Limit theory for high frequency sampled MCARMA models

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We consider a multivariate continuous-time ARMA (MCARMA) process sampled at a high-frequency time-grid $\{h_n, 2h_n, \dots, nh_n\}$ where $h_n \downarrow 0$ and $nh_n \to \infty$ as $n \to \infty$, or at a constant time-grid where $h_n = h$. For this model we present the asymptotic behavior of the properly normalized partial sum to a multivariate stable or a multivariate normal random vector depending on the domain of attraction of the driving Lévy process. Further, we derive the asymptotic behavior of the sample autocovariance. In the case of finite second moments of the driving Lévy process the sample autocovariance is a consistent estimator. Moreover, we embed the MCARMA process in a cointegrated model. For this model we propose a parameter estimator and derive its asymptotic behavior. The results are given for more general processes than MCARMA processes and contain some asymptotic properties of stochastic integrals.

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1. Introduction

Multivariate continuous-time ARMA (MCARMA) processes $\mathbf{V} = (\mathbf{V}(t))_{t\geq 0}$ are the continuous-time versions of the well known multivariate ARMA processes in discrete time having short memory. They are important for stochastic modelling in many areas of application as, e.g., signal processing and control (cf. [20, 26]), econometrics (cf. [2, 32]), high-frequency financial econometrics (cf. [45]), and financial mathematics (cf. [1]). Starting at least with Doob [13] in 1944, Gaussian CARMA processes under the name Gaussian processes with rational spectral density appeared, where the driving force is a Brownian motion. To obtain more flexible marginal distributions and dynamics Brockwell (cf. [6, 7]) analyzed Lévy driven CARMA models, which were extended by Marquardt and Stelzer [28] to the multivariate setting; see [8] for an overview and a comprehensive list of references.

Lévy processes are defined to have independent and stationary increments, and are characterized by their Lévy-Khintchine representation. An $\mathbb{R}^m$-valued Lévy process $(\mathbf{L}(t))_{t\geq 0}$ has the Lévy-Khintchine representation $\mathbb{E}(e^{i\Theta'\mathbf{L}(t)}) = \exp(-t\Psi(\Theta))$ for $\Theta \in \mathbb{R}^m$, where Θ' is the transpose of Θ and

$$\Psi(\Theta) = -i\gamma_{\mathbf{L}}\Theta + \frac{1}{2}\Theta'\Sigma_{\mathbf{L}}\Theta + \int_{\mathbb{R}^m}\left(1 - e^{i\mathbf{x}'\Theta} + i\mathbf{x}'\Theta\mathbb{1}_{\{\|\mathbf{x}\|^2\leq 1\}}\right)\nu_{\mathbf{L}}(d\mathbf{x})$$

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with $\gamma_{\mathbf{L}} \in \mathbb{R}^m$, $\Sigma_{\mathbf{L}}$ a positive semi-definite matrix in $\mathbb{R}^{m\times m}$ and $\nu_{\mathbf{L}}$ a measure on $(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m))$, called the *Lévy measure*, which satisfies $\int_{\mathbb{R}^m} \min\{\|\mathbf{x}\|^2, 1\}\, \nu_{\mathbf{L}}(\mathrm{d}\mathbf{x}) < \infty$ and $\nu_{\mathbf{L}}(\{\mathbf{0}_m\}) = 0$. The triplet $(\gamma_{\mathbf{L}}, \Sigma_{\mathbf{L}}, \nu_{\mathbf{L}})$ is called the *characteristic triplet*, because it characterizes completely the distribution of the Lévy process. A two-sided Lévy process $(\mathbf{L}(t))_{t\in\mathbb{R}}$ is then a composition of two independent and identically distributed Lévy processes $(\mathbf{L}^{(1)}(t))_{t\geq 0}$ and $(\mathbf{L}^{(2)}(t))_{t\geq 0}$ in

$$\mathbf{L}(t) = \begin{cases} \mathbf{L}^{(1)}(t) & \text{for } t \geq 0, \\ \mathbf{L}^{(2)}(-t-) & \text{for } t < 0. \end{cases}$$

We refer to the excellent monograph of Sato [42] for more details on Lévy processes. In this paper the driving Lévy process is very general. It is allowed to have both a finite variance $\mathbb{E}\|\mathbf{L}(1)\|^2 < \infty$ and an infinite variance $\mathbb{E}\|\mathbf{L}(1)\|^2 = \infty$, which is modelled by a multivariate regularly varying Lévy process. CARMA processes driven by infinite variance Lévy processes are particularly relevant in modelling energy markets, see Garcia et al. [19], for instance. We will investigate MCARMA processes (see Definition 2.1) observed not only at a constant frequency h but also, and especially for high frequencies as found in finance and turbulence. Then the observation grid is $\{h_n, 2h_n, \ldots, nh_n\}$, where $h_n \downarrow 0$ and $\lim_{n\to\infty} nh_n = \infty$. The behavior of the spectral density of a high frequency sampled CARMA model and kernel density estimation was recently explored in Brockwell et al. [5, 9]. The estimation of the spectral density and the model parameters is topic of Fasen and Fuchs [17, 18]. For the statistical inference of a MCARMA process, e.g., parameter estimation and hypothesis testing, it is crucial to know the asymptotic behavior of the partial sum (cf. [17, 18]). We will show the convergence of the properly normalized partial sum to an α-stable random vector, where $\alpha = 2$ reflects the multivariate normