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LIMIT THEOREMS FOR THE DISTRIBUTIONS OF THE SUMS OF A RANDOM NUMBER OF RANDOM VARIABLES[1]

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A necessary condition is given for the convergence of distributions of the sums of a random number of independent random variables. This is made on the basis of a theorem which gives sufficient conditions for the convergence of distributions of randomly stopped stochastic processes. The random indices are supposed to be independent of the sequence of summands.

0. Introduction. For every n, let $\xi_{n1}, \cdots, \xi_{nk}, \cdots$ be a sequence of independent random variables, and ν_n a random index *independent* of the sequence $\{\xi_{nk}\}$. Suppose that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_k P(|\xi_{nk}| > \varepsilon) = 0 ,$$

and that $P \lim_{n\to\infty} \nu_n = \infty$. Put

$$S_k^{(n)} = \xi_{n1} + \cdots + \xi_{nk} .$$

Following the classical work of Robbins [7] many authors investigated the limiting behavior of the distributions of the sum $S_{\nu_n}^{(n)}$. The aim of our investigations is to extend the theory of limit distributions of sums of independent random variables (see [4]) to the case, when the number of summands depends also on the chance. Our starting point is a result of Gnedenko and Fahim [3], which gives sufficient conditions for the existence of limit distribution of the sum $S_{\nu_n}^{(n)}$ in the case, when the summands $\xi_{n1}, \cdots, \xi_{nk}, \cdots$ are identically distributed $(n = 1, 2, \cdots)$. For the same case B. Freyer and the author have obtained necessary conditions under strong additional assumptions [10], while in [8] the author was able not only to get rid of these conditions but to obtain necessary as well as necessary and sufficient conditions in the case of identically distributed summands.

In the recent work we shall give both sufficient and necessary conditions for the convergence of the distribution of $S_{\nu_n}^{(n)}$ in the case of non-identically distributed summands. In Section 1 we give a sufficient condition, which is a consequence of a limit theorem on stochastic processes stopped at random. Section 2 contains necessary conditions for the convergence. The results of Section 2 will be proved in Section 4, while in Section 3 we examine some simple properties of Doob's centers.

We remark that applying the obtained results we proved necessary and sufficient conditions for the stability and the law of large numbers for the sum. These

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conditions were published in [9]. We also remark that our results remain valid if the summands are m-dimensional vectors.

1. The sufficient condition. First we shall prove a lemma. If the distribution function of the random variable X is $F(x)$, then denote by $l_F(q) = l_X(q)$ an arbitrary q-quantile of $F(x)$, and by $\underline{l}_F(q) = \underline{l}_X(q)$ ($\bar{l}_F(q) = \bar{l}_X(q)$) the greatest lower bound (least upper bound) of all q-quantiles of $F(x)$ ($0 \leqq q \leqq 1$).

LEMMA 1. *The distribution functions $F_n(x)$ converge weakly to the distribution function $F(x)$* iff *for every q*

$$(1.1) \qquad \underline{l}_F(q) \leqq \liminf_{n\to\infty} \underline{l}_{F_n}(q) \leqq \limsup_{n\to\infty} \bar{l}_{F_n}(q) \leqq \bar{l}_F(q) .$$

PROOF. *Necessity.* Suppose that the distribution functions $F_n(x)$ converge weakly to the distribution function $F(x)$, and for some q_0

$$\liminf_{n\to\infty} \underline{l}_{F_n}(q_0) < \underline{l}_F(q_0) .$$

Let x be a point of continuity of $F(x)$, such that

$$\liminf_{n\to\infty} \underline{l}_{F_n}(q_0) < x < \underline{l}_F(q_0) .$$

It follows from this inequality that for an infinite set of indices n

$$F_n(x) \geqq q_0 .$$

Therefore

$$F(x) = \lim_{n\to\infty} F_n(x) \geqq q_0 ,$$

whence

$$\underline{l}_F(q_0) \leqq x .$$

This inequality contradicts the choice of x.

