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LIMIT THEOREMS FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE[1]

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Let $\{S_n\}$ be a random walk on the integers with negative drift, and let $A_n = \{S_k \geq 0,\ 1 \leq k \leq n\}$ and $A = A_\infty$. Conditioning on A is troublesome because $P(A) = 0$ and there is no natural sigma-field of events "like" A. A natural definition of $P(B|A)$ is $\lim_{n\to\infty} P(B|A_n)$. The main result here shows that this definition makes sense, at least for a large class of events B: The finite-dimensional conditional distributions for the process $\{S_k\}_{k\geq 0}$ given A_n converge strongly to the finite-dimensional distributions for a measure Q. This distribution Q is identified as the distribution for a stationary Markov chain on $\{0, 1, \dots\}$.

1. Introduction. Let $\{S_n = \sum_1^n X_i\}$ be a random walk on the integers with negative drift. Standard results in large deviation theory assert that as $n \to \infty$ the conditional distributions for S_n given $\{S_n \geq 0\}$ converge weakly to a geometric distribution. The goal in this paper is to investigate properties of the process with the conditioning events $\{S_n \geq 0\}$ replaced by the event $A = \{S_k \geq 0,\ k \geq 1\}$. Of course $P(A) = 0$ so this requires some care. A natural definition for conditional probabilities given A would be $P(B|A) = \lim_{n\to\infty} P(B|A_n)$, where $A_n = \{S_k \geq 0,\ 1 \leq k \leq n\}$. Our two main results show that this definition makes sense for a large class of events B.

The first result deals with simple random walks. $X, X_1, X_2, \dots,$ are i.i.d. with $0 < P(X = 1) = p < 1/2$ and $P(X = -1) = q = 1 - p$. The process $\{Y_n\}_{n\geq 0}$ will be a stationary Markov chain on $\{0, 1, \dots\}$ with $Y_0 = 0$ and transition probabilities

$$P(Y_{n+1} = Y_n + 1|Y_n) = \frac{1}{2}\frac{Y_n + 2}{Y_n + 1} = 1 - P(Y_{n+1} = Y_n - 1|Y_n).$$

Our first theorem asserts that the finite-dimensional conditional distributions for the random walk $\{S_n\}$ converge to the finite-dimensional distributions for the Markov chain $\{Y_n\}$. The Markov chain $\{Y_n\}$ also arises in the work of Pitman (1975) studying the conditional distribution of $\{S_n\}$ with $p = 1/2$ given that it hits some positive constant before it becomes negative.

THEOREM 1.1. *For any* $B \subset \mathbb{Z}^k$,

$$\lim_{n\to\infty} P[(S_1, \dots, S_k) \in B|A_n] = P[(Y_1, \dots, Y_k) \in B].$$

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Our second results deals with random walks from a discrete exponential family. Under P_ω, $X, X_1, X_2, \ldots$ will be i.i.d. from a discrete exponential family,

$$P_\omega(X = x) = P_0(X = x)\exp\{\omega x - \psi(\omega)\}$$

for $\omega \in \Omega$, where $\psi(0) = \psi'(0) = 0$. Then $E_0 X = 0$ and $E_\omega X < 0$ for $\omega < 0$. We will assume the parameter space Ω contains some neighborhood of the origin and that the support of the P_0 distribution of X is contained in $\mathbb{Z}$ but is not contained in any sublattice of $\mathbb{Z}$. The exponential family formulation is slightly unusual. It is more common in the large deviation literature to begin with a fixed distribution of interest, but to impose conditions on the moment generating function sufficient to embed the target distribution in an exponential family. If P_ω were the distribution of interest, embedding in a family satisfying the requirements above could be accomplished provided $E_\omega X < 0$, the P_ω distribution of X is arithmetic with unit span and $0 < dE_\omega e^{tX}/dt < \infty$ for some t, where $E_\omega e^{tX} < \infty$. One useful feature of the exponential family formulation is that the P_ω-conditional distribution of $X_1, \ldots, X_n$ given $S_n = j$ does not depend on ω. This is called *sufficiency* in statistics, a term we will use later.

Let $\tau(a) = \inf\{n \geq 1\colon S_n < -a\}$ and $\tau = \tau(0)$. Also, let $\tau^+(b) = \inf\{n \geq 1\colon S_n > b\}$ and $\tau^+ = \tau^+(0)$. The process $\{Y_n\}_{n \geq 0}$ will again be a stationary Markov chain on $\{0, 1, \ldots\}$ with $Y_0 = 0$, but now the appropriate transition probabilities are given by

$$P(Y_{n+1} = z | Y_n = y) = P_0(X = z - y)\frac{E_0|S_{\tau(z)}|}{E_0|S_{\tau(y)}|}$$

for $z, y \geq 0$.

THEOREM 1.2. *Under the conditions just stated, for any negative $\omega \in \Omega$ and for any $B \subset \mathbb{Z}^k$,*

$$\lim_{n \to \infty} P_\omega[(S_1, \ldots, S_k) \in B | A_n] = P[(Y_1, \ldots, Y_k) \in B].$$

This result is not quite a generalization of Theorem 1.1 since the distribution of X there is concentrated on the sublattice $2\mathbb{Z} + 1$ of $\mathbb{Z}$. Although Theorem 1.2 is the deeper of these two results, some of the key ideas in its proof are easier to follow in the context of Theorem 1.1 where a few crucial calculations can be done explicitly using the reflection principle. We suspect Theorem 1.2 can be extended to other lattice distributions and to absolutely continuous distributions using similar methods.

A number of other authors have obtained conditional limit theorems with the same conditioning events A_n. Kao (1978), Iglehart (1974b, 1975) and Durrett (1980) consider the broken line process in continuous time. They rescale time to lie in (0, 1) and rescale space to obtain weak convergence for the conditional distributions of the entire process in various function space topologies. These results complement the results here by providing information about the process at much later times. Durrett's results are particularly interesting as he arrives at a completely different limit when the tails of X decay algebraically. This suggests the exponential family structure assumed here may be necessary. Daley (1968) and Iglehart (1974a) show that $\mathscr{L}(S_n|A_n)$ converges weakly as $n \to \infty$. Iglehart identifies the limiting distribution in the general case through its moment generating function. The following theorem gives a probabilistic expression for the limiting mass function, thus inverting Iglehart's moment generating function.

THEOREM 1.3. *Under the conditions of Theorem* 1.2, *for any negative* $\omega \in \Omega$,

$$P_\omega(S_n = b|A_n) \to \frac{e^{b\omega}E_0|S_{\tau^+(b)}|}{\sum_{k=0}^\infty e^{k\omega}E_0|S_{\tau^+(k)}|}$$

as $n \to \infty$.

A number of conditional limit theorems for Markov chains on an infinite countable state space have been given by Seneta and Vere-Jones (1966). If the chain is "r-positive," the conditional limit theorems obtained are the same as results for chains with a finite state space established in Darroch and Seneta (1965). Viewing $\{S_n\}$ as a Markov chain, two limits they study (called quasistationary distributions) are $\lim_{n\to\infty} P(S_n = x|A_n)$ and $\lim_{m\to\infty}\lim_{n\to\infty} P(S_m = x|A_n)$. The first limit is that considered in Theorem 1.3, and the second without letting $m \to \infty$ is similar to the limits in Theorems 1.1 and 1.2. These results have been extended to a general state space by Tweedie (1974a, b, c), again for r-positive chains. Unfortunately, the r-positive condition fails for random walks [this is noted in the discussion of simple random walks in the last section of Seneta and Vere-Jones (1966)], so these general results are not directly relevant here. The methods used in these papers are more algebraic than the methods used here. It may be possible to give a simpler proof of Theorem 1.2 using algebraic arguments, especially since the limiting process $\{Y_n\}$ is an h-transform [see Kemeny, Snell and Knapp (1976), Section 8.2] of the random walk $\{S_n\}$ without drift. There is also a growing literature on conditional limit theorems for Markov processes in continuous time. In particular, quasistationary distributions for simple random walks in continuous time are given in Seneta (1966) and Pollett (1986).

Some of the results established to prove Theorem 1.2 may be of independent interest. The approach taken uses saddle-point approximations to derive asymptotic properties of pinned random walks. In the relevant limits, the process has small drift and the theory developed is similar in some respects to the results for corrected diffusion approximations given by Siegmund (1979, 1985a) and Hogan (1984), and similar to the results about the maximum of a random walk with small negative drift given by Klass (1983). Reversing time, the results here about the initial behavior of the process, given that it stays below a boundary for a long time, could be recast as results about the last steps of a process, given that it stays below a boundary until a late stage n. This problem with near constant drift has received considerable interest in the nonlinear renewal theory literature [see, e.g., Woodroofe (1982), Chapter 5] and the results here are similar but with drift tending to 0.

2. Simple random walks. In addition to P, it is convenient to introduce measures P_0, $P^{(a)}$ and $P_0^{(a)}$. Under P_0, $X, X_1, X_2, \dots$ are i.i.d. with $P_0(X = 1) = P_0(X = -1) = 1/2$. Under $P^{(a)}$ and $P_0^{(a)}$, $X_1, X_2, \dots$ are i.i.d. with the same marginal distributions as they have under P and P_0 respectively, but $S_n = a + \Sigma_1^n X_i$. The events A_n are related to the stopping time $\tau = \inf\{n: S_n < 0\}$ by $A_n = \{\tau > n\}$.

THEOREM 2.1. *For $a \geq 0$ and $j = j_n \geq 0$,*

$$P^{(a)}(\tau > n | S_n = j) \sim \frac{2(j+1)(a+1)}{n}$$

as $n \to \infty$ with $j + n - a$ even provided $j^2/n \to 0$.

