*Limit Theorems for the Split Times of Branching Processes**

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0. Introduction. Consider a continuous time one dimensional strong Markov branching process $\{X(t); t \geqq 0\}$ with the non-negative integers as state space defined on a probability space $(\Omega, \mathfrak{F}, P)$ and standardized to have right continuous sample paths. Let the associated infinitesimal generating function be $u(z) = a[h(z) - z]$ where

$$h(z) = \sum_{i=0}^{\infty} p_i z^i, \qquad p_i \geqq 0, i = 0, 1, \cdots, \sum_{i=0}^{\infty} p_i = 1$$

and $0 < a < \infty$. As usual, we assume henceforth that $h'(1) < \infty$.

It is very suggestive to regard $X(t)$ as the total number of particles at time t in a system where we start with $X(0)$ particles at $t = 0$, each particle lives an exponential length of time with mean a^{-1} and on death creates (or splits into) a random number of new particles whose generating function is $h(z)$, and all particles behave independently of each other and identically. The value $h'(1)$ is then the expected number at each split. A full discussion of definitions and elementary properties of continuous time Markov branching processes is given in [5, Chap. 5], see also [6, Chap. 11].

We will assume henceforth unless stated explicitly to the contrary that $p_0 = 0$ so that extinction of the population is impossible. Because of the Markov property we assume without loss of generality that $p_1 = 0$ as this just amounts to ignoring the case where an individual replicates himself at a split.

It is well known that $\{X(t)e^{-\lambda t}; t \geqq 0\}$ $(\lambda = u'(1))$ is a non-negative martingale with respect to the family of σ fields $\mathfrak{F}(t) = \sigma\{X(s, \omega); s \leqq t\}$ and hence

$$(1) \qquad \lim_{t \to \infty} X(t, \omega)e^{-\lambda t} = W(\omega) \qquad \text{exists with probability one (w.p.1)}$$

and by Fatou's lemma (E denotes the expectation operation)

$$EW(\omega) \leqq \lim_{t \to \infty} E(X(t, \omega)e^{-\lambda t}) = X(0).$$

(Frequently, as customary, we suppress the ω variable.)

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In order to formulate the objectives of this paper it is necessary to introduce the following notation and terminology. Let $N(t, \omega)$, $t \geqq 0$ denote for each sample point $\omega \in \Omega$ the number of discontinuities of $X(s, \omega)$ for $s \leqq t$ (or what is the same, since $p_1 = 0$, the number of splits up to time t). Let $\tau_n(\omega)$ denote the time of the n^{th} discontinuity of $X(t, \omega)$, *i.e.*, the n^{th} split time, for $n \geqq 1$, and set for convenience $\tau_0 = 0$. Then it is easy to see that the random variables τ_n, $n = 0, 1, \cdots$ form a sequence of increasing stopping times adapted to the family of σ-fields $\mathfrak{F}(t)$. Furthermore, let

$$
\begin{aligned}
T_i &= \tau_i - \tau_{i-1} \\
\xi_i &= X(\tau_i + 0) - X(\tau_i - 0) \qquad\qquad i \geqq 1 \\
S_i &= X(\tau_i) = X(0) + \xi_1 + \xi_2 + \cdots + \xi_i \\
Y_n &= \sum_{i=1}^{n} \left(T_i - \frac{1}{aS_{i-1}}\right)
\end{aligned}
\tag{2}
$$

and $\mathfrak{F}_n = \mathfrak{F}_{\tau_n}$, the σ-field associated with the stoping time τ_n (see [4]). Note that ξ_i, $i = 1, 2, \cdots$ are independently distributed non-negative valued random variables possessing the common probability generating function $f(z) = h(z)/z$. The values of ξ_i can be interpreted as the net contribution from each split.

This paper investigates the growth properties of the increasing process $\{N(t, \omega), t \geqq 0\}$ and the family of successive splitting times $\{\tau_n\}_{n=0}^{\infty}$. The motivation for a study of the sequence of splitting times τ_n comes from the fact that many processes like Friedman's and Pólya's familiar urn schemes are recovered by examining a multitype continuous time Markov branching process at precisely the time points τ_n. We refer to [1] for further details. We will now describe and interpret a number of the theorems. Detailed presentation of the proofs and discussion of these and other results are relegated to the later sections.

Owing to the assumptions $p_0 = 0$ and $h'(1) < \infty$ it is easily proved that

$$
\Pr\{\lim_{n\to\infty} \tau_n = \infty\} = 1. \tag{3}
$$

An application of the strong law for sums of independent identically distributed random variables in conjunction with another martingale argument will establish

$$
\frac{\tau_n}{\log n} \to \frac{1}{\lambda} \qquad \text{w.p.1} \tag{4}
$$

(see Corollary 1 in Section 2).

Next combining appropriately the fundamental convergence property (1) and the strong law of large numbers leads to the result

$$
\lim_{n\to\infty} \mu n e^{-\lambda \tau_n} = W \qquad \text{w.p.1} \tag{5}
$$

where $\lambda = u'(1)$, $\mu = \lambda/a = h'(1) - 1$ (see Proposition 1 of section 2).

If $W > 0$ with positive probability then the following refinement of (5) is available. We have

Theorem 3. *The random variable* $\tau_n - (\log n)/\lambda$ *converges* w.p.1 *to a finite valued random variable if and only if*

$$\sum_{k=1}^{\infty} p_k k \log k < \infty. \tag{6}$$

The above theorem is intimately related to a theorem recently obtained by Kesten and Stigum [7] for a discrete time multidimensional Galton-Watson process (see also Corollary 3 below). Our approach, however, is different from theirs and works via a study of the convergence nature of the series $\sum_{i=1}^{\infty} (1/S_i)$, properly centered where S_i is defined in (2).

The following more general proposition concerning reciprocals of sums of non-negative i.i.d. (independent, identically distributed) random variables emerges from these considerations.

Theorem 1. *Let* $\xi_1, \xi_2, \xi_3, \cdots$ *be i.i.d. non-negative (not necessarily integer valued) random variables. Define* $S_n = c + \xi_1 + \xi_2 + \cdots + \xi_n$ *where c is a non-negative constant and assume that* $0 < E(\xi_i) = \mu \leqq \infty$. *Then*

(i) *The family of random variables* $\{(n/S_n)^r, n = 1, 2, \cdots,\}$ *are equi-integrable for each* $r > 0$, *provided one of the following conditions prevail:*

$$\text{(a)} \quad c > 0, \qquad \text{(b)} \quad E\left(\frac{1}{\xi_i}\right) < \infty.$$

(ii) *Under the conditions of* (i) *for each* $r > 0$

$$\lim_{n\to\infty} E\left[\left(\frac{n}{S_n}\right)^r\right] = \frac{1}{\mu^r}.$$

(iii) $\sum_{i=1}^{n} ((1/S_i) - (1/\mu i))$ *converges* w.p.1 *iff* $E(\xi\,|\log \xi|) < \infty$, *provided* $\mu < \infty$ *and either* $c > 0$ *or* $E(1/\xi_i) < \infty$.

The result stated in part (iii) is sharp and perhaps the most striking assertion of the theorem. Actually, in the necessary part of (iii) it is enough to require the convergence of $\sum_{i=1}^{n} ((1/S_i) - (1/i\mu))$ on a set of positive probability. We will decisively use the conclusions of parts (ii) and (iii) of Theorem 1 in determining the growth behavior of the τ_n process.

Consider next the counting process $N(t)$ associated with the τ_i's *viz.*, $N(t) = i$, iff $\tau_i \leqq t < \tau_{i+1}$. We can verify for a general one dimensional age dependent branching process $\{X(t), t \geqq 0\}$ with $p_0 = 0$ and $G(0+) = 0$ where $G(x)$ is the lifetime distribution of each individual, the following identity.

Assume

$$\mu = \left(\sum_{j=0}^{\infty} jp_j\right) - 1 < \infty, \qquad X(0) = 1.$$

Then

(7) $1 + \mu EN(t) = EX(t)$ (The formal proof appears in Section 3).

The same techniques producing (7) can be refined to yield formulas connecting higher moments and relations among the random variables $N(t)$ and $X(t)$. Specifically, by exploiting the obvious representation

$$X(t) = X(0) + \xi_1 + \xi_2 + \cdots + \xi_{N(t)}$$

we will establish the following theorem.

Theorem 4. *Consider a one-dimensional age dependent branching process with lifetime distribution $G(x)(G(0+) = 0)$ possessing a finite second moment. Assume that at each split the number of progeny produced is a random variable whose probability law has a generating function $h(z) = \sum_{j=0}^{\infty} p_j z^j$ with $p_0 = p_1 = 0$, $\mu = h'(1)$ and $h''(1) < \infty$. We further postulate that $G(x)$ possesses a density $g(x)$ satisfying $\int_0^\infty [g(x)]^r \, dx < \infty$ for some $r > 0$. Then*

(i) $$\lim_{t\to\infty} N(t)\mu e^{-\lambda t} = W \quad \text{w.p.1}$$

where

$$W = \lim_{t\to\infty} X(t)e^{-\lambda t} \quad \text{w.p.1}$$

(ii) *Moreover,*

$$\frac{X(t) - N(t)\mu}{(N(t)\sigma^2)^{1/2}} \xrightarrow{\text{law}} N(0, 1) \quad as \quad t \to \infty .$$

(8) (iii) Pr {ω: set of limit points of

$$\zeta(t, \omega) = \frac{X(t) - N(t)\mu}{(2N(t)\sigma^2 \log \log N(t))^{1/2}}$$

as t varies in $[0, \infty)$ coincides with $[-1, +1]\} = 1$ *where*

$$\sigma^2 = \sum_{j=2}^{\infty} j^2 p_j - \mu^2 .$$

The validity of the limit relation $W = \lim_{t\to\infty} X(t)e^{-\lambda t}$ in the case of a one dimensional age dependent branching process under the hypothesis stated is proved in [5, page 147].

We next describe the analogs of Theorems 1 and 3 for the multitype branching process. Consider a multidimensional Markov branching process $\{\mathbf{X}(t); t \geqq 0\}$ $(\mathbf{X}(t) = (X_1(t), \cdots, X_p(t)))$, where $X_i(t)$ denotes the population size of particles of the i^{th} type at time t and $p \geqq 2$. Let the set of infinitesimal generating functions of the process be

(9) $$u_i(\mathbf{s}) = a_i[h_i(\mathbf{s}) - s_i], \qquad i = 1, 2, \cdots, p.$$

where

$$\mathbf{s} = (s_1, s_2, \cdots, s_p) \qquad 0 < a_i < \infty \quad \text{for} \quad i = 1, 2, \cdots, p;$$

$h_i(\mathbf{s})$ is a probability generating function.

As before it is suggestive to associate with $\{\mathbf{X}(t); t \geqq 0\}$ a system consisting of p types of particles where a type i particle lives an exponential length of time with mean a_i^{-1} and on death creates particles of all types with a joint generating function $h_i(\mathbf{s})$. All particles determine family histories independently of each other and particles of the same type are governed by the same laws. We postulate

$$\text{(10a)} \qquad \left.\frac{\partial^2 u_k(\mathbf{s})}{\partial s_j \partial s_i}\right|_{\mathbf{s}=1} < \infty \quad \text{for all} \quad i, j \quad \text{and} \quad k = 1, 2, \cdots, p;$$

and that the extinction probabilities are zero for any nontrivial initial population makeup. Furthermore assume that there exists a $t_0 > 0$ such that each element of the matrix $\exp(t_0\mathbf{M})$ is positive where $\mathbf{M} = ||m_{ij}||_{p \times p}$ is the infinitesimal expectation matrix, *i.e.*,

$$\text{(10b)} \qquad m_{ij} = \left.\frac{\partial u_i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=1}$$

We refer the reader to [5, Chaps. 2 and 5] for additional discussions concerning the formulation of multi-type branching processes.

Let $N_i(n)$ denote the number of splits, among the first n, which are of type i. Subject to the stipulations imposed above the multi-dimensional version of Theorems 1 and 3 achieves the following expression.

Theorem 5. (i) *For each* i

$$\text{(11a)} \qquad \lim_{n\to\infty} \frac{N_i(n)}{n} \textit{ exists } \text{w.p.1} \textit{ and equals } \frac{a_i u_i}{\sum_{i=1}^{p} a_i u_i} = q_i$$

where $\mathbf{u} = (u_1, \cdots, u_p)$ *is a left eigenvector (with positive components) of* $\mathbf{M}$ *corresponding to the maximal positive eigenvalue* λ.

(ii)

$$\text{(11b)} \qquad \lim_{n\to\infty} n\mu e^{-\lambda\tau_n} = W \qquad \text{w.p.1}$$

where W *is the a.s. limit of the martingale* $(\sum_{i=1}^{p} v_i X_i(t))e^{-\lambda t}$ *and* $\mathbf{v} = (v_1, \cdots, v_p)$ *is the unique right eigenvector of* $\mathbf{M}$ *associated with* λ *(i.e.,* $\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$*) obeying the normalization condition* $\sum_{i=1}^{p} u_i v_i = 1$, *and* $\mu = \lambda(\sum_{j=1}^{p} a_j u_j)^{-1}$.

We close this introduction with a brief outline of the various sections of this paper. In Section 1 some properties on sums of reciprocals of sums of independently and identically distributed non-negative random variables (not necessarily integer valued) are described. In particular we elaborate the proof of Theorem 1.

Section 2 studies the τ_n process including the proof of relation (5) and Theorem 3.

Section 3 deals with the $\{N(t);\ t \geqq 0\}$ process for general age dependent branching processes.

Section 4 is concerned with some multidimensional generalizations as exemplified by Theorem 5.

In the last section we indicate extensions of the results set forth in Sections 2 and 3 to the case of Markov branching process where extinction of the population in finite time happens with non-zero probability.

1. Some results on reciprocals of sums of independently identically distributed non-negative random variables. This section is principally devoted to the proof of Theorem 1 (see the introduction). For clarity of exposition we decompose the proof into a series of lemmas of some independent interest.

Let ξ_i, $i = 1, 2, \cdots$ be a sequence of i.i.d. non-negative random variables such that $E\xi_i = \mu$, $0 < \mu \leqq \infty$. Let $c \geqq 0$ and define

$$S_n = c + \xi_1 + \cdots + \xi_n = c + S_n' \quad \text{for} \quad n \geqq 1, \tag{12}$$

c being a constant. The above notation prevails throughout this section.