Sufficiency. If the distribution functions $F_n(x)$ do not converge weakly to $F(x)$, then there exists a point x_0 of continuity of the function $F(x)$ such that for a suitable subsequence $\{n'\}$ of indices and $\delta > 0$

$$(1.2) \qquad F_{n'}(x_0) > F(x_0) + \delta \qquad \text{(say)} .$$

Set $\beta = F(x_0) + \delta$. It follows from (1.2) that for every n'

$$\bar{l}_{F_{n'}}(\beta) \leqq x_0 .$$

Now, since $F(x)$ is continuous at the point x_0, there exists an $a > 0$ such that for every $x_0 \leqq x \leqq x_0 + a$ we have

$$F(x) \leqq F(x_0) + \delta/2 .$$

Thus

$$\underline{l}_F(\beta) \geqq x_0 + a$$

and consequently,

$$\underline{l}_F(\beta) \geqq x_0 + a > x_0 \geqq \bar{l}_{F_{n'}}(\beta) \geqq \underline{l}_{F_{n'}}(\beta) ,$$

that contradicts (1.1).

The lemma is proved.

In the case of identically distributed summands Gnedenko and Fahim gave the following sufficient condition:

PROPOSITION 1 [3]. *Suppose that for every n* $\xi_{n1}, \cdots, \xi_{nk}, \cdots$ *are identically distributed. If there exists a sequence* $\{k_n\}_n$ *of positive integers* ($\lim_{n\to\infty} k_n = \infty$) *and distribution functions* $\Phi(x)$ *and* $A(x)$ *such that for* $n \to \infty$

$$\text{(A)} \qquad P(S_{k_n}^{(n)} < x) \to \Phi(x) ,$$
$$\text{(B)} \qquad P(\nu_n/k_n < x) \to A(x) ,$$

then

$$P(S_{\nu_n}^{(n)} < x) \to \Psi(x) .$$

The distribution function $\Psi(x)$ is determined by the characteristic function

$$\psi(t) = \int_{-\infty}^{+\infty} e^{ixt}\, d\Psi(x) = \int_0^\infty [\varphi(t)]^y\, dA(y) ,$$

where

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{ixt}\, d\Phi(x) \cdot$$

In view of Lemma 1, it follows from condition (B) that for almost every q (namely, for every q satisfying $\underline{l}_A(q) = \bar{l}_A(q)$) we have

$$\lim_{n\to\infty} \frac{l_n(q)}{k_n} = l(q) ,$$

where $l_n(q) = l_{\nu_n}(q)$ and $l(q) = l_A(q)$. Thus, taking into account condition (A), for almost every q

$$\lim_{n\to\infty} E \exp iuS_{l_n(q)}^{(n)} = [\varphi(u)]^{l(q)} . \tag{1.3}$$

Let us define the stochastic processes $\chi_n(t)$ in the interval [0, 1], as follows

$$\chi_n(t) = S_{l_n(t)}^{(n)} . \tag{1.4}$$

In view of (1.3) there exists a stochastic process $\{\chi(t) : t \in [0, 1]\}$ with independent stationary increments such that for almost every t

$$\chi_n(t) \to \chi(t)$$

in distribution, as $n \to \infty$.

Let us introduce the following notation: π always denotes a random variable, uniformly distributed in [0, 1] and independent of all stochastic processes, for which π is a random stopping time. It is easy to see that the distributions of the random variables $\chi_n(\pi)$ and $S_{\nu_n}^{(n)}$ are identical. Thus the assertion of Proposition 1 can be formulated so:

$$\chi_n(\pi) \to \chi(\pi)$$

in distribution, as $n \to \infty$. In this form, Proposition 1 is a particular case of the following Theorem 1. We remark that in the sequel measurability of stochastic processes will be understood as measurability with respect to the product σ-algebra of the σ-algebra of Borel sets of the domain of the stochastic process and of the σ-algebra of measurable subsets of the underlying probability space.