PROOF. By sufficiency,

$$P^{(a)}(\tau > n | S_n = j) = P_0^{(a)}(\tau > n | S_n = j).$$

These probabilities can be computed exactly by the reflection principle as follows. Since $S_\tau = -1$ on $\{\tau \leq n\}$ and since $X \sim -X$ under $P_0^{(a)}$, conditioning on the process prior to time τ,

$$\begin{aligned} P_0^{(a)}(\tau < n, S_n = j) &= P_0^{(a)}(\tau < n, S_n = -2 - j) \\ &= P_0^{(a)}(S_n = -2 - j). \end{aligned}$$

Since $(S_n - a + n)/2$ has a binomial distribution under $P_0^{(a)}$ with sample size n and success probability $1/2$,

$$\begin{aligned} P_0^{(a)}(\tau < n | S_n = j) &= \frac{P_0^{(a)}(S_n = -2 - j)}{P_0^{(a)}(S_n = j)} \\ &= \frac{\left(\frac{n + j - a}{2}\right)! \left(\frac{n - j + a}{2}\right)!}{\left(\frac{n - a - j - 2}{2}\right)! \left(\frac{n + a + j + 2}{2}\right)!}. \end{aligned}$$

The theorem then follows using the identity

$$\frac{(z+c)!}{z^c z!} = 1 + \frac{c(c+1)}{2z} + o\left(\frac{c^2}{z}\right)$$

as $z \to \infty$, provided $c^2/z \to 0$. To establish this identity when $c > 0$, note that $c^2/z \to 0$ implies $c/z \to 0$, so Taylor expansion of the log function about 1 gives

$$\log\left[\frac{(z+c)!}{z^c z!}\right] = \sum_{j=1}^{c} \log\left(1+\frac{j}{z}\right)$$
$$= \sum_{j=1}^{c}\left[\frac{j}{z}+o\left(\frac{j}{z}\right)\right]$$
$$= \frac{c(c+1)}{2z} + o\left(\frac{c(c+1)}{z}\right).$$

Taking exponentials verifies the identity since $e^{\varepsilon} = 1 + \varepsilon + o(\varepsilon)$ as $\varepsilon \to 0$ and $c(c+1) \le 2c^2$. The case $c < 0$ is similar. □

PROOF OF THEOREM 1.1. Let $y_1, \dots, y_k$ be nonnegative integers with $y_1 = 1$ and $|y_{i+1} - y_i| = 1$ for $1 \le i \le k-1$, and let

$$B = \{S_1 = y_1, \dots, S_k = y_k\}.$$

By the DeMoivre–Laplace central limit theorem for the binomial distribution,

$$P_0(S_n = j) \sim \sqrt{\frac{2}{n\pi}} \tag{2.1}$$

as $n \to \infty$ with $n - j$ even, uniformly for $|j| = o(\sqrt{n})$. Also,

$$P(S_n = j) = P_0(S_n = j)(4pq)^{n/2}(p/q)^{j/2}.$$

Using sufficiency and the Markov property, with $y = y_k$,

$$\begin{aligned} P(B, \tau > n, S_n = j) &= P_0(B, \tau > n | S_n = j) P(S_n = j) \\ &= (4pq)^{n/2}(p/q)^{j/2} P_0(B, \tau > n, S_n = j) \\ &= (4pq)^{n/2}(p/q)^{j/2} 2^{-k} P_0^{(y)}(\tau > n-k, S_{n-k} = j) \\ &= (4pq)^{n/2}(p/q)^{j/2} 2^{-k} P_0^{(y)}(\tau > n-k | S_{n-k} = j) \\ &\quad \times P_0(S_{n-k} = j - y). \end{aligned}$$

Then

$$\begin{aligned} \frac{n^{3/2} P(B, \tau > n)}{(4pq)^{n/2}} = \sum_{\substack{n-j \text{ even} \\ j \ge 0}} & 2^{-k}(p/q)^{j/2} n P_0^{(y)}(\tau > n-k | S_{n-k} = j) \\ & \times \sqrt{n} P_0(S_{n-k} = j - y). \end{aligned}$$

The sum over $j > n^{1/4}$ approaches 0 since the summand is bounded by $2^{-k}(p/q)^{j/2} n^{3/2}$. By Theorem 2.1 and (2.1), the summand is asymptotic to

$2^{-k}(p/q)^{j/2}2(y+1)(j+1)\sqrt{2/\pi}$ as $n \to \infty$, uniformly for $0 \le j \le n^{1/4}$. Hence

$$\frac{n^{3/2}P(B, \tau > n)}{(4pq)^{n/2}} - \sum_{\substack{n-j \text{ even} \\ j \ge 0}} 2^{-k}(p/q)^{j/2}2(y+1)(j+1)\sqrt{2/\pi} \to 0$$

as $n \to \infty$. Similarly,

$$\frac{n^{3/2}P(\tau > n)}{(4pq)^{n/2}} - \sum_{\substack{n-j \text{ even} \\ j \ge 0}} (p/q)^{j/2}2(j+1)\sqrt{2/\pi} \to 0$$

as $n \to \infty$. Using these relations,

$$P(B|\tau > n) \to 2^{-k}(y+1) \tag{2.2}$$

as $n \to \infty$. If $\{Y_n\}$ is a process with these finite-dimensional distributions, then

$$P(Y_{n+1} = Y_n + 1 | Y_1, \dots, Y_n) = \frac{(Y_n + 2)2^{-n-1}}{(Y_n + 1)2^{-n}} = \frac{1}{2}\left(\frac{Y_n + 2}{Y_n + 1}\right),$$

so $\{Y_n\}$ is a stationary Markov chain with the transition probabilities stated in Section 1. Equation (2.2) asserts that the conditional finite-dimensional mass functions converge pointwise, and the theorem follows. □

3. Discrete exponential families. For this section, under P_ω, $X, X_1, X_2, \dots$ will be i.i.d. from a discrete exponential family,

$$P_\omega(X = x) = P_0(X = x)\exp\{\omega x - \psi(\omega)\}$$

for $\omega \in \Omega$, where $\psi(0) = \psi'(0) = 0$. We will assume the parameter space Ω contains some neighborhood of the origin and that the support of the P_0 distribution of X is contained in $\mathbb{Z}$ but is not contained in any sublattice of $\mathbb{Z}$. A conditional version of Theorem 1.2 approximating $P_\omega(X_1 = x_1, \dots, X_k = x_k | \tau > n,\ S_n = b)$ will be obtained, and from this Theorem 1.2 will follow by summation. If we let $P_\omega^{(a)}$ denote a measure under which $X, X_1, X_2, \dots$ are i.i.d. with the same marginal distribution as they have under P_ω, but with $S_n = a + \sum_1^n X_i$, then we can express this quantity as

$$\begin{aligned} &P_\omega(X_1 = x_1, \dots, X_k = x_k | \tau > n,\ S_n = b) \\ &\quad = \frac{P_\omega(X_1 = x_1, \dots, X_k = x_k)P_\omega^{(a)}(\tau > n-k,\ S_{n-k} = b)}{P_\omega(\tau > n,\ S_n = b)} \\ &\quad = \frac{P_\omega(X_1 = x_1, \dots, X_k = x_k)P_\omega^{(a)}(\tau > n-k | S_{n-k} = b)P_\omega^{(a)}(S_{n-k} = b)}{P_\omega(\tau > n | S_n = b)P_\omega(S_n = b)}, \end{aligned} \tag{3.1}$$

where $a = \sum_1^k x_i$. Since the distribution of S_n is "almost known" by central limit theory, the key expression is $P_\omega^{(a)}(\tau > n | S_n = b)$, which is approximated in the next theorem. Of course for fixed a and b, this probability will approach 0 as $n \to \infty$, but since a ratio is involved in (3.1), the rate of approach is crucial. Let $P_n^{(a,b)}(B) = P_\omega^{(a)}(B|S_n = b)$ for events $B \in \sigma(X_1, \dots, X_n)$ (by sufficiency, these measures do not depend on ω).

THEOREM 3.1. *With fixed* a,

$$P_n^{(a,b)}(\tau > n) = \frac{2}{n\sigma_0^2} E_0^{(a)}|S_\tau - a| E_0^{(-b)}|S_{\tau+} + b| + o\left(\frac{1+b}{n}\right)$$

as $n \to \infty$, *uniformly for* $0 \le b \le n^{1/8}$.

To prove Theorems 1.2 and 1.3 from this result, we will need the following two lemmas which are relatively standard results in large deviations theory. Let $g_k(x) = P_0(S_k = x)$ and $g_k(x, \omega) = P_\omega(S_k = x) = g_k(x)\exp\{\omega x - k\psi(\omega)\}$. The first lemma is a saddle-point approximation for g_n, and the second gives bounds for the tails of the distributions of S_n. In exponential families, means and variances are given by $E_\omega X = \psi'(\omega)$ and $\operatorname{Var}_\omega(X) = \psi''(\omega) > 0$ for $\omega \in \Omega^o$ (the interior of Ω). Then ψ' is strictly increasing on Ω^o and for $x \in \psi'(\Omega^o)$ we can define $\hat\omega(x)$ by $\psi'(\hat\omega(x)) = x$. Then $\hat\omega(x/n)$ maximizes $g_n(x, \omega)$. On Ω^o, ψ is infinitely differentiable, so on $\psi'(\Omega^o)$, $\hat\omega$ is infinitely differentiable. For proofs of these assertions, see Brown (1986). Let $\hat\psi = \psi \circ \hat\omega$ and $\hat\psi'' = \psi'' \circ \hat\omega$.

LEMMA 3.2. *As* $n \to \infty$,

$$g_n(x) = \frac{\exp\{-x\hat\omega(x/n) + n\hat\psi(x/n)\}}{\sqrt{2\pi n \hat\psi''(x/n)}}\left(1 + O\left(\frac{1}{n}\right)\right),$$

uniformly for integer-valued x *with* x/n *in a sufficiently small neighborhood of* 0. *Also,*

$$\sup_{x \in \mathbb{Z}} \sqrt{n}\left| g_n(x, \omega) - \frac{\exp\{-[x - n\psi'(\omega)]^2/(2n\psi''(\omega))\}}{\sqrt{2\pi n \psi''(\omega)}}\right| \to 0$$

as $n \to \infty$, *uniformly for* ω *sufficiently near* 0.

This approximation for the distribution of S_n was originally derived by Daniels (1954). A careful account is given by Barndorff-Nielsen and Cox (1979), and their regularity conditions are satisfied here. The second assertion follows from the first by Taylor expansion. The fact that $\Sigma_x g_n(x, \omega) = 1$ can be used to take care of the supremum over very large $|x|$. Alternatively, one could use our next lemma for large $|x|$ or refer directly to the proofs in Barndorff-Nielsen and Cox (1979).