Lemma 1. *Suppose $E(1/\xi_i) < \infty$. Then*

$$\sup_n E\left(\frac{n}{S_n}\right) \leqq E\left(\frac{1}{\xi_i}\right).$$

Proof. Invoke the harmonic mean arithmetic mean inequality, *viz.*,

$$\frac{n}{S_n} = \frac{n}{c + \xi_1 + \cdots + \xi_n} \leqq \frac{n}{\xi_1 + \xi_2 + \cdots + \xi_n} \leqq \frac{1}{n}\left(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \cdots + \frac{1}{\xi_n}\right)$$

and take expectations using the hypothesis of the lemma.

Remark. If $c = 0$ and $E(1/\xi_i) = \infty$, examples can be constructed to show that $E(n/S_n) = \infty$ for all $n \geqq 1$. However, if $c > 0$ this last contingency does not occur as attested to by the next lemma.

Lemma 2. *If $c > 0$ then $\sup_n E(n/S_n) < \infty$.*

Proof. If $P\{\xi_1 < c\} = 0$, we trivially have $n/S_n \leqq 1/c$ w.p.1 and so we need to consider only the case $P\{\xi_1 < c\} > 0$. Let

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda x}\, dP\{\xi_1 \leqq x\}.$$

Then

$$E\left(\frac{n}{S_n}\right) = n\int_0^\infty \frac{1}{c+x}\, dP\{S_n' \leqq x\} = n\int_0^\infty \left(\int_0^\infty e^{-(c+x)y}\, dy\right) dP\{S_n' \leqq x\} \tag{13}$$

$$= n\int_0^\infty e^{-cy}\varphi^n(y)\, dy = \int_0^\infty e^{-u} f\left(\frac{u}{n}\right) du$$

by making the change of variable $u = -n \log \varphi(y)$, and where

$$f(u) = e^{-cy} \frac{\varphi(y)}{-\varphi'(y)}.$$

Notice since $P\{\xi_1 > 0\} > 0$, the map y to u is strictly monotone and hence invertible. Now

$$(14) \quad \frac{e^{-cy}\varphi(y)}{(-\varphi'(y))} \leqq \frac{1}{e^{cy}(-\varphi'(y))} \leqq \frac{1}{\int_0^c xe^{(c-x)\xi}\, dP\{\xi_1 \leqq x\}} \leqq \frac{1}{\int_0^c x\, dP\{\xi_1 \leqq x\}} < \infty$$

since $P\{\xi_1 < c\} > 0$. Thus $f(u)$ is bounded on $[0, \infty)$ and

$$(15) \qquad \lim_{u \downarrow 0} f(u) = \frac{1}{(-\varphi'(0))} = \frac{1}{\mu} < \infty \quad \text{since} \quad \mu > 0.$$

Therefore by the dominated convergence theorem we obtain $\lim_{n\to\infty} E(n/S_n) = 1/\mu < \infty$.

We need one more preliminary lemma.

Lemma 3. *Let S_n be defined as in* (12) *and assume that either $c > 0$ or $E(1/\xi_i) < \infty$. Then for each $r > 0$,*

$$(16) \qquad K_r = \sup_n E\left[\left(\frac{n}{S_n}\right)^r\right] < \infty.$$

Proof. We first prove the lemma with $r = 2$. To this end, write

$$S_{2n} = \left(\frac{c}{2} + \xi_1 + \cdots + \xi_n\right) + \left(\frac{c}{2} + \xi_{n+1} + \cdots + \xi_{2n}\right) = S_n^* + S_n^{**}.$$

Since ξ_i are mutually independent non-negative random variables and consequently S_n^* and S_n^{**} are likewise independent, we obtain

$$E\left(\frac{(2n)^2}{S_{2n}^2}\right) \leqq E\left(\frac{n}{S_n^*}\frac{n}{S_n^{**}}\right) \leqq E\left(\frac{n}{S_n^*}\right)E\left(\frac{n}{S_n^{**}}\right) \leqq K_1^2$$

where the last inequality ensues by virtue of Lemmas 1 and 2 and the first uses the fact $(a + b)^2 \geqq 4ab$. Furthermore, obviously

$$E\left[\frac{(2n+1)^2}{S_{2n+1}^2}\right] \leqq 4E\left(\frac{(2n)^2}{S_{2n}^2}\right) \leqq 4K_1^2.$$

Thus the inequality (16) is established when $r = 2$. The cases for arbitrary r are handled by similar means with the additional help of Hölder's inequality.

An easy application of (16) and the strong law of large numbers validates the conclusions of parts (i) and (ii) of Theorem 1. The result of part (iii), perhaps, the most striking assertion of Theorem 1, is restated here to emphasize its importance.

Theorem 1′. *Let* $S_n = c + \xi_1 + \cdots + \xi_n$, $c \geqq 0$ *where* ξ_i *are i.i.d. non-negative random variables with* $0 < E(\xi_i) = \mu < \infty$. *Suppose either* $c > 0$ *or* $E(1/\xi_i) < \infty$. *Then*

$$\Pr\left\{\lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{S_n} - \frac{1}{n\mu}\right) \text{exists}\right\} > 0 \tag{17}$$

if and only if

$$E(\xi_i |\log \xi_i|) < \infty. \tag{18}$$

When (17) *holds its value is* 1.

The proof of Theorem 1′ will occupy the bulk of the remaining discussion of this section. The next 3 lemmas comprise the essential steps of the analysis.

Lemma 4. *If* $E(\xi |\log \xi|) < \infty$ *then*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{(n\mu - S_n)^2}{n^2 S_n}$$

exists w.p.1.

Proof. Since the ξ_i's are non-negative and $n/S_n \to 1/\mu > 0$ almost surely it suffices to show

$$\Pr\left\{\lim_{N\to\infty} \sum_{n=1}^{N} \frac{(n\mu - S_n)^2}{n^3} \text{exists}\right\} = 1.$$

Let $X_i = \mu - \xi_i$ and $\tilde{S}_n = X_1 + X_2 + \cdots + X_n$ and thus $n\mu - S_n = \tilde{S}_n - c$. It is clearly enough to prove

$$\Pr\left\{\lim_{N\to\infty} \sum_{1}^{N} \frac{\tilde{S}_n^2}{n^3} < \infty\right\} = 1. \tag{19}$$

Employing a standard truncation technique we introduce the random variables

$$U_i = \begin{cases} X_i & \text{if } |X_i| < i, \\ 0 & \text{otherwise}, \end{cases} \tag{20}$$

and form the partial sums $R_n = U_1 + \cdots + U_n$. Since $E|X_i| < \infty$, by tail equivalence (see [9]) the verification of (19) reduces to proving

$$\Pr\left\{\lim_{N\to\infty} \sum_{n=1}^{N} \frac{R_n^2}{n^3} < \infty\right\} = 1.$$