THEOREM 1. *Let* $\{\chi_n(t) : t \in [0, 1]\}$ *and* $\{\chi(t) : t \in [0, 1]\}$ *be measurable stochastic processes and* τ *be a random variable, independent of the processes* $\{\chi_n(t)\}$ *and* $\{\chi(t)\}$. *If for* μ*-almost every* t ($\mu(H) = P(\tau \in H)$)

$$\chi_n(t) \to \chi(t)$$

in distribution, as $n \to \infty$, *then*

$$\chi_n(\tau) \to \chi(\tau)$$

in distribution, as $n \to \infty$.

PROOF. The theorem follows from the limit relation

$$Ee^{iu\chi_n(\tau)} = \int_0^1 Ee^{iu\chi_n(t)}\mu(dt) \to \int_0^1 Ee^{iu\chi(t)}\mu(dt)$$

and the Lebesgue dominated convergence theorem.

The following Corollary of this theorem shows that the study of stochastic processes stopped at random may be useful in the limit theory of sums of a random number of random variables (cf. also [1] Section 17).

COROLLARY. *Suppose that for almost every* $q \in [0, 1]$ *there exists a distribution function* $\Phi_q(x)$ *such that*

$$\lim_{n\to\infty} P(S^{(n)}_{l_n(q)} < x) = \Phi_q(x) .$$

Then there exists a measurable stochastic process $\{\chi(t) : t \in [0, 1]\}$ *with independent increments such that for almost every* t

$$P(\chi(t) < x) = \Phi_t(x)$$

and

$$S^{(n)}_{\nu_n} \to \chi(\pi)$$

in distribution, as $n \to \infty$.

PROOF. It is sufficient to define the processes $\{\chi_n(t)\}$ according to (1.4) and then apply Theorem 1.

2. Necessary conditions. In [8] we proved that if the sums of a random number of identically distributed random variables have a limit distribution, then there exists a decomposition of the indices n for a finite or infinite number of subsequences such that the sufficient conditions for the existence of limit distribution (i.e. in case of identically distributed summands the conditions of Proposition 1) are satisfied for all of these subsequences. In case of non-identically distributed summands the situation is similar. In the formulation of our results we shall use the notion of Doob's center.

For an arbitrary random variable X there exists a unique real number $\Delta(X)$ such that

$$E \operatorname{arctg} (X - \Delta(X)) = 0 .$$

This number $\Delta(X)$ is called the *Doob's center* of X or briefly the center of X. If $\Delta(X) = 0$, then we say that the random variable X is centered. A stochastic process $\{\chi(t) : t \in T\}$ is centered, if for *all* $t \in T$ the random variable $\chi(t)$ is centered (cf. [8]).

THEOREM 2. *If the sums* $S^{(n)}_{\nu_n}$ *have a limit distribution, as* $n \to \infty$, *then there exist step functions* $m_n(t)$ $(n = 1, 2, \cdots; t \in [0, 1])$, *a decomposition of the indices for a finite or infinite number of subsequences* $\mathcal{N}_1, \cdots, \mathcal{N}_j$, *and centered, measurable stochastic processes* $\{\chi^{(i)}(t) : t \in [0, 1]\}$ *with independent increments* $(1 \leqq i \leqq j)$, *such that for almost every* t

$$S^{(n)}_{l_n(t)} - m_n(t) \to \chi^{(i)}(t)$$

in distribution, and also

$$S^{(n)}_{\nu_n} - m_n(A_n(\nu_n)) \to \chi^{(i)}(\pi)$$

in distribution, as $n \to \infty$, $n \in \mathcal{N}_i$. *Here* $A_n(x) = P(\nu_n < x)$.