LEMMA 3.3. *For any* $\omega_0 \ge \omega$,

$$P_\omega(S_n \ge x) \le \exp\{(\omega - \omega_0)x - n[\psi(\omega) - \psi(\omega_0)]\}$$

and for any $\omega_0 \le \omega$,

$$P_\omega(S_n \le x) \le \exp\{(\omega - \omega_0)x - n[\psi(\omega) - \psi(\omega_0)]\}.$$

PROOF. Introducing the likelihood ratio dP_ω/dP_{ω_0} [or more precisely the likelihood ratio for the restriction of these measures to $\sigma(X_1, \dots, X_n)$],

$$P_\omega(S_n \ge x) = E_{\omega_0}[\exp\{(\omega - \omega_0)S_n - n(\psi(\omega) - \psi(\omega_0))\}; S_n \ge x]$$

and the first bound follows easily. The other bound is similar. □

The following corollary of Theorem 3.1 is the promised conditional version of Theorem 1.2.

COROLLARY 3.4. *Let* $y_1, \dots, y_k$ *be nonnegative integers*, $y_0 = 0$, $x_j = y_j - y_{j-1}$ *for* $1 \le j \le k$ *and* $a = y_k$. *Then*

$$P_n^{(0,b)}(S_1 = y_1, \dots, S_k = y_k | \tau > n) \to P_0(X_1 = x_1, \dots, X_k = x_k) \frac{E_0^{(a)}|S_\tau - a|}{E_0|S_\tau|}$$

as $n \to \infty$, *uniformly for* $0 \le b \le n^{1/8}$.

PROOF. By the second assertion in Lemma 3.2,

$$P_0^{(a)}(S_{n-k} = b) = \frac{\exp\left\{-\dfrac{(b-a)^2}{2(n-k)\psi''(0)}\right\}}{\sqrt{2\pi(n-k)\psi''(0)}} + o\left(\frac{1}{\sqrt{n-k}}\right)$$

$$\sim \frac{1}{\sqrt{2\pi n \psi''(0)}}$$

as $n \to \infty$, uniformly for $0 \le b \le n^{1/8}$. So

$$\frac{P_0^{(a)}(S_{n-k} = b)}{P_0(S_n = b)} \to 1$$

as $n \to \infty$, uniformly for $0 \le b \le n^{1/8}$. Using this and using Theorem 3.1 to approximate $P_0^{(a)}(\tau > n - k | S_{n-k} = b)$ and $P_0(\tau > n | S_n = b)$, the corollary follows taking $\omega = 0$ in (3.1). □

PROOF OF THEOREM 1.3. Using Theorem 3.1 and Lemmas 3.2 and 3.3 (with $\omega_0 = 0$),

$$\begin{aligned} P_\omega(S_n > n^{1/8} | \tau > n) &\le \frac{P_\omega(S_n > n^{1/8})}{P_\omega(\tau > n)} \\ &\le \frac{P_\omega(S_n > n^{1/8})}{P_\omega(S_n = 0) P_n^{(0,0)}(\tau > n)} \\ &\le \frac{\exp\{\omega n^{1/8} - n\psi(\omega)\}}{\exp\{-n\psi(\omega)\} g_n(0) P_n^{(0,0)}(\tau > n)} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Hence $P_\omega(\tau > n) \sim P_\omega(\tau > n,\ S_n \le n^{1/8})$. By the second assertion in Lemma 3.3 with $\omega = 0$, $g_n(b) \sim 1/\sqrt{2\pi\sigma_0^2}$, so using Theorem 3.1,

$$\begin{aligned} P_\omega(\tau > n,\ S_n = b) &= g_n(b)\exp\{\omega b - n\psi(\omega)\}P_n^{(0,b)}(\tau > n) \\ &= \frac{2\exp\{\omega b - n\psi(\omega)\}}{\sqrt{2\pi}\,\sigma_0^3 n^{3/2}}\left\{E_0|S_\tau|E_0^{(-b)}|S_{\tau^+} + b| + o\left(\frac{1+b}{n}\right)\right\}\end{aligned}$$

as $n \to \infty$, uniformly for $0 \le b \le n^{1/8}$. Summing over $0 \le b \le n^{1/8}$,

$$P_\omega(\tau > n) \sim \frac{2e^{-n\psi(\omega)}E_0|S_\tau|}{\sqrt{2\pi}\,\sigma_0^3 n^{3/2}} \sum_{b=0}^{\infty} e^{b\omega}E_0^{(-b)}|S_{\tau^+} + b|$$

as $n \to \infty$, and the theorem follows by division. □

PROOF OF THEOREM 1.2. A simple calculation shows that the limiting distribution in Corollary 3.4 agrees with the distribution for the Markov chain $\{Y_n\}$ defined in Section 1. With the notation of Corollary 3.4,

$$\begin{aligned} P_\omega(S_1 = y_1, \dots, S_k = y_k|\tau > n) = \sum_{b=0}^{\infty} & P_n^{(0,b)}(S_1 = y_1, \dots, S_k = y_k|\tau > n) \\ & \times P_\omega(S_n = b|\tau > n).\end{aligned}$$

Since $P_\omega(S_n > n^{1/8}|\tau > n) \to 0$, the theorem follows from Corollary 3.4. □

The main task remaining is to establish Theorem 3.1. Intuition suggests that if $\{\tau > n\}$ fails under $P_n^{(a,b)}$, it most likely fails because $S_k < 0$ for values of k near 0 or n. We can take advantage of this by conditioning on S_m, where $m = \lfloor n/2 \rfloor$ (the greatest integer $\le n/2$). This gives

$$(3.2)\quad P_n^{(a,b)}(\tau > n) = \sum_{c=0}^{\infty} P_n^{(a,b)}(S_m = c)P_m^{(a,c)}(\tau > m)P_{n-m}^{(c,b)}(\tau > n - m).$$

The relevant values of c in this expression are of order $\sqrt{n}$. Theorem 3.11 below shows that the conditional probabilities $P_m^{(a,c)}(\tau > m)$ are close to the unconditional probabilities $P_\eta^{(a)}(\tau = \infty)$, where η is chosen to make the process drift to c, that is, with $\psi'(\eta) = E_\eta X = c/m$. Our next result approximates these unconditional probabilities. This result and its proof are related to Lemma 2 of Siegmund (1979).

LEMMA 3.5. *As* $\omega \downarrow 0$,

$$P_\omega^{(a)}(\tau = \infty) \sim 2\omega E_0^{(a)}|S_\tau - a|.$$

PROOF. Since Ω contains a neighborhood of the origin, by Taylor expansion $\psi(\omega) \sim \omega^2\sigma_0^2/2$ as $\omega \downarrow 0$, where $\sigma_\omega^2 = \mathrm{Var}_\omega(X) = \psi''(\omega)$. From this and convexity of ψ, for ω positive and sufficiently small there will exist a unique value $\omega^* = \omega^*(\omega) < 0$ with $\psi(\omega) = \psi(\omega^*)$. Also, $\omega^* \sim -\omega$ as $\omega \downarrow 0$. Let $\mathscr{F}_k$ be the sigma field generated by X_i for $1 \le i \le k$ and let $\mathscr{F}_\tau = \{A\colon A \cap \{\tau = k\} \in \mathscr{F}_k$, for $k \ge 1\}$. If P and Q are measures on $\mathscr{F}_\infty$ with restrictions P_n

and Q_n to $\mathscr{F}_n$ and if $Q_n \ll P_n$ for all n, with $\Lambda_n = dQ_n/dP_n$, then Wald's fundamental identity [see Woodroofe (1982), Theorem 1.1] asserts that

$$Q(A, \tau < \infty) = \int_{A, \tau < \infty} \Lambda_\tau \, dP$$

for all $A \in \mathscr{F}_\tau$. Since the likelihood ratio between the restrictions of $P_\omega^{(a)}$ and $P_{\omega^*}^{(a)}$ to $\mathscr{F}_k$ is given by

$$\left.\frac{dP_\omega^{(a)}}{dP_{\omega^*}^{(a)}}\right|\mathscr{F}_k = \exp\{(\omega - \omega^*)(S_k - a)\},$$

Wald's fundamental identity implies

$$P_\omega^{(a)}(\tau < \infty) = E_{\omega^*}^{(a)} \exp\{(\omega - \omega^*)(S_\tau - a)\}$$

[we have used the fact $P_{\omega^*}(\tau < \infty) = 1$ since $E_{\omega^*}X < 0$]. Hence

$$P_\omega^{(a)}(\tau = \infty) = E_{\omega^*}^{(a)}[1 - \exp\{(\omega - \omega^*)(S_\tau - a)\}].$$

Since $[1 - \exp\{(\omega - \omega^*)(S_\tau - a)\}]/\omega$ is nonnegative, bounded by $(\omega^* - \omega) \times (S_\tau - a)/\omega$ and converges almost surely to $-2(S_\tau - a)$ as $\omega \downarrow 0$, the lemma follows provided S_τ is uniformly integrable under $P_\omega^{(a)}$ for ω negative and sufficiently close to 0. In fact, for any $k > 0$, $E_\omega^{(a)}|S_\tau|^k$ is uniformly bounded for $\omega \le 0$ but sufficiently close to 0. To see this, pick a value $\omega_0 < 0$. Then for $\omega_0/2 \le \omega \le 0$, $E_\omega^{(a)}|S_\tau|^k = E_0^{(a)}|S_\tau|^k \exp\{\omega(S_\tau - a) - \tau\psi(\omega)\}$ (by Wald's identity) which is bounded by a suitable constant plus $E_0^{(a)} \exp\{\omega_0(S_\tau - a)\}$. By the next lemma, for some $\omega < 0$, $E_0 \exp(\omega S_\tau) < \infty$, so the descending ladder variables have exponential left tails and $E_0^{(a)} \exp(\omega_0 S_\tau)$ will be finite for some $\omega_0 < 0$. □

Lemma 3.6 appears as Proposition 1 of Siegmund (1975). Problem 8.9 of Siegmund (1985b) describes another method of proof. The result can also be obtained from analytic continuation in the Wiener–Hopf factorization formula.