We will establish the stronger result that $\sum_{n=1}^{\infty} ER_n^2/n^3 < \infty$. Now $ER_n^2 = E(R_n - \mu_n)^2 + \mu_n^2$ where $\mu_n = E(R_n) = \sum_{i=1}^{n} EU_i$. Moreover, since the U_i's are independent, we have $E(R_n - \mu_n)^2 = \text{Var}(R_n) = \sum_{i=1}^{n} \text{Var}(U_i) \leqq \sum_{i=1}^{n} E(U_i^2)$, and hence

$$\sum_{n=1}^{\infty} \frac{E(R_n - \mu_n)^2}{n^3} \leqq \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{i=1}^{n} EU_i^2 \leqq \sum_{i=1}^{\infty} \frac{(EU_i^2)}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} \sum_{k=0}^{i-1} a_k \leqq \sum_{k=0}^{\infty} \frac{a_k}{k+1}$$

where $a_k = \int_{k \leqq |x| < k+1} x^2 dF(x)$ and $F(x) = Pr\{X_i \leqq x\}$. Now

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} = \sum_{k=0}^{\infty} \frac{1}{k+1} \int_{k \leqq |x| < k+1} x^2\, dF(x) \leqq \sum_{k=0}^{\infty} \int_{k \leqq |x| < k+1} |x|\, dF(x) = E(|X|)$$

and consequently $\sum_{n=1}^{\infty} [E(R_n - \mu_n)^2/n^3] < \infty$. It remains to establish the convergence of the series $\sum_1^{\infty} \mu_n^2/n^3$. To complete this task essential use of the assumption $E(\xi\, |\log \xi|) < \infty$ is made. Observe that

$$\mu_n^2 = \left(\sum_{i=1}^{n} \int_{|x|<i} x\, dF(x)\right)^2 = \left(\sum_{i=1}^{n} \int_{|x|>i} x\, dF(x)\right)^2 \quad \text{since} \quad EX_i = 0.$$

Now

$$(21) \qquad \int_{|x| \geqq i} |x|\, dF \leqq \frac{1}{\log i} \int_{|x| \geqq i} |x| \log |x|\, dF \leqq \frac{E(\xi\, |\log \xi|)}{\log i} = \frac{C}{\log i}.$$

On the basis of the estimate in (21) we conclude that $\mu_n^2 = 0((n/\log n)^2)$ and therefore $\sum_{n=1}^{\infty} \mu_n^2/n^3 < \infty$. The proof of Lemma 4 is complete.

Lemma 5. *For U_i defined in Lemma* 4 (see (20)), *$\lim_{N\to\infty} \sum_{i=1}^{N} E(U_i)/i$ exists if and only if $E(\xi\, |\log \xi|) < \infty$.*

Proof. Since $E(\mu - \xi_i) = 0$ we have

$$EU_i = \int_{|\mu - \xi_i| < i} (\mu - \xi_i) P(d\omega) = \int_{|\xi_i - \mu| \geqq i} (\xi_i - \mu) P(d\omega).$$

Now ξ_i is a nonnegative random variable and thus for $i > [\mu] + 1$, $|\xi_i - \mu| \geqq i$ iff $\xi_i - \mu \geqq i$ and therefore

$$EU_i = \int_{(\xi_i - \mu) \geqq i} (\xi_i - \mu) P(d\omega) \geqq 0 \quad \text{for} \quad i > [\mu] + 1.$$

Now set $b_k = \int_{k \leqq \xi - \mu < k+1} (\xi - \mu)\, P(d\omega)$. The foregoing discussion also entails the inequalities $b_k \geqq 0$ for $k > [\mu] + 1$. Then for $r = [\mu] + 1$, we have

$$\begin{aligned}\sum_{i=r}^{N} \frac{E(U_i)}{i} &= \sum_{i=r}^{N} \frac{1}{i} \sum_{k=i}^{\infty} b_k = \sum_{k=r}^{N} b_k \sum_{i=r}^{k} \frac{1}{i} + \left(\sum_{i=r}^{N} \frac{1}{i}\right)\left(\sum_{k=N+1}^{\infty} b_k\right) \\ &= \sum_{k=r}^{N} b_k \log (k+1) + \log (N+1) \sum_{k=N+1}^{\infty} b_k + C_N\end{aligned}$$

where C_N tends to a finite limit as $N \to \infty$ since the series $\sum b_k$ converges. It follows that $\lim_{N\to\infty} \sum_{i=1}^{N} E(U_i)/i$ exists if and only if $\lim_{N\to\infty} \sum_{k=r}^{N} b_k \log (k+1)$ exists. But for $k > r = [\mu] + 1$,

$$\int_{k \leqq \xi - \mu < k+1} (\xi - \mu) \log (\xi - \mu) P(d\omega) < b_k \log (k+1)$$

$$\leqq 2 \int_{k \leqq \xi - \mu \leqq k+1} (\xi - \mu) \log (\xi - \mu) P(d\omega)$$

Consequently $\lim_{N\to\infty}\sum_{i=1}^{N} EU_i/i$ exists iff $\int_{\xi-\mu>r}(\xi-\mu)\log(\xi-\mu)\,P(d\omega)<\infty$ or equivalently $E(\xi\,|\log\xi|)<\infty$ since ξ is a non-negative random variable.

Lemma 6. *The moment condition $E\xi\,|\log\xi|<\infty$ implies*

$$\Pr\left\{\lim_{N\to\infty}\sum_{n=1}^{N}\frac{n\mu-S_n}{n^2}\text{ exists}\right\}=1.$$

Proof. We employ the same notation as before. It suffices to prove

$$\Pr\left\{\lim_{N\to\infty}\sum_{1}^{N}\frac{\tilde{S}_n}{n^2}\text{ exists}\right\}=1. \tag{22}$$

Use the expression $\tilde{S}_n=\sum_{i=1}^{n}X_i$, interchange the order of summation, and finally appeal to the strong law, to conclude that (22) holds iff

$$\Pr\left\{\lim_{N\to\infty}\sum_{i=1}^{N}\frac{X_i}{i}\text{ converges}\right\}=1.$$

This is easy *via* Kolmogorov's three series theorem (see [9]). Indeed, since $E(X_i)<\infty$, by tail equivalence, it suffices to prove

$$\Pr\left\{\lim_{N\to\infty}\sum_{1}^{N}\frac{U_i}{i}\text{ converges}\right\}=1 \tag{23}$$

where as defined earlier

$$U_i=\begin{cases}X_i & \text{if } |X_i|<i\\ 0 & \phantom{\text{if }} |X_i|\geqq i\end{cases}.$$

By Kolmogorov's three series criterion we deduce that

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{(U_i-EU_i)}{i}\quad\text{exists}\quad\text{w.p.1.} \tag{24}$$

and this requires only the moment condition $E\,|X_i|<\infty$. According to Lemma 5 $\lim_{N\to\infty}\sum_{i=1}^{N}EU_i/i$ exists under the hypothesis $E\xi_i\,|\log\xi_i|<\infty$.

The proof of the convergence in (23) is concluded and thereby the demonstration of the lemma is complete.

The ingredients are now available to finish the proof of Theorem 1′.