We remark that the assertion of the theorem can not be improved. In [8] we gave an example, where j is necessarily greater than 1. It would be preferable to get rid of the functions $m_n(t)$. The following example shows, however, that in general this is impossible. Let

$$P(\nu_n = i) = \frac{1}{2^{4n}} \qquad \text{if} \quad 1 \leqq i \leqq 2^{4n}$$

and

$$\begin{aligned}
\xi_{nk} &= 0\,, && \text{if} \quad 1 \leqq k \leqq 2^{3n} - 2^{2n} \\
\xi_{nk} &= 0\,, && \text{if} \quad j2^{3n} + 2^{2n} < k \leqq (j+1)2^{3n} - 2^{2n} \\
& && \qquad\qquad (1 \leqq j < 2^n - 1) \\
\xi_{nk} &= 0\,, && \text{if} \quad (2^n - 1)2^{3n} + 2^{2n} < k \leqq 2^{4n} \\
\xi_{nk} &= 2^{-(2n+1)}\,, && \text{if} \quad (2j+1)2^{3n} - 2^{2n} < k \leqq (2j+1)2^{3n} + 2^{2n} \\
& && \qquad\qquad (0 \leqq j \leqq 2^{n-1} - 1) \\
\xi_{nk} &= -2^{-(2n+1)}\,, && \text{if} \quad 2j2^{3n} - 2^{2n} < k \leqq 2j2^{3n} + 2^{2n} \\
& && \qquad\qquad (1 \leqq j \leqq 2^{n-1} - 1)\,.
\end{aligned}$$

In this case the functions $S^{(n)}_{l_n(t)}$ are not random and for almost every $t \in [0, 1]$ and large n the value of $m_n(t)$ equals the value of the following function $r_n(t)$ of Rademacher type

$$\begin{aligned}
r_n(t) &= 0 \qquad && \frac{2j}{2^n} < t < \frac{2j+1}{2^n} \\
&= 1 && \frac{2j+1}{2^n} < t < \frac{2j+2}{2^n}\,.
\end{aligned}$$

Now from the sequence $r_n(t)$ it is impossible to choose a subsequence, that is almost everywhere convergent. Therefore, in the assertion of Theorem 2 the functions $m_n(t)$ play an important role; we will see, however that under additional assumptions it is possible to get rid of them.

COROLLARY 1. *Suppose that the random variables* ξ_{nk} *are symmetric for every* n *and* k *and the sums* $S^{(n)}_{\nu_n}$ *have a limit distribution, as* $n \to \infty$. *Then there exist a* *decomposition of indices for a finite or infinite number of subsequences* $\mathscr{N}_1, \cdots, \mathscr{N}_j$, *and centered, measurable stochastic processes* $\{\chi^{(i)}(t): t \in [0, 1]\}$ *with independent increments* $(1 \leqq i \leqq j)$, *such that for almost every* t

$$S^{(n)}_{l_n(t)} \to \chi^{(i)}(t) \tag{2.1}$$

in distribution, as $n \to \infty$, $n \in \mathscr{N}_i$, *and also*

$$S^{(n)}_{\nu_n} \to \chi^{(i)}(\pi) \tag{2.2}$$

in distribution, as $n \to \infty$ (i *is arbitrary*, $1 \leqq i \leqq j$).

COROLLARY 2. *Suppose that the random variables* ξ_{nk} *are nonnegative for every* n *and* k *and that the sums* $S^{(n)}_{\nu_n}$ *have a limit distribution, as* $n \to \infty$. *Then there exist a decomposition of indices for a finite or infinite number of subsequences* $\mathscr{N}_1, \cdots, \mathscr{N}_j$ *and measurable stochastic processes* $\{\chi^{(i)}(t): t \in [0, 1]\}$ *with independent increments* $(1 \leqq i \leqq j)$ *such that* (2.1) *and* (2.2) *are satisfied. The processes* $\{\chi^{(i)}(t): t \in [0, 1]\}$ *are of the form*

$$\chi^{(i)}(t) = \hat{\chi}^{(i)}(t) + M_i(t)$$

where $\{\hat{\chi}^{(i)}(t): t \in [0, 1]\}$ *is a centered process, and* $M_i(t)$ *is a non-decreasing function* $(1 \leqq i \leqq j)$.