LEMMA 3.6. *For some $\omega < 0$,*

$$E_0 e^{-\omega S_\tau} < \infty.$$

The next four results provide information about the conditional measures $P_n^{(a,c)}$ needed to establish Theorem 3.11. The first result is a corollary of Lemma 3.2 bounding the tails of the distribution of X_1.

COROLLARY 3.7. *For some $\varepsilon > 0$, if $a = a_n$ and $c = c_n$ are integer-valued sequences with $a = o(n)$ and $c = o(n)$ as $n \to \infty$, then*

$$P_n^{(a,c)}(|X_1| \ge x) = O(\sqrt{n}\, e^{-\varepsilon x})$$

as $n \to \infty$, uniformly for $x \ge 0$.

PROOF. For any $\omega \in \Omega$,

$$P_n^{(a,c)}(|X_1| \ge x) = \frac{P_\omega(|X_1| \ge x,\ S_n = c - a)}{P_\omega(S_n = c - a)} \le \frac{P_\omega(|X_1| \ge x)}{P_\omega(S_n = c - a)}.$$

This bound will be used with $\omega = \omega_n = \hat{\omega}((c-a)/n)$. Then $c - a = n\psi'(\omega_n)$, so the second assertion in Lemma 3.2 implies

$$\sqrt{n}\left[P_{\omega_n}(S_n = c - a) - \frac{1}{\psi''(\omega_n)}\right] \to 0$$

as $n \to \infty$. Since $\omega_n \to 0$ as $n \to \infty$,

$$\liminf_{n\to\infty} \sqrt{n}\, P_{\omega_n}(S_n = c - a) > 0.$$

To bound the numerator, apply Lemma 3.3 with $n = 1$ and a positive and negative value for ω_0. Since $\omega_n \to 0$ and ψ is strictly convex with a minimum at 0, for both choices of ω_0, $\psi(\omega_n) - \psi(\omega_0) \le 0$ for all n sufficiently large, and hence with ε the smaller of the magnitudes of the two values of ω_0,

$$P_{\omega_n}(|X_1| \ge x) \le 2e^{\varepsilon x}$$

for n sufficiently large. The corollary follows. □

LEMMA 3.8. *Suppose $m = m_n \sim \varepsilon n$ as $n \to \infty$ for some $\varepsilon \in (0,1)$. Then there exist neighborhoods N_0 and N_1 (that may depend on ε) of the origin such that*

$$E_n^{(0,c)} e^{tS_m} \sim \sqrt{\frac{\hat{\psi}''(c/n)}{\varepsilon\psi''(\omega) + (1-\varepsilon)\psi''(\omega - t)}} \exp\{c[t + \hat{\omega}(c/n) - \omega] + (n-m)\psi(\omega - t) + m\psi(\omega) - n\hat{\psi}(c/n)\}$$

as $n \to \infty$, uniformly for $c/n \in N_0$ and $t \in N_1$, where $\omega = \omega_n$ for n sufficiently large is the unique solution of

$$\frac{m}{n}\psi'(\omega) + \frac{(n-m)}{n}\psi'(\omega - t) = \frac{c}{n}. \tag{3.3}$$

PROOF. We will assume throughout that $0 \le m \le n$. Then the left-hand side of (3.3) is strictly increasing as a function of ω. Let $[-\delta, \delta]$ be a neighborhood of 0 contained in Ω. If $t \in [-\delta/3, \delta/3]$ and $\omega = -2\delta/3$, then the left-hand side of (3.3) is bounded above by $\psi'(-\delta/3)$. Similarly, if $t \in [-\delta/3, \delta/3]$ and $\omega = 2\delta/3$, the left-hand side of (3.3) is bounded below by $\psi'(\delta/3)$. Hence the range of the left-hand side of (3.3) (viewed as a function of ω) contains $[\psi'(-\delta/3), \psi'(\delta/3)]$ and a unique solution is guaranteed if $t \in [-\delta/3, \delta/3]$ and $c/n \in [\psi'(-\delta/3), \psi'(\delta/3)]$. Since $g_k(x, \omega) = g_k(x)\exp\{\omega x - k\psi(\omega)\}$,

$$e^{tS_m} g_{n-m}(c - S_m, \omega) = \exp\{tc - (n-m)[\psi(\omega) - \psi(\omega - t)]\} \times g_{n-m}(c - S_m, \omega - t).$$

Conditioning on S_m,

$$
\begin{aligned}
E_\omega\big[e^{tS_m}; S_n = c\big] &= E_\omega\big[e^{tS_m} g_{n-m}(c - S_m, \omega)\big] \\
&= \exp\{tc - (n-m)[\psi(\omega) - \psi(\omega - t)]\} \\
&\quad \times E_\omega g_{n-m}(c - S_m, \omega - t).
\end{aligned} \tag{3.4}
$$

From the second assertion of Lemma 3.2, and using (3.3),

$$
\begin{aligned}
g_{n-m}(c - S_m, \omega - t) &= \frac{\exp\left\{-\dfrac{[c - S_m - (n-m)\psi'(\omega - t)]^2}{2(n-m)\psi''(\omega - t)}\right\}}{\sqrt{2\pi(n-m)\psi''(\omega - t)}} + o\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{\exp\left\{-\dfrac{[S_m - m\psi'(\omega)]^2}{2(n-m)\psi''(\omega - t)}\right\}}{\sqrt{2\pi(n-m)\psi''(\omega - t)}} + o\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

as $n \to \infty$. Since the P_ω distributions of $(S_m - m\psi'(\omega))/\sqrt{m\psi''(\omega)}$ converge weakly to $N(0,1)$,

$$
\sqrt{n}\, E_\omega g_{n-m}(c - S_m, \omega - t) = 1/\sqrt{2\pi\{\varepsilon\psi''(\omega) + (1-\varepsilon)\psi''(\omega - t)\}} + o(1)
$$

as $n \to \infty$. The lemma follows using this in (3.4) and approximating $P_\omega(S_n = c)$ using Lemma 3.2 by

$$
\begin{aligned}
P_\omega(S_n = c) &= g_n(c)\exp\{\omega c - n\psi(\omega)\} \\
&\sim \frac{\exp\{(\omega - \hat{\omega}(c/n))c + n(\hat{\psi}(c/n) - \psi(\omega))\}}{\sqrt{2\pi n \hat{\psi}''(c/n)}}.
\end{aligned}
\qquad \square
$$

Corollary 3.9. *If $a = a_n$ and $c = c_n$ satisfy $a = o(n)$ and $c = o(n)$ as $n \to \infty$ and if $m = m_n \sim \varepsilon n$ as $n \to \infty$ for some value $\varepsilon \in (0,1)$, then the $P_n^{(a,c)}$ distributions of $(S_m - a - (c-a)m/n)/\sqrt{n}$ converge weakly to $N(0, \sigma_0^2\varepsilon(1-\varepsilon))$ as $n \to \infty$. Also, the moment generating functions converge pointwise to the moment generating function for the limiting distribution.*

Proof. If the conditions on a and c were strengthened to $a = o(\sqrt{n})$ and $c = o(\sqrt{n})$, weak convergence would follow from the invariance principle for pinned random walks due to Liggett (1968). With the additional uniformity in a and c, weak convergence could be obtained using Lemma 3.2 to show pointwise convergence of the conditional mass functions scaled up by $\sqrt{n}$. Since we are concerned with tail behavior, it seems more convenient to use the previous lemma to establish pointwise convergence of the moment generating function. Fix $\delta \in (-\infty, \infty)$, $\delta \neq 0$ and assume without loss of generality that $a_n = 0$ for all n. It is sufficient to show that as $n \to \infty$,

$$
E_n^{(0,c_n)} \exp\{\delta(S_m - c_n m_n/n)/\sqrt{n}\} \to \exp\{\delta^2\sigma_0^2\varepsilon(1-\varepsilon)/2\},
$$

the moment generating function for $N(0, \sigma_0^2\varepsilon(1-\varepsilon))$. Hence, applying Lemma 3.8, we should take $t = \delta/\sqrt{n}$. The corresponding parameter ω_n solves

$$\varepsilon_n\psi'(\omega) + (1-\varepsilon_n)\psi'(\omega - \delta/\sqrt{n}) = c_n/n,$$

where $\varepsilon_n = m_n/n \to \varepsilon$ as $n \to \infty$. As noted in the proof of Lemma 3.8, the left-hand side of this equation is increasing in ω. By Taylor expansion about $\hat{\omega}_n = \hat{\omega}(c_n/n)$, the left-hand side of the equation evaluated at $\omega = \hat{\omega}_n \pm 2\delta/\sqrt{n}$ [after simplification using $\psi'(\hat{\omega}_n) = c_n/n$] equals

$$\frac{c_n}{n} + \frac{\delta}{\sqrt{n}}\psi''(\hat{\omega}_n)(\pm 2 - (1-\varepsilon_n)) + O(1/n)$$

as $n \to \infty$. For all sufficiently large n, one of the values will be less than c_n/n, the other greater than c_n/n, implying $\omega_n \in (\hat{\omega}_n - 2|\delta|/\sqrt{n}, \hat{\omega}_n + 2|\delta|/\sqrt{n})$. Hence $\omega_n - \hat{\omega}_n = O(1/\sqrt{n})$ as $n \to \infty$. Using this, Taylor expansion about $\hat{\omega}_n$ in the equation defining ω_n gives

$$\frac{c_n}{n} + \psi''(\hat{\omega}_n)\left[\omega_n - \hat{\omega}_n - (1-\varepsilon_n)\delta/\sqrt{n}\right] + O(1/n) = c_n/n,$$

so

$$\omega_n - \hat{\omega}_n = (1-\varepsilon_n)\delta/\sqrt{n} + O(1/n)$$

as $n \to \infty$. Since

$$\sqrt{\frac{\hat{\psi}''(c_n/n)}{\varepsilon\psi''(\omega_n) + (1-\varepsilon)\psi''(\omega_n - \delta/\sqrt{n})}} \to 1$$