Proof of Theorem 1′. The "if" part is a consequence of Lemmas 4 and 6 applied appropriately to the right hand terms of the identity

$$\sum_{n=1}^{N}\left(\frac{1}{S_n}-\frac{1}{n\mu}\right)=\sum_{n=1}^{m}\frac{(n\mu-S_n)^2}{n^2\mu^2S_n}+\sum_{n=1}^{N}\frac{(n\mu-S_n)}{n^2\mu^2}.$$

In order to prove the converse we define the event

$$A=\left\{\omega:\lim_{N\to\infty}\sum_{1}^{N}\left(\frac{1}{S_n}-\frac{1}{n\mu}\right)\text{ exists}\right\}.$$

Now

$$n\mu - S_n = \sum_{i=1}^{n} X_i - c = \tilde{S}_n - c.$$

Hence on A

$$\lim_{N\to\infty}\left(\sum_{1}^{N}\frac{(n\mu - S_n)^2}{n^2 S_n} + \sum_{1}^{N}\frac{\tilde{S}_n}{n^2}\right)$$

exists. But

$$\sum_{1}^{N}\frac{\tilde{S}_n}{n^2} = \sum_{i=1}^{N}\frac{X_i}{i} + \delta(N)$$

where $\delta(N)$ converges w.p.1. since $E|X_i| < \infty$. Moreover, $\sum_{i=1}^{N}(X_i - U_i)/i$ converges w.p.1 since by tail equivalence $X_i \neq U_i$ only finitely often (see (20)). Thus on A

$$\lim_{N\to\infty}\left(\sum_{1}^{N}\frac{(n\mu - S_n)^2}{n^2 S_n} + \sum_{1}^{N}\frac{U_i}{i}\right)$$

exists w.p.1.

Again as observed earlier in (24) $\lim_{N\to\infty}\sum_{i=1}^{N}(U_i - EU_i)/i$ exists w.p.1. Thus

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{(n\mu - S_n)^2}{n^2 S_n} + \sum_{n=1}^{N}\frac{EU_n}{n}\right) \quad \text{exists w.p.1 on } A \tag{25}$$

We pointed out in the proof of Lemma 4 that $EU_i \geqq 0$ for all $i > [\mu] + 1$. It follows since $\Pr\{A\} > 0$ that $\sum_{n=1}^{N} EU_n/n$ converges. Now appealing to Lemma 5, the desired result is established.

We close this section by recording some further convergence facts concerning reciprocals of sums of non-negative random variables.

Lemma 7. *Let ξ_i, $i = 1, 2, \cdots$ be i.i.d. positive random variables. Define $S_n = c + \xi_1 + \xi_2 + \cdots + \xi_n$, $c \geqq 0$.*

(a) *If $0 < E\xi_i = \mu \leqq \infty$ then $(1/\log n)\sum_{i=1}^{n} 1/S_i \to 1/\mu$* w.p.1.

(b) *Assume either of the hypothesis of Lemmas 1 or 2. If furthermore $E\xi_i^2 < \infty$ then $\sum_{n=1}^{\infty}|(1/S_n) - (1/n\mu)|$ converges* w.p.1.

Proof. (a) We merely invoke a standard summability argument in conjunction with the strong law of large numbers.

(b) The Schwartz inequality coupled with the identity $E(S_n - n\mu)^2 = n\sigma^2$ gives

$$\sum_{n=1}^{\infty} E\left|\frac{1}{S_n} - \frac{1}{n\mu}\right| \leqq \sum_{n=1}^{\infty}\frac{1}{n}[E(S_n - n\mu)^2]^{1/2}E\left(\frac{1}{S_n^2}\right)^{1/2} \tag{26}$$

$$= \sigma\sum_{n=1}^{\infty}\frac{1}{(n)^{1/2}}\left[E\left(\frac{1}{S_n^2}\right)\right]^{1/2}.$$

The inequality $E(n^2/S_n^2) \leqq K_2$ (Lemma 3) used on the right of (26) yields the bound $\sigma(K)^{1/2} \sum_{n=1}^{\infty} 1/n^{3/2}$ for the final series. Finally, apply the monotone convergence theorem to complete the proof.

Lemma 8. *Let ξ_i be i.i.d. non-negative random variable and $c \geqq 0$. Define $S_n = c + \xi_1 + \xi_2 + \cdots + \xi_n$ where either $c > 0$ or $E(1/\xi_i) < \infty$. Assume that $E(\xi_i^2) < \infty$. Then*

$$\sup_{k>0} E \left| \sum_{i=n}^{n+k} \sum_{j=n}^{n+k} \left(\frac{1}{S_i} - \frac{1}{i\mu}\right)\left(\frac{1}{S_j} - \frac{1}{j\mu}\right) \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{27}$$

We omit the proof.

2. Split times process. We adhere to the notation of the introduction and start with the proof of (5).

Proposition 1. *Let τ_n be defined as in (2). Then $\lim_{n\to\infty} n\mu\, e^{-\lambda\tau_n} = W$ w.p.1 where $\lambda = u'(1)$, $\mu = \lambda/a = h'(1) - 1$ and $W = \lim_{t\to\infty} X(t)\, e^{-\lambda t}$, the a.s. limit of the non-negative martingale $\{X(t)\, e^{-\lambda t}; \mathfrak{F}(t)\,, t \geqq 0\}$.*

Proof. Notice that $X(\tau_n) = X(0) + \xi_1 + \cdots + \xi_n$. Hence by the strong law $\lim_{n\to\infty} X(\tau_n)/n\mu = 1$ since $E(\xi_i) = \mu$. Moreover, clearly $\Pr\{\lim_n \tau_n = \infty\} = 1$. Therefore $\lim_{n\to\infty} X(\tau_n)\, e^{-\lambda\tau_n} = W$ w.p.1. Combining these limit relations, we infer that $\lim_{n\to\infty} n\mu\, e^{-\lambda\tau_n} = W$ w.p.1. Q.E.D.

The proof of (4) is based on the following theorem of some independent interest.

Theorem 2. *Assume that $X(0) > 0$. The sequence $\{Y_n\,; \mathfrak{F}_n\,; n = 1, 2, \cdots\}$* (see (2)) *constitutes a square integrable martingale such that*

$$\sup_n E(Y_n^2) < \infty \tag{28}$$

and hence $\lim_n Y_n = Y$ exists w.p.1 *and in mean square.*

Proof. By definition $Y_n = \sum_{i=1}^{n} (T_i - 1/aS_{i-1})$, where $S_j = X(0) + \xi_1 + \cdots + \xi_j\,, j = 0, 1, \cdots$. Now $T_i\,, S_i$ are $\mathfrak{F}_i$ measurable and therefore

$$E(Y_n \mid \mathfrak{F}_{n-1}) = Y_{n-1} + E\left(T_n - \frac{1}{aS_{n-1}} \,\middle|\, \mathfrak{F}_{n-1}\right) \quad \text{a.s.}$$

But by the strong Markov property, we obtain

$$E(T_n \mid \mathfrak{F}_{n-1}) = \frac{1}{aS_{n-1}} \quad \text{a.s.}$$

Thus it follows that $\{Y_n\,; \mathfrak{F}_n\,, n = 1, 2, \cdots\}$ is a martingale. A similar computation conditioning appropriately on $\mathfrak{F}_{i-1}$ and appealing to the strong Markov property yields the formula

$$EY_n^2 = \sum_{i=0}^{n-1} \frac{1}{a^2} E\left(\frac{1}{S_i^2}\right). \tag{29}$$

Consulting Lemma 3 we have

$$\sup_i E\left[\left(\frac{i}{S_i}\right)^2\right] = K_2 < \infty. \tag{30}$$

From (30) and (29) we have $\sup_n E(Y_n^2) = K' < \infty$. To finish the proof we appeal to Doob's martingale convergence theorem [3].

