ω act by shifts, i.e. $(f_\omega\xi)_j = \xi_{j+1}$ for each $\xi \in \Xi^\omega$. The spaces Ξ^ω are imbedded into the compact space $\Xi = \bar{\mathbb{Z}}^+ \times \bar{\mathbb{Z}}^+ \times \cdots$,

which is the infinite product of the one-point compactifications $\bar{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{\infty\}$ of $\mathbb{Z}^+$ with a metric on Ξ given by

$$(2.19) \qquad d(\xi, \tilde{\xi}) = \sum_{i=0}^{\infty} 2^{-i} \left| \frac{1}{\xi_i} - \frac{1}{\tilde{\xi}_i} \right|.$$

For any measurable function $g = g(\xi, \omega) = g_\omega(\xi)$ on $\Xi \times \Omega$ set

$$(2.20) \qquad \mathrm{var}_n g_\omega = \sup\{ |g_\omega(\xi) - g_\omega(\tilde{\xi})| : \xi_i = \tilde{\xi}_i \,\forall i = 0, 1, ..., n-1 \}.$$

Consider g such that there exist $\kappa > 0$ and a random variable $K_g(\omega) > 0$ satisfying

$$(2.21) \qquad E_P |\log K_g(\omega)| < \infty$$

and, for any $n = 0, 1, 2, ...,$

$$(2.22) \qquad \mathrm{var}_n g_\omega \le K_g(\omega) e^{-\kappa n}.$$

Suppose also that

$$(2.23) \qquad E_P \sup_\xi |g_\omega(\xi)| < \infty.$$

The random Ruelle-Perron-Frobenius (RPF) operator $\mathcal{L}_\gamma^\omega$ corresponding to a function g maps functions on Ξ^ω to functions on $\Xi^{\theta\omega}$ by the formula

$$(2.24) \qquad \mathcal{L}_g^\omega q(\xi) = \sum_{\zeta \in f_\omega^{-1}\xi} e^{g_\omega(\zeta)} q(\zeta).$$

If g satisfies (2.21)-(2.23), then the random RPF theorem for a topologically mixing random subshift of finite type yields (see [KK] and in a more restricted form [BG2]) that there exist a unique positive random variable $\lambda = \lambda_\omega$ with $E_P |\log \lambda| < \infty$, a positive measurable in (ξ, ω) and continuous in ξ function $h = h(\xi, \omega) = h_\omega(\xi)$, and probability measures ν^ω on Ξ^ω such that

$$(2.25) \qquad \mathcal{L}_g^\omega h_\omega = \lambda_\omega h_{\theta\omega}, \ (\mathcal{L}_g^\omega)^* \nu^{\theta\omega} = \lambda_\omega \nu^\omega, \text{ and } \int_{\Xi^\omega} h_\omega d\nu^\omega = 1.$$

Then probability measures μ^ω (called random Gibbs measures) such that

$$(2.26) \qquad \frac{d\mu^\omega}{d\nu^\omega}(\xi) = h_\omega(\xi)$$

satisfy

$$(2.27) \qquad f_\omega \mu^\omega = \mu^{\theta\omega},$$

and the corresponding measure μ with the disintegration (1.1) is invariant with respect to the corresponding skew product transformation τ.

Let $\mathcal{F}_{m,n}^\omega$, $n < \infty$, be the finite $\sigma-$algebra generated by cylinder sets

$$C_{\eta_m, \eta_{m+1}, ..., \eta_n} = \{ \xi = (\xi_0, \xi_1, ...) : \xi_i = \eta_i \text{ for } i = m, m+1, ..., n \},$$

and let $\mathcal{F}_{m,\infty}^\omega$ be the minimal $\sigma-$algebra containing $\bigcup_{n \ge m} \mathcal{F}_{m,n}^\omega$.

2.5. Theorem. *Suppose that a random subshift of finite type is topologically mixing, (2.18) is satisfied, the $\sigma-$algebras $\mathcal{F}_{m,n}^\omega$ are defined as above, and a measurable family of probability measures μ^ω is constructed by a function g satisfying (2.21)-(2.23) so that (2.25)-(2.27) hold true. Then one can choose a set Q in the form $Q = Q_L = \{\omega : L_\omega \le L\}$ with $P(Q) > 0$, where L_ω is a random variable constructed explicitly in the proof (see Remark 6.5), so that the condition (2.5) will be satisfied. Thus, by Theorem 2.1, if (2.6) and (2.7) are satisfied, then (2.8) and (2.9) hold*

true together with the characterization (2.10) of the case $\sigma = 0$. The corresponding LIL follows, as well.

If I set $\mathcal{T}_l^\omega = \mathcal{F}_{l,\infty}^\omega$, then it follows from Section 6 that the conditions of Theorem 2.3 are also satisfied for random subshifts of finite type as above, and so Theorem 2.5 can be derived from Theorem 2.3, as well, though for a more restricted class of functions similar to the one in Theorem 2.6 below.

Finally, I shall consider expanding in average smooth random maps studied in [KK]. In this set-up the Ξ^ω's for all ω coincide with one compact connected $d-$dimensional C^2 Riemannian manifold M (though a more general case of manifolds M^ω depending on ω may be considered, as well) and all $f_\omega : M \longrightarrow M$ are C^2 endomorphisms of M such that

$$(2.28) \qquad \log \|Df_\omega^{-1}\|, \log \|Df_\omega\| \in L^1(\Omega, P),$$

and

$$(2.29) \qquad \alpha = E_P \log \|Df_\omega^{-1}\| < 0,$$

where Df is the differential of f and $\|\cdot\|$ is the supremum norm. Again, I define a random RPF operator $\mathcal{L}_g^\omega$ by (2.24), and if the g_ω's are Hölder continuous, i.e.

$$(2.30) \qquad |g_\omega(x) - g_\omega(y)| \le K_g(\omega)(d(x,y))^\kappa \,\forall x,y \in M$$

(where $d(\cdot, \cdot)$ is the distance function on M) for some $\kappa > 0$ and a random variable $K_g(\omega) > 0$ satisfying (2.21), then (see [KK]) the random RPF theorem holds true, yielding a random variable $\lambda^\omega > 0$, a function $h = h_\omega(x) > 0$ on $M \times \Omega$, and probability measures ν^ω, μ^ω on M and μ on $M \times \Omega$ satisfying (2.25)-(2.27).

2.6. Theorem. *Assume that the random transformation f_ω satisfies (2.28) and (2.29), the measures μ^ω with properties (2.25)-(2.27) are constructed via the random RPF theorem from [KK] by a function g satisfying (2.30), and a function $\varphi = \varphi_\omega(x)$ (for which the CLT is going to be proved) is Hölder continuous in ξ, i.e. it satisfies (2.30) with, say, the same exponent $\kappa > 0$ and a random variable $K_\varphi(\omega) > 0$ such that*

$$(2.31) \qquad E_{P_Q}\left(\sum_{j=0}^{k_1(\omega)-1} (\|\varphi_{\theta^j\omega}\| + K_\varphi(\theta^j\omega)) \right)^2 < \infty,$$

with k_1 given by (2.3). Then one can choose a set Q in the form $Q = Q_L = \{\omega : L_\omega \le L\}$ with $P(Q) > 0$, where L_ω is a random variable constructed explicitly in the proof (see Remark 6.5), so that the conditions (i) and (ii) of Theorem 2.3 will be satisfied, and so (2.8) and (2.9) together with the characterization of the case $\sigma = 0$ hold true, as well as the corresponding LIL.

If the closure of $\{f_\omega, \omega \in \Omega\}$ is a compact set of uniformly expanding endomorphisms, then one can construct a random Markov partition (see [BG1]) and reduce the problem to the case of random subshifts of finite type considered in Theorem 2.5. In the general case of random transformations expanding only in average (and in further generalizations) when symbolic representations are not available it is not clear how to construct the $\sigma-$algebras $\mathcal{F}_{m,n}^\omega$ and to exhibit the uniform mixing condition of Theorem 2.1, and one has to use Theorem 2.3 instead. If a version of the RPF theorem holds true so that an RPF operator (2.24) together with $h_\omega, \lambda_\omega, \nu^\omega$ satisfying (2.25) exist, then it is possible to write an explicit formula for the operator u_ω^* (see (6.29) in Section 6); but such a formula does not exist in more general

situations, and then it is not clear how to verify the condition (ii) of Theorem 2.3, so in a number of cases Theorem 2.1 may be more readily applicable. Also the class of functions for which one can derive a CLT and LIL via Theorem 2.3 is usually more restrictive than what one gets via Theorem 2.1.

Observe that Theorem 2.6 yields fiber-wise CLT and LIL for some deterministic skew product transformations. For instance, consider an expanding map of the 3-dimensional torus $\mathbb{T}^3 = \mathbb{T}^1 \times \mathbb{T}^2$ given by the formula $\tau(x, y) = (f_y x, \theta y)$, where θ is an ergodic automorphism of $\mathbb{T}^2$ and $f_y x = \gamma(y) + n(y)x \,(\mathrm{mod}\, 1)$, where $\gamma(y) \in \mathbb{R}$, $n(y) \in \mathbb{Z}_+$ are measurable functions with $0 < \int_{\mathbb{T}^2} \log n(y) dy < \infty$. Since both θ and the f_y's preserve the Lebesgue measures (denoted Leb below) on $\mathbb{T}^2$ and on $\mathbb{T}^1$, respectively, I can view the f_y's as "random" expanding maps of $\mathbb{T}^1$ with $\Omega = \mathbb{T}^2$, $P =$Leb, $M = \mathbb{T}^1$, and $\mu^y =$Leb (which is a "random" Gibbs measure corresponding to the function $g_y = \log n(y)$). Thus I arrive at a very particular case of the above situation, and Theorem 2.6 yields that for Leb-a.a.y, Leb$\{x : \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \psi \circ \tau^l(x, y) \le a\}$ converges as $n \to \infty$ to the right hand side of (2.9), and the corresponding LIL follows, as well.

As another application of Theorem 2.6 I can write a CLT and an LIL for sums of digits in "random base" expansions (see [Ki4]).

2.7. Remark. An easy extension of the proofs of Theorems 2.1 and 2.3 yields a random functional CLT saying that for $P-$a.a.ω the process $\zeta_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]} \psi \circ \tau^i(\xi, \omega)$ on the space (Ξ^ω, μ^ω) weakly converges as $n \to \infty$ to the process σw_t, where w_t is the standard Wiener process (see Remark 2 on p. 510 in [Sh] and its proof on p. 517). Furthermore, combining methods of [PS] and the present paper, one can derive some invariance principle type results which also yield a $P-$a.s. law of the iterated logarithm for $\sum_{i=0}^n \psi \circ \tau^i(\xi, \omega)$ with respect to μ^ω. Note also that, using analyticity properties of the random RPF operators (see [Gu]), one can derive similarly to [Ki2] relativized (fiberwise) moderate deviations type results.

Important classes of continuous time dynamical systems can be modelled by suspension flows constructed by a well-mixing transformation and a sufficiently regular function. I introduce next the notion of random suspensions and formulate a corresponding CLT for them. The set-up consists of a measurable function $l = l_\omega(\xi)$ such that $C^{-1}(\omega) < l_\omega(\xi) < C(\omega)$ for some random variable $C(\omega) > 0$ on Ω, the space $Y = \{(s, \xi, \omega) : (\xi, \omega) \in \Xi, 0 \le s < l_\omega(\xi)\}$, and a continuous time (random) dynamical system (semi-flow) σ^t defined by $\sigma^t(s, \xi, \omega) = (s + t, \xi, \omega)$ if $s + t < l_\omega(\xi)$ and $\sigma^t(s, \xi, \omega) = (u, \tau^k(\xi, \omega))$ if $0 \le u = s + t - \sum_{0 \le i \le k-1}(l \circ \tau^i)_\omega(\xi)$, where $k = k(s, t, \xi, \omega) = \max\{j \ge 1 : s + t - \sum_{0 \le i \le j-1}(l \circ \tau^i)_\omega(\xi) \ge 0\}$. The semi-flow σ^t is called a suspension over τ with the ceiling function l. One can speak also of a family of random transformations g_ω^t defined by $g_\omega^t(s, \xi) = (s + t, \xi)$ if $\xi \in \Xi^\omega, s + t < l_\omega(\xi)$ and $g_\omega^t(s, \xi) = (u, f_\omega^k \xi)$ if u and $k = k(s, t, \xi, \omega)$ are as above. Then $g_{\theta^k \omega}^u g_\omega^t(s, \xi) = g_\omega^{u+t}$.

Let μ be a $\tau-$invariant probability measure with the disintegration (1.1); then $\eta \in \mathcal{P}(Y)$ defined by $d\eta(s, \xi, \omega) = \frac{1}{\bar{l}} d\mu^\omega(\xi) dP(\omega) ds$, where $\bar{l} = \int l_\omega(\xi) d\mu^\omega(\xi) dP(\omega)$, is $\sigma-$invariant. Let $\varphi = \varphi_\omega(\xi, s)$ be a measurable function on $Y, \psi = \psi_\omega = \varphi_\omega - \int \varphi_\omega d\mu^\omega, \varphi_\omega^{(l)}(\xi) = \int_0^{l_\omega(\xi)} \varphi_\omega(\xi, s) ds$, and $\psi_\omega^{(l)}(\xi) = \varphi_\omega^{(l)}(\xi) - \int \varphi_\omega^{(l)} d\mu^\omega$. Let $\sigma-$algebras $\mathcal{F}_{m,n}^\omega$, a set Q, and hitting times k_i be as in Theorem 2.1, and set $\Psi_\omega^{(l)} = \sum_{i=0}^{k_1(\omega)-1}(\psi^{(l)} \circ \tau^i)_\omega(\xi)$ and $R_\omega^{(l)} = \sum_{i=0}^{k_1(\omega)-1} |\psi^{(l)} \circ \tau^i(\xi, \omega)|$.

2.8. Theorem. *Suppose that the conditions of Theorem 2.1 are satisfied both for the function $\varphi^{(l)}$ and for the function l in place of φ there, and, in addition, $\sup_\omega C(\omega) < \infty$,*

$$(2.32) \qquad E_{\mu_Q}(R^{(l)})^8 < \infty \quad and \quad E_{P_Q}\int_0^{l_\omega(\xi)} |\psi_\omega(\xi,s)|ds < \infty,$$

$$(2.33) \qquad \sum_{j=1}^\infty j^4 \phi_j^{1/8} < \infty,$$

and

$$(2.34) \qquad \sum_{j=1}^\infty j^4 \beta_j^{(8)} < \infty$$

where $\beta_j^{(8)} = (E_{P_Q}(D_{k_j}^{(8)})^8)^{1/8}$ and $D_n^{(8)}(\omega) = (E_{\mu^\omega}(\Psi_\omega^{(l)} - E_{\mu^\omega}(\Psi_\omega^{(l)}|\mathcal{F}_{0,n}^\omega))^8)^{1/8}$. Then, $P-$a.s. for any number a,

$$(2.35) \quad \lim_{t\to\infty} \mu^\omega\left\{\xi \in \Xi^\omega : \frac{1}{\sqrt{t}}\int_0^t (\psi \circ \sigma^s)_\omega(0,\xi)ds \le a\right\} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^a e^{-\frac{x^2}{2\sigma^2}}\,dx$$

where, $P-$a.s.,

$$(2.36) \qquad \sigma^2 = \lim_{t\to\infty} \frac{1}{t}E_{\mu^\omega}\left(\int_0^t (\psi \circ \sigma^s)_\omega ds\right)^2$$

$$= \bar{l}^{-1}P(Q)\left(E_{\mu_Q}(\Psi^{(l)})^2 + \sum_{l=1}^\infty E_{\mu_Q}(\Psi^{(l)}(\Psi^{(l)} \circ T^j))\right).$$

If $\sigma > 0$, then, combining methods of [DP] and the present paper, a corresponding LIL follows, as well.