as $n \to \infty$, Lemma 3.8 implies

$$\begin{aligned}
&E_n^{(0,c_n)} \exp\{\delta(S_m - c_n m_n/n)/\sqrt{n}\} \\
&\quad \sim \exp\Big\{c_n(\delta/\sqrt{n} + \hat{\omega}_n - \omega_n) + n(1-\varepsilon_n)\psi(\omega_n - \delta/\sqrt{n}) \\
&\qquad\qquad + n\varepsilon_n\psi(\omega_n) - n\psi(\hat{\omega}_n) - \varepsilon_n c_n\delta/\sqrt{n}\Big\}
\end{aligned}$$

as $n \to \infty$. By Taylor expansion, the argument of the exponential here is

$$\begin{aligned}
&c_n(\delta/\sqrt{n} + \hat{\omega}_n - \omega_n) \\
&\quad + n(1-\varepsilon_n)\psi(\hat{\omega}_n) + c_n(1-\varepsilon_n)(\omega_n - \hat{\omega}_n - \delta/\sqrt{n}) \\
&\quad + \tfrac{1}{2}n(1-\varepsilon_n)\psi''(\hat{\omega}_n)(\omega_n - \hat{\omega}_n - \delta/\sqrt{n})^2 \\
&\quad + n\varepsilon_n\psi(\hat{\omega}_n) + c_n\varepsilon_n(\omega_n - \hat{\omega}_n) + \tfrac{1}{2}n\varepsilon_n\psi''(\hat{\omega}_n)(\omega_n - \hat{\omega}_n)^2 \\
&\quad - n\psi(\hat{\omega}_n) - c_n\varepsilon_n/\sqrt{n} + o(1) \\
&= \tfrac{1}{2}\psi''(\hat{\omega}_n)\left[(1-\varepsilon_n)\varepsilon_n^2\delta^2 + \varepsilon_n(1-\varepsilon_n)^2\delta^2\right] + o(1) \\
&\to \tfrac{1}{2}\sigma_0^2\varepsilon(1-\varepsilon)
\end{aligned}$$

as $n \to \infty$, which proves the corollary. □

LEMMA 3.10. *If* $a = a_n = O(n^{9/16})$ *and* $c = c_n = O(n^{1/8})$ *as* $n \to \infty$ *and if* $\liminf_{n\to\infty} a_n/\sqrt{n} > 0$, *then*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} P_n^{(a,c)}(\tau < \varepsilon n) = 0.$$

PROOF. This result almost follows from the invariance principle in Liggett (1968). Let $\hat{\omega}_n = \hat{\omega}(-a/n)$. Removing the conditioning on S_n followed by conditioning on $\mathscr{F}_\tau$,

$$\begin{aligned} P_n^{(a,c)}(\tau < \varepsilon n) &= \frac{P_{\hat{\omega}_n}^{(a)}(\tau < \varepsilon n, S_n = c)}{g_n(c-a, \hat{\omega}_n)} \\ &= \frac{E_{\hat{\omega}_n}^{(a)}[g_{n-\tau}(c - S_\tau, \hat{\omega}_n); \tau < \varepsilon n]}{g_n(c-a, \hat{\omega}_n)} \\ &\le \frac{\sup_{(1-\varepsilon)n<k<n} \sup_{x\in\mathbb{Z}} g_k(x, \hat{\omega}_n)}{g_n(c-a, \hat{\omega}_n)} P_{\hat{\omega}_n}^{(a)}(\tau < \varepsilon n). \end{aligned}$$

By the second assertion in Lemma 3.2,

$$\begin{aligned} g_n(c-a, \hat{\omega}_n) &= \frac{\exp\{-c^2/(2n\psi''(\hat{\omega}_n))\}}{\sqrt{2\pi n \psi''(\hat{\omega}_n)}} + o\left(\frac{1}{\sqrt{n}}\right) \\ &\sim \frac{1}{\sqrt{2\pi n \sigma_0^2}} \end{aligned}$$

as $n \to \infty$ and

$$\begin{aligned} \sup_{x\in\mathbb{Z}} g_k(x, \hat{\omega}_n) &= \frac{1}{\sqrt{2\pi k \psi''(\hat{\omega}_n)}} + o\left(\frac{1}{\sqrt{k}}\right) \\ &\sim \frac{1}{\sqrt{2\pi k \sigma_0^2}} \end{aligned}$$

as $k \to \infty$, $n \to \infty$. So

$$\sup_{(1-\varepsilon)n<k<n} \sup_{x\in\mathbb{Z}} g_k(x, \hat{\omega}_n) \sim \frac{1}{\sqrt{2\pi n(1-\varepsilon)\sigma_0^2}}$$

as $n \to \infty$ and

$$\limsup_{n\to\infty} P_n^{(a,c)}(\tau < \varepsilon n) \le \limsup_{n\to\infty} P_{\hat{\omega}}^{(a)}(\tau < \varepsilon n)/\sqrt{1-\varepsilon}.$$

By Kolmogorov's inequality, if $\hat\omega_n \le 0$,

$$\begin{aligned} P^{(a)}_{\hat\omega_n}(\tau < \varepsilon n) &= P_{\hat\omega_n}(S_k < -a \text{ for some } 0 < k < \varepsilon n) \\ &\le P_{\hat\omega_n}(S_k - k\psi'(\hat\omega_n) < -a - \varepsilon n\psi'(\hat\omega_n) \text{ for some } 0 < k < \varepsilon n) \\ &\le \frac{\varepsilon n\psi''(\hat\omega_n)}{(a + \varepsilon n\psi'(\hat\omega_n))^2} \\ &= \frac{\varepsilon n\psi''(\hat\omega_n)}{(a(1-\varepsilon))^2}. \end{aligned}$$

So

$$\begin{aligned} \limsup_{n\to\infty} P^{(a,c)}_n(\tau < \varepsilon n) &\le \limsup_{n\to\infty} \frac{\varepsilon n\psi''(\hat\omega_n)}{\sqrt{1-\varepsilon}\,(a(1-\varepsilon))^2} \\ &= \frac{\varepsilon\sigma_0^2}{(1-\varepsilon)^{5/2}\left(\liminf_{n\to\infty} a_n/\sqrt{n}\right)^2}. \end{aligned}$$

The lemma follows. □

The next result is the pinned version of Lemma 3.5. Theorem 3.1 will be proved using this result to approximate the summands in (3.2). In Theorem 3.11, the primary goal is an approximation for $P^{(a,c)}_n(\tau > n)$ when $a = O(1)$ and $c = O(\sqrt{n})$. Since the approximation will be summed in (3.2), it is technically convenient to strive for uniformity over $0 \le c \le n^{9/16}$ (the choice of the exponent, 9/16, is somewhat arbitrary). With this level of uniformity, for some values of a and c, the error rate can exceed the leading term in magnitude. Although the relative error could be large in these regions, the result still provides useful information by giving an asymptotic upper bound for $P^{(a,c)}_n(\tau > n)$.

THEOREM 3.11. *As $n \to \infty$,*

$$P^{(a,c)}_n(\tau \ge n) = (2c/n\sigma_0^2)E^{(a)}_0|S_\tau - a| + o\{(1+a)/\sqrt{n} + ac/n + c^3/n^2\},$$

uniformly for $0 \le c \le n^{9/16}$ and $0 \le a \le n^{1/8}$.

PROOF. The proof given will mimic the proof of Lemma 3.5, with the tilting to $P^{(a,-c)}_n$ instead of $P^{(a)}_{\omega^*}$. Since

$$\begin{aligned} \frac{P^{(a,c)}_n(S_k = x)}{P^{(a,-c)}_n(S_k = x)} &= \frac{P^{(a)}_0(S_k = x, S_n = c)}{P^{(a)}_0(S_k = x, S_n = -c)}\,\frac{P^{(a)}_0(S_n = -c)}{P^{(a)}_0(S_n = c)} \\ &= \frac{g_{n-k}(c - x)g_n(-c-a)}{g_{n-k}(-c-x)g_n(c-a)}, \end{aligned}$$

Wald's identity implies

$$P_n^{(a,c)}(\tau < n, B) = E_n^{(a,-c)}[L; B, \tau < n] \tag{3.5}$$

for any $B \in \mathscr{F}_\tau$, where

$$L = \frac{g_{n-\tau}(c - S_\tau) g_n(-c-a)}{g_{n-\tau}(-c - S_\tau) g_n(c-a)}.$$

Fix $\varepsilon > 0$ and let $B_n = \{\tau \le (1-\varepsilon)n, |S_\tau| \le n^{1/8}\}$. Using the approximation

$$g_n(x) = \frac{\exp\{-x\hat{\omega}(x/n) + n\hat{\psi}(x/n)\}}{\sqrt{2\pi n \hat{\psi}''(x/n)}}\left(1 + O\left(\frac{1}{n}\right)\right),$$

from Lemma 3.2, on B_n we have

$$\begin{aligned} L &= \frac{g_{n-\tau}(c - S_\tau) g_n(-c-a)}{g_{n-\tau}(-c - S_\tau) g_n(c-a)} \\ &= \sqrt{\frac{\hat{\psi}''\left(\dfrac{-c-S_\tau}{n-\tau}\right)\hat{\psi}''\left(\dfrac{c-a}{n}\right)}{\hat{\psi}''\left(\dfrac{c-S_\tau}{n-\tau}\right)\hat{\psi}''\left(\dfrac{-c-a}{n}\right)}} \\ &\quad \times \exp\left\{-(c-S_\tau)\hat{\omega}\left(\frac{c-S_\tau}{n-\tau}\right) + (n-\tau)\hat{\psi}\left(\frac{c-S_\tau}{n-\tau}\right)\right. \\ &\qquad + (-c-S_\tau)\hat{\omega}\left(\frac{-c-S_\tau}{n-\tau}\right) - (n-\tau)\hat{\psi}\left(\frac{-c-S_\tau}{n-\tau}\right) \\ &\qquad - (-c-a)\hat{\omega}\left(\frac{-c-a}{n}\right) + n\hat{\psi}\left(\frac{-c-a}{n}\right) \\ &\qquad \left. + (c-a)\hat{\omega}\left(\frac{c-a}{n}\right) - n\hat{\psi}\left(\frac{c-a}{n}\right) + o\left(\frac{1}{\sqrt{n}}\right)\right\}, \end{aligned}$$