Corollary 1.

$$\frac{\tau_n}{\log n} \to \frac{1}{\lambda} \qquad \text{w.p.1.}$$

Proof. Observe that

$$Y_n = \sum_{i=1}^{n}\left(T_i - \frac{1}{aS_{i-1}}\right) = \tau_n - \frac{1}{a}\sum_{i=1}^{n}\frac{1}{S_{i-1}} \tag{31}$$

and Y_n converges to a finite limit w.p.l in accordance with Theorem 2. Taking account of the result of Lemma 7 part (a) and recognizing that $\lambda = a\mu$ we infer the desired conclusion.

Proposition 2.

$$\frac{E(\tau_n)}{\log n} \to \frac{1}{\lambda}. \tag{32}$$

Proof. In fact, since $E(T_i) = (1/a)E(1/S_{i-1})$, the limit relation (32) is a consequence of Theorem 1 part (ii).

Theorem 3 (see introduction) presents a refinement of the result of Corollary 1. To prove this theorem, we develop one further lemma.

Lemma 9. *The two ω-sets*

$$\{\omega : W > 0\} \quad \textit{and} \quad \left\{\omega : \lim_{N\to\infty} \sum_{i=1}^{N}\left(\frac{1}{S_i} - \frac{1}{i\mu}\right) \quad \text{exists}\right\}$$

coincide w.p.1.

Proof. Consider $\lambda\tau_n - \log n$. After appropriate rearrangement it takes the form

$$\begin{aligned}\lambda\tau_n - \log n &= \lambda \sum_{i=1}^{n}\left(T_i - \frac{1}{aS_{i-1}}\right) + \mu \sum_{i=1}^{n}\left(\frac{1}{S_{i-1}} - \frac{1}{\mu i}\right) + \sum_{i=1}^{n}\frac{1}{i} - \log n \\ &= \lambda Y_n + \left(\sum_{i=1}^{n}\frac{1}{i} - \log n\right) + \mu \sum_{i=1}^{n}\left(\frac{1}{S_{i-1}} - \frac{1}{\mu i}\right).\end{aligned}$$

According to Theorem 2, $\lim Y_n = Y$ exists w.p.1. Hence

$$\{\omega : \lambda\tau_n - \log n \text{ converges}\} = \left\{\omega : \lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{1}{S_i} - \frac{1}{i\mu}\right) \text{exists}\right\} \tag{33}$$

w.p.1. Consulting Proposition 1 we see that

$$\{\omega : W > 0\} = \{\omega : \lambda\tau_n - \log n \text{ converges}\} \quad \text{w.p.1.} \tag{34}$$

The relations (33) and (34) together imply the statement of the lemma.

Proof of Theorem 3. Simply combine the results of Lemma 9, the identity (33) and the conclusion of Theorem 1′ part (iii).

This yields us the following important

Corollary 3. *Either* $W \equiv 0$ *or* $W > 0$ w.p.1. *The latter event occurs if and only if* $\sum_{j=2}^{\infty} p_j\, j \log j < \infty$.

A final result on the τ_n process based on Lemma 9 is

Lemma 10. *Assume* $E(\xi_1^2) < \infty$.

$$\sup_{k \geq 0} \operatorname{Var}\,[\tau_{n+k} - \tau_n] \to 0 \quad as \quad n \to \infty.$$

Proof. In the notation of Theorem 2 we have

$$\tau_{n+k} - \tau_n = Y_{n+k} - Y_n + \frac{1}{a} \sum_{i=n}^{n+k-1} \frac{1}{S_i}$$

and therefore

$$\operatorname{Var}\,[\tau_{n+k} - \tau_n] \leq 2\left\{E(Y_{n+k} - Y_n)^2 + CE\left[\sum_{i=n}^{n+k}\left(\frac{1}{S_i} - \frac{1}{i\mu}\right)\right]^2\right\}$$

where C is a constant. The assertion of the lemma is validated by appealing to the results of Theorem 2 and Lemma 8.

3. The stochastic process $\{N(t);\ t \geq 0\}$. In this section we assume $\{X(t);\ t \geq 0\}$ is an arbitrary age dependent one dimensional continuous time branching process with life time distribution function $G(x)$ and offspring generating function $h(s) = \sum_{i=0}^{\infty} p_i s^i$. We assume $G(0+) = 0$.

Let

$$m(t) = E(X(t)), \qquad r(t) = EN(t), \qquad \psi(z, t) = E(z^{N(t)})$$

with $X(0) = 1$. The usual renewal arguments involving the first split time show than $\psi(z, t)$ is the unique solution of the functional equation

$$\psi(z, t) = 1 - G(t) + \int_0^t zh(\psi(z, t - u))\, dG(u)$$

with initial condition $\psi(z, 0) = 1$. This implies that $EN(t) = r(t)$ solves the integral equation

$$r(t) = G(t) + \int_0^t (\mu + 1)r(t - u)\, dG(u), \tag{35}$$

with $r(0) = 0$ where $\mu = h'(1) - 1 < \infty$. It is also a familiar fact that $m(t) = EX(t)$ satisfies the integral equation

$$m(t) = 1 - G(t) + \int_0^t (\mu + 1)m(t - u)\, dG(u) \tag{36}$$

subject to the initial condition $m(0) = 1$.

Invoking a standard uniqueness criterion for the general renewal equation and comparing (35) and (36) we infer that $m(t) = 1 + \mu r(t)$ which is precisely equation (7).

We conclude this section with the proof of Theorem 4 stated in the introduction.

Proof of Theorem 4: Recall the identity

$$X(t) = X(0) + \xi_1 + \cdots + \xi_{N(t)}$$

which exhibits $X(t)$ as a sum of a random number of i.i.d. non-negative random variables of finite variance. Further, since by hypothesis $\lim_{t\to\infty} X(t)e^{-\lambda t} = W$ exists and > 0 w.p.1. we infer from the strong law the limit relation

$$N(t)\mu e^{-\lambda t} \to W \qquad \text{w.p.1.}$$

Now the general formulation of the central limit theorem for random sums of random variables ([10], [13]) implies assertion (ii).

Again since w.p.1, $N(t) \to \infty$ through all integer values, by appealing to the standard law of the iterated logarithm [12] the assertion of (iii) follows The proof of Theorem 4 is hereby complete.

4. Splitting times process for multitype continuous time branching processes. Let $\mathbf{X}(t) = (X_1(t), \cdots, X_p(t))$, $t \geqq 0$ be a p dimensional strong Markov, multitype, continuous time, branching process. The precise moment conditions, notation, terminology and relevant functionals under consideration are set forth in the introduction (*cf.*, Theorem 5 and its accompanying discussion). As previously, τ_n represents the time of the n^{th} split of the process, $N_i(n)$, $i = 1, 2, \cdots, p$; designates the number of splits, among the first n, which are of type i and let $\tau_r^{(i)}$ denote the time of occurrence of the r^{th} split of type i. Lastly let $\mathfrak{F}_n$ be the σ-algebra associated with the stopping time τ_n.

The following variant of the strong law for martingales due to P. Levy ([8] pp. 250) is needed for the proof of Theorem 5 and is now stated explicitly for the convenience of the reader.