3. Limit theorems under mixing conditions

Set $N_n(\omega) = \max\{l : k_l(\omega) < n\}$; then

$$(3.1) \qquad \left|\sum_{l=0}^{n-1} \psi \circ \tau^l(\xi,\omega) - \sum_{j=0}^{N_n(\omega)-1} \Psi \circ T^j(\xi,\omega)\right| \le R \circ T^{N_n(\omega)}(\xi,\omega),$$

where $R_\omega(\xi) = R(\xi,\omega) = \sum_{j=0}^{k_1(\omega)-1} |\psi \circ \tau^j(\xi,\omega)|$. Next,

$$(3.2) \qquad (E_{\mu^\omega}R_\omega^2)^{1/2} \le \sum_{j=0}^{k_1(\omega)-1} c(\theta^j\omega) \stackrel{\text{def}}{=} I(\omega).$$

By (2.7) and the ergodic theorem, $P-$a.s.,

$$(3.3) \qquad \lim_{n\to\infty} \frac{1}{n}N_n(\omega) = P(Q) \quad and \quad \lim_{n\to\infty} \frac{1}{\sqrt{n}}I(\Theta^{N_n(\omega)}\omega) = 0.$$

Hence by (3.1) and (3.2), $P-$a.s.,

$$(3.4) \qquad \lim_{n\to\infty} \frac{1}{n}E_{\mu^\omega}\left(\sum_{j=0}^{n-1}(\psi \circ \tau^j)_\omega\right)^2 = \lim_{n\to\infty} \frac{1}{n}E_{\mu^\omega}\left(\sum_{j=0}^{N_n(\omega)-1}(\Psi \circ T^j)_\omega\right)^2,$$

and for each $\varepsilon > 0$ and $P-$a.a.ω,

$$(3.5) \qquad \lim_{n \to \infty} \mu^\omega \left\{ \xi : \frac{1}{\sqrt{n}} \left| \sum_{l=0}^{n-1} \psi \circ \tau^l(\xi, \omega) - \sum_{j=0}^{N_n(\omega)-1} \Psi \circ T^j(\xi, \omega) \right| > \varepsilon \right\} = 0.$$

Thus it remains to deal with the asymptotical behavior of $\frac{1}{\sqrt{n}} \sum_{j=0}^{N_n(\omega)-1} \Psi \circ T^j(\xi, \omega)$, which is the same as of $\frac{1}{\sqrt{n}} \sum_{j=1}^{N_n(\omega)-1} \Psi \circ T^j(\xi, \omega)$, and so one has to prove results only for P_Q-a.s.ω. First, I shall study the variance

$$(3.6) \qquad E_{\mu^\omega} \left(\sum_{i=0}^{N_n(\omega)-1} (\Psi \circ T^i)_\omega \right)^2 = \sum_{i,j=0}^{N_n(\omega)-1} E_{\mu^\omega} \left((\Psi \circ T^i)_\omega (\Psi \circ T^j)_\omega \right)$$

$$= \sum_{i=0}^{N_n(\omega)-1} E_{\mu^{\Theta^i \omega}} \Psi^2_{\Theta^i \omega} + 2 \sum_{i=0}^{N_n(\omega)-2} \Phi_{N_n(\omega)-i}(\Theta^i \omega),$$

where $\Phi_k(\omega) = \sum_{l=1}^{k-1} E_{\mu^\omega}(\Psi_\omega(\Psi \circ T^l)_\omega)$, and I use the equality $E_{\mu^\omega}(g \circ T)_\omega = E_{\mu^{\Theta \omega}} g_{\Theta \omega}$, which follows since $F_\omega \mu^\omega = \mu^{\Theta \omega}$. By Lemma 1 from Section 20 in [Bi1], whenever the function V is $\mathcal{F}^\omega_{0,k_i(\omega)}-$measurable and the function W is $\mathcal{F}^\omega_{k_{i+j}(\omega),\infty}-$measurable, for any $r, s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, one has

$$(3.7) \qquad |E_{\mu^\omega}(VW) - E_{\mu^\omega} V E_{\mu^\omega} W| \le 2\phi_j^{1/r}(E_{\mu^\omega}|V|^r)^{1/r}(E_{\mu^\omega}|W|^s)^{1/s}.$$

Since $(\Psi \circ T^l)_\omega$ is $\mathcal{F}^\omega_{k_l(\omega),\infty}-$measurable, then by (3.7) and the Cauchy-Schwarz inequality

$$(3.8) \qquad \left| E_{\mu^\omega} \left(\Psi_\omega(\Psi \circ T^l)_\omega \right) \right| \le \left| E_{\mu^\omega} \left((\Psi_\omega - E_{\mu^\omega}(\Psi_\omega | \mathcal{F}^\omega_{0,k_{[l/2]}(\omega)}))(\Psi \circ T^l)_\omega \right) \right|$$

$$+ \left| E_{\mu^\omega} \left(E_{\mu^\omega}(\Psi_\omega | \mathcal{F}^\omega_{0,k_{[l/2]}(\omega)})(\Psi \circ T^l)_\omega \right) \right|$$

$$\le \left(E_{\mu^\omega}(\Psi_\omega - E_{\mu^\omega}(\Psi_\omega | \mathcal{F}^\omega_{0,k_{[l/2]}(\omega)}))^2 \right)^{1/2} \left(E_{\mu^\omega}(\Psi \circ T^l)^2_\omega \right)^{1/2}$$

$$+ 2\phi_{[l/2]}^{1/2} \left(E_{\mu^\omega}(E_{\mu^\omega}(\Psi_\omega | \mathcal{F}^\omega_{0,k_{[l/2]}(\omega)}))^2 \right)^{1/2} \left(E_{\mu^\omega}(\Psi \circ T^l)^2_\omega \right)^{1/2}$$

$$\le D_{k_{[l/2]}(\omega)}(\omega)C(\Theta^l \omega) + 2\phi_{[l/2]}^{1/2}C(\omega)C(\Theta^l \omega),$$

where $[a]$ denotes the integral part of a. This together with the Cauchy-Schwarz inequality gives

$$(3.9) \qquad E_{P_Q} \left| E_{\mu^\omega} \left(\Psi_\omega(\Psi \circ T^l)_\omega \right) \right| \le \beta_{[l/2]}(E_{P_Q}C^2)^{1/2} + 2\phi_{[l/2]}^{1/2}E_{P_Q}C^2.$$

It follows from (2.4)-(2.7) and (3.9) that the series

$$(3.10) \qquad \Phi(\omega) = \sum_{l=1}^\infty E_{\mu^\omega}(\Psi_\omega(\Psi \circ T^l)_\omega)$$

converges absolutely P_Q−a.s. and in $L^1(Q, P_Q)$. Next, I claim that, P_Q−a.s. and in $L^1(Q, P_Q)$,

$$(3.11) \qquad \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-2} \Phi_{m-i}(\Theta^i\omega) = E_{P_Q}\Phi,$$

which by the ergodic theorem is the same as

$$(3.12) \qquad \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-2} (\Phi(\Theta^i\omega) - \Phi_{m-i}(\Theta^i\omega)) = 0.$$

Indeed,

$$(3.13) \qquad |\Phi(\Theta^i\omega) - \Phi_{m-i}(\Theta^i\omega)| \le \delta_{m-i}(\Theta^i\omega),$$

where

$$\delta_j(\omega) = \sum_{l=j}^{\infty} \left(D_{k_{[l/2]}(\omega)}(\omega) C(\Theta^l\omega) + 2\phi_{[l/2]}^{1/2} C(\omega) C(\Theta^l\omega) \right).$$

Since δ_j is monotone decreasing in j, then

$$(3.14) \qquad \sum_{i=0}^{m-2} \delta_{m-i}(\Theta^i\omega) \le \sum_{i=0}^{m-j} \delta_j(\Theta^i\omega) + \sum_{i=m-j+1}^{m-2} \delta_2(\Theta^i\omega).$$

By (2.5)–(2.7), $\delta_2 \in L^1(Q, P_Q)$, and so $\lim_{m\to\infty} \frac{1}{m}\delta_2(\Theta^{m-i}\omega) = 0$ P−a.s. and in $L^1(Q, P_Q)$. Thus (3.14) together with the ergodic theorem yield that, P−a.s. and in $L^1(Q, P_Q)$,

$$(3.15) \qquad \limsup_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-2} \delta_{m-i}(\Theta^i\omega) \le \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-j} \delta_j(\Theta^i\omega) = E_{P_Q}\delta_j.$$

Since $\delta_j \to 0$ as $j \to \infty$ both P_Q−a.s. and in $L^1(Q, P_Q)$ and the left hand side of (3.15) does not depend on j, I derive that the left hand side of (3.15) is zero, which by (3.13) gives (3.12), and so (3.11) holds true. Finally, (3.1)-(3.4), (3.6), (3.11), and the ergodic theorem yield both (2.8) and

$$(3.16) \qquad \lim_{m\to\infty} \frac{1}{m} E_{\mu^\omega} \left(\sum_{i=0}^{m-1} (\Psi \circ T^i)_\omega \right)^2 = E_{\mu_Q}\Psi^2 + E_{P_Q}\Phi = \sigma^2/P(Q).$$

Next, I shall show that, P_Q−a.s. for any number a,

$$(3.17) \quad \lim_{m\to\infty} \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} \Psi \circ T^i(\xi, \omega) \le a \right\} = \frac{\sqrt{P(Q)}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{P(Q)x^2}{2\sigma^2}} dx$$

with $\sigma = 0$ if and only if there exists a function η on Ξ_Q from $L^2(\Xi_Q, \mu_Q)$ such that μ_Q−a.s. (2.10) holds true. Actually, this last statement (i.e. the last assertion of Theorem 2.1) follows already from (3.16) (see §§2–5 from Ch.18 in [IL]). If $\sigma = 0$ then (2.8) says that $\frac{1}{\sqrt{m}} \sum_{j=0}^{n-1} \psi_{\theta^j\omega} \circ f_\omega^j$ converges in $L^2(\Xi^\omega, \mu^\omega)$ (and so in μ^ω−measure) to zero, which yields the assertion of Theorem 2.1 for this case, so in what follows I assume that $\sigma^2 > 0$. Observe also that it suffices to prove (3.17) for each a P−a.s. and not P−a.s. simultaneously for all a, since the latter follows from the former in view of the monotonicity in a of the left hand side of (3.17) and continuity in a of the right hand side of (3.17), and so everything is determined already by rational a's.

Set $\Psi_\omega^{(N)} = E_{\mu^\omega}(\Psi_\omega | \mathcal{F}_{0,k_N(\omega)}^\omega)$. I want to show first that if $\delta^{(N)} = \Psi - \Psi^{(N)}$, then, $P-$a.s. for any $\varepsilon > 0$,

$$(3.18) \qquad \lim_{N \to \infty} \limsup_{m \to \infty} \mu^\omega \left\{ \xi : \left| \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \delta^{(N)} \circ T^j(\xi, \omega) \right| \geq \varepsilon \right\} = 0.$$

To do this I estimate

$$(3.19) \qquad E_{\mu^\omega} \left(\sum_{j=0}^{m-1} (\delta^{(N)} \circ T^j)_\omega \right)^2 = \sum_{j=0}^{m-1} E_{\mu^{\Theta^j \omega}} (\delta_{\Theta^j \omega}^{(N)})^2 + 2 \sum_{i=0}^{m-2} H_{m-i}^{(N)}(\Theta^i \omega),$$

where $H_k^{(N)} = \sum_{l=1}^{k-1} E_{\mu^\omega}(\delta_\omega^{(N)}(\delta^{(N)} \circ T^l)_\omega)$. Taking into account that

$$(3.20) \qquad \left(E_{\mu^\omega} \left(\delta_\omega^{(N)} - E_{\mu^\omega}(\delta_\omega^{(N)} | \mathcal{F}_{0,k_{[l/2]}(\omega)}^\omega) \right)^2 \right)^{1/2}$$

$$\leq \left(E_{\mu^\omega} \left(\Psi_\omega - E_{\mu^\omega}(\Psi_\omega | \mathcal{F}_{0,k_{[l/2]}(\omega)}^\omega) \right)^2 \right)^{1/2}$$

$$+ \left(E_{\mu^\omega} \left(\Psi_\omega^{(N)} - E_{\mu^\omega}(\Psi_\omega^{(N)} | \mathcal{F}_{0,k_{[l/2]}(\omega)}^\omega) \right)^2 \right)^{1/2} \leq 2D_{k_{[l/2]}(\omega)}(\omega),$$

I derive in the same way as in (3.8) that

$$(3.21) \quad \left| E_{\mu^\omega} \left(\delta_\omega^{(N)}(\delta^{(N)} \circ T^l)_\omega \right) \right| \leq \left(E_{\mu^\omega}(\delta_\omega^{(N)} - E_{\mu^\omega}(\delta_\omega^{(N)} | \mathcal{F}_{0,k_{[l/2]}(\omega)}^\omega))^2 \right)^{1/2}$$

$$\times \left(E_{\mu_\omega}(\delta^{(N)} \circ T^l)_\omega^2 \right)^{1/2} + 2\phi_{[l/2]}^{1/2} \left(E_{\mu^\omega}(\delta_\omega^{(N)})^2 \right)^{1/2} \left(E_{\mu_\omega}(\delta^{(N)} \circ T^l)_\omega^2 \right)^{1/2}$$

$$\leq 2D_{k_{[l/2]}(\omega)}(\omega)D_{k_N(\omega)}(\Theta^l \omega) + 2\phi_{[l/2]}^{1/2} D_{k_N(\omega)}(\omega)D_{k_N(\omega)}(\Theta^l \omega).$$

This together with the Cauchy-Schwarz inequality gives

$$(3.22) \qquad E_{P_Q} \left| E_{\mu^\omega} \left(\delta_\omega^{(N)}(\delta^{(N)} \circ T^l)_\omega \right) \right| \leq 2\beta_{[l/2]}\beta_N + 2\phi_{[l/2]}^{1/2}\beta_N^2.$$

It follows from (2.5) and (2.6) that the series

$$(3.23) \qquad H^{(N)}(\omega) = \sum_{l=1}^{\infty} E_{\mu^\omega}(\delta_\omega^{(N)}(\delta^{(N)} \circ T^l)_\omega)$$

converges $P-$a.s. and in $L^1(Q, P_Q)$, and so I derive similarly to (3.16) that, $P-$a.s. and in $L^1(Q, P_Q)$,

$$(3.24) \qquad \lim_{m \to \infty} \frac{1}{m} E_{\mu^\omega} \sum_{j=0}^{m-1} (\delta^{(N)} \circ T_\omega^j)^2 = \beta_N^2 + E_{P_Q} H^{(N)}.$$

Now (3.18) follows from (3.21)-(3.23) together with the Chebyshev inequality. Similarly to above I derive also that, $P-$a.s. and in $L^1(Q, P_Q)$,

$$(3.25) \quad \lim_{m \to \infty} \frac{1}{m} E_{\mu^\omega} \left(\sum_{j=0}^{m-1} (\Psi^{(N)} \circ T^j)_\omega \right)^2 = E_{\mu_Q}(\Psi^{(N)})^2 + E_{P_Q} \Phi^{(N)} \stackrel{\text{def}}{=} (\sigma^{(N)})^2,$$

where $\Phi^{(N)}(\omega) = \sum_{l=1}^{\infty} E_{\mu^\omega}(\Psi_\omega^{(N)}(\Psi^{(N)} \circ T^l)_\omega)$, and it follows from (2.5), (2.6), and (3.22) that

$$(3.26) \qquad \lim_{N \to \infty} \lim_{m \to \infty} \frac{1}{m} E_{\mu^\omega} \left(\sum_{j=0}^{m-1} (\Psi^{(N)} \circ T^j)_\omega \right)^2 = \lim_{N \to \infty} (\sigma^{(N)})^2 = \frac{\sigma^2}{P(Q)}.$$

Since I am assuming that $\sigma^2 > 0$, then for all N large enough $(\sigma^{(N)})^2 > 0$; and only such N will be considered in what follows. Therefore, in order to obtain (3.17) it remains to show that, P_Q−a.s. for each number a and any N large enough,

(3.27)

$$\lim_{m \to \infty} \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \Psi^{(N)} \circ T^j(\xi, \omega) \le a \right\} = \frac{\sqrt{P(Q)}}{\sigma^{(N)} \sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{P(Q)x^2}{2(\sigma^{(N)})^2}} \, dx.$$

Set $Z^{(n)} = Z_\omega^{(n)} = \sum_{j=0}^{n-1} (\Psi^{(N)} \circ T^j)_\omega$, $Z^{(m,n)} = Z_\omega^{(m,n)} = Z_\omega^{(n)} - Z_\omega^{(m)}$ for $n \ge m$, and $\chi^{(m,n)}(u) = \chi_\omega^{(m,n)}(u) = E_{\mu^\omega} e^{iuZ_\omega^{(m,n)}}$, where $i = \sqrt{-1}$. Observe that

(3.28)

$$(\Psi^{(N)} \circ T^j)_\omega = E_{\mu^{\Theta^j \omega}}(\Psi_{\Theta^j \omega} | \mathcal{F}_{0,k_N(\Theta^j \omega)}^{\Theta^j \omega}) \circ F_\omega^j = E_{\mu^\omega}(\Psi \circ T^j | \mathcal{F}_{k_j(\omega), k_{j+N}(\omega)}^\omega),$$

and so $Z_\omega^{(m,n)}$ is $\mathcal{F}_{k_m(\omega), k_{n+N-1}(\omega)}^\omega$−measurable. Set

$$A_j(\omega) = E_{\mu^\omega} \left((e^{iuZ_\omega^{((k-1)l, kl)}} - 1)(e^{iu(\Psi^{(N)} \circ T^{kl+j})_\omega} - 1)e^{iuZ_\omega^{(kl+j+1, ml)}} \right)$$
$$- E_{\mu^\omega}(e^{iuZ_\omega^{((k-1)l, kl)}} - 1)E_{\mu^\omega} \left((e^{iu(\Psi^{(N)} \circ T^{kl+j})_\omega} - 1)e^{iuZ_\omega^{(kl+j+1, ml)}} \right).$$

Then the Cauchy-Schwarz inequality and $|e^{iux} - 1| \le |ux|$ yield that for any $j = 0, 1, \dots,$

$$(3.29) \qquad A_j(\omega) \le 2u^2 \left(E_{\mu^\omega}(Z_\omega^{((k-1)l, kl)})^2 \right)^{1/2} C(\Theta^{kl+j} \omega),$$

and using, in addition, (3.7), I derive for $j = N, N+1, \dots$ that

$$(3.30) \qquad A_j(\omega) \le 2u^2 \phi_{j-N+1}^{1/2} \left(E_{\mu^\omega}(Z_\omega^{((k-1)l, kl)})^2 \right)^{1/2} C(\Theta^{kl+j} \omega).$$

Thus, similarly to Lemma 3 from [Co], it follows that, for any $k = 1, 2, \dots, m-1$ and $J = (m-k)l$,

(3.31)

$$|\chi_\omega^{((k-1)l, ml)}(u) - \chi_\omega^{((k-1)l, kl)}(u)\chi_\omega^{(kl, ml)}(u)| \le |\sum_{j=0}^{J-1} A_j(\omega)|$$
$$\le 2u^2 \left(E_{\mu^\omega}(Z_\omega^{((k-1)l, kl)})^2 \right)^{1/2} \left(\sum_{j=0}^{N-1} C(\Theta^{kl+j} \omega) + \sum_{j=N}^{J-1} \phi_{j-N+1}^{1/2} C(\Theta^{kl+j} \omega) \right).$$

Therefore by the Cauchy-Schwarz inequality

(3.32)

$$|\chi_\omega^{(0,ml)}(u) - \prod_{k=1}^{m} \chi_\omega^{((k-1)l,kl)}(u)|$$

$$\leq \Big| \sum_{k=1}^{m-1} (\chi_\omega^{((k-1)l,ml)}(u) - \chi_\omega^{((k-1)l,kl)}(u)\chi_\omega^{(kl,ml)}(u)) \prod_{j=1}^{k-1} \chi_{((j-1)l,jl)}(u) \Big|$$

$$\leq 2u^2 \Big(\sum_{k=1}^{m-1} (E_{\mu^\omega}(Z_\omega^{((k-1)l,kl)})^2) \Big)^{1/2}$$

$$\times \Big(\sum_{k=1}^{m} \Big(\sum_{j=0}^{N} C(\Theta^{kl+j}\omega) + \sum_{j=N}^{\infty} \phi_{j-N+1}^{1/2}(C(\Theta^{kl+j}\omega)) \Big)^2 \Big)^{1/2}$$

$$\leq 2u^2 \Big(\sum_{k=1}^{m-1} (E_{\mu^\omega}(Z_\omega^{((k-1)l,kl)})^2) \Big)^{1/2} (\sum_{j=1}^{\infty} \phi_j^{1/2})^{1/2} \Big(\sum_{k=1}^{m} G(\Theta^{kl+N-1}\omega) \Big)^{1/2},$$

where $G(\omega) = \sum_{j=1}^{\infty} \phi_j^{1/2} C^2(\Theta^j \omega)$. Since Θ preserves P_Q, then by (2.4), (2.5), (2.7), and the monotone convergence theorem I obtain that

(3.33)
$$E_{P_Q} G = E_{P_Q} C^2 \sum_{j=1}^{\infty} \phi_j^{1/2} < \infty.$$

By the ergodic theorem

(3.34)
$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m-1} G \circ \Theta^{kl+N-1}(\omega) = E_{P_Q}(G \circ \Theta^{N-1} | J_l),$$

where J_l is the σ-algebra of Θ^l-invariant subsets of Q.