What concerns the class of the limit distributions, Mogyoródi [6] remarked that an arbitrary distribution can be the limit distribution of the sums $S^{(n)}_{\nu_n}$. And what is more, this is possible even in the particular case, when all the summands are nonrandom, i.e. they are constant with probability 1. In fact, if $F(x)$, is an arbitrary distribution function, then it is easy to define random variables ν_n and constants ξ_{nk} $(n, k = 1, 2, \cdots)$ in such a way that

$$\lim_{n\to\infty} \sup_k |\xi_{nk}| = 0$$

and for almost every $t \in [0, 1)$

$$S^{(n)}_{l_n(t)} \to F^{-1}(t) \ .$$

It is obvious that in this case the limit distribution of $S^{(n)}_{\nu_n}$ equals $F(x)$. So we can state that the richness of limit distributions is caused by the circumstance that the summands are not centered (not necessarily in Doob's sense). In fact, by the aid of suitable centering it is possible to make the class of limit distributions narrower. For example, if the summands are centered in such a way that the processes $\{\chi_n(t) = S^{(n)}_{l_n(t)}: t \in [0, 1)\}$ are centered in Doob's sense, (this will be done in the proof of Theorem 2), then the limit processes $\{\chi^{(i)}(t): t \in [0, 1)\}$ are also centered. In this case the random variables $\chi^{(i)}(\pi)$ are centered as well; however, not every centered distribution can occur as the distribution of $\chi^{(i)}(\pi)$, that can be seen easily in the example, when the distribution is concentrated to the points -1 and 1 with masses $\frac{1}{2} - \frac{1}{2}$.

3. Some simple properties of Doob's centers.

LEMMA 2. *Let* X *be an arbitrary random variable. If for some* $\alpha (0 \leqq \alpha \leqq \frac{1}{2})$

there exist a and b such that

$$P(a \leqq X \leqq b) \geqq 1 - \alpha$$

and

$$\varepsilon \geqq \operatorname{tg}\left(\frac{\pi}{2} \cdot \frac{\alpha}{1 - \alpha}\right) \tag{3.1}$$

then

$$\Delta(X) \in [a - \varepsilon, b + \varepsilon] .$$

PROOF. Suppose that the assertion of the lemma is not valid, and

$$\Delta(X) > b + \varepsilon \qquad \text{(say)} . \tag{3.2}$$

Then, denoting the distribution function of X by $F(x)$, we have

$$0 = \int_I \operatorname{arctg}(u - \Delta(X))\, dF(u) + \int_{\bar{I}} \operatorname{arctg}(u - \Delta(X))\, dF(u) ,$$

where $I = [a, b]$. This equality and (3.2) imply that

$$0 < - \operatorname{arctg} \varepsilon \cdot (1 - \alpha) + \frac{\pi}{2} \cdot \alpha ,$$

that contradicts (3.1).

LEMMA 3. *If the sequence $\{X_n\}_n$ of random variables is weakly compact, then the sequence $\{\Delta(X_n)\}_n$ is bounded.*

PROOF. For a fixed $\alpha (0 < \alpha < \frac{1}{2})$ there exists a u_α such that for every n

$$P(|X_n| > u_\alpha) < \alpha .$$

Choose ε in accordance with (3.1). Then, on the basis of Lemma 2, for every n

$$|\Delta(X_n)| \leqq u_\alpha + \varepsilon .$$

LEMMA 4. *If the distributions of the random variables X_n converge weakly to the distribution of the random variable X, as $n \to \infty$, then*

$$\lim_{n\to\infty} \Delta(X_n) = \Delta(X) .$$

PROOF. It is obvious from the previous lemma that the sequence $\{\Delta(X_n)\}_n$ is bounded. Let us choose from this sequence a convergent subsequence

$$\lim_{j\to\infty} \Delta(X_{n_j}) = c .$$

This, and the equality

$$E \operatorname{arctg}(X_{n_j} - \Delta(X_{n_j})) = 0$$

given that

$$E \operatorname{arctg}(X - c) = 0$$

whence $c = \Delta(X)$.