Lemma 11. *Let δ_i, $i = 1, 2, \cdots$ be a sequence of random variables whose only possible values are 0 and 1 and adapted to a sequence $\mathfrak{F}_i$, $i = 1, 2, \cdots$ of increasing σ-fields. Define $p_n = E(\delta_n \mid \mathfrak{F}_{n-1})$, $n = 1, 2, \cdots$. Then*

$$\frac{1}{n}\sum_{i=1}^{n} (\delta_i - p_i) \to 0 \qquad \text{w.p.1.}$$

With these preparations we can now prove Theorem 5 (consult the introduction for its precise statement).

Proof of Theorem 5. Fix i and define

$$\delta^{(i)}(n) = \begin{cases} 1 & \text{if the } n^{\text{th}} \text{ split is of a type } i \text{ particle} \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Then obviously

$$N_i(n) = \sum_{j=1}^{n} \delta^{(i)}(j).$$

Consider the family of random variables and σ-fields $\{\delta^{(i)}(n); \mathfrak{F}_n, n = 1, 2, \cdots\}$. From the nature of the process and by virtue of the strong Markov property, we have

$$p_n^{(i)} = E(\delta^{(i)}(n) \mid \mathfrak{F}_{n-1}) = \frac{a_i X_i(\tau_{n-1})}{\sum_{j=1}^{p} a_j X_j(\tau_{n-1})}. \tag{38}$$

Using a recent result of Kesten and Stigum [7] it can be shown (see [1]) that

$$\lim_{t\to\infty} \mathbf{X}(t) e^{-\lambda t} = W \cdot \mathbf{u} \qquad \text{w.p.1} \tag{39}$$

where λ, W, $\mathbf{u}$ are defined in the statement of Theorem 5 and the random variable W is positive w.p.1.

Owing to the moment assumptions (10a) it follows from (39) that

$$\lim_{n\to\infty} p_n^{(i)} \quad \text{exists w.p.1 and its value is} \quad \frac{a_i u_i}{\sum_{j=1}^{p} a_j u_j}. \tag{40}$$

Now (40) trivially implies

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} p_n^{(i)} = \frac{a_i u_i}{\sum_{j=1}^{p} a_j u_j} \qquad \text{w.p.1.} \tag{41}$$

Next in view of (41), Lemma 11 manifestly implies the conclusion of Theorem 5, part (i). The proof of the assertion in (ii) uses the result of (11a) in conjunction with the strong law of large numbers for sums of independent random variables. Indeed, considering the contributions at each split, we may write

$$\sum_{j=1}^{p} v_j X_j(\tau_n) = \sum_{j=1}^{p} \sum_{r=1}^{N_j(n)} \eta_r^{(j)}$$

where

$$\eta_r^{(j)} = \sum_{k=1}^{p} v_k [X_k(\tau_r^{(j)} + 0) - X_k(\tau_r^{(j)} - 0)], \quad j = 1, 2, \cdots, p; \ r = 1, 2, \cdots .$$

Here, as previously, the positive vector $\mathbf{v}$ is determined as the eigenvector of $\mathbf{M}$ corresponding to the largest eigenvalue λ (*i.e.*, $\mathbf{Mv} = \lambda\mathbf{v}$) satisfying the normalization condition $\sum_{i=1}^{p} v_i u_i = 1$. Obviously $\{\eta_r^{(i)}\}$ consist of independent random variables with the property that for each j the elements of the set $\{\eta_1^{(i)}, \eta_2^{(i)}, \cdots\}$ are identically distributed. Note that

$$q_i = \frac{a_i u_i}{\sum_{j=1}^{p} a_j u_j} \quad \text{is} > 0 \quad \text{for all} \quad i,$$

and hence because of (i) certainly $N_j(n)$ increases to ∞ w.p.1 as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{p} v_i X_i(\tau_n)\right) = \frac{\lambda}{\left(\sum_{j=1}^{p} a_j u_j\right)}.$$

The proof of Theorem 5 is concluded using (39), and the fact $P\{\tau_n \to \infty\} = 1$.

Remark. Note that the limit relation (41) is a consequence of (39) provided $W > 0$. The property that $W > 0$ w.p.1 subject to the moment conditions

$$\left.\frac{\partial^2 u^{(k)}(\mathrm{s})}{\partial s_i \partial s_j}\right|_{\mathrm{s}=1} < \infty, \qquad i, j, k = 1, 2, \cdots, p$$

can be established using a corresponding result by Harris [5] for discrete time process. Also from the results on discrete time process, due to Kesten and Stigum [7] one can demonstrate the remarkable fact that $W > 0$ w.p.1 if and only if

$$\sum_{i_1,\cdots,i_p} |b_{i_1,\cdots,i_p}^{(k)}|\, i_r \log i_r < \infty, \quad \text{for each} \quad k, r = 1, 2, \cdots, p$$

where

$$u^{(k)}(\mathrm{s}) = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}^{(k)} s_1^{i_1} s_2^{i_2} \cdots s_p^{i_p}.$$

The multi-dimensional analog of Theorems 2 and 3 is embodied in the following theorem.

Theorem 6. *Assume the hypothesis of Theorem 5 prevails. Define*

$$Y_n = \tau_n - \sum_{j=0}^{n-1} \frac{1}{Z(\tau_j)}, \qquad n = 1, 2, \cdots$$

where $\tau_0 = 0$ *and* $Z(\tau_j) = \sum_{i=1}^{p} a_i X_i(\tau_j)$. *Then*

(i) $\{Y_n, \mathfrak{F}_n, n = 1, 2, \cdots\}$ *constitutes a uniformly square integrable Martingale.*

(ii)

(42) $$\tau_n - \frac{1}{\lambda} \log n$$

converges w.p.1 to a finite valued random variable and

(iii)

$$\lim_{n\to\infty}\left(\sum_{j=0}^{n-1}\frac{1}{Z(\tau_j)}-\frac{\log n}{\lambda}\right) \quad \textit{exists w.p.1.} \tag{43}$$

Proof. (i) Let T_i denote the time between the i^{th} and $i-1^{st}$ split. By definition, we have

$$Y_n = \tau_n - \sum_{j=0}^{n-1}\frac{1}{Z(\tau_j)} = \sum_{j=0}^{n-1}\left[T_{j+1}-\frac{1}{Z(\tau_j)}\right]$$

and by strong Markov property

$$E\left(T_{j+1}-\frac{1}{Z(\tau_j)}\,\middle|\,\mathfrak{F}_j\right) = 0.$$

Consequently

$$E(Y_{n+1}\mid\mathfrak{F}_n) = Y_n\,.$$

A straightforward calculation invoking the strong Markov property yields

$$E[Y_n]^2 = \sum_{j=0}^{n-1}E\left(\frac{1}{[Z(\tau_j)]^2}\right).$$

We will now prove

$$\sum_{n=1}^{\infty}E\left(\frac{1}{Z(\tau_n)^2}\right) < \infty.$$

It suffices to show that $\sum_{n=1}^{\infty}E(1/\tilde{Z}(\tau_n)^2) < \infty$ where $\tilde{Z}(t) = X_1(t)+\cdots+X_p(t)$.