Now let $m = [n/l]$. Since

(3.35)
$$\frac{1}{\sqrt{n}} \Big(E_{\mu^\omega}(Z_\omega^{(n)} - Z_\omega^{(ml)})^2 \Big)^{1/2} \leq \frac{\sum_{j=0}^{l} C(\Theta^{ml+j}\omega)}{\sqrt{ml}},$$

then it follows by (2.4) and (2.7) that, for P_Q-a.a.ω and any $\varepsilon > 0$,

(3.36)
$$\mu^\omega\{\xi : \frac{1}{\sqrt{n}} |Z_\omega^{(n)}(\xi) - Z_\omega^{(ml)}(\xi)| > \varepsilon\} \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

while l remains fixed. Next, applying (3.32) I obtain

(3.37)

$$|\chi_\omega^{(0,ml)}(\frac{u}{\sqrt{n}}) - \prod_{k=1}^{m} \chi_\omega^{((k-1)l,kl)}(\frac{u}{\sqrt{n}})| \leq \frac{2u^2}{l} \Big(\frac{1}{m} \sum_{k=1}^{m-1} E_{\mu^\omega}(Z_\omega^{((k-1)l,kl)})^2 \Big)^{\frac{1}{2}}$$

$$\times \Big(\sqrt{N} \Big(\frac{1}{m} \sum_{k=1}^{m-1} \sum_{j=0}^{N-1} C^2(\Theta^{kl+j}\omega) \Big)^{\frac{1}{2}} + (\sum_{j=1}^{\infty} \phi_j^{\frac{1}{2}})^{\frac{1}{2}} \Big(\frac{1}{m} \sum_{k=1}^{m-1} G(\Theta^{kl+N-1}\omega) \Big)^{\frac{1}{2}} \Big).$$

Since for any Borel function q,

(3.38)
$$E_{\mu^\omega} q(Z_\omega^{((k-1)l,kl)}) = E_{\mu^{\Theta^{(k-1)l}\omega}} q(Z_{\Theta^{(k-1)l}\omega}^{(l)}),$$

then by the ergodic theorem as $m \to \infty$ the right hand side of (3.37) converges P_Q-a.s. and in $L^1(Q, P_Q)$ to

$$(3.39) \quad \frac{2u^2}{\sqrt{l}} \left\{ \frac{1}{l} E_{P_Q} \left(E_{\mu^\omega} (Z_\omega^{(l)})^2 \Big| J_l \right) \right\}^{1/2}$$
$$\times \left\{ \left(E_{P_Q} (N \sum_{j=0}^{N-1} C^2 \circ \Theta^j | J_l) \right)^{1/2} + (\sum_{j=1}^{\infty} \phi_j^{\frac{1}{2}})^{\frac{1}{2}} \left(E_{P_Q} (G \circ \Theta^{N-1} | J_l) \right)^{\frac{1}{2}} \right\}.$$

By (3.25), in $L^1(Q, P_Q)$,

$$(3.40) \qquad \lim_{l \to \infty} \frac{1}{l} E_{P_Q} (E_{\mu^\omega} (Z_\omega^{(l)})^2 | J_l) = (\sigma^{(N)})^2.$$

Choose $l = l_\nu = \nu!$ for $\nu = 1, 2, ...$; then $J_{l_{\nu+1}} \supset J_{l_\nu}$, and so, by the martingale convergence theorem the term in the second braces in (3.39) converges P_Q-a.s. as $l_\nu \to \infty$. By (3.40) the term in the first braces in (3.39) converges in $L^1(Q, P_Q)$, and so I can choose a subsequence ν_j so that along the subsequence l_{ν_j} this term converges P_Q-a.s. In view of the coefficient $l^{-1/2}$ in front, the whole expression in (3.39) converges P_Q-a.s. to zero along $l = l_{\nu_j} \to \infty$, and so P_Q-a.s.,

$$(3.41) \qquad \chi_\omega^{(0,n)}(\frac{u}{\sqrt{n}}) - \prod_{k=1}^{n} \chi_\omega^{((k-1)l_{\nu_j}, kl_{\nu_j})}(\frac{u}{\sqrt{n}}) \longrightarrow 0$$

as, first, $m \to \infty$ and, then, $l = l_{\nu_j} \to \infty$. Since $\chi_\omega^{(0,n)}(\frac{u}{\sqrt{n}})$ does not depend directly on m and l_{ν_j}, while $m = m_{\nu_j} = [\frac{n}{l_{\nu_j}}]$, and the product in (3.41) is the characteristic function of a sum of (not identically distributed) independent random variables then in order to complete the proof of Theorem 2.1 I can use a CLT for arrays (see Theorem 7.1 in [Bi1], Theorem 27.2 in [Bi2], or §4 of Ch.III in [Sh]) provided the Lindenberg condition saying that $P-$a.s. for any $\varepsilon > 0$ (recall that I assume $(\sigma^{(N)})^2 > 0$ in (3.25)),

(3.42)

$$\lim_{l_{\nu_j} \to \infty} \lim_{m_{\nu_j} \to \infty} \frac{1}{m_{\nu_j} l_{\nu_j}} \sum_{k=1}^{m_{\nu_j}} \int_{\{|Z_\omega^{((k-1)l_{\nu_j}, kl_{\nu_j})}| > \varepsilon \sqrt{m_{\nu_j} l_{\nu_j}}\}} (Z_\omega^{((k-1)l_{\nu_j}, kl_{\nu_j})})^2 d\mu^\omega = 0$$

holds true. In order to obtain (3.42), observe that in view of (3.38) by the ergodic theorem for any $K > 0$, P_Q-a.s. and in $L^1(Q, P_Q)$,

$$(3.43) \qquad \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} E_{\mu^\omega} \left((Z_\omega^{((k-1)l, kl)})^2 \mathbb{I}_{|Z_\omega^{((k-1)l, kl))}| > K} \right)$$
$$= E_{P_Q} \left(E_{\mu^\omega} ((Z_\omega^{(l)})^2 \mathbb{I}_{|Z_\omega^{(l)}| > K} \Big| J_l \right).$$

If m_{ν_j} is large enough then $\varepsilon \sqrt{m_{\nu_j} l_{\nu_j}} > K$, and so, P_Q-a.s.,

$$(3.44) \qquad \limsup_{m_{\nu_j} \to \infty} \frac{1}{m_{\nu_j} l_{\nu_j}} \sum_{k=1}^{m_{\nu_j}} \int_{\{|Z_\omega^{((k-1)l_{\nu_j}, kl_{\nu_j})}| > \varepsilon \sqrt{m_{\nu_j} l_{\nu_j}}\}} (Z_\omega^{((k-1)l_{\nu_j}, kl_{\nu_j})})^2 d\mu^\omega$$
$$\leq E_{P_Q} \left(E_{\mu^\omega} (Z^{(l_{\nu_j})})^2 \mathbb{I}_{|Z^{(l_{\nu_j})}| > K} \Big| J_{l_{\nu_j}} \right)$$

for any $K > 0$. Since the left hand side of (3.44) does not depend on K and the right hand side of (3.44) tends to zero as $K \to \infty$, I conclude that both the limit in (4.44) and the limit in (3.42) equal zero.

The invariance principle for the LIL part of Theorem 2.1 will be derived in the next section via a martingale differences representation by some modification of methods from [HH], which will provide another proof of the CLT part of Theorem 2.1, as well.

In order to prove Proposition 2.2, observe that

$$(3.45) \qquad E_{P_Q}\left(\sum_{j=0}^{k_1-1} c \circ \theta^j\right)^2 \leq c^2 E_{P_Q} k_1^2 = \frac{1}{P(Q)} \sum_{j=0}^{\infty} P\{k_1 > j\}$$

if $\sup_\omega c(\omega) = c < \infty$, and

$$(3.46) \qquad E_{P_Q}\left(\sum_{j=0}^{k_1-1} c \circ \theta^j\right)^2 \leq \sum_{j=0}^{\infty} (E_{P_Q} \mathbb{I}_{k(\omega)>j} c^2 \circ \theta^j)^{1/2}$$

$$\leq (E_{P_Q} c^4)^{1/4} (P(Q))^{-1/4} \sum_{j=0}^{\infty} (P\{k_1(\omega) > j\})^{1/4}.$$

Employing Lemma 1.1 from [VR] (or applying repeatedly Lemma 2 from Section 27 of [Bi2]), I obtain

$$(3.47) \quad P\{k_1 > n\} = P\{\theta^j \notin Q, j = 1, 2, ..., n\} \leq E_P \prod_{\frac{1}{3}\sqrt{n} \geq l \geq 0} \mathbb{I}_{\Omega \setminus \theta^{-(1+3l[\sqrt{n}])}Q}$$

$$\leq E_P \prod_{\frac{1}{3}\sqrt{n} \geq l \geq 0} E_P\left(I_{\Omega \setminus \theta^{-(1+3l[\sqrt{n}])}Q} \middle| \mathcal{G}_{1+(3l-1)[\sqrt{n}], 1+(3l+1)[\sqrt{n}]}\right)$$

$$+ \sum_{\frac{1}{3}\sqrt{n} \geq l \geq 0} E_P\left| \mathbb{I}_{\theta^{-(1+3l[\sqrt{n}])}Q} - E_P(I_{\theta^{-(1+3l[\sqrt{n}])}Q} | \mathcal{G}_{1+(3l-1)[\sqrt{n}], 1+(3l+1)[\sqrt{n}]}) \right|$$

$$\leq 6\sqrt{n} \alpha_{([\sqrt{n}])} + (1-P(Q))^{[\frac{1}{3}\sqrt{n}]} + \sum_{\frac{1}{3}\sqrt{n} \geq l \geq 0} E_P\left| \mathbb{I}_Q - E_P(\mathbb{I}_Q | \mathcal{G}_{-[\sqrt{n}],[\sqrt{n}]}) \right|$$

$$\leq 6\sqrt{n}(\alpha_{([\sqrt{n}])} + \gamma_{([\sqrt{n}])}) + (1-P(Q))^{[\frac{1}{3}\sqrt{n}]}$$

with α_n and γ_n defined in (2.11) and (2.12). This together with (3.45) and (3.46) yields Proposition 2.2. $\qquad \square$

4. Limit theorems via martingale differences

In order to obtain martingale difference representations in the framework of Theorem 2.1 I shall pass first to an invertible case by means of the "random natural extension construction". Namely, introduce the spaces

$$\hat{\Xi}^\omega = \{\hat{\xi} : \hat{\xi} = (..., \xi_{-1}, \xi_0, \xi_1, ...), \, \xi_i \in \Xi^{\theta^i \omega}, \, \xi_{i+1} = f_{\theta^i \omega} \xi_i \, \forall i \in \mathbb{Z}\},$$

$\hat{\Xi} = \{(\hat{\xi}, \omega) : \omega \in \Omega, \hat{\xi} \in \hat{\Xi}^\omega\}$, and $\hat{\Xi}_Q = \{(\hat{\xi}, \omega) : \omega \in Q, \hat{\xi} \in \hat{\Xi}^\omega\}$. Then the two-sided shift $\hat{f}_\omega : \hat{\Xi}^\omega \to \hat{\Xi}^{\theta\omega}$ defined by $(\hat{f}_\omega \hat{\xi})_i = \xi_{i+1}$ is invertible and $f_\omega \iota_\omega = \iota_{\theta\omega} \hat{f}_\omega$ where, $\iota_\omega : \hat{\Xi}^\omega \to \Xi^\omega$ acting by $\iota_\omega(\hat{\xi}) = \xi_0$, $\hat{\xi} = (..., \xi_{-1}, \xi_0, \xi_1, ...)$ is the natural projection. Set $\hat{\mathcal{F}}_{m,n}^\omega = \iota_\omega^{-1} \mathcal{F}_{m,n}^\omega$, $n \geq m \geq 0$, and $\hat{\mathcal{F}}_{-k,n}^\omega = \hat{f}_{\theta^{-k}\omega}^k \hat{\mathcal{F}}_{0,n+k}^{\theta^{-k}\omega}$ for $k \geq 0$.

Observe that $\hat{\mathcal{F}}^\omega_{m,n} \subset \hat{\mathcal{F}}^\omega_{m',n'}$ if $m' \leq m$, and $n' \geq n$ and $\hat{f}_\omega^{-1}\hat{\mathcal{F}}^{\theta\omega}_{m,n} = \hat{\mathcal{F}}^\omega_{m+1,n+1}$, and so for all $k \geq 1$,

$$\hat{\mathcal{F}}^\omega_{-(k-1),n} = \hat{f}^k_{\theta^{-k}\omega}\hat{f}^{-1}_{\theta^{-k}\omega}\hat{\mathcal{F}}^{\theta^{-(k-1)}\omega}_{0,n+k-1} = \hat{f}^k_{\theta^{-k}\omega}\hat{\mathcal{F}}^{\theta^{-k}\omega}_{1,n+k} \subset \hat{f}^k_{\theta^{-k}\omega}\hat{\mathcal{F}}^{\theta^{-k}\omega}_{0,n+k} = \hat{\mathcal{F}}^\omega_{-k,n}.$$

By $\hat{\mathcal{F}}^\omega_{-\infty,n}$ I denote the minimal $\sigma-$algebra containing $\bigcup_{k\geq 0}\hat{\mathcal{F}}^\omega_{-k,n}$. Let μ be a $\tau-$invariant probability measure with the disintegration (1.1). For any $\Gamma \in \hat{\mathcal{F}}^\omega_{m,n}$ with $0 \leq m \leq n \leq \infty$ there exists $G \in \mathcal{F}^\omega_{m,n}$ such that $\Gamma = \iota_\omega^{-1}G$, and I set $\hat{\mu}^\omega(\Gamma) = \mu^\omega(G)$. If $\Gamma \in \hat{\mathcal{F}}^\omega_{m,n}$ with $m < 0$ and $-\infty < m \leq n \leq \infty$, then there exists $G \in \mathcal{F}^{\theta^m\omega}_{0,n-m}$ such that $\Gamma = (\hat{f}^m_\omega)^{-1}\iota^{-1}_{\theta^m\omega}G$. For such Γ I define $\hat{\mu}^\omega(\Gamma) = \mu^{\theta^m\omega}(G)$ and I extend $\hat{\mu}^\omega$ to the whole $\hat{\mathcal{F}}^\omega_{-\infty,\infty}$ by monotone limits. It follows that $\hat{f}_\omega\hat{\mu}^\omega = \hat{\mu}^{\theta\omega}$ and $\hat{f}_\omega^{-1}\hat{\mu}^{\theta\omega} = \hat{\mu}^\omega$. Set also $\hat{\varphi}_\omega = \varphi_\omega \circ \iota_\omega$, $\hat{\psi}_\omega = \psi_\omega \circ \iota_\omega$, $\hat{\Psi}_\omega = \Psi_\omega \circ \iota_\omega$, $\hat{\tau}(\hat{\xi},\omega) = (\hat{f}_\omega\hat{\xi}, \theta\omega)$, $\hat{F}_\omega = \hat{f}^{k_1(\omega)}_\omega$ and $\hat{T}(\hat{\xi},\omega) = \hat{\tau}^{k_1(\omega)}(\hat{\xi},\omega)$.