LEMMA 5. *Let, for every n, the random variables X_n and Y_n be independent. Suppose that*

(a) $\Delta(X_n) = \Delta(X_n + Y_n) = 0 \quad (n = 1, 2, \cdots)$;

(b) *for some distribution function* $F(x)$

$$F_n(u) = P(X_n < u) \to F(u)$$

as $n \to \infty$;

(c) *for suitable constants* a_n

$$P \lim_{n\to\infty} (Y_n - a_n) = 0 .$$

Then

$$P \lim_{n\to\infty} Y_n = 0 .$$

PROOF. It is sufficient to prove that

$$\lim_{n\to\infty} a_n = 0 .$$

If this equality is not true, then, without the restriction of generality, we may assume that for every n

$$a_n \geqq c > 0 .$$

Denote

$$G_n(u) = P(Y_n < u) , \qquad \varepsilon(u) = 0 \quad u \leqq 0$$
$$= 1 \quad u > 0 .$$

Then

$$\begin{aligned} 0 &= \iint \operatorname{arctg} (u + v)\, dF_n(u)\, dG(v) \\ &= \iint \operatorname{arctg} (u + v + a_n)\, dF_n(u)\, dG_n(v + a_n) \\ &\geqq \iint \operatorname{arctg} (u + v + c)\, dF_n(u)\, dG_n(v + a_n) . \end{aligned}$$

Passing to the limit as $n \to \infty$, and using the conditions of the lemma we get

$$\begin{aligned} 0 &\geqq \iint \operatorname{arctg} (u + v + c)\, dF(u)\, d\varepsilon(v) \\ &= \int \operatorname{arctg} (u + c)\, dF(u) . \end{aligned}$$

Now

$$\int \operatorname{arctg} (u + c)\, dF(u) > \int \operatorname{arctg} u\, dF(u) = 0$$

contradicting the previous inequality.

4. Proof of the results of Section 2. In this paragraph $l_{\nu_n}(t)$ will be denoted by $l_n(t)$ (see Section 1). Concerning the properties of processes with independent increments, we refer to monographs [5] and [2].

We shall work with the method of symmetrization. Namely, we shall use the following one of the inequalities of symmetrization: if X and Y are independent, symmetrically distributed random variables, then for every $t > 0$

$$P(|X + Y| > t) \geqq \tfrac{1}{2} P(|X| > t) . \tag{4.1}$$

Moreover, we shall need the following two simple remarks:

REMARK 1. If for every n, X_n and Y_n are independent random variables, and

$$P \lim_{n\to\infty} (X_n + Y_n) = 0$$

then there exists a sequence of constants a_n such that

$$P \lim_{n\to\infty} (X_n - a_n) = 0 .$$

REMARK 2. If for every n, X_n and Y_n are independent random variables, and the sequence $\{X_n + Y_n\}_n$ of random variables is weakly compact, then there exists a sequence of constants a_n such that the sequence $\{X_n - a_n\}_n$ is also weakly compact.

PROOF OF THEOREM 2. Let for every n $\{\eta_{nk}\}_k$ be a sequence of independent random variables such that it is independent of the sequence $\{\xi_{nk}\}_k$ and of the random variable ν_n, and that

$$P(\xi_{nk} < x) = P(\eta_{nk} < x) \qquad (n, k = 1, 2, \cdots) .$$

Put

$$\zeta_{nk} = \xi_{nk} - \eta_{nk}$$
$$Z_k^{(n)} = \zeta_{n1} + \cdots + \zeta_{nk} . \quad .$$

From the convergence of the distributions of $S_{\nu_n}^{(n)}$ it follows that the sequence $\{Z_{\nu_n}^{(n)}\}_n$ is weakly compact. We assert that the sequence $\{Z_{l_n(t)}^{(n)}\}_n$ is also weakly compact. Suppose, on the contrary, that

$$\lim_{x\to\infty} \sup_n P(|Z_{l_n(t)}^{(n)}| > x) > 0 .$$

By (4.1) we have

$$P(|Z_{\nu_n}^{(n)}| > x) \geqq \sum_{k \geqq l_n(t)} P(|Z_k^{(n)}| > x) P(\nu_n = k)$$
$$\geqq (1-t)\tfrac{1}{2} P(|Z_{l_n(t)}^{(n)}| > x) ;$$