as $n \to \infty$, uniformly for $|c| \le 2n^{9/16}$. By Taylor expansion $\psi'(x) = x\sigma_0^2 + \psi'''(0)x^2/2 + O(|x|^3)$ as $x \to 0$. So $\hat{\omega}(x) = x/\sigma_0^2 + K_1 x^2 + O(|x|^3)$ as $x \to 0$, where $K_1 = -\psi'''(0)/(2\sigma_0^6)$. Also, since $\psi(x) = \sigma_0^2 x^2/2 + \psi'''(0)x^3/6 + O(x^4)$ and $\psi''(x) = \sigma_0^2 + \psi'''(0)x + O(x^2)$, we have $\hat{\psi}(x) = x^2/(2\sigma_0^2) + K_2 x^3 + O(x^4)$ and $\hat{\psi}''(x) = \sigma_0^2 + K_3 x + O(x^2)$, where $K_2 = -\psi'''(0)/(3\sigma_0^6)$ and $K_3 = \psi'''(0)/\sigma_0^2$. Using these relations, the expression for L just given can be simplified. On B_n we have

$$-(c-S_\tau)\hat{\omega}\left(\frac{c-S_\tau}{n-\tau}\right) = \frac{-c^2}{\sigma_0^2(n-\tau)} + \frac{2cS_\tau}{\sigma_0^2(n-\tau)} - \frac{K_1 c^3}{(n-\tau)^2} + o\left(\frac{1}{\sqrt{n}}\right),$$

$$(n-\tau)\hat{\psi}\left(\frac{c-S_\tau}{n-\tau}\right) = \frac{c^2}{2\sigma_0^2(n-\tau)} - \frac{cS_\tau}{\sigma_0^2(n-\tau)} + \frac{K_2 c^3}{(n-\tau)^2} + o\left(\frac{1}{\sqrt{n}}\right)$$

and

$$\hat{\psi}''\left(\frac{-c-S_\tau}{n-\tau}\right) = \sigma_0^2 - \frac{K_3 c}{(n-\tau)} + o\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$, uniformly for $|c| \le 2n^{9/16}$. The other terms that arise have c changed to $-c$, S_τ changed to a, and/or $n - \tau$ changed to n. Also, it is easy to check that these relations hold uniformly for $|a| \le n^{1/8}$. After some algebra,

$$
\begin{aligned}
L = &\sqrt{\left[\sigma_0^4 + \sigma_0^2 K_3 c\left(\frac{1}{n} - \frac{1}{n-\tau}\right) + o\left(\frac{1}{\sqrt{n}}\right)\right] \Big/ \left[\sigma_0^4 - \sigma_0^2 K_3 c\left(\frac{1}{n} - \frac{1}{n-\tau}\right) + o\left(\frac{1}{\sqrt{n}}\right)\right]} \\
&\times \exp\left\{\frac{2cS_\tau}{(n-\tau)\sigma_0^2} - \frac{2ac}{n\sigma_0^2} + 2c^3(K_1 - K_2)\left[\frac{1}{n^2} - \frac{1}{(n-\tau)^2}\right] + o\left(\frac{1}{\sqrt{n}}\right)\right\} \\
(3.6)\quad &= 1 + \frac{2cS_\tau}{(n-\tau)\sigma_0^2} - \frac{2ac}{n\sigma_0^2} + 2c^3(K_1 - K_2)\left(\frac{1}{n^2} - \frac{1}{(n-\tau)^2}\right) \\
&\quad + \frac{K_3 c}{\sigma_0^2}\left(\frac{1}{n} - \frac{1}{n-\tau}\right) + o\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

as $n \to \infty$ on B_n, uniformly for $|c| \le 2n^{9/16}$ and $|a| \le n^{1/8}$. Before we use this approximation for L in (3.5), we will derive approximations for the $P_n^{(a,-c)}$ expectations of a few of the terms that arise. Suppose Y is $\mathscr{F}_\tau$ measurable and $Y = 0$ on $\{\tau > n\}$. Conditioning on $\mathscr{F}_\tau$,

$$
\begin{aligned}
E_n^{(a,-c)} Y &= \frac{E_0^{(a)}[Y; S_n = -c]}{g_n(-c-a)} \\
&= \frac{E_0^{(a)}[Y g_{n-\tau}(-c - S_\tau)]}{g_n(-c-a)},
\end{aligned}
$$

so the $P_n^{(a,-c)}$ expectation of Y is the same as the $P_0^{(a)}$ expectation of Yl, where $l = g_{n-\tau}(-c - S_\tau)/g_n(-c-a)$. By the invariance principle for random walks, the $P_0^{(a)}$ distributions of τ/a^2 converge weakly as $a \to \infty$. Therefore, $\tau/\sqrt{n} \to 0$ in $P_0^{(a)}$-probability as $n \to \infty$, uniformly for $|a| \le n^{1/8}$. Also, by Lemma 3.6, S_τ has finite moments under P_0 and consequently the $P_0^{(a)}$ distributions of S_τ are uniformly integrable for $a \ge 0$ [see Lorden (1970)]. Hence $S_\tau/\sqrt{n} \to 0$ in $P_0^{(a)}$-probability as $n \to \infty$, uniformly for $0 \le a \le n^{1/8}$. From these facts, using Lemma 3.2, $l \to 1$ as $n \to \infty$ in $P_0^{(a)}$-probability, uniformly for $|c| \le 2n^{9/16}$ and $0 \le a \le n^{1/8}$. Using Lemma 3.2 and the Taylor approximations for $\hat{\omega}$, $\hat{\psi}$ and $\hat{\psi}''$ given above, on B_n,

$$
\begin{aligned}
l = &\left(1 + O\left(\frac{1}{n}\right)\right)\sqrt{\frac{n}{n-\tau}}\sqrt{\hat{\psi}''\left(\frac{-c-a}{n}\right) \Big/ \hat{\psi}''\left(\frac{-c-S_\tau}{n-\tau}\right)} \\
&\times \exp\left\{-(-c - S_\tau)\hat{\omega}\left(\frac{-c-S_\tau}{n-\tau}\right) + (n-\tau)\hat{\psi}\left(\frac{-c-S_\tau}{n-\tau}\right)\right. \\
&\qquad\left. + (-c-a)\hat{\omega}\left(\frac{-c-a}{n}\right) - n\hat{\psi}\left(\frac{-c-a}{n}\right)\right\} \\
= &\left(1 + O(n^{-7/16})\right)\sqrt{\frac{n}{n-\tau}}\exp\left\{-\frac{c^2}{2\sigma_0^2(n-\tau)} + \frac{c^2}{2\sigma_0^2 n} + O(n^{-5/16})\right\}
\end{aligned}
$$

as $n \to \infty$, uniformly for $|a| \le n^{1/8}$ and $|c| \le 2n^{9/16}$. Consequently, l is bounded on B_n, uniformly for $|a| \le n^{1/8}$ and $|c| \le 2n^{9/16}$. Since $l \to 1$, $\tau/\sqrt{n} \to 0$ and $S_\tau/\sqrt{n} \to 0$ in $P_0^{(a)}$-probability,

$$P_n^{(a,-c)}(B_n) = E_0^{(a)}[l; B_n] \to 1$$

as $n \to \infty$, uniformly for $|a| \le n^{1/8}$ and $|c| \le 2n^{9/16}$. Also, since

$$n\left(\frac{1}{n} - \frac{1}{n-\tau}\right) \to 0 \quad \text{and} \quad n^2\left(\frac{1}{n^2} - \frac{1}{(n-\tau)^2}\right) \to 0$$

in $P_0^{(a)}$-probability and these remain bounded on B_n, we have

$$nE_n^{(a,-c)}\left[\frac{1}{n} - \frac{1}{n-\tau}; B_n\right] = nE_0^{(a)}\left[l\left(\frac{1}{n} - \frac{1}{n-\tau}\right); B_n\right] \to 0$$

and

$$n^2E_n^{(a,-c)}\left[\frac{1}{n^2} - \frac{1}{(n-\tau)^2}; B_n\right] = n^2E_0^{(a)}\left[l\left(\frac{1}{n^2} - \frac{1}{(n-\tau)^2}\right); B_n\right] \to 0$$

as $n \to \infty$, uniformly for $0 \le c \le 2n^{9/16}$ and $0 \le a \le n^{1/8}$. Similarly, but using the fact that the $P_0^{(a)}$ distributions of S_τ are uniformly integrable for $a \ge 0$,

$$nE_n^{(a,-c)}\left[\frac{S_\tau}{n-\tau}; B_n\right] - E_0^{(a)}S_\tau \to 0$$

as $n \to \infty$, uniformly for $0 \le c \le 2n^{9/16}$ and $0 \le a \le n^{1/8}$. Using these limits to approximate $P_n^{(a,-c)}$ expectations of terms in the approximation (3.6) for L, (3.5) gives

$$\begin{aligned} P_n^{(a,c)}(B_n) &= E_n^{(a,-c)}(L; B_n) \\ &= P_n^{(a,-c)}(B_n) + \frac{2c}{n}E_0^{(a)}[S_\tau - a] + o\left(\frac{1}{\sqrt{n}} + \frac{c^3}{n^2}\right) \end{aligned}$$

as $n \to \infty$, uniformly for $0 \le c \le 2n^{9/16}$ and $0 \le a \le n^{1/8}$. By Corollary 3.7, for some $\varepsilon_0 > 0$,

$$P_n^{(a,-c)}\{|X_i| \ge n^{1/8} \quad \text{for some} \quad 1 \le i \le n\} = O\{n^{3/2}\exp(-\varepsilon_0 n^{1/8})\},$$

so we have

$$\begin{aligned} P_n^{(a,c)}\{\tau > (1-\varepsilon)n\} &= \frac{2c}{n\sigma_0^2}E_0^{(a)}|S_\tau - a| \\ &\quad + P_n^{(a,-c)}\{\tau > (1-\varepsilon)n\} + o\left(\frac{1}{\sqrt{n}} + \frac{c^3}{n^2}\right). \end{aligned} \tag{3.7}$$

The proof will be finished by showing $P_n^{(a,c)}\{(1-\varepsilon)n < \tau < n\}$ and $P_n^{(a,-c)}\{\tau > (1-\varepsilon)n\}$ are both small. Unfortunately, this calculation is rather delicate. Define $h_n(a,c) = P_n^{(a,c)}(\tau > n)$. Let us begin by showing that $h_n(a,c)$

is not too large. Define

$$d_n = \sup_{0 \le c \le 2n^{9/16}} \sup_{0 \le a \le n^{1/8}} \frac{\sqrt{n}\, h_n(a,c)}{1 + a + ac/\sqrt{n} + c^3/n^{3/2}}.$$