Let $k_i(\omega)$ be as before for $i \geq 0$ and $k_i(\omega) = \max\{k < k_{i+1}(\omega) : \theta^k\omega \in Q\}$ for $i < 0$. Take arbitrary $A \in \hat{\mathcal{F}}^\omega_{k_l(\omega),k_i(\omega)}$ and $B \in \hat{\mathcal{F}}^\omega_{k_{i+j}(\omega),\infty}$ with $l < 0$ and $l \leq i \leq j$; then there exist $G \in \mathcal{F}^{\theta^{k_l(\omega)}\omega}_{0,k_i(\omega)-k_l(\omega)} = \mathcal{F}^{\Theta^l\omega}_{0,k_{i-l}(\Theta^l\omega)}$ and $H \in \mathcal{F}^{\Theta^l\omega}_{k_{i+j-l}(\Theta^l\omega),\infty}$ such that $A = \hat{F}^{-l}_{\Theta^l\omega}\iota^{-1}_{\Theta^l\omega}G$ and $B = \hat{F}^{-l}_{\Theta^l\omega}\iota^{-1}_{\theta^l\omega}H$. It follows that

$$(4.1) \qquad \left|\frac{\hat{\mu}^\omega(A \cap B)}{\hat{\mu}^\omega(A)} - \hat{\mu}^\omega(B)\right| = \left|\frac{\mu^{\Theta^l\omega}(G \cap H)}{\mu^{\Theta^l\omega}(G)} - \mu^{\Theta^l\omega}(H)\right| \leq \phi_j.$$

Since (4.1) is preserved under monotone limits, I conclude that it remains true for any $A \in \hat{\mathcal{F}}^\omega_{-\infty,k_i(\omega)}$. Set

$$I_i^{(l)}(\omega) = E_{\hat{\mu}^\omega}(\hat{\Psi} \circ \hat{T}^l|\hat{\mathcal{F}}^\omega_{-\infty,k_{-i}(\omega)})$$

and

$$J_i^{(l)}(\omega) = \hat{\Psi} \circ \hat{T}^l - E_{\hat{\mu}^\omega}(\hat{\Psi} \circ \hat{T}^l|\hat{\mathcal{F}}^\omega_{-\infty,k_{-i}(\omega)}),$$

where $\hat{T}^l(\hat{\xi},\omega) = ((\hat{F}^{-l}_{\Theta^l\omega})^{-1}\hat{\xi}, \Theta^l\omega)$ for $l < 0$, and define

$$(4.2) \qquad \begin{aligned} Y_\omega &= \sum_{l=0}^\infty(I_0^{(l)}(\omega) - I_1^{(l)}) + \sum_{l=-\infty}^{-1}(J_1^{(l)}(\omega) - J_0^{(l)}(\omega)), \\ g_\omega &= \sum_{l=0}^\infty I_1^{(l)}(\omega) - \sum_{l=-\infty}^{-1} J_1^{(l)}(\omega). \end{aligned}$$

I claim that for P_Q-a.a.ω all series above converge in $L^2(\hat{\Xi}^\omega, \hat{\mu}^\omega)$. Indeed, for $l \geq 0$,

$$I_0^{(l)}(\omega) = E_{\hat{\mu}^{\Theta^l\omega}}(\hat{\Psi}_{\Theta^l\omega}|\hat{\mathcal{F}}^{\Theta^l\omega}_{-\infty,k_{-l}(\Theta^l\omega)}) \circ \hat{F}^l_{\Theta^l\omega},$$

and so

$$(4.3) \qquad E_{\hat{\mu}^\omega}(I_0^{(l)}(\omega))^2 = E_{\hat{\mu}^{\Theta^l\omega}}\left(\hat{\Psi}_{\Theta^l\omega}E_{\hat{\mu}^{\Theta^l\omega}}(\hat{\Psi}_{\Theta^l\omega}|\hat{\mathcal{F}}^{\Theta^l\omega}_{-\infty,k_{-l}(\Theta^l\omega)})\right).$$

Since $\hat{\Psi}_{\Theta^l\omega}$ is $\hat{\mathcal{F}}^{\Theta^l\omega}_{0,\infty}-$measurable, in view of (4.1) I can apply to the right hand side of (4.3) Lemma 1 from Section 20 in [Bi1], which in the same way as in (3.7) yields that

$$E_{\hat{\mu}^\omega}(I_0^{(l)}(\omega))^2 \leq 2\phi_l^{1/2}C(\Theta^l\omega)(E_{\hat{\mu}^\omega}(I_0^{(l)}(\omega))^2)^{1/2},$$

and so

$$(4.4) \qquad (E_{\hat{\mu}^\omega}(I_0^{(l)}(\omega))^2)^{1/2} \leq 2\phi_l^{1/2}C(\Theta^l\omega).$$

Next, for $l < 0$,

$$J_0^{(l)}(\omega) = \left(\hat{\Psi}_{\Theta^l\omega} - E_{\hat{\mu}^{\Theta^l\omega}}(\hat{\Psi}_{\Theta^l\omega}|\hat{\mathcal{F}}_{-\infty,k_{-l}(\Theta^l\omega)}^{\Theta^l\omega})\right) \circ (\hat{F}_{\Theta^l\omega}^{-l})^{-1},$$

and so

$$(4.5) \qquad E_{\hat{\mu}^\omega}(J_0^{(l)})^2 = D_{k_{-l}(\Theta^l\omega)}^2(\Theta^l\omega).$$

Now (4.4), (4.5) and similar estimates for $I_1^{(l)}$ and for $J_1^{(l)}$ together with (2.5) and (2.6) yield that for P_Q-a.a. ω the series in (4.2) converge in $L^2(\hat{\Xi}^\omega, \hat{\mu}^\omega)$.

It is easy to see that $g_\omega \circ \hat{T} = \sum_{l=1}^{\infty} I_0^{(l)} - \sum_{l=-\infty}^{0} J_0^{(l)}(\omega)$, and so

$$Y_\omega + g_\omega - (g \circ \hat{T})_\omega = I_0^{(0)}(\omega) + J_0^{(0)} = \hat{\Psi}_\omega.$$

For $l = 0, 1, ...$ set $Y^{(l)} = Y \circ \hat{T}^l$; then $Y_\omega^{(l)}$ is $\hat{\mathcal{F}}_{-\infty,k_l(\omega)}^\omega-$measurable,

$$(4.6) \qquad \hat{\Psi} \circ \hat{T}^l = Y^{(l)} + g \circ \hat{T}^l - g \circ \hat{T}^{l+1},$$

and

$$(4.7) \qquad E_{\hat{\mu}^\omega}(Y_\omega^{(l)}|\hat{\mathcal{F}}_{-\infty,k_{l-1}(\omega)}^\omega) = E_{\hat{\mu}^{\Theta^l\omega}}(Y_{\Theta^l\omega}|\hat{\mathcal{F}}_{-\infty,k_{-1}(\Theta^l\omega)}^{\Theta^l\omega}) \circ \hat{F}_\omega^l = 0,$$

which provides the required martingale difference representation. Thus

$$(4.8) \qquad \frac{1}{\sqrt{m}}\sum_{l=0}^{m-1}\Psi \circ T^l = \frac{1}{\sqrt{m}}\sum_{l=0}^{m-1}Y^{(l)} + \frac{1}{\sqrt{m}}(g - g \circ T^m).$$

Since $g \in L^2(\hat{\Xi}_Q, \hat{\mu}_Q)$, where $\hat{\mu}_Q$ is the normalized restriction of $\hat{\mu}$ to $\hat{\Xi}_Q$, then the last term in the right hand side of (4.8) tends to 0 as $m \to \infty$ both μ_Q-a.s. and in $L^2(\hat{\Xi}^\omega, \hat{\mu}^\omega)$ for P_Q-a.a.ω. Note also that in view of (2.7), μ_Q-a.s. $\frac{1}{\sqrt{n}}R \circ T^n \to 0$ as $n \to \infty$. It follows that in order to prove Theorem 2.1 it suffices to derive the CLT and the invariance principle for the LIL for martingales $\sum_{l=0}^{m-1}Y_\omega^{(l)}$.

First, observe that by (4.6)-(4.8) and by the ergodic theorem, P_Q-a.s.,

$$(4.9) \qquad \lim_{m\to\infty}\frac{1}{m}E_{\hat{\mu}^\omega}\left(\sum_{l=0}^{m-1}(\hat{\Psi}\circ T^l)_\omega\right)^2 = \lim_{m\to\infty}\frac{1}{m}E_{\hat{\mu}^\omega}\left(\sum_{l=0}^{m-1}Y_\omega^{(l)}\right)^2$$

$$= \lim_{m\to\infty}\frac{1}{m}\sum_{l=0}^{m-1}E_{\hat{\mu}^{\Theta^l\omega}}(Y_{\Theta^l\omega})^2 = E_{\hat{\mu}_Q}(\hat{\Psi} + g \circ \hat{T} - g)^2.$$

This together with (3.16) yields that $\sigma = 0$ if and only if $\hat{\Psi} = g - g \circ \hat{T}$ $\hat{\mu}_Q-$a.s.

Assume now that $\sigma > 0$. In order to complete the CLT part of the proof of Theorem 2.1 by means of the central limit theorem for nonstationary martingale differences (see Section 4.1 in [HH] or §8 of Ch.VII from [Sh]) it remains only to verify the Lindenberg condition saying that, P_Q-a.s.,

$$(4.10) \qquad \lim_{m\to\infty}\frac{1}{m}\sum_{l=0}^{m-1}E_{\mu^\omega}\left((Y_\omega^{(l)})^2\mathbb{I}_{|Y_\omega^{(l)}|>\varepsilon\sqrt{m}}\right) = 0.$$

In order to derive (4.10) observe that by the ergodic theorem, P_Q—a.s. and in $L^1(Q, P_Q)$ for any $K > 0$,

$$(4.11) \quad \limsup_{m \to \infty} \frac{1}{m} \sum_{l=0}^{m-1} E_{\hat{\mu}^\omega} \left((Y \circ T^l)_\omega^2 \mathbb{I}_{|(Y \circ T^l)_\omega| > \varepsilon \sqrt{m}} \right)$$

$$= \limsup_{m \to \infty} \frac{1}{m} \sum_{l=0}^{m-1} E_{\hat{\mu}^{\Theta^l \omega}} \left((Y_{\Theta^l \omega})^2 \mathbb{I}_{|Y_{\Theta^l \omega}| > \varepsilon \sqrt{m}} \right)$$

$$\leq \lim_{m \to \infty} \frac{1}{m} \sum_{l=0}^{m-1} E_{\hat{\mu}^{\Theta^l \omega}} \left((Y_{\Theta^l \omega})^2 \mathbb{I}_{|Y_{\Theta^l \omega}| > K} \right) = E_{P_Q} E_{\hat{\mu}^\omega} \left((Y_\omega)^2 \mathbb{I}_{|Y_\omega| > K} \right).$$

Since the last expression in (4.11) tends to zero as $K \to \infty$ and the first expression in (4.11) does not depend on K, then (4.10) follows.

In order to derive the invariance principle for the LIL for sums of the martingale differences $Y^{(l)}$ I shall employ Theorem 1 from [HS] (see also Section 4 in [HH]). The last condition of this theorem follows from (3.3), (3.4), (3.16) and (4.9), and the other two conditions hold true since

$$(4.12) \quad E_{P_Q} \sum_{n=1}^{\infty} n^{-2} E_{\hat{\mu}^\omega} \left((Y_\omega^{(n)})^4 \mathbb{I}_{|Y_\omega^{(n)}| < \delta n^{1/2}} \right) = E_{\hat{\mu}_Q} \left(Y^4 \sum_{n=1}^{\infty} n^{-2} \mathbb{I}_{|Y| < \delta n^{1/2}} \right)$$

$$\leq E_{\hat{\mu}_Q} \left(\delta^4 + Y^4 \int_{|Y|^2 \delta^{-2}}^{\infty} x^{-2} dx \right) = \delta^4 + \delta^2 E_{\mu_Q} Y^2 < \infty$$

and

$$(4.13) \quad E_{P_Q} \sum_{n=1}^{\infty} n^{-1/2} E_{\hat{\mu}^\omega} \left(|Y_\omega^{(n)}|_\omega \mathbb{I}_{|Y_\omega^{(n)}| \geq \varepsilon n^{1/2}} \right) = E_{\hat{\mu}_Q} \left(|Y| \sum_{n=1}^{\infty} n^{-1/2} \mathbb{I}_{|Y| \geq \varepsilon n^{1/2}} \right)$$

$$\leq E_{\hat{\mu}_Q} \left(|Y| \int_0^{|Y|^2 \varepsilon^{-2}} x^{-1/2} dx \right) = 2 \varepsilon^{-1} E_{\hat{\mu}_Q} |Y|^2 < \infty.$$

Observe that, for any Borel set $U \subset \mathbb{R}$,

$$\{\hat{\xi} \in \hat{\Xi}^\omega : \sum_{l=0}^{m-1} \hat{\Psi} \circ \hat{T}^l(\hat{\xi}, \omega) \in U\} = \{\hat{\xi} \in \hat{\Xi}^\omega : \sum_{l=0}^{m-1} \Psi_{\Theta^l \omega} \circ \iota_{\Theta^l \omega} \circ \hat{F}_\omega^l(\hat{\xi}) \in U\}$$

$$= \{\hat{\xi} \in \hat{\Xi}^\omega : \sum_{l=0}^{m-1} \Psi_{\Theta^l \omega} \circ F_\omega^l \circ \iota_\omega(\hat{\xi}) \in U\} = \iota_\omega^{-1} (S_\omega^{(m)})^{-1} U$$

where $S_\omega^{(m)} = \sum_{l=0}^{m-1} \Psi_{\Theta^l \omega} \circ F_\omega^l$ is an $\mathcal{F}_{0,\infty}^\omega$—measurable function on Ξ^ω. It follows that

$$\hat{\mu}^\omega \{\hat{\xi} \in \hat{\Xi}^\omega : \sum_{l=0}^{m-1} \hat{\Psi} \circ \hat{T}^l(\hat{\xi}, \omega) \in U\} = \mu^\omega ((S_\omega^{(m)})^{-1} U)$$

$$= \mu^\omega \{\xi \in \Xi^\omega : \sum_{l=0}^{m-1} \Psi \circ T^l(\xi, \omega) \in U\}.$$

Hence the CLT and the LIL derived above for $\sum_{l=0}^{m-1} \hat{\Psi} \circ \hat{T}^l$ with respect to $\hat{\mu}^\omega$ imply the CLT and the LIL for $\sum_{l=0}^{m-1} \Psi \circ T^l$ with respect to μ^ω, yielding Theorem 2.1. $\square$

4.1. Remark. It is possible to prove Theorem 2.1 and even stronger invariance principles without passing to random natural extension constructions as above by modifying methods from [PS] which lead to martingale difference representations of the form

$$\Psi^{(N(n))} \circ T^n = Y^{(n)} + g^{(n)} - g^{(n+1)}$$

where $\Psi_\omega^{(N)} = E_{\mu^\omega}(\Psi_\omega | \mathcal{F}_{0,k_N(\omega)}^\omega)$,

$$g_\omega^{(n)} = \sum_{l=0}^\infty E_{\mu^\omega}(\Psi^{(N(n+l))} \circ T^{n+l} | \mathcal{F}_{0,k_{n-1}(\omega)+N(n-1)}^\omega),$$

and

$$Y_\omega^n = \sum_{l=0}^\infty \bigg(E_{\mu^\omega}(\Psi^{(N(n+l))} \circ T^{n+l} | \mathcal{F}_{0,k_n(\omega)+N(n)}^\omega)$$
$$- E_{\mu^\omega}(\Psi^{(N(n+l))} \circ T^{n+l} | \mathcal{F}_{0,k_n(\omega)+N(n-1)}^\omega) \bigg).$$

Here one chooses $N(j)$ to be small relative to $j^{1/2}$ and such that

$$n^{-1/2} \sum_{j=0}^n (\Psi_\omega - \Psi_\omega^{(N(n))}) \circ T^j \to 0$$

as $n \to \infty$. The corresponding estimates here are more technical than above, and they can be done similarly to [PS].

Next, I shall prove Theorem 2.3 working in the original spaces without using the above natural extension construction. Since (3.4) and (3.5) require only (2.7), which I assume also in Theorem 2.3, then one has to obtain only (3.16) and (3.17), which already yield (2.8) and (2.9). I shall find functions $Y_\omega^{(l)} = Y_\omega^{(l)}(\xi)$ and $g_\omega = g_\omega(\xi)$ belonging to $L^2(\Xi_Q, \mu_Q)$ such that for P_Q−a.a.ω and any $l = 1, 2, ...,$

(4.14) $$E_{\mu^\omega}(Y_\omega^{(l)} | \mathcal{T}_{k_l(\omega)}^\omega) = 0, \qquad Y_\omega^{(l)} \text{ is } \mathcal{T}_{k_{l-1}(\omega)}^\omega - \text{measurable}$$

and

(4.15) $$(\Psi \circ T^l)_\omega = Y_\omega^{(l)} + (g \circ T^l)_\omega - (g \circ T^{l-1})_\omega,$$

which will provide the required backwards martingale difference representation. If (4.15) holds true, then

(4.16) $$\frac{1}{\sqrt{m}} \sum_{l=1}^{m-1} (\Psi \circ T^l)_\omega = \frac{1}{\sqrt{m}} \sum_{l=1}^{m-1} Y_\omega^{(l)} + \frac{1}{\sqrt{m}}((g \circ T^m)_\omega - g_\omega) + \frac{1}{\sqrt{m}} \Psi_\omega.$$

Since $g \in L^2(\Xi_Q, \mu_Q)$, the last two terms in the right hand side of (4.16) tend to zero as $m \to \infty$ in $L^2(\Xi^\omega, \mu^\omega)$ for P_Q−a.a.ω. Thus it will suffice to show that, P_Q−a.s.,

(4.17) $$\lim_{m \to \infty} \frac{1}{m} E_{\mu^\omega} \bigg(\sum_{l=1}^{m-1} Y_\omega^{(l)} \bigg)^2 = \lim_{m \to \infty} \frac{1}{m} E_{\mu^\omega} \bigg(\sum_{i=0}^{m-1} (\Psi \circ T^i)_\omega \bigg)^2 = \sigma^2 / P(Q)$$

and to derive the P−a.s. CLT and LIL with respect to μ^ω for backwards martingales $\sum_{l=1}^{m-1} Y_\omega^{(l)}$.