consequently

$$\lim_{x\to\infty} \sup_n P(|Z_{\nu_n}^{(n)}| > x) \geqq \tfrac{1}{2}(1-t) \lim_{x\to\infty} \sup_n P(|Z_{l_n(t)}^{(n)}| > x) > 0 ;$$

that contradicts the weak compactness of the sequence $\{Z_{\nu_n}^{(n)}\}_n$. Since $\{Z_{l_n(t)}^{(n)}\}_n$ is weakly compact, in consequence of Remark 2 for every $t \in [0, 1)$ there exists a sequence $\{m_n(t)\}_n$ of constants such that the sequence $\{S_{l_n(t)}^{(n)} - m_n(t)\}_n$ is weakly compact. Now it follows from Lemma 4 that for every n and t the constants $m_n(t)$ can be chosen to be equal to $\Delta(S_{l_n(t)}^{(n)})$.

Define for every n the stochastic process $\{\tilde{\chi}_n(t) : t \in [0, 1)\}$ as follows

$$\tilde{\chi}_n(t) = S_{l_n(t)}^{(n)} - m_n(t) . \tag{4.2}$$

Then for every n and t

$$\Delta(\tilde{\chi}_n(t)) = 0 . \tag{4.3}$$

Moreover, the random variables

$$\tilde{\chi}_n(\pi) \qquad \text{and} \qquad S_{\nu_n}^{(n)} - m_n(A_n(\nu_n)) \tag{4.4}$$

are identically distributed.

By the usual diagonal method of Cantor we can choose a subsequence $\mathscr{N}_1$ of natural numbers, such that $1 \in \mathscr{N}_1$ and for every $t_1 < t_2$; $t_1, t_2 \in \Theta$ (Θ is the set of dyadically rational numbers of $[0, 1)$)

$$P(\tilde{\chi}_n(t_2) - \tilde{\chi}_n(t_1) < x) \to \Phi_{t_1, t_2}(x) , \tag{4.5}$$

as $n \to \infty$, $n \in \mathcal{N}_1$. Here $\Phi_{t_1,t_2}(x)$ is an infinitely divisible distribution function. Let $\{\vartheta(t) : t \in \Theta\}$ be a stochastic process with independent increments such that

$$\vartheta(0) = 0$$

and for $t_1 < t_2$; $t_1, t_2 \in \Theta$

$$P(\vartheta(t_2) - \vartheta(t_1) < x) = \Phi_{t_1,t_2}(x) . \tag{4.6}$$

There exists a stochastic process $\{\chi^{(1)}(t) : t \in [0, 1)\}$ with independent increments such that $\chi^{(1)}(0) = 0$ and for every $t_1 < t_2$; $t_1, t_2 \in \Theta$

$$P(\chi^{(1)}(t_2) - \chi^{(1)}(t_1) < x) = P(\vartheta(t_2) - \vartheta(t_1) < x) . \tag{4.7}$$

We assert that for every $t \in [0, 1)$

$$\tilde{\chi}_n(t) \to \chi^{(1)}(t) \tag{4.8}$$

in distribution, as $n \to \infty$, $n \in \mathcal{N}_1$. From (4.3) and Lemma 4 it follows that for $t \in [0, 1)$

$$\Delta(\chi^{(1)}(t)) = 0 . \tag{4.9}$$

Therefore the process $\{\chi^{(1)}(t) : t \in [0, 1)\}$ is stochastically continuous, except for at most a countable number of points [5]. Let t be a point of stochastic continuity of the process $\{\chi^{(1)}(t) : t \in [0, 1)\}$. We prove that in this point (4.8) holds.