Now since $\mathbf{X}(\tau_n)\to\infty$ as $n\to\infty$ w.p.1. there exists an $r>0$ such that $P\{\tilde{Z}(\tau_r)-\tilde{Z}(0)=0\}<1$ for any initial population make up. We will consider only the case $r=1$ as the extension to the general case is straightforward. Set

$$\xi_i = \begin{cases}1 & \text{if } \tilde{Z}(\tau_i)-\tilde{Z}(\tau_i-0)>0,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}P\{\tilde{Z}(\tau_n)\leqq n^{2/3}\} &\leqq P\left\{\sum_{i=1}^{n}\xi_i\leqq n^{2/3}\right\}\\ &\leqq P\{\text{at least } [n/2]+1 \text{ of } \xi_i \text{ mustbe } 0\},\quad \text{for}\quad n\geqq n_0\,,\\ &\leqq p^{(n+1)/2}\quad \text{where}\quad P\{\xi_i=0\}<p<1.\end{aligned}$$

Hence $\sum_{n=1}^{\infty}P\{\tilde{Z}(\tau_n)<n^{2/3}\}<\infty$. Now

$$\sum_{n=1}^{\infty}E\left(\frac{1}{\tilde{Z}(\tau_n)^2}\right)\leqq\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}P\{Z(\tau_n)>n^{2/3}\}+\frac{1}{(\tilde{Z}(0))^2}\sum_{n=1}^{\infty}P\{\tilde{Z}(\tau_n)\leqq n^{2/3}\}<\infty$$

(ii) We know according to Theorem 5, part (ii) that

$$\lim_{n\to\infty} ne^{-\lambda\tau_n} = cW$$

w.p.1 where c is a positive constant. Taking logarithms and recalling that $\Pr\{W > 0\} = 1$ yields (42).

(iii) Observing the identity

$$\tau_n - \frac{1}{\lambda}\log n = Y_n + \left(\sum_{j=0}^{n-1} \frac{1}{Z(\tau_j)} - \frac{\log n}{\lambda}\right)$$

in conjunction with the results of parts (i) and (ii) proved above clearly implies (43).

Remark. It is not difficult to ascertain that the vector family of random variables

$$\boldsymbol{\delta}(n) = (\delta^{(1)}(n), \delta^{(2)}(n), \cdots, \delta^{(p)}(n)), \qquad n = 1, 2, \cdots.$$

become asymptotically independent. On the basis of this fact, it is suggestive to conjecture that $(N_1(n), N_2(n), \cdots, N_p(n))$ properly normalized is asymptotically normal. This problem is open.

5. General remarks. We describe briefly the range of validity of the discussions of the preceding section in the case of the general one-dimensional continuous time branching process where $u(0) = ap_0 > 0$, *i.e.*, the possibility of extinction of the population occurs with positive probability.

I. In the subcritical or critical case where extinction in finite time is a sure event, it is clear that (since $N(t)$ and τ_n are monotone increasing)

$$\lim_{t\to\infty} N(t) = N(\infty) \quad \text{and} \quad \lim_{n\to\infty} \tau_n = \tau_\infty$$

exist and $N(\infty)$ and τ_∞ are finite valued w.p.1. Let

$$U(x) = \sum_{k=0}^{\infty} x^k \Pr\{N(\infty) = k\}. \tag{44}$$

It is readily verified that $U(x)$ may be evaluated as the smallest positive solution of the functional equation

$$U(x) = xh(U(x))$$

(for the definition of h see the Introduction).

The determination of the distribution of $N(\infty)$ reduces to a problem concerning sums of i.i.d. random variables.

Let

$$\eta_i = \zeta_i - 1, \qquad i = 1, 2, \cdots$$

and consider $S_n^* = \eta_1 + \cdots + \eta_n + X(0)$ where ζ_i are independent random variables with probability generating function $h(s)$.

Let N^* = smallest integer n for which $S_n^* = 0$. The variable N^* is finite valued owing to the fact that $E(\eta_i) \leqq 0$ for the critical or subcritical situations. A little reflection reveals that N^* and $N(\infty)$ possess the same distribution The literature on fluctuation theory for sums of independent random variables embraces the study of N^*, (*e.g.*, see Baxter [2], Spitzer [11]). The above considerations apply, *mutatis mutandis*, to the case of age dependent branching processes.

II. This paragraph treats cursorily generalizations of the results of Sections 2–4 in the supercritical case $u'(0) = \lambda > 0$ allowing for the contingency of extinction (*i.e.*, $u(0) = ap_0 > 0$).

A standard device in the theory of Markov branching processes permits us to reduce the analysis of general supercritical branching processes to the case where $u(0) = 0$. Specifically if $f_t(x)$ denotes the probability generation function of $X(t)$, $t > 0$, for an initial population of one individual, *i.e.*,

$$f_t(x) = \sum_{k=0}^{\infty} \Pr\{X(t) = k \mid X(0) = 1\}x^k \tag{45}$$

then

$$\hat{f}_t(x) = \frac{f_t((1-q)x + q) - q}{1 - q} \tag{46}$$

determines a continuous time branching process $\hat{X}(t)$, $t \geqq 0$ with $\hat{f}_t(0) = 0$ for all $t > 0$. Here, q denotes the probability of ultimate extinction of the $X(t)$ process, *i.e.*,

$$q = \Pr\{X(t) = 0 \text{ for some } t \mid X(0) = 1\}$$

which can be calculated as the smallest positive root of the equation $u(x) = 0$. Notice that $f_t(q) = q$ for all $t > 0$.

The infinitesimal generating function of $\hat{X}(t)$ is $\hat{u}(x) = u((1-q)x + q)/(1-q)$ where $u(x)$ is the infinitesimal generating function associated with $f_t(x)$. The processes $X(t)$ and $\hat{X}(t)$ are simply related. In fact, for each $t > 0$, let $\tilde{X}(t)$ denote those among the $X(t)$ particles at time t which produce an infinite line of descent. It is easy to convince oneself that $\{\tilde{X}(t), t \geqq 0\}$ engenders a continuous time Markov process. The following computation establishes the equivalence of the processes $\tilde{X}(t)$ and $\hat{X}(t)$. We have for $k = 1, 2, \cdots$

$$\begin{aligned} \Pr\{\tilde{X}(t) = k \mid \tilde{X}(0) = X(0) = 1\} &\\ = \frac{\sum_{l=k}^{\infty} \Pr\{\tilde{X}(t) = k \mid X(t) = l\} \Pr\{X(t) = l \mid X(0) = 1\}}{\Pr\{\tilde{X}(0) = 1 \mid X(0) = 1\}} &\\ = \frac{1}{1-q} \sum_{l=k}^{\infty} \binom{l}{k} q^k (1-q)^{l-k} \Pr\{X(t) = l \mid X(0) = 1\}. & \end{aligned} \tag{47}$$

Notice that (47) is not valid for $k = 0$. With the aid of the above formula a straightforward calculation yields

$$\sum_{k=1}^{\infty} \Pr \{\tilde{X}(t) = k \mid \tilde{X}(0) = X(0) = 1\} x^k = \hat{f}_t(x)$$

The preceding identification signifies that the sample paths of the $X(t)$, $t \geqq 0$ process possessing an infinite line of descent may be made to correspond to the sample paths of the process $\hat{X}(t)$. The measure of these paths is $1 - q$.

Invoking the conclusions of the theory of Sections 1–4 in terms of the $\hat{X}(t)$ process it appears that $N(t)$ and τ_n enjoy the same asymptotic properties as before except that all limit relations now hold with probability $1 - q$. The rigorous development of the above discussion is deferred to a future publication.

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