Set

$$(4.18) \qquad g_\omega = \sum_{n=0}^{\infty} U^{*n}_{\Theta^{-n}\omega} \Psi_{\Theta^{-n}\omega};$$

then by the assumption (ii) of Theorem 2.3 g belongs to $L^2(\Xi_Q, \mu_Q)$. Clearly,

$$(4.19) \qquad \Psi_{\Theta\omega} = g_{\Theta\omega} - U^*_\omega g_\omega.$$

Observe that for any bounded $\mathcal{T}^{\Theta\omega}_0$−measurable function q on $\Xi^{\Theta\omega}$,

$$(4.20) \qquad E_{\mu^{\Theta\omega}}\left(q U^*_\omega E^\omega_\mu(g_\omega | \mathcal{T}^\omega_{k_1(\omega)}) \right) = E_{\mu^\omega}\left((U_\omega q) g_\omega \right) = E_{\mu^{\Theta\omega}}\left(q U^*_\omega g_\omega \right),$$

since $U_\omega q$ is $\mathcal{T}^\omega_{k_1(\omega)}$−measurable. Hence $U^*_\omega g_\omega = U^*_\omega E^\omega_\mu(g_\omega | \mathcal{T}^\omega_{k_1(\omega)})$ $\mu^{\Theta\omega}$−a.s., which together with (4.19) gives

$$(4.21) \qquad \Psi_{\Theta\omega} = g_{\Theta\omega} - U^*_\omega E^\omega_\mu(g_\omega | \mathcal{T}^\omega_{k_1(\omega)}).$$

Applying U_ω to both parts of (4.21) and taking into account that $U_\omega U^*_\omega q = q$ for any $\mathcal{T}^\omega_{k_1(\omega)}$−measurable q, I obtain

$$(4.22) \qquad \Psi_{\Theta\omega} \circ F_\omega = g_{\Theta\omega} \circ F_\omega - E_{\mu^\omega}(g_\omega | \mathcal{T}^\omega_{k_1(\omega)}).$$

Considering (4.22) for $\Theta^{l-1}\omega$ in place of ω and applying U^{l-1}_ω to both parts, I derive

$$(4.23) \qquad \Psi_{\Theta^l\omega} \circ F^l_\omega = g_{\Theta^l\omega} \circ F^l_\omega - E_{\mu^{\Theta^{l-1}\omega}}\left(g_{\Theta^{l-1}\omega} \middle| \mathcal{T}^{\Theta^{l-1}\omega}_{k_1(\omega)} \right) \circ F^{l-1}_\omega$$

$$= (g \circ T^l)_\omega - E_{\mu^\omega}((g \circ T^{l-1})_\omega | \mathcal{T}^\omega_{k_l(\omega)}).$$

Taking conditional expectations in (4.23), I have

$$(4.24) \quad E_{\mu^\omega}((\Psi \circ T^l)_\omega | \mathcal{T}^\omega_{k_l(\omega)}) = E_{\mu^\omega}((g \circ T^l)_\omega | \mathcal{T}^\omega_{k_l(\omega)}) - E_{\mu^\omega}((g \circ T^{l-1})_\omega | \mathcal{T}^\omega_{k_l(\omega)}).$$

Set $Y^{(l)}_\omega = (\Psi \circ T^l)_\omega - (g \circ T^l)_\omega + (g \circ T^{l-1})_\omega$. Then (4.24) implies (4.14) and

$$(4.25) \qquad Y^{(l)}_\omega = (Y^{(1)} \circ T^{l-1})_\omega.$$

In fact, one can reverse the above arguments and conclude that such g_ω and $Y^{(l)}_\omega$ are the only possible choices for the functions satisfying (4.14) and (4.15).

Next I shall prove (4.17). It follows from assumption (i) of Theorem 2.3 that the series (3.10) converges absolutely P_Q−a.s. and in $L^1(Q, P_Q)$. One has

$$(4.26) \qquad |\Phi(\Theta^l\omega) - \Phi_{m-l}(\Theta^l\omega)| \le \tilde{\delta}_{m-l}(\Theta^l\omega),$$

where

$$\tilde{\delta}_l(\omega) = \sum_{j=l}^{\infty} |E_{\mu^\omega}(\Psi_\omega (\Psi \circ T^l)_\omega)|.$$

In the same way as in (3.14)-(3.15) I derive (3.16) from this and assumption (i) of Theorem 2.3. Thus the second equality in (4.17) holds true, and the first equality there follows from (4.16) and the fact that

$$(4.27) \qquad \lim_{n\to\infty} \frac{1}{n} E_{\mu^\omega}(g \circ T^n)^2_\omega = \lim_{n\to\infty} \frac{1}{n} E_{\mu^{\Theta^n\omega}} g^2_{\Theta^n\omega} = 0,$$

since $E^\omega_\mu g_\omega \in L^2(Q, P_Q)$. Observe that by (4.14), for any $l_2 > l_1 \ge 1$,

$$E_{\mu^\omega}(Y^{(l_1)}_\omega Y^{(l_2)}_\omega) = E_{\mu^\omega}(Y^{(l_2)}_\omega E_{\mu^\omega}(Y^{(l_1)}_\omega | \mathcal{T}^\omega_{k_{l_1}(\omega)})) = 0,$$

and so, by (4.15), (4.25) and the ergodic theorem, P_Q-a.s.,

$$(4.28) \qquad \lim_{m\to\infty} \frac{1}{m} E_{\mu^\omega} \left(\sum_{l=1}^{m-1} Y_\omega^{(l)} \right)^2 = \lim_{m\to\infty} \frac{1}{m} \sum_{l=1}^{m-1} E_{\mu^\omega} (Y_\omega^{(l)})^2$$

$$= \lim_{m\to\infty} \frac{1}{m} \sum_{l=1}^{m-1} E_{\mu^{\Theta^l \omega}} (Y_{\Theta^l \omega}^{(1)})^2 = E_{\mu_Q} (\Psi \circ T - g \circ T + g)^2.$$

This together with (4.17) yields that $\sigma = 0$ if and only if $\Psi \circ T = g \circ T - g$ μ_Q-a.s. Finally, assuming that $\sigma > 0$, I verify the Lindenberg condition (4.10) for martingale differences $Y_\omega^{(l)}$ required for application of the CLT results from [HH] and [Sh] and the conditions for the LIL from [HS] and [HH] exactly in the same way as in (4.11)-(4.13) above, which completes the proof of Theorem 2.3.

5. Markov chains in random environments

In this section I shall derive Theorem 2.4 from Theorem 2.1, verifying the conditions of the latter. Define inductively $\tilde{k}_{i+1}(\omega) = \min\{k > \tilde{k}_i(\omega) : \theta^{-k}\omega \in Q\}$ with $\tilde{k}_0(\omega) \equiv 0$ and $\tilde{n}_1 = \tilde{n}_1(\omega) = \min\{\tilde{k}_i(\omega) : \tilde{k}_i(\omega) \geq L\}$ and $\tilde{n}_{j+1}(\omega) = \min\{\tilde{k}_i(\omega) : \tilde{k}_i(\omega) \geq \tilde{n}_j(\omega) + L\}$, $j = 1, 2, \ldots$. Set $\tilde{l}^\omega(n) = \tilde{l}_L^\omega(n) = \max\{j : \tilde{n}_j(\omega) < n\}$. By (2.12) from [Ki3] it follows that for any $x \in X^{\theta^{-n}\omega}$, a measurable $\Gamma \subset X^\omega$, and $P-$a.a. $\tilde{\omega} \in \Omega$,

$$(5.1) \qquad |P^{\theta^{-n}\tilde{\omega}}(n, x, \Gamma) - \eta^{\tilde{\omega}}(\Gamma)| \leq 2(1 - L^{-1})^{\tilde{l}^{\tilde{\omega}}(n)},$$

where η^ω is the unique probability measure satisfying (2.15). Take $\tilde{\omega} = \theta^{k_{i+j}(\omega)}\omega$, $n = k_{i+j}(\omega) - k_i(\omega) = k_j(\theta^{k_i(\omega)}\omega)$, and set

$$(5.2) \qquad \zeta_n^{x,\omega}(\Gamma) = P^\omega(n, x, \Gamma) - \eta^{\theta^n \omega}(\Gamma)$$

Then by (5.1), for any $y \in X^{\theta^{k_i(\omega)}\omega}$ and a measurable $\Gamma \subset X^{\theta^{k_{i+j}(\omega)}\omega}$,

$$(5.3) \qquad \left| \zeta_{k_{i+j}(\omega)-k_i(\omega)}^{y, \theta^{k_i(\omega)}\omega}(\Gamma) \right| \leq 2(1 - L^{-1})^{\tilde{l}^{\theta^{k_{i+j}(\omega)}\omega}(k_{i+j}(\omega)-k_i(\omega))} \leq 2(1 - L^{-1})^{j/L}.$$

In order to verify (2.5) it suffices to consider the right hand side of (2.2) for $A \in \mathcal{F}_{0,k_i(\omega)}^\omega$ of the form $A = \{\xi : Z_1^\omega(\xi) \in \Gamma_1, \ldots, Z_{k_i(\omega)}^\omega(\xi) \in \Gamma_{k_i(\omega)}\}$ and an arbitrary $B \in \mathcal{F}_{k_{i+j}(\omega),\infty}^\omega$. By (2.14), (2.15), (5.2), (5.3), and the Markov property it follows that

(5.4)

$$\mu^\omega(A \cap B) = \int_{X^\omega} d\eta^\omega(x) \int_{\Gamma_1} \ldots \int_{\Gamma_{k_i(\omega)}} P^\omega(x, dz_1) \cdots P^{\theta^{k_i(\omega)-1}\omega}(z_{k_i(\omega)-1}, dz_{k_i(\omega)})$$

$$\times \int_{X^{\theta^{k_{i+j}(\omega)}\omega}} P^{\theta^{k_i(\omega)}\omega}(k_{i+j}(\omega) - k_i(\omega), z_{k_i}, dv) \mu^\omega(B|Z_{k_{i+j}(\omega)}^\omega = v)$$

$$= \mu^\omega(A)\mu^\omega(B) + I_{A,B}^\omega,$$

where

$$(5.5) \qquad |I_{A,B}^\omega| \leq \sup_{y \in X^{\theta^{k_i(\omega)}\omega}} \left| \int_{X^{\theta^{k_{i+j}(\omega)}\omega}} \zeta_{k_{i+j}(\omega)-k_i(\omega)}^{y,\theta^{k_i(\omega)}\omega}(dv) \mu^\omega(B|Z_{k_{i+j}(\omega)}^\omega = v) \right|$$

$$\leq \sup_{y \in X^{\theta^{k_i(\omega)}\omega}} \zeta_{k_{i+j}(\omega)-k_i(\omega)}^{y,\theta^{k_i(\omega)}\omega}(\Gamma_y^+) \leq 2(1 - L^{-1})^{j/L}$$

and $\Gamma_y^+ \subset X^{\theta^{k_{i+j}(\omega)}\omega}$ is the positive set in the Hahn decomposition (see [Bi2]) for the signed measure $\zeta_{k_{i+j}(\omega)-k_i(\omega)}^{y,\theta^{k_i(\omega)}\omega}$. Hence

$$(5.6) \qquad \phi_j \leq 2(1 - L^{-1})^{j/L},$$

and so (2.5) holds true. Thus if (2.6) and (2.7) are satisfied, then (2.8) and (2.9) follow by Theorem 2.1. If $\varphi_\omega(\xi) = \varphi_\omega(Z^\omega(\xi))$ is, in fact, a function on X and $\psi_\omega(z) = \varphi_\omega(z) - E_{\mu^\omega}\varphi_\omega(Z^\omega(\xi)) = \varphi_\omega(z) - E_{\eta^\omega}\varphi_\omega$, then the second equality in (2.17) follows from Theorem 2.1.

In order to obtain the first equality in (2.17), set

$$(5.7) \qquad S_n^\omega(\xi) = \sum_{k=0}^{n-1} \psi(Z_k^\omega(\xi), \theta^k\omega), \qquad S_{m,n}^\omega(\xi) = S_n^\omega(\xi) - S_m^\omega(\xi) \text{ for } n \geq m,$$

$\mu_x^\omega\{\cdot\} = \mu^\omega\{\cdot \,|\, Z_0^\omega = x\}$, and observe that

$$(5.8) \qquad \mu_x^\omega\left\{\xi : \frac{1}{\sqrt{n}}S_{k_i(\omega),n}^\omega(\xi) \leq a - \delta\right\} - \mu_x^\omega\left\{\xi : \frac{1}{\sqrt{n}}|S_{0,k_i(\omega)}^\omega(\xi)| > \delta\right\}$$

$$\leq \mu_x^\omega\left\{\xi : \frac{1}{\sqrt{n}}S_{0,n}^\omega(\xi) \leq a\right\}$$

$$\leq \mu_x^\omega\left\{\xi : \frac{1}{\sqrt{n}}S_{k_i(\omega),n}^\omega(\xi) \leq a + \delta\right\} + \mu_x^\omega\left\{\xi : \frac{1}{\sqrt{n}}|S_{0,k_i(\omega)}^\omega(\xi)| > \delta\right\}.$$

By (2.14), (2.15), (5.2), (5.3), and the Markov property it follows that

$$(5.9) \qquad \mu_x^\omega\left\{\xi : \frac{1}{\sqrt{n}}S_{k_i(\omega),n}^\omega(\xi) \leq b\right\}$$

$$+ \int_{X^{\theta^{k_i(\omega)}\omega}} P^\omega(k_i(\omega), x, dy)\mu^\omega\left\{\xi : \frac{1}{\sqrt{n}}S_{k_i(\omega),n}^\omega(\xi) \leq b \,\Big|\, Z_{k_i(\omega)}^\omega = y\right\}$$

$$= \mu^\omega\left\{\xi : \frac{1}{\sqrt{n}}S_{k_i(\omega),n}^\omega(\xi) \leq b\right\} + J_{x,i}^\omega,$$

where

$$(5.10) \qquad |J_{x,i}^\omega| \leq \sup_{x \in X^\omega} \zeta_{k_i(\omega)}^{x,\omega}(\tilde{\Gamma}_x^+) \leq 2(1 - L^{-1})^{i/L}$$

and $\tilde{\Gamma}_x^+$ is the positive set in the Hahn decomposition for the signed measure $\zeta_{k_i(\omega)}^{x,\omega}$. Letting, first, $n \to \infty$, then $i \to \infty$, and finally $\delta \to 0$, I conclude from (5.7)-(5.10) that both limits in (2.17) are the same, provided the second limit in (2.17) exists and equals the right hand side of (2.17), which I have proved already.

6. Random subshifts and random expanding transformations

In this section I shall verify the conditions of Theorem 2.1 for random subshifts of finite type and of Theorem 2.3 for random expanding in average transformations, extending to my more general situation the deterministic arguments from Lemmas 1.9–1.12 and 1.14 in [Bo]. Whenever possible I shall deal with both cases simultaneously. Set

$$(6.1) \qquad R_\omega = \sum_{l=1}^\infty K_g(\theta^{-l}\omega)e^{-\kappa l}, \quad \rho(\xi, \tilde{\xi}) = e^{-\min\{j \geq 0: \xi_j \neq \tilde{\xi}_j\}},$$

provided $\xi, \tilde{\xi} \in \Xi^\omega$, in the case of random subshifts,

$$(6.2) \quad R_\omega = \sum_{l=1}^\infty K_g(\theta^{-l}\omega)(\|Df_{\theta^{-l}\omega}^{-1}\| \cdots \|Df_{\theta^{-2}\omega}^{-1}\| \|Df_{\theta^{-1}\omega}^{-1}\|)^\kappa, \rho(\xi, \tilde{\xi}) = d(\xi, \tilde{\xi})$$

in the case of random expanding transformations, and in both cases

$$(6.3) \qquad\qquad\qquad R_\omega(\xi, \tilde{\xi}) = R_\omega(\rho(\xi, \tilde{\xi}))^\kappa.$$

By (2.21) and (2.29) it follows that $R_\omega < \infty$ $P-$a.s.

In the expanding random transformations case set

$$Z_\omega = \sum_{l=1}^\infty \|Df_{\theta^{-l}\omega}^{-1}\| \cdots \|Df_{\theta^{-2}\omega}^{-1}\| \|Df_{\theta^{-1}\omega}^{-1}\|$$

and $\varepsilon_\omega = \frac{1}{2}\min(1, \rho_M)\min(1, Z_\omega^{-1})$, where ρ_M is the injectivity radius of the Riemannian manifold M. It follows (see Exercise 2.3 and its solution in Ch. 2 of [Ha]) that, for any $z \in M$, if $f_\omega^{-1}z = \{z_1, ..., z_m\}$ then the balls $B_{z_i}(\frac{\varepsilon_{\theta\omega}}{\|Df_\omega\|})$ are disjoint and there exist closed domains $U_{z_i} \supset B_{z_i}(\frac{\varepsilon_{\theta\omega}}{\|Df_\omega\|})$ which are mapped by f_ω diffeomorphically onto $B_z(\varepsilon_{\theta\omega})$, and so each U_{z_i} may contain only one preimage of each point from $B_z(\varepsilon_{\theta\omega})$. In particular, distances between different preimages under f_ω of any point in M are not less than $\frac{\varepsilon_{\theta\omega}}{\|Df_\omega\|}$, and so the number ℓ_ω of preimages under f_ω of any point of M is bounded by $C(\frac{\varepsilon_{\theta\omega}}{\|Df_\omega\|})^d$, where $C > 0$ depends only on M. If $d(z, \tilde{z}) \le \varepsilon_{\theta\omega}$, then for each i there is a unique $\tilde{z}_i \in f_\omega^{-1}\tilde{z}$ belonging to U_{z_i} and

$$(6.4) \qquad d(z_i, \tilde{z}_i) \le \|Df_\omega^{-1}\|d(z, \tilde{z}) \le \frac{1}{2}\min(1, \rho_M)\min(\|Df_\omega^{-1}\|, \frac{\|Df_\omega^{-1}\|}{Z_{\theta\omega}})$$

$$\le \frac{1}{2}\min(1, \rho_M)(1 + Z_\omega)^{-1} \le \varepsilon_\omega.$$

In order to deal simultaneously both with random expanding transformations and with random subshifts I set also in the second case $\varepsilon_\omega = e^{-1}$ and define in both cases a family of cones of continuous functions on Ξ^ω by

$$(6.5) \quad \Lambda_\gamma^\omega = \{q : q \ge 0, \int q d\nu^\omega = 1 \text{ and } q(\xi) \le e^{\gamma R_\omega(\xi, \tilde{\xi})} q(\tilde{\xi}) \text{ if } \rho(\xi, \tilde{\xi}) \le \varepsilon_\omega\},$$

where ν^ω is the same as in (2.25) and $\gamma \ge 1$ is a constant. By (2.25) and (6.4) it is easy to check directly (cf. [Ki1]) that

$$(6.6) \qquad\qquad \lambda_\omega^{-1}\mathcal{L}_g^\omega \Lambda_\gamma^\omega \subset \Lambda_\gamma^{\theta\omega} \text{ for any } \gamma \ge 1,$$

where λ_ω is the same as in (2.25), and by the construction, the functions h_ω from (2.25) belong to Λ_1^ω. (see [KK]). It is easy to see that there exists a constant $\tilde{C} > 0$ depending only on M such that for any C^1 endomorphism $f : M \to M$ and any $x \in M$ the set $f^{-1}x$ is $\tilde{C}\|Df^{-1}\|-$net, i.e. for any $y \in M$ there exists $z \in f^{-1}x$ such that $d(y, z) \le \tilde{C}\|Df^{-1}\|$. Applying this to f_ω^n and taking into account that, by (2.29) and the ergodic theorem,

$$(6.7) \qquad \lim_{n\to\infty} \frac{1}{n}\log\|(Df_\omega^n)^{-1}\| \le \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\log\|Df_{\theta^j\omega}^{-1}\| = \alpha < 0 \, P-\text{a.s.},$$

I conclude that there exists a random variable $N = N_\omega < \infty$ on $(\Omega, \mathcal{G}, P)$ with values in $\mathbb{Z}^+$ such that $P-$a.s. for any $x \in M$ the set $(f_\omega^N)^{-1}x$ is an $\varepsilon_\omega-$net. In the case of random subshifts I denote by $N = N_\omega$ a $\mathbb{Z}^+-$valued random variable such

that $A(\omega)A(\theta\omega)\cdots A(\theta^n\omega)$ is a positive matrix for all $n \geq N$, where the $A(\omega)$'s are $\ell(\omega) \times \ell(\theta\omega)$ matrices apearing in the definition of a random subshift in Section 2. Such an N_ω exists by the topological mixing assumption in Theorem 2.5. Observe that this definition of $N = N_\omega$ is equivalent to saying that for $P-$a.a.ω and any $\eta \in \Xi^{\theta^N\omega}$ the set $(f_\omega^N)^{-1}\eta$ is an $\varepsilon(\omega)-$net in Ξ^ω with respect to the metric $\rho(\cdot, \cdot)$ defined in (6.1), since both characterizations of N_ω are equivalent to the third one saying that for any $j \in \{1, ..., \ell(\omega)\}$ the set $(f_\omega^N)^{-1}$ contains a point $\xi = (\xi_0, \xi_1, ...)$ from Ξ^ω with $\xi_0 = j$. Thus the characterization of N_ω is, essentially, the same in both cases under consideration, which enables me to give their simultaneous treatment.