Denote the distribution function of the random variable X by F_X, and the Lévy-metric, defined in the space of distribution functions, by $L(F, G)$. Let us choose for every $r = 1, 2, \cdots$ two points $t^1(r)$, $t^2(r)$; $t^i(r) \in \Theta$, $i = 1, 2$, $t^1(r) \leqq t \leqq t^2(r)$ in such a way that

$$L(F_{\chi^{(1)}(t)}, F_{\chi^{(1)}(t^i(r))}) < r^{-1} \qquad (i = 1, 2) .$$

It follows, however, from (4.5), (4.6) and (4.7) that for sufficiently large values of $n \in \mathcal{N}_1$ $(n \geqq n_r)$

$$L(F_{\chi^{(1)}(t^i(r))}, F_{\tilde{\chi}_n(t^i(r))}) < r^{-1} \qquad (i = 1, 2) .$$

We can suppose that $n_r \leqq n_{r+1}$ and $\lim_{r\to\infty} n_r = \infty$. For $n_r \leqq n < n_{r+1}$, $n \in \mathcal{N}_1$ put $t_n{}^i = t^i(r)$. From the previous inequalities we get that

$$\lim_{n\to\infty, n\in\mathcal{N}_1} L(F_{\tilde{\chi}_n(t_n{}^i)}, F_{\chi^{(1)}(t)}) = 0 \qquad (i = 1, 2) .$$

Since in the decomposition

$$\tilde{\chi}_n(t_n{}^2) = \tilde{\chi}_n(t_n{}^1) + [\tilde{\chi}_n(t_n{}^2) - \tilde{\chi}_n(t_n{}^1)]$$

the summands on the right-hand side are independent, the last equality yields to

$$P \lim_{n\to\infty, n\in\mathcal{N}_1} (\tilde{\chi}_n(t_n{}^2) - \tilde{\chi}_n(t_n{}^1)) = 0 .$$

Thus, on the basis of Remark 1, there exist constants a_n $(n \in \mathcal{N}_1)$ such that

$$P \lim_{n\to\infty, n\in\mathcal{N}_1} (\tilde{\chi}_n(t) - \tilde{\chi}_n(t_n{}^1) - a_n) = 0 .$$

Putting in Lemma 5

$$X_n = \tilde{\chi}_n(t_n{}^1) \qquad \text{and} \qquad Y_n = \tilde{\chi}_n(t) - \tilde{\chi}_n(t_n{}^1) \qquad (n \in \mathcal{N}_1)$$

the conditions of the lemma are satisfied, and thus

$$\tilde{\chi}_n(t) \to \chi^{(1)}(t)$$

in distribution, as $n \to \infty$, $n \in \mathcal{N}_1$. This convergence, together with (4.2), (4.4), (4.9) and Theorem 1, shows that for the subsequence $\mathcal{N}_1$ the assertion of the theorem is valid.

If the subsequences $\mathcal{N}_1, \cdots, \mathcal{N}_{i-1}$ have already been determined, and the set

$$\{1, 2, \cdots\} - (\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_{i-1}) \tag{4.10}$$

is finite, then we can add it to any of the sets $\mathcal{N}_1, \cdots, \mathcal{N}_{i-1}$, and the assertion of the theorem is proved. If the set (4.10) is infinite, then we can define a subsequence $\mathcal{N}_i$ of the sequence (4.10) analogously as $\mathcal{N}_1$ has been defined. We can also assume that $i \in \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_i$. In this way we get, at last, the desired decomposition $\mathcal{N}_1, \cdots, \mathcal{N}_j$. The theorem is proved.

From the proved theorem Corollary 1 follows immediately, while for the proof of Corollary 2 it is sufficient to remark that the functions $m_n(t)$ defined in the proof of Theorem 2, are not decreasing. In fact, if X and Y are arbitrary random variables and $Y \geqq 0$, then

$$\Delta(X) \leqq \Delta(X + Y) .$$

Note. In this paper we supposed that the summands satisfy the condition of asymptotic negligibility. Actually, this condition is not necessary, thus our results, and the limit theory for sums without the condition of asymptotic negligibility, worked out by V. M. Zolotariov, give conditions for the convergence of the distributions of sums of a random number of independent random variables, not necessarily satisfying the condition of asymptotic negligibility.

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