With $m = \lfloor (1-\varepsilon)n \rfloor$, conditioning on $\mathscr{F}_m$ gives the bound

$$\begin{aligned} P_n^{(a,-c)}\{\tau > (1-\varepsilon)n\} &= E_n^{(a,-c)}[h_m(a, S_m); S_m \ge 0] \\ &\le d_m E_n^{(a,-c)}\left[\frac{1+a}{\sqrt{n}} + \frac{aS_m}{n} + \frac{S_m^3}{n^2}; S_m \ge 0\right] \\ &\quad + P_n^{(a,-c)}\big(S_m > 2n^{9/16}\big). \end{aligned} \tag{3.8}$$

Let $Z \sim N(0,1)$ and $\delta \ge 0$. Using Corollary 3.9, it is not hard to show that

$$\sup_{\delta\sqrt{n} \le c \le 2n^{9/16}} \sup_{0 \le a \le n^{1/8}} P_n^{(a,-c)}(S_m \ge 0) \to P\left\{Z > \frac{\delta}{\sigma_0}\sqrt{\frac{1-\varepsilon}{\varepsilon}}\right\}, \tag{3.9}$$

$$\begin{aligned} &\sup_{\delta\sqrt{n} \le c \le 2n^{9/16}} \sup_{0 \le a \le n^{1/8}} E_n^{(a,-c)}[S_m; S_m \ge 0]/\sqrt{n} \\ &\quad \to \sigma_0\sqrt{\varepsilon(1-\varepsilon)}\, E\left(Z - \frac{\delta}{\sigma_0}\sqrt{\frac{1-\varepsilon}{\varepsilon}}\right)^+, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\sup_{\delta\sqrt{n} \le c \le 2n^{9/16}} \sup_{0 \le a \le n^{1/8}} E_n^{(a,-c)}\big[S_m^3; S_m \ge 0\big]/n^{3/2} \\ &\quad \to \sigma_0^3\sqrt{\varepsilon^3(1-\varepsilon)^3}\, E\left[\left(Z - \frac{\delta}{\sigma_0}\sqrt{\frac{1-\varepsilon}{\varepsilon}}\right)^+\right]^3 \end{aligned} \tag{3.11}$$

and

$$\sup_{0 \le c \le 2n^{9/16}} \sup_{0 \le a \le n^{1/8}} P_n^{(a,-c)}\big(S_m > n^{9/16}\big) = o\big(1/\sqrt{n}\big) \tag{3.12}$$

as $n \to \infty$. With $\delta = 0$ this gives

$$\begin{aligned} &\sup_{0 \le c \le 2n^{9/16}} \sup_{0 \le a \le n^{1/8}} \frac{\sqrt{n}\, P_n^{(a,-c)}\{\tau > (1-\varepsilon)n\}}{1+a} \\ &\quad \le \left\{\frac{1}{2} + \sqrt{\frac{\varepsilon(1-\varepsilon)\sigma_0^2}{2\pi}} + 2\sqrt{\frac{\varepsilon^3(1-\varepsilon)^3\sigma_0^6}{2\pi}} + o(1)\right\} d_m + o(1) \end{aligned} \tag{3.13}$$

as $n \to \infty$. We will assume that ε has been chosen small enough that the right-hand side of this inequality is less than $(3/4 + o(1))d_m + o(1)$. Now

$$\begin{aligned} \frac{2cE_0^{(a)}|S_\tau - a|/n}{1/\sqrt{n} + ac/n + c^3/n^2} &\le 2 + \frac{2cE_0^{(a)}|S_\tau|}{\sqrt{n} + c^3/n} \\ &\le 2 + 2E_0^{(a)}|S_\tau|, \end{aligned}$$

so (3.7) and (3.13) give

$$d_n \le \frac{2}{\sigma_0^2} + 2\sup_{a\ge 0}\frac{E_0^{(a)}|S_\tau|}{\sigma_0^2} + \left(\frac{3}{4} + o(1)\right)d_m + o(1)$$

as $n \to \infty$. From this difference equation it is easy to verify that d_n remains bounded as $n \to \infty$.

The next step is to show that $P_n^{(a,c)}\{(1-\varepsilon)n < \tau < n\}$ and $P_n^{(a,-c)}\{\tau > (1-\varepsilon)n\}$ are small when $c > \delta\sqrt{n}$. Since d_n remains bounded and since the limits in (3.9)–(3.11) vanish as $\varepsilon \downarrow 0$ when $\delta > 0$, using (3.9)–(3.12) with $\delta > 0$ in (3.8), we have

$$\text{(3.14)}\quad \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\sup_{\delta\sqrt{n}\le c\le n^{9/16}}\sup_{0\le a\le n^{1/8}}\frac{\sqrt{n}\,P_n^{(a,-c)}\{\tau > (1-\varepsilon)n\}}{1+a} = 0.$$

Let $\bar{d} = \sup_{n\ge 1} d_n$. Using the bounds

$$|S_m| \le (1-\varepsilon)c + |S_m - (1-\varepsilon)c|$$

and

$$|S_m|^3 \le 8(1-\varepsilon)^3c^3 + 8|S_m - (1-\varepsilon)c|^3,$$

we have

$$\begin{aligned}
&P_n^{(a,c)}\{\tau > (1-\varepsilon)n, |S_m - (1-\varepsilon)c| > \delta\sqrt{n}/2\}\\
&\quad\le \bar{d}E_n^{(a,c)}\left\{\frac{1+a}{\sqrt{m}} + \frac{a|S_m|}{m} + \frac{|S_m|^3}{m^2}; |S_m - (1-\varepsilon)c| > \frac{\delta\sqrt{n}}{2}\right\}\\
&\qquad + P_n^{(a,c)}\{S_m > 2n^{9/16}\}\\
&\quad\le \bar{d}E_n^{(a,c)}\left\{\frac{1+a}{\sqrt{m}} + \frac{a|S_m - (1-\varepsilon)c|}{m} + \frac{ac(1-\varepsilon)}{m} + \frac{8(1-\varepsilon)^3c^3}{m^2}\right.\\
&\qquad\qquad\left. + \frac{8|S_m - (1-\varepsilon)c|^3}{m^2}; |S_m - (1-\varepsilon)c| > \frac{\delta\sqrt{n}}{2}\right\}\\
&\qquad + P_n^{(a,c)}\{S_m > 2n^{9/16}\},
\end{aligned}$$

so Corollary 3.9 gives

$$\text{(3.15)}\quad \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\sup_{0\le c\le n^{9/16}}\sup_{0\le a\le n^{1/8}}\frac{P_n^{(a,c)}\{\tau > (1-\varepsilon)n, |S_m - (1-\varepsilon)c| > \delta\sqrt{n}/2\}}{(1+a)/\sqrt{n} + ac/n + c^3/n^2} = 0.$$

If $\varepsilon < 1/4$ and $n \ge 2$, then $m > n/2$. If $\varepsilon < 1/4$, then for $c \ge \sqrt{n}\,\delta$ on $\{|S_m - (1-\varepsilon)c| < \delta\sqrt{n}/2\}$ we will have $|S_m| \le 2c$, which implies (for $n \ge 2$)

$$\frac{1+a}{\sqrt{m}} + \frac{a|S_m|}{m} + \frac{|S_m|^3}{m^2} \le 32\left[\frac{1+a}{\sqrt{n}} + \frac{ac}{n} + \frac{c^3}{n^2}\right].$$

Conditioning on S_m, for $\varepsilon < 1/4$ and n sufficiently large

$$\begin{aligned} P_n^{(a,c)}&\{(1-\varepsilon)n < \tau < n, |S_m - (1-\varepsilon)c| \le \delta\sqrt{n}/2\} \\ &\le 32\bar{d}\big[(1+a)/\sqrt{n} + ac/n + c^3/n^2\big] \\ &\quad \times P_n^{(a,c)}\{S_k \le 0 \text{ for some } (1-\varepsilon)n < k < n\}. \end{aligned}$$

Since $X_1, \ldots, X_n$ have the same joint distribution under $P_n^{(a,c)}$ and $P_n^{(-c,-a)}$ and are exchangeable, reversing time gives

$$\begin{aligned} P_n^{(a,c)}&\{S_k \le 0 \text{ for some } (1-\varepsilon)n < k < n\} \\ &= P_n^{(-c,-a)}\{-S_k \le 0 \text{ for some } 0 < k < \varepsilon n\} \\ &= P_n^{(-c,-a)}(\tau^+ < \varepsilon n). \end{aligned} \tag{3.16}$$

Since τ^+ is just τ with the random walk $\{S_n\}_{n\ge 0}$ changed to $\{-S_n\}_{n\ge 0}$ (both are hitting times for $\{0, -1, \ldots\}$) and since the random walk $\{-S_n\}_{n\ge 0}$ satisfies the same regularity conditions as $\{S_n\}_{n\ge 0}$, by Lemma 3.10,

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P_n^{(-c,-a)}(\tau^+ < \varepsilon n) = 0.$$

Hence

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\sup_{\delta\sqrt{n}\le c\le n^{9/16}}\sup_{0\le a\le n^{1/8}} \frac{P_n^{(a,c)}\{(1-\varepsilon)n < \tau < n, |S_m - (1-\varepsilon)c| \le \delta\sqrt{n}/2\}}{(1+a)/\sqrt{n} + ac/n + c^3/n^2} = 0.$$

Combining this with (3.15),

$$\lim_{\varepsilon\downarrow 0}\sup_{\delta\sqrt{n}\le c\le n^{9/16}}\sup_{0\le a\le n^{1/8}} \frac{P_n^{(a,c)}\{(1-\varepsilon)n < \tau < n\}}{(1+a)/\sqrt{n} + ac/n + c^3/n^2} = 0.$$