I claim that in both cases, for any $q \in \Lambda_\gamma^\omega$,

$$(6.8) \qquad \inf_\xi (\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1} \mathcal{L}_g^{\omega, N_\omega} q(\xi) \geq G_\omega^{-1} e^{-\gamma H_\omega^{(1)}},$$

where $G_\omega = \lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega} \exp(\sum_{j=0}^{N_\omega-1} \|g_{\theta^j\omega}\|)$ with $\|\cdot\|$ being the supremum norm, $H_\omega^{(a)}(\omega) = R_\omega \varepsilon_\omega^a$, $\mathcal{L}_g^{\omega,n} = \mathcal{L}_g^{\theta^{n-1}\omega} \circ \cdots \circ \mathcal{L}_g^{\theta\omega} \circ \mathcal{L}_g^\omega$, and I take the infinum in (6.8) over all $\xi \in \Xi^{\theta^{N_\omega}\omega}$ where in the random expanding transformations case $\Xi^\omega = M$ for all $\omega \in \Omega$. Indeed, if $\xi \in \Xi^{\theta^N\omega}, \eta \in \Xi^\omega$ then there exists $\zeta' \in (f_\omega^N)^{-1}\xi \in \Xi^\omega$ with $\rho(\eta, \zeta') \leq \varepsilon_\omega$, and so by (2.24),

$$(6.9) \qquad \mathcal{L}_g^{\omega, N_\omega} q(\xi) = \sum_{\zeta \in (f_\omega^{N_\omega})^{-1}\xi} \exp(\sum_{j=0}^{N_\omega-1} g \circ \tau^j(\zeta, \omega)) q(\zeta)$$

$$\geq e^{-\sum_{j=0}^{N_\omega-1} \|g_{\theta^j\omega}\|} q(\zeta') \geq e^{-\sum_{j=0}^{N_\omega-1} \|g_{\theta^j\omega}\|} e^{-\gamma R_\omega \varepsilon_\omega} q(\eta).$$

Since $\int_{\Xi^\omega} q d\nu^\omega = 1$ and $q \geq 0$, then there exists $\eta \in \Xi^\omega$ with $q(\eta) \geq 1$, and for such η (6.9) implies (6.8). Since $\int \mathcal{L}_g^{\omega, N_\omega} q d\nu^{\theta^{N_\omega}\omega} = \lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega}$, (6.9) yields also that for any $q \in \Lambda_\gamma^\omega$,

$$(6.10) \qquad q \leq G_\omega e^{\gamma H_\omega^{(1)}}.$$

Next I shall need

6.1. Lemma. *For $P-$a.a.ω and any $\gamma \geq 2$ there exists $a_\omega = a_{\omega, \gamma} \leq \frac{1}{2}$ such that for any $\gamma \geq 2$ and $q \in \Lambda_\gamma^\omega$ one can find $q' \in \Lambda_\gamma^{\theta^{N_\omega}\omega}$ satisfying*

$$(6.11) \qquad (\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1} \mathcal{L}_g^{\omega, N_\omega} q = a_\omega h_{\theta^{N_\omega}\omega} + (1 - a_\omega) q',$$

where h_ω is the same as in (2.25).

Proof. I have to choose a_ω so that

$$(6.12) \qquad (\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1} \mathcal{L}_g^{\omega, N_\omega} q - a_\omega h_{\theta^{N_\omega}\omega} \in \Lambda_\gamma^{\theta^{N_\omega}\omega},$$

which requires

$$(\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1} \mathcal{L}_g^{\omega, N_\omega} q(\xi) - a_\omega h_{\theta^{N_\omega}\omega}(\xi)$$

$$\leq e^{\gamma R_{\theta^{N_\omega}\omega}(\xi, \tilde\xi)} \left((\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1} \mathcal{L}_g^{\omega, N_\omega} q(\tilde\xi) - a_\omega h_{\theta^{N_\omega}\omega}(\tilde\xi) \right),$$

provided $\rho(\xi, \tilde{\xi}) \leq \varepsilon_{\theta^{N}\omega}$, or, which is the same,

(6.13)

$$a_\omega\left(e^{\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}h_{\theta^{N_\omega}\omega}(\tilde{\xi}) - h_{\theta^{N_\omega}\omega}(\xi)\right)$$

$$\leq e^{\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}(\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1}\mathcal{L}_g^{\omega,N_\omega}q(\tilde{\xi}) - (\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1}\mathcal{L}_g^{\omega,N_\omega}q(\xi).$$

By (6.6),

(6.14)

$$\tilde{q} \overset{\text{def}}{=} (\lambda_\omega \cdots \lambda_{\theta^{N_\omega-2}\omega})^{-1}\mathcal{L}_g^{\omega,N_\omega-1}q \in \Lambda_\gamma^{\theta^{N_\omega-1}\omega}.$$

Observe that if $\xi, \tilde{\xi} \in \Xi^{\theta^{N_\omega}\omega}$ satisfy $\rho(\xi, \tilde{\xi}) \leq \varepsilon_{\theta^{N_\omega}\omega}$, then there exists a one-to-one correspondence between points $\xi' \in f_{\theta^{N_\omega-1}\omega}^{-1}\xi$ and $\tilde{\xi}' \in f_{\theta^{N_\omega-1}\omega}^{-1}\tilde{\xi}$ so that

(6.15)

$$\rho(\xi', \tilde{\xi}') \leq \min(\varepsilon_{\theta^{N_\omega-1}\omega}, D_{\theta^{N_\omega-1}\omega}\rho(\xi, \tilde{\xi})),$$

where $D_\omega = e^{-1}$ for random subshifts and $D_\omega = \|Df_\omega^{-1}\|$ for random expanding transformations. Set $D_\omega(\xi, \tilde{\xi}) = K_g(\omega)D_\omega^\kappa(\rho(\xi, \tilde{\xi}))^\kappa$; then

$$\gamma D_{\theta^{N_\omega-1}\omega}(\xi, \tilde{\xi}) + D_{\theta^{N_\omega-1}\omega}^\kappa R_{\theta^{N_\omega-1}\omega}(\xi, \tilde{\xi}) = R_{\theta^{N_\omega}\omega}(\xi, \tilde{\xi}),$$

and so by (2.22), (2.30), (6.14), and (6.15), for any $\xi, \tilde{\xi} \in \Xi^{\theta^{N_\omega}\omega}$ as above

$$(\lambda_{\theta^{N_\omega-1}\omega})^{-1}\mathcal{L}_g^{\theta^{N_\omega-1}}\tilde{q}(\xi) = (\lambda_{\theta^{N_\omega-1}\omega})^{-1}\sum_{\xi' \in f_{\theta^{N_\omega-1}\omega}^{-1}\xi}e^{g_{\theta^{N_\omega-1}\omega}(\xi')}\tilde{q}(z')$$

$$\leq e^{-(\gamma-1)D_{\theta^{N_\omega-1}\omega}(\xi,\tilde{\xi})+\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}(\lambda_{\theta^{N_\omega-1}\omega})^{-1}\mathcal{L}_g^{\theta^{N_\omega-1}\omega}\tilde{q}(\tilde{\xi}).$$

So in order to obtain (6.13) it suffices to have

$$a_\omega\left(e^{\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}h_{\theta^{N_\omega}\omega}(\tilde{\xi}) - h_{\theta^{N_\omega}\omega}(\xi)\right)$$

$$\leq \left(e^{\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})} - e^{-(\gamma-1)D_{\theta^{N_\omega-1}\omega}(\xi,\tilde{\xi})+\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}\right)(\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1}\mathcal{L}_g^{\omega,N_\omega}q(\tilde{\xi}).$$

Since by the construction $h_{\theta^{N_\omega}\omega} \in \Lambda_1^{\theta^{N_\omega}\omega}$, and so $h_{\theta^{N_\omega}\omega}(\tilde{\xi}) \leq e^{R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}h_{\theta^{N_\omega}\omega}(\xi)$, then taking into account (6.8) and (6.10) (the latter for $q = h_{\theta^{N_\omega}\omega}$) it suffices to obtain that

(6.16)

$$a_\omega\left(e^{\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})} - e^{-R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}\right)G_{\theta^{N_\omega}\omega}e^{H_{\theta^{N_\omega}\omega}^{(1)}}$$

$$\leq \left(e^{\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})} - e^{-(\gamma-1)D_{\theta^{N_\omega-1}\omega}(\xi,\tilde{\xi})+\gamma R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi})}\right)G_\omega^{-1}e^{-H_\omega^{(1)}}.$$

Next, for $\xi, \tilde{\xi}$ as above,

$$0 \leq R_{\theta^{N_\omega}\omega}(\xi, \tilde{\xi}) \leq R_\omega \varepsilon_\omega^\kappa = H_\omega^{(\kappa)}$$

and

$$0 \leq \gamma R_{\theta^{N_\omega}\omega}(\xi, \tilde{\xi}) - (\gamma-1)D_{\theta^{N_\omega-1}\omega}(\xi, \tilde{\xi}) \leq \gamma R_{\theta^{N_\omega}\omega}(\xi, \tilde{\xi}).$$

For any $x, y \in [-H_\omega^{(\kappa)}, \gamma H_\omega^{(\kappa)}]$ one has

$$e^{-H_\omega^{(\kappa)}}(x - y) \leq e^x - e^y \leq e^{\gamma H_\omega^{(\kappa)}}(x - y).$$

Hence in order to obtain (6.16) it suffices to have

$$(\gamma + 1)a_\omega e^{(\gamma+1)H_{\theta^{N_\omega}\omega}^{(\kappa)}} R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi}) G_{\theta^{N_\omega}\omega} \leq (\gamma - 1) D_{\theta^{N_\omega-1}\omega}(\xi,\tilde{\xi}) G_\omega^{-1} e^{-\gamma H_\omega^{(1)}},$$

and since

$$D_{\theta^{N_\omega-1}\omega}(\xi,\tilde{\xi}) R_{\theta^{N_\omega}\omega} = R_{\theta^{N_\omega}\omega}(\xi,\tilde{\xi}) K_g(\theta^{N_\omega-1}\omega) D_{\theta^{N_\omega-1}\omega}^\kappa$$

and I always can assume that $K_g \geq 1$, I only need that

$$(6.17) \qquad a_\omega \leq \frac{1}{3} e^{-\gamma(2H_{\theta^{N_\omega}\omega}^{(\kappa)} + H_\omega^{(1)})} \Upsilon_\omega$$

where $\Upsilon_\omega = G_{\theta^{N_\omega}\omega}^{-1} G_\omega^{-1} D_{\theta^{N_\omega-1}\omega}^\kappa R_{\theta^{N_\omega}\omega}^{-1}$. Observe that in view of (6.1), (6.2), (6.8), and (6.10) the inequality (6.17) implies that $a_\omega h_{\theta^{N_\omega}\omega} \leq (\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega})^{-1} \mathcal{L}_g^{\omega,N_\omega} q$, i.e. the other condition for (6.12) is also satisfied. Thus, defining a_ω by the right hand side of (6.17), I complete the proof of Lemma 6.1. $\quad\square$

Let Q be a measurable set in Ω such that $P(Q) > 0$ and

$$(6.18) \qquad Q \subset Q_L = \left\{ \omega : \max(N_\omega, 2H_{\theta^{N_\omega}\omega}^{(\kappa)} + H_\omega^{(1)}, G_\omega, \Upsilon_\omega^{-1}) \leq L \right\},$$

and define the hitting times $k_i(\omega)$ by (2.3).

6.2. Lemma. *For any $\omega \in Q$, $q \in \Lambda_\gamma^\omega$, $\gamma \geq 2$, and n satisfying $k_j(\omega) \leq n$ one has*

$$(6.19) \qquad \|(\lambda_\omega \cdots \lambda_{\theta^n\omega})^{-1} \mathcal{L}_g^{\omega,n} q - h_{\theta^n\omega}\| \leq 2L e^{\gamma L} \left(1 - \frac{1}{3} L^{-1} e^{-\gamma L}\right)^{[j/L]}.$$

Proof. It follows from (6.11) that for any $\tilde{\omega} = \theta^{k_i(\omega)}\omega$, $q \in \Lambda_\gamma^{\tilde{\omega}}$, and $l \geq L$ there exists $q' \in \Lambda_\gamma^{\theta^l\omega}$ such that

$$(6.20) \qquad (\lambda_{\tilde{\omega}} \cdots \lambda_{\theta^{l-1}\tilde{\omega}})^{-1} \mathcal{L}_g^{\tilde{\omega},l} q = a_{\tilde{\omega}} h_{\theta^l\tilde{\omega}} + (1 - a_{\tilde{\omega}}) q'.$$

By induction it is easy to derive from (6.20) that for any $i = 1, 2, \dots$ and $q \in \Lambda_\gamma^{\tilde{\omega}}$ there exists $q'_{iL} \in \Lambda_\gamma^{\theta^{k_{iL}(\omega)}\omega}$ such that

$$(\lambda_\omega \cdots \lambda_{\theta^{k_{iL}(\omega)-1}\omega})^{-1} \mathcal{L}_g^{\omega,k_{iL}(\omega)} q = \left(1 - \prod_{l=0}^{i-1}(1 - a_{\theta^{k_{lL}(\omega)}\omega})\right) h_{\theta^{k_{iL}(\omega)}\omega}$$

$$+ q'_{iL} \prod_{l=0}^{i-1}(1 - a_{\theta^{k_{lL}(\omega)}\omega}).$$

This together with (6.10), (6.18), and the choice of a_ω yields (6.19). $\quad\square$

6.3. Lemma. *Let q and r be measurable functions on Ξ^ω and $\Xi^{\theta^{k_j(\omega)}\omega}$, respectively, such that $\|q\|, \|r\| < \infty$ and there exists $K_q \geq 1$ with*

$$(6.21) \qquad |q(\xi) - q(\tilde{\xi})| \leq K_q (\rho(\xi,\tilde{\xi}))^\kappa \; \forall \xi, \tilde{\xi} \in \Xi^\omega,$$

where $k_j(\omega)$ are the same as in Lemma 6.2. Then

$$(6.22) \qquad \left| \int q(r \circ f_\omega^{k_j(\omega)}) d\mu^\omega - \int q d\mu^\omega \int r d\mu^{\theta^{k_j(\omega)}\omega} \right|$$

$$\leq 4L e^{2L}(\|q\| + K_q R_\omega^{-1}) \|r\| \left(1 - \frac{1}{3} L^{-1} e^{-2L}\right)^{[j/L]}$$

and

$$(6.23) \quad \|(\lambda_\omega \cdots \lambda_{\theta^{k_j(\omega)}\omega})^{-1} h^{-1}_{\theta^{k_j(\omega)}\omega} \mathcal{L}^{\omega, k_j(\omega)}_g q - \int q d\mu^\omega\|,$$

$$\leq 4Le^{2L}(\|q\| + K_q R_\omega^{-1})\|h^{-1}_{\theta^{k_j(\omega)}\omega}\| \left(1 - \frac{1}{3}L^{-1}e^{-2L}\right)^{[j/L]},$$

where μ^ω is the same as in (2.26) and $\gamma = \gamma_\omega = 1 + K_q R_\omega^{-1}$.

Proof. Observe that, by (2.28),

$$(6.24) \quad R_\omega \geq K_g(\theta^{-1}\omega)\|Df^{-1}_{\theta^{-1}\omega}\| \geq \|Df^{-1}_{\theta^{-1}\omega}\| \geq \|Df_{\theta^{-1}\omega}\|^{-1} > 0 \, P-\text{a.s.},$$

since I assume (without loss of generality) that $K_g(\omega) \geq 1$. Suppose, first, that $q \geq 0$, and set $\hat{q} = (q + K_q R_\omega^{-1})(\int q d\mu^\omega + K_q R_\omega^{-1})^{-1}$. Then $\int \hat{q} h_\omega d\nu^\omega = 1$ and, by (6.21), for any $\xi, \tilde{\xi} \in \Xi^\omega$,

$$(6.25) \quad \hat{q}(\xi) \leq \hat{q}(\tilde{\xi})\left(1 + \frac{K_q(\rho(\xi, \tilde{\xi}))^\kappa}{q(\tilde{\xi}) + K_q R_\omega^{-1}}\right) \leq \hat{q}(\tilde{\xi})(1 + R_\omega(\xi, \tilde{\xi})) \leq \hat{q}(\tilde{\xi})e^{R_\omega(\xi, \tilde{\xi})},$$

and since $h_\omega \in \Lambda_1^\omega$ I derive that $qh_\omega \in \Lambda_2^\omega$. Thus by (2.27) and (6.19),

$$(6.26) \quad \left| \int q(r \circ f^{k_j(\omega)}_\omega) d\mu^\omega - \int q d\mu^\omega \int r d\mu^{\theta^{k_j(\omega)}\omega} \right|$$

$$= (\int q d\mu^\omega + \frac{K_q}{R_\omega}) \left| \int \hat{q}(r \circ f^{k_j(\omega)}_\omega) h_\omega d\nu^\omega - \int r d\mu^{\theta^{k_j(\omega)}\omega} \right|$$

$$= (\int q d\mu^\omega + \frac{K_q}{R_\omega})$$

$$\times \left| (\lambda_\omega \cdots \lambda_{\theta^{k_j(\omega)}\omega})^{-1} \int r \mathcal{L}^{\omega, k_j(\omega)}_g (\hat{q} h_\omega) d\nu^{\theta^{k_j(\omega)}\omega} - \int r h_{\theta^{k_j(\omega)}\omega} d\nu^{\theta^{k_j(\omega)}\omega} \right|$$

$$\leq (\|q\| + K_q R_\omega^{-1})\|r\| \|(\lambda_\omega \cdots \lambda_{\theta^{k_j(\omega)}\omega})^{-1} \mathcal{L}^{\omega, k_j(\omega)}_g (\hat{q} h_\omega) - h_{\theta^{k_j(\omega)}\omega}\|$$

$$\leq 2Le^{2L}(\|q\| + K_q R_\omega^{-1})\|r\| \left(1 - \frac{1}{3}L^{-1}e^{-2L}\right)^{[j/L]},$$

proving (6.22) for a nonnegative q. Similarly, by (6.19),

$$(6.27) \quad \|(\lambda_\omega \cdots \lambda_{\theta^{k_j(\omega)}\omega})^{-1} h^{-1}_{\theta^{k_j(\omega)}\omega} \mathcal{L}^{\omega, k_j(\omega)}_g q - \int q d\mu^\omega\|$$

$$\times \|h^{-1}_{\theta^{k_j(\omega)}\omega}\| \|(\lambda_\omega \cdots \lambda_{\theta^{k_j(\omega)}\omega})^{-1} \mathcal{L}^{\omega, k_j(\omega)}_g \hat{q} - h_{\theta^{k_j(\omega)}\omega}\|$$

$$\leq 2Le^{2L}(\|q\| + K_q R_\omega^{-1})\|h^{-1}_{\theta^{k_j(\omega)}\omega}\| \left(1 - \frac{1}{3}L^{-1}e^{-2L}\right)^{[j/L]},$$

proving (6.23) for a nonnegative q. For a general q, represent it as the difference of its positive q_+ and negative q_- parts and apply (6.26) and (6.27) to each of them; then (6.22) and (6.23) follow via the triangle inequality. □

I shall deal first with the random expanding maps case. In order to employ the estimate (6.23) I first need an upper bound for $\|h_\omega^{-1}\|$. Observe that as $n \to \infty$,

$$\|(Df^n_{\theta^{-n}\omega})^{-1}\| Z_{\theta^{-n}\omega} \leq \|Df^{-1}_{\theta^{-n}\omega}\| \cdots \|Df^{-1}_{\theta^{-1}\omega}\| Z_{\theta^{-n}\omega}$$

$$= \sum_{l=n+1}^\infty \|Df^{-1}_{\theta^{-l}\omega}\| \cdots \|Df^{-1}_{\theta^{-1}\omega}\| \to 0$$

as the remainder of a converging series. It follows that there exists a random variable $\tilde{N} = \tilde{N}_\omega$ such that $\forall x \in M$ the set $(f_{\theta-\tilde{N}_\omega}^{\tilde{N}})^{-1}x$ is an $\varepsilon_{\theta-\tilde{N}_\omega}$−net. Thus for any $x, y \in M$ there exists $z \in (f_{\theta-\tilde{N}_\omega}^{\tilde{N}})^{-1}x$ with $d(y, z) \le \varepsilon_{\theta-\tilde{N}_\omega}$, and so

$$\lambda_{\theta-\tilde{N}_\omega} \cdots \lambda_{\theta-1_\omega} h_\omega(x) = \mathcal{L}_g^{\theta-\tilde{N}_\omega, \tilde{N}} h_{\theta-\tilde{N}_\omega}(x) \ge h_{\theta-\tilde{N}_\omega}(z) \exp\left(-\sum_{j=1}^{\tilde{N}} \|g_{\theta-j_\omega}\|\right)$$

$$\ge h_{\theta-\tilde{N}_\omega} \exp\left(-\gamma R_{\theta-\tilde{N}_\omega} \varepsilon_{\theta-\tilde{N}_\omega} - \sum_{j=1}^{\tilde{N}} \|g_{\theta-j_\omega}\|\right).$$

Since $\int h_{\theta-\tilde{N}_\omega} d\nu^{\theta-\tilde{N}_\omega} = 1$, I can choose y so that $h_{\theta-\tilde{N}_\omega}(y) \ge 1$. Taking into account also that

$$\lambda_{\theta-\tilde{N}_\omega} \cdots \lambda_{\theta-1_\omega} = \int \mathcal{L}_g^{\theta-\tilde{N}_\omega, \tilde{N}} 1 d\nu^\omega \le \prod_{j=1}^{\tilde{N}} \ell(\theta^{-j}\omega) e^{\|g_{\theta-j_\omega}\|},$$

I derive that for any $x \in M$,

$$h_\omega(x) \ge e^{-\gamma R_{\theta-\tilde{N}_\omega} \varepsilon_{\theta-\tilde{N}_\omega}} \prod_{j=1}^{\tilde{N}} \ell^{-1}(\theta^{-j}\omega) e^{-2\|g_{\theta-j_\omega}\|} \stackrel{\text{def}}{=} \iota(\omega),$$

and so $\|h_\omega^{-1}\| \le \iota^{-1}(\omega)$.

Now take $Q = Q_L \cap \{\omega : \max(R_\omega^{-1}, \iota^{-1}(\omega)) \le L\}$ with Q_L defined by (6.18), and apply (6.22) and (6.23) with $q = \Psi_\omega$ and $r = \Psi_{\theta^{k_j(\omega)}\omega}$. Since $\int \Psi_\omega d\mu^\omega = \int \Psi_{\theta^{k_j(\omega)}\omega} d\mu^{\theta^{k_j(\omega)}\omega} = 0$, it follows by (2.31), (6.22), and the Cauchy-Schwarz inequality that

$$(6.28) \qquad E_{P_Q} \sum_{n=0}^\infty \left| E_{\mu^\omega}(\Psi_\omega(\Psi \circ T^n)_\omega) \right| \le 12L^4 e^{4L} E_{P_Q}(\|\Psi_\omega\| + K_\Psi(\omega))^2 < \infty,$$

establishing condition (i) of Theorem 2.3. In order to derive condition (ii) I use the formula

$$(6.29) \qquad u_\omega^* q_\omega = \lambda_\omega^{-1} h_{\theta\omega}^{-1} \mathcal{L}_g^\omega(q_\omega h_\omega),$$

which holds true since for any $q \in L^2(\Xi^\omega, \mu^\omega)$ and $r \in L^2(\Xi^{\theta\omega}, \mu^{\theta\omega})$,

$$\lambda_\omega^{-1} \int r h_{\theta\omega}^{-1} \mathcal{L}_g^\omega(q h_\omega) d\mu^{\theta\omega} = \lambda_\omega^{-1} \int r \mathcal{L}_g^\omega(q h_\omega) d\nu^{\theta\omega}$$

$$= \lambda_\omega^{-1} \int \mathcal{L}_g^\omega(q h_\omega u_\omega r) d\nu^{\theta\omega} = \int (u_\omega r) q d\mu^\omega.$$

Applying (6.29) successively, I obtain that

$$U_{\Theta^{-n}\omega}^{*n} \Psi_{\Theta^{-n}\omega} = (\lambda_{\tilde{\omega}} \cdots \lambda_{\theta^{k_n(\tilde{\omega})}\tilde{\omega}})^{-1} h_{\theta^{k_n(\tilde{\omega})}\tilde{\omega}}^{-1} \mathcal{L}_g^{\tilde{\omega}, k_n(\tilde{\omega})} \Psi_{\tilde{\omega}}, \text{ with } \tilde{\omega} = \Theta^{-n}\omega.$$

Therefore by (2.31), (6.23), and the Cauchy-Schwarz inequality,

$$E_{P_Q} E_{\mu^\omega} \Big(\sum_{n=0}^{\infty} |U_{\Theta^{-n}\omega}^{*n} \Psi_{\Theta^{-n}\omega}| \Big)^2$$

(6.30)

$$\leq 16 L^4 e^{4L} E_{P_Q} \Big(\sum_{n=0}^{\infty} (\|\Psi_{\Theta^{-n}\omega}\| + K_\Psi(\Theta^{-n}\omega))(1 - \frac{1}{3} L^{-1} e^{-2L})^{[n/L]} \Big)^2$$

$$\leq 48 L^6 e^{6L} E_{P_Q} \sum_{n=0}^{\infty} (\|\Psi_{\Theta^{-n}\omega}\| + K_\Psi(\Theta^{-n}\omega))^2 (1 - \frac{1}{3} L^{-1} e^{-2L})^{[n/L]}$$

$$= 144 L^8 e^{8L} E_{P_Q} (\|\Psi_\omega\| + K_\Psi(\omega))^2 < \infty,$$

establishing the condition (ii) of Theorem 2.3, and so completing the proof of Theorem 2.6.

In order to complete the proof of Theorem 2.5 I shall need a few additional arguments which can be applied only in the random subshifts case. Denote by $\mathcal{C}_n^\omega$ the set of functions r on Ξ^ω such that $\mathrm{var}_n^\omega r = 0$, with var_n^ω defined by (2.20).

6.4. Lemma. *If* $q \in \Lambda_\gamma^\omega, \gamma \geq 1, r \in \mathcal{C}_n^\omega, r \geq 0$, *and* $rq \not\equiv 0$, *then*

(6.31)
$$(\int qr d\nu^\omega)^{-1} (\lambda_\omega \cdots \lambda_{\theta^{n-1}\omega})^{-1} \mathcal{L}_g^{\omega,n}(qr) \in \Lambda_\gamma^{\theta^n \omega}.$$

Proof. Let $\xi, \tilde\xi \in \Xi^\omega$ and $\rho(\xi, \tilde\xi) = e^{-m}$, then

(6.32)
$$\mathcal{L}_g^{\omega,n}(qr)(\xi) = \sum_{\zeta \in (f_\omega^n)^{-1}\xi} \exp(\sum_{j=0}^{n-1} g \circ \tau^j(\zeta, \omega)) q(\zeta) r(\zeta)$$

$$\leq \sum_{\tilde\zeta \in (f_\omega^n)^{-1}\tilde\xi} \exp\Big(\sum_{j=0}^{n-1} (g \circ \tau^j(\tilde\zeta, \omega) + \mathrm{var}_{m+n-j}^{\theta^j \omega} g) \Big) e^{\gamma R_\omega(\zeta,\tilde\zeta)} q(\tilde\zeta) r(\tilde\zeta),$$

where $\rho(\zeta, \tilde\zeta) = e^{-(m+n)}$. By (2.22), (6.1), and (6.3),

$$R_\omega(\zeta, \tilde\zeta) + \sum_{j=0}^{n-1} \mathrm{var}_{m+n-j}^{\theta^j \omega} g \leq \sum_{j=1}^{\infty} K_g(\theta^{-j}\omega) e^{-\kappa(m+n+j)}$$

$$+ \sum_{l=0}^{n-1} K_g(\theta^l \omega) e^{-\kappa(m+n-l)} = R_{\theta^n \omega}(\xi, \tilde\xi).$$

This together with (6.32) yields that for $\gamma \geq 1$,

(6.33)
$$\mathcal{L}_g^{\omega,n}(qr)(\xi) \leq e^{\gamma R_{\theta^n \omega}(\xi,\tilde\xi)} \mathcal{L}_g^{\omega,n}(qr)(\tilde\xi).$$

Next, in the same way as in (6.9),

(6.34)
$$\int qr d\nu^\omega$$

$$= (\lambda_{\theta^n \omega} \cdots \lambda_{\theta^{N_{\theta^n \omega}-1}(\theta^n \omega)})^{-1} \int \mathcal{L}_g^{\theta^n \omega, N_{\theta^n \omega}}(\lambda_\omega \cdots \lambda_{\theta^{n-1}\omega})^{-1} \mathcal{L}_g^{\omega,n}(qr) d\nu^\omega$$

$$\geq G_\omega^{-1} e^{-\gamma B_1(\omega)} (\lambda_\omega \cdots \lambda_{\theta^{n-1}\omega})^{-1} \mathcal{L}_g^{\omega,n}(qr)(\xi)$$

for any $\xi \in \Xi^{\theta^n \omega}$. Since $qr \not\equiv 0$, there exists ξ so that the right hand side of (6.34) is not equal to 0, and the left hand side of (6.34) is positive. This together with (6.33) yields (6.31). $\qquad \square$

Now I can complete the proof of Theorem 2.5. Take $Q = Q_L$ with Q_L defined by (6.18), and consider cylinder sets $A = \{\xi = (\xi_0, \xi_1, ...) \in \Xi^\omega : \xi_i = a_i \text{ for } i = 0, 1, ..., l\}$ and $B = \{\zeta = (\zeta_0, \zeta_1, ...) \in \Xi^{\theta^{l+n}\omega} : \zeta_i = b_i \text{ for } i = 0, 1, ..., m\}$. Then

$$(6.35) \qquad \mu^\omega(A \cap (f_\omega^{l+n})^{-1}B) = \int \mathbb{I}_A(\mathbb{I}_B \circ f_\omega^{l+n})h_\omega d\nu^\omega$$

$$= (\lambda_\omega \cdots \lambda_{\theta^{l+n-1}\omega})^{-1} \int \mathbb{I}_B \mathcal{L}_g^{\omega,l+n}(\mathbb{I}_A h_\omega)d\nu^{\theta^{l+n}\omega}.$$

Since $\mathbb{I}_A \in \mathcal{C}_l^\omega, \mathbb{I}_A \geq 0$, $h_\omega \in \Lambda_1^\omega$, and $h_\omega > 0$, then by Lemma 6.4,

$$q_{\theta^l\omega} \overset{\text{def}}{=} (\int h_\omega \mathbb{I}_A d\nu^\omega)^{-1}(\lambda_\omega \cdots \lambda_{\theta^l\omega})^{-1}\mathcal{L}_g^{\omega,l}(\mathbb{I}_A h_\omega) \in \Lambda_1^{\theta^l\omega},$$

and so by (6.35) and by (6.19) considered with $\tilde{\omega} = \theta^l\omega$ in place of ω and $n \geq k_j(\tilde{\omega})$ it follows that

$$(6.36) \qquad |\mu^\omega(A \cap (f_\omega^{l+n})^{-1}B) - \mu^\omega(A)\mu^{\theta^{l+n}\omega}(B)|$$

$$= \left| \int \left((\lambda_\omega \cdots \lambda_{\theta^{l+n-1}\omega})^{-1}\mathcal{L}_g^{\omega,l+n}(\mathbb{I}_A h_\omega) \right. \right.$$

$$\left. \left. - (\int h_\omega \mathbb{I}_A d\nu^\omega)h_{\theta^{l+n}\omega} \right) \mathbb{I}_B d\nu^{\theta^{l+n}\omega} \right|$$

$$\leq \mu^\omega(A)\nu^{\theta^{l+n}\omega}(B)\|(\lambda_{\tilde{\omega}} \cdots \lambda_{\theta^n\tilde{\omega}})^{-1}\mathcal{L}_g^{\tilde{\omega},n}q_{\tilde{\omega}} - h_{\theta^n\tilde{\omega}}\|$$

$$\leq 2\mu^\omega(A)\nu^{\theta^{l+n}\omega}(B)Le^L\left(1 - \frac{1}{3}L^{-1}e^{-L}\right)^{[j/L]}.$$

Take $l = k_i(\omega)$ and $n = k_j(\theta^l\omega)$; then, by the choice of $Q = Q_L$, $\nu^{\theta^{l+n}\omega}(B) \leq \mu^{\theta^{l+n}\omega}(B)\|h_{\theta^{l+n}\omega}^{-1}\| \leq L\mu^{\theta^{l+n}\omega}(B)$, which together with (6.36) yields (2.5). This yields (2.5) and completes the proof of Theorem 2.5. $\qquad \square$

6.5. Remark. It is possible to define the set Q via the function g and intrinsic characterictics of random subshifts and random expanding transformations. In order to do that it is necessary to obtain appropriate upper bounds for G_ω and $\|h_\omega\|$. Since $h_\omega \in \Lambda_1^\omega$, then by (6.10) one needs only an upper bound on G_ω. In view of the definition of G_ω one has to obtain an upper bound for $\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega}$, which follows by

$$\lambda_\omega \cdots \lambda_{\theta^{N_\omega-1}\omega} = \int \mathcal{L}_g^{\omega,N_\omega} 1 d\nu^{\theta^{N_\omega}\omega} \leq \prod_{j=0}^{N_\omega-1} \ell(\theta^j\omega) \exp(\sum_{j=0}^{N_\omega-1} \|g_{\theta^j\omega}\|).$$

Thus I can take $Q = \tilde{Q}_L = \{\omega : \max(N_\omega, \tilde{G}_\omega, \tilde{G}_{\theta^{N_\omega}\omega}, R_\omega, R_{\theta^{N_\omega}\omega}) \leq L\}$ with $\tilde{G}_\omega = \prod_{j=0}^{N_\omega-1} \ell(\theta^j\omega)e^{\|g_{\theta^j\omega}\|}$ in the random subshift case, and $Q = \tilde{Q}_L \cap \{\omega : \max(R_\omega^{-1}, \iota^{-1}(\omega)) \leq L\}$ in the random expanding maps case.