Using this equation and (3.14) in (3.7),

$$\begin{aligned} P_n^{(a,c)}(\tau \ge n) &= \big(2c/n\sigma_0^2\big)E_0^{(a)}|S_\tau - a| \\ &\quad + o\big\{(1+a)/\sqrt{n} + ac/n + c^3/n^2\big\} \end{aligned} \tag{3.17}$$

as $n \to \infty$, uniformly for $\delta\sqrt{n} \le c \le n^{9/16}$ and $0 \le a \le n^{1/8}$. Since δ is arbitrary, to finish the proof it is sufficient to show that $P_n^{(a,c)}(\tau > n) = o\{(1+a)/\sqrt{n}\}$ as $n \to \infty$, uniformly for $0 \le a \le n^{1/8}$ when $c = o(\sqrt{n})$ as $n \to \infty$. If we condition on S_m (with $m = \lfloor 1-\varepsilon \rfloor n$ still) and use (3.17),

$$\begin{aligned} P_n^{(a,c)}(\tau \ge m) &\le P_n^{(a,c)}\big(S_m \ge m^{9/16}\big) \\ &\quad + P_n^{(a,c)}\big(0 \le S_m < \delta\sqrt{m}\big)\frac{1 + a + a\delta + \delta^3}{\sqrt{m}}\bar{d} \\ &\quad + \frac{2}{m\sigma_0^2}E_n^{(a,c)}\big[S_m; S_m \ge \delta\sqrt{m}\big]E_0^{(a)}|S_\tau - a| \\ &\quad + o\left\{\frac{1+a}{\sqrt{m}} + E_n^{(a,c)}\left[\frac{aS_m}{m} + \frac{S_m^3}{m^2}; S_m \ge \delta\sqrt{m}\right]\right\} \end{aligned}$$

as $n \to \infty$. Using Corollary 3.9, if $c = o(\sqrt{n})$, then $P_n^{(a,c)}(0 \le S_m < \delta/\sqrt{m}) \to P(0 \le Z < \delta/(\sigma_0\sqrt{\varepsilon}))$,

$$E_n^{(a,c)}\big[S_m/\sqrt{m}\,;\, S_m \ge 0\big] \to \sigma_0\sqrt{\varepsilon(1-\varepsilon)}\,E\big[Z; Z \ge \delta/\big(\sigma_0\sqrt{\varepsilon}\big)\big]$$

and $E_n^{(a,c)}[S_m^3/m^{3/2}; S_m \ge 0]$ remains bounded as $n \to \infty$. So

$$\begin{aligned}\limsup_{n\to\infty} &\frac{\sqrt{n}\,P_n^{(a,c)}(\tau > n)}{1+a} \\ &\le \frac{(1+\delta+\delta^3)\bar{d}}{\sqrt{1-\varepsilon}} P\left\{0 \le Z \le \frac{\delta}{\big(\sigma_0\sqrt{\varepsilon}\big)}\right\} \\ &\quad + \frac{2}{\sigma_0}\sqrt{\frac{\varepsilon}{1-\varepsilon}}\, E\left[Z; Z \ge \frac{\delta}{\big(\sigma_0\sqrt{\varepsilon}\big)}\right] \sup_{a\ge 0}\frac{E_0^{(a)}|S_\tau - a|}{1+a}.\end{aligned}$$

Choosing δ and ε small, this expression can be made arbitrarily small. This completes the proof of Theorem 3.11. □

PROOF OF THEOREM 3.1. We now take $m = \lfloor n/2 \rfloor$. Using the time reversal argument that led to (3.16),

$$P_{n-m}^{(c,b)}(\tau > n-m) = P_{n-m}^{(-b,-c)}(\tau^+ > n-m).$$

Since τ^+ is the first time the random walk $\{-S_n\}_{n\ge 0}$ hits $\{0, -1, \dots\}$, this probability can be approximated using Theorem 3.11. This gives

$$P_{n-m}^{(-b,-c)}(\tau^+ > n-m) = \frac{2c}{(n-m)\sigma_0^2} E_0^{(-b)}|S_{\tau^+} + b| + o\left\{\frac{1+b}{\sqrt{n}} + \frac{bc}{n} + \frac{c^3}{n^2}\right\}$$

as $n \to \infty$, uniformly for $0 \le c \le m^{9/16}$ and $0 \le b \le n^{1/8}$. Using Theorem 3.11 to approximate $P_m^{(a,c)}(\tau > m)$ (taking advantage that a is fixed),

$$\begin{aligned} P_m^{(a,c)}&(\tau > m) P_{n-m}^{(c,b)}(\tau > n-m) \\ &= \left[\frac{2c}{m\sigma_0^2}E_0^{(a)}|S_\tau - a| + o\left\{\frac{1}{\sqrt{n}} + \frac{c^3}{n^2}\right\}\right] \\ &\quad \times \left[\frac{2c}{(n-m)\sigma_0^2}E_0^{(-b)}|S_{\tau^+} + b| + o\left\{\frac{1+b}{\sqrt{n}} + \frac{bc}{n} + \frac{c^3}{n^2}\right\}\right] \\ &= \frac{4c^2}{m(n-m)\sigma_0^4}E_0^{(a)}|S_\tau - a| E_0^{(-b)}|S_{\tau^+} + b| \\ &\quad + o\left\{\frac{1+b}{n} + \frac{c(1+b)}{n\sqrt{n}} + \frac{bc^2}{n^2} + \frac{c^3}{n^2\sqrt{n}} + \frac{c^4(1+b)}{n^3} + \frac{c^6}{n^4}\right\}\end{aligned}$$

as $n \to \infty$, uniformly for $0 \le c \le m^{9/16}$ and $0 \le b \le n^{1/8}$. Using (3.2) and

Corollary 3.9 (the sum over $c \geq m^{9/16}$ can be ignored by Corollary 3.7),

$$P_n^{(a,b)}(\tau > n) = \frac{4}{m(n-m)\sigma_0^4} E_n^{(a,b)}[S_m^2; S_m \geq 0]$$

$$\times E_0^{(a)}|S_\tau - a| E_0^{(-b)}|S_{\tau^+} + b| + o\left\{\frac{1+b}{n}\right\}$$

$$= \frac{2}{n\sigma_0^2} E_0^{(a)}|S_\tau - a| E_0^{(-b)}|S_{\tau^+} + b| + o\left\{\frac{1+b}{n}\right\}$$

as $n \to \infty$, proving the theorem. □

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REFERENCES

BARNDORFF-NIELSEN, O. and COX, D. R. (1979). Edgeworth and saddle-point approximations with statistical applications. *J. Roy. Statist. Soc. Ser. B* **41** 279–312.

BROWN, L. (1986). *Fundamentals of Statistical Exponential Families with Applications in Statistical Decision Theory*. IMS, Hayward, Calif.

DALEY, D. (1968). Quasi-stationary behavior of a left-continuous random walk. *Ann. Math. Statist.* **40** 532–539.

DANIELS, H. E. (1954). Saddlepoint approximations in statistics. *Ann. Math. Statist.* **25** 631–650.

DARROCH, J. and SENETA, E. (1965). On quasi-stationary distributions in absorbing discrete-time finite Markov chairs. *J. Appl. Probab.* **2** 88–100.

DURRETT, R. (1980). Conditioned limit theorems for random walks with negative drift. *Z. Wahrsch. Verw. Gebiete* **52** 277–287.

HOGAN, M. (1984). Comment on "Corrected diffusion approximations in certain random walk problems." *J. Appl. Probab.* **23** 89–96.

IGLEHART, D. L. (1974a). Random walks with negative drift conditioned to stay positive. *J. Appl. Probab.* **11** 742–751.

IGLEHART, D. L. (1974b). Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probab.* **2** 608–619.

IGLEHART, D. L. (1975). Conditioned limit theorems for random walks. In *Stochastic Processes and Related Topics. Proceedings of the Summer Research Institute on Statistical Inference for Stochastic Processes* (M. L. Puri, ed.) **1** 167–194. Academic, New York.

KAO, P. (1978). Limiting diffusion for random walks with drift conditioned to stay positive. *J. Appl. Probab.* **15** 280–291.

KEMENY, J., SNELL, J. and KNAPP, A. (1976). *Denumerable Markov Chains*. Springer, New York.

KLASS, M. (1983). On the maximum of a random walk with small negative drift. *Ann. Probab.* **11** 491–505.

LIGGETT, T. M. (1968). An invariance principle for conditioned sums of independent random variables. *J. Math. Mech.* **18** 559–570.

LORDEN, G. (1970). On the excess over the boundary. *Ann. Math. Statist.* **41** 520–527.

PITMAN, J. (1975). One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. in Appl. Probab.* **7** 511–526.

POLLETT, P. K. (1986). On the equivalence of μ-invariant measures for the minimal process and its q-matrix. *Stochastic Process Appl.* **22** 203–221.

SENETA, E. (1966). Quasi-stationary behavior in the random walk with continuous time. *Austral. J. Statist.* **8** 92–98.

SENETA, E. and VERE-JONES, D. (1966). On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.* **3** 403–434.

SIEGMUND, D. (1975). Error probabilities and average sample number of the sequential probability ratio test. *J. Roy. Statist. Soc. Ser. B* **37** 394–401.

SIEGMUND, D. (1979). Corrected diffusion approximations in certain random walk problems. *Adv. in Appl. Probab.* **11** 701–719.

SIEGMUND, D. (1985a). Corrected diffusion approximations and their applications. In *Proceedings of the Berkeley Conference in Honor of Jerzy Neyman and Jack Kiefer* (L. Le Cam and R. Olshen, eds.). Wadsworth, Belmont, Calif.

SIEGMUND, D. (1985b). *Sequential Analysis: Tests and Confidence Intervals.* Springer, New York.

TWEEDIE, R. L. (1974a). *R*-theory for Markov chains on a general state space. I: Solidarity properties and *r*-recurrent chains. *Ann. Probab.* **2** 840–864.

TWEEDIE, R. L. (1974b). *R*-theory for Markov chains on a general state space. II: *r*-subinvariant measures for *r*-transient chains. *Ann. Probab.* **2** 865–878.

TWEEDIE, R. L. (1974c). Quasi-stationary distributions for Markov chains on a general state space. *J. Appl. Probab.* **11** 726–741.

WOODROOFE, M. (1982). *Non Linear Renewal Theory in Sequential Analysis.* SIAM, Philadelphia.

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