7. RANDOM SUSPENSIONS

In order to prove Theorem 2.8, set $\bar{l}_\omega = \int l_\omega(\xi)d\mu^\omega(\xi)$ and apply Theorem 2.1 to the function l. Then for any $\alpha > 0$, $P-$ a.s.,

$$(7.1) \qquad \lim_{n \to \infty} \mu^\omega \left\{ \xi : n^{-(\frac{1}{2}+\alpha)}|\sum_{i=0}^{n-1}(l_{\theta^i\omega} - \bar{l}_{\theta^i\omega})(f_{\theta^i\omega}^i(\xi))| > \delta \right\} = 0 \ \forall \delta > 0.$$

Set $t_\omega(k) = \sum_{i=0}^k \bar{l}_{\theta^i\omega}$ and $n_\omega(\xi, s) = \max\{n \geq 0 : \sum_{i=0}^n (l \circ \tau^i)_\omega(\xi) \leq s\}$. Then

$$(7.2) \qquad \{\xi : |n_\omega(\xi, t_\omega(k)) - k| \geq b\sqrt{k}\} = \{\xi : \sum_{0 \leq i \leq k+b\sqrt{k}} (l \circ \tau^i)_\omega(\xi) \leq \sum_{k=0}^k \bar{l}_{\theta^i\omega}\}$$

$$\cup \{\xi : \sum_{0 \leq i \leq j} (l \circ \tau^i)_\omega(\xi) > \sum_{i=0}^k \bar{l}_{\theta^i\omega} \text{ for some } j = 0, ..., [k - b\sqrt{k}] + 1\}$$

$$\subset \{\xi : k^{-1/2}|\sum_{i=0}^k (l_{\theta^i\omega} - \bar{l}_{\theta^i\omega})(f_\omega^i\xi)| \geq (b-1)C_1^{-1}\},$$

where $C_1 = \sup_\omega C(\omega)$. By (7.1) and (7.2), $P-$a.s. for any $\alpha > 0$,

$$(7.3) \qquad \lim_{k\to\infty} \mu^\omega\{\xi : |n_\omega(\xi, t_\omega(k)) - k| > k^{\frac{1}{2}+\alpha}\} = 0.$$

Observe that

$$(7.4) \qquad \int_0^t (\psi \circ \sigma^s)_\omega(0, \xi)ds = \sum_{i=0}^{n_\omega(\xi,t)} (\psi^{(l)} \circ \tau^i)_\omega(\xi) + \int_{s_\omega(\xi,t)}^t (\psi \circ \sigma^s)_\omega(0, \xi)ds,$$

where $s_\omega(\xi, t) = \sum_{i=0}^{n_\omega(\xi,t)} (l \circ \tau^i)_\omega(\xi)$ and $0 \leq t - s_\omega(\xi, t) \leq C_1$, and

$$(7.5) \qquad \left| \int_{s_\omega(\xi,t)}^t (\psi \circ \sigma^s)_\omega(0, \xi)ds \right| \leq (q \circ \tau^{n_\omega(\xi,t)})_\omega(\xi),$$

where by assumption

$$(7.6) \qquad q_\omega(\xi) = \int_0^{l_\omega(\xi)} |\psi_\omega(\xi, s)|ds \in L^1(\Xi_Q, \mu_Q).$$

In view of (7.3)-(7.6) it is easy to see that in order to obtain (2.35) it suffices to show that, $P-$a.s. for any a,

$$(7.7) \qquad \lim_{k\to\infty} \mu^\omega\left\{\xi \in \Xi^\omega : \frac{1}{\sqrt{k}} \sum_{i=0}^k (\psi^{(l)} \circ \tau^i)_\omega(\xi) \leq a\right\} = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{x^2}{2\tilde{\sigma}^2}} dx,$$

where

$$(7.8) \qquad \tilde{\sigma}^2 = \lim_{k\to\infty} \frac{1}{k} E_{\mu^\omega}\left(\sum_{i=0}^{k-1} (\psi^{(l)} \circ \tau^i)_\omega \right)^2$$

$$= P(Q)\left(E_{\mu_Q}(\Psi^{(l)})^2 + \sum_{j=1}^\infty E_{\mu_Q}(\Psi^{(l)}(\Psi^{(l)} \circ T^j)) \right).$$

By the ergodic theorem, $P-$a.s.,

$$(7.9) \qquad\qquad\qquad\qquad \lim_{k\to\infty} \frac{t_\omega(k)}{k} = \bar{l},$$

and so

$$(7.10) \quad \lim_{k\to\infty} \mu^\omega\left\{\xi \in \Xi^\omega : \frac{1}{\sqrt{t_\omega(k)}} \sum_{i=0}^k (\psi^{(l)} \circ \tau^i)_\omega(\xi) \leq a\right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{x^2}{2\sigma^2}} dx.$$

Theorem 2.8 will follow if I show that for any $\varepsilon > 0$ and $P-$a.a.ω,

(7.11)

$$\lim_{k \to \infty} \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{k}} \left| \sum_{i=0}^{k} (\psi^{(l)} \circ \tau^i)_\omega(\xi) - \sum_{i=0}^{n_\omega(\xi, t_\omega(k))} (\psi^{(l)} \circ \tau^i)_\omega(\xi) \right| > \varepsilon \right\} = 0.$$

Actually, it suffices to obtain (7.11) for P_Q-a.a.ω, since $n^{-1/2} \sum_{i=0}^{k_1(\omega)} (\psi^{(l)} \circ \tau^i)_\omega \to 0$ as $n \to \infty$. Taking into account (7.3), this will hold true if, P_Q-a.s.,

(7.12) $$\lim_{k \to \infty} \sum_{-k^{\frac{1}{2}+\alpha} \leq j \leq k^{\frac{1}{2}+\alpha}} \mu^\omega \left\{ \xi \in \Xi^\omega : \frac{1}{\sqrt{k}} |L_{k,j}(\xi, \omega)| > \varepsilon \right\} = 0$$

where $L_{k,j}(\xi, \omega) = r_{|j|} \circ \tau^{k+j}(\xi, \omega)$ and $r_n(\xi, \omega) = \sum_{i=0}^{n} (\psi^{(l)} \circ \tau^i)_\omega(\xi)$. It suffices to show that there exists C_2 such that for any $n = 1, 2, ...$,

(7.13) $$E_{\mu_Q} r_n^8 \leq C_2 n^4.$$

Indeed, by the Chebyshev inequality the sum in (7.12) does not exceed

(7.14) $$M_k(\omega) \stackrel{\text{def}}{=} \varepsilon^{-8} k^{-4} \sum_{-k^{\frac{1}{2}+\alpha} \leq j \leq k^{\frac{1}{2}+\alpha}} E_{\mu^{\theta^{k+j}\omega}} (r_{|j|}^8)_{\theta^{k+j}\omega}.$$

If (7.13) holds true, then

(7.15) $$E_{P_Q} M_k \leq 2C_2 \varepsilon^{-8} k^{-\frac{3}{2}+5\alpha}.$$

If $\alpha < \frac{1}{10}$ then the right hand side of (7.15) is a $k-$th term of a converging series, and so employing the Borel-Cantelli lemma I conclude that P_Q-a.s. $\lim_{k \to \infty} M_k = 0$, which proves (7.12).

In order to establish (7.13) I observe that

(7.16) $$(E_{\mu_Q} r_n^8)^{1/8} \leq (E_{\mu_Q} V_n^8)^{1/8} + (E_{\mu_Q} (R^{(l)})^8)^{1/8},$$

where $V_n = \sum_{j=0}^{N_n(\omega)-1} \Psi^{(l)} \circ T^j(\xi, \omega)$ with $N_n(\omega)$ and $R^{(l)}(\xi, \omega)$ being the same as in (3.1) and (2.32), respectively. I have to estimate only the first term in the right hand side of (7.16), since the last term there is bounded by (2.32). Since $N_n(\omega) \leq n$, then

(7.17) $$E_{\mu_Q} V_n^8 \leq 8! \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_8} E_{P_Q} \left| E_{\mu^\omega} \prod_{j=1}^{8} \Psi^{(l)} \circ T^{i_j} \right|,$$

and so one obtains (7.13) from the following result.

7.1. Lemma. *Let* $\Psi = \Psi_\omega(\xi)$ *be a measurable function on* Ξ *such that* Ψ_ω *is* $\mathcal{F}_{0,\infty}^\omega-$*measurable function on* Ξ^ω *with* $E_{\mu^\omega} \Psi_\omega = 0$. *Set*

$$D_n^{(m)} = (E_{\mu^\omega} (\Psi_\omega - E_{\mu^\omega} (\Psi_\omega | \mathcal{F}_{0,n}^\omega))^{2m})^{\frac{1}{2m}}$$

and $\beta_j^{(m)} = (E_{P_Q} (D_{k_j}^{(m)})^{2m})^{\frac{1}{2m}}$. *Suppose that*

(7.18) $$E_{\mu_Q} \Psi^{2m} < \infty,$$

(7.19) $$\sum_{j=1}^{\infty} j^m \phi_j^{\frac{1}{2m}} < \infty,$$

and

$$\sum_{j=1}^{\infty} j^m \beta_j < \infty. \tag{7.20}$$

Then there is a constant $C = C_m$ such that for any $n = 1, 2, ...,$

$$\sum_{0 \le i_1 \le \cdots \le i_{2m} \le n} E_{P_Q} \left| E_{\mu^\omega} \prod_{j=1}^{2m} \Psi \circ T^{i_j} \right| \le Cn^m, \tag{7.21}$$

and so

$$E_{\mu_Q} \big(\sum_{i=0}^{n} \Psi \circ T^i \big)^{2m} \le 2m! Cn^m. \tag{7.22}$$

Proof. The argument resembles the proofs of Lemma 2.1 from [Kh], of Lemma 4 from §20 in [Bi], and of Theorem 2 from Section 1.4.1 in [Do] (see also Remark 4 there), and it proceeds in the following way. Let

$$s = s_{i_1,...,i_{2m}} = \max_{1 \le q < m} (i_{2q+1} - i_{2q}) = i_{2r+1} - i_{2r}.$$

Then by (3.7) and the Cauchy-Schwarz inequality,

$$(7.23)$$

$$I_{i_1,...,i_{2m}}^\omega \stackrel{\text{def}}{=} \left| E_{\mu^\omega} \left(\prod_{j=1}^{2m} \Psi \circ T^{i_j} \right) \right.$$

$$\left. - E_{\mu^\omega} \left(\prod_{j=1}^{2r} E_{\mu^\omega} \left(\Psi \circ T^{i_j} \Big| \mathcal{F}_{0,k_{i_{2r}+[s/2]}}^\omega \right) \right) E_{\mu^\omega} \left(\prod_{j=2r+1}^{2m} \Psi \circ T^{i_j} \right) \right|$$

$$\le \left(2\phi_{[s/2]}^{1/2m} + \sum_{u=1}^{2r} D_{[s/2]}^{(2m)}(\Theta^{i_u}\omega) \left(E_{\mu^{\Theta^{i_u}\omega}} \Psi_{\Theta^{i_u}\omega}^{2m} \right)^{-1/2m} \right) \prod_{j=1}^{2m} \left(E_{\mu^{\Theta^{i_j}\omega}} \Psi_{\Theta^{i_j}\omega}^{2m} \right)^{1/2m}.$$

Furthermore, by the same reasons, for $t = i_{2m} - i_{2m-1}$ and each $v = 1, 2, ..., 2m-2$,

$$(7.24)$$

$$\left| E_{\mu^\omega} \prod_{j=v}^{2m} \Psi \circ T^{i_j} \right|$$

$$\le \left(2\phi_{[t/2]}^{1/2m} + \sum_{u=v}^{2m-1} D_{[t/2]}^{(2m)}(\Theta^{i_u}\omega) \left(E_{\mu^{\Theta^{i_u}\omega}} \Psi_{\Theta^{i_u}\omega}^{2m} \right)^{-1/2m} \right) \prod_{j=v}^{2m} \left(E_{\mu^{\Theta^{i_j}\omega}} \Psi_{\Theta^{i_j}\omega}^{2m} \right)^{1/2m}.$$

Using (7.24) with $v = 1$ and $v = 2r + 1$, it follows from (7.18), (7.23), and the Cauchy-Schwarz inequality that

$$E_{P_Q} I_{i_1,...,i_{2m}} \le 2\phi_{[\frac{1}{2}\max(s,t)]}^{1/2m} E_{\mu_Q} \Psi^{2m} + 2m\beta_{[\frac{1}{2}\max(s,t)]}^{(2m)} (E_{\mu_Q}\Psi^{2m})^{7/8}, \tag{7.25}$$

which by (7.19) and (7.20) yields that

$$\sum_{0 \le i_1 \le ... \le i_{2m} \le n} E_{P_Q} I_{i_1,...,i_{2m}} \le \tilde{C}n^m \tag{7.26}$$

for some constant $\tilde{C} > 0$ independent of n. Estimating

$$
E_{\mu^\omega}\left(\prod_{j=1}^{2r} E_{\mu^\omega}\left(\Psi \circ T^{i_j} | \mathcal{F}^\omega_{0,k_{i_{2r}+[s/2]}} \right) \right) \quad \text{and} \quad E_{\mu^\omega}\left(\prod_{j=2r+1}^{2m} \Psi \circ T^{i_j} \right)
$$

similarly to (7.23)-(7.26) and continuing in the same way, one eventually arrives at (7.21). $\qquad\square$

References

[Bi1]   P. Billingsley, *Convergence of Probability Measures*, J.Wiley, New York, 1968. MR **38:**1718

[Bi2]   P. Billingsley, *Probability and Measure*, J.Wiley, New York, 1986. MR **87f:**60001

[Bo]   R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lect. Notes in Math. 470, Springer–Verlag, Berlin, 1975. MR **56:**1364

[BG1]   T. Bogenschutz and V. M. Gundlach, *Symbolic dynamics for expanding random dynamical systems*, Random & Comp. Dyn. **1** (1992), 219-227. MR **93j:**58042

[BG2]   T. Bogenschutz and V. M. Gundlach, *Ruelle's transfer operator for random subshifts of finite type*, Ergod. Th. & Dynam. Sys. **15** (1995), 413-447. MR **96m:**58133

[Co]   R. Cogburn, *On the central limit theorem for Markov chains in random environments*, Ann. Probab. **19** (1991), 587-604. MR **92h:**60029

[CFS]   I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Springer-Verlag, Berlin, 1982. MR **87f:**28019

[De]   M. Denker, *The central limit theorem for dynamical systems*, Dynamical Systems and Ergodic Theory, Banach Center Publ., vol. 23, PWN-Polish Sci. Publ., Warszawa, 1989, pp. 33–62. MR **92d:**28007

[Do]   P. Doukhan, *Mixing: Properties and Examples*, Lecture Notes in Stat. 85, Springer-Verlag, New York, 1994. MR **96b:**60090

[DP]   M. Denker and W. Philipp, *Approximation by Brownian motion for Gibbs measures and flows under a function*, Ergod. Th. & Dynam. Sys. **4** (1984), 541-552. MR **86g:**28025

[Go]   M. I. Gordin, *The central limit theorem for stationary processes*, Soviet Math. Dokl. **10** (1969), 1174-1176. MR **40:**5012

[Gu]   V. M. Gundlach, *Thermodynamic formalism for random subshifts of finite type*, Preprint, 1996.

[Ha]   P. Hartman, *Ordinary Differential Equations*, J.Wiley, New York, 1964. MR **30:**1270

[HH]   P. Hall and C. C. Heyde, *Martingale Limit Theory and its Applications*, Acad. Press, New York, 1980. MR **83a:**60001

[HS]   C. C. Heyde and D. J. Scott, *Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments*, Ann. Probab. **1** (1973), 428-436. MR **50:**5886

[IL]   I.A.Ibragimov and Y.V.Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Nordhoff, Groningen, 1971. MR **48:**1287

[Kh]   R.Z.Khasminskii, *On stochastic processes defined by differential equations with a small parameter*, Theory Probab. and its Appl. **11** (1966), 211-228. MR **34:**3636

[Ki1]   Y. Kifer, *Equilibrium states for random expanding transformations*, Random & Comp. Dyn. **1** (1992), 1-31. MR **93i:**58075

[Ki2]   Y. Kifer, *Limit theorems in averaging for dynamical systems*, Ergod. Th. & Dynam. Sys. **15** (1995), 1143-1172. MR **97e:**58087

[Ki3]   Y. Kifer, *Perron-Frobenius theorem, large deviations, and random perturbations in random environments*, Math. Zeitschrift **222** (1996), 677-698. MR **97f:**60131

[Ki4]   Y. Kifer, *Fractal dimensions and random transformations*, Trans. Amer. Math. Soc. **348** (1996), 2003-2038. MR **96i:**28009

[KK]   K. Khanin and Y. Kifer, *Thermodynamic formalism for random transformations and statistical mechanics*, in: Sinai's Moscow Seminar on Dynamical Systems (L.A.Bunimovich, B.M.Gurevich, Ya.B.Pesin, eds.), American Mathematical Society Translations (2) **171** (1995), 107–140. MR **96j:**58136

[Li]   C. Liverani, *Central limit theorem for deterministic systems*, in: Proc. Int. Congr. in
          Dynam. Sys. (F.Ledrappier, J.Lewowicz, and S.Newhouse, eds.), Pitman Research Notes
          in Mathematics, Longman, Harlow, 1996.

[Or]   S. Orey, *Markov chains with stochastically stationary transition probabilities*, Ann.
          Probab. **19** (1991), 907-928. MR **92i:**60065

[PS]   W. Philipp and W. F. Stout, *Almost sure invariance principles for sums of weakly
          dependent random variables*, Memoirs of Amer. Math. Soc. 161, Amer. Math. Soc.,
          Providence, 1975. MR **55:**6570

[Ru]   B.-Z. Rubshtein, *A central limit theorem for conditional distributions*, in: Convergence
          in Ergodic Theory and Probability (Bergelson, March, Rosenblatt, eds.), Walter de
          Gruyter, Berlin, 1996. CMP 97:02

[Se]   T. Seppäläinen, *Large deviations for Markov chains with random transitions*, Ann.
          Probab. **22** (1994), 713-748. MR **95j:**60044

[Sh]   A.N.Shiryaev, *Probability*, Springer-Verlag, New York, 1984. MR **85a:**60007

[VR]   V.A.Volkonskii and Yu.A.Rozanov, *Some limit theorems for random functions I*, Th.
          Probab. Appl. **4** (1959), 178-197. MR **22:**12586; MR **21:**4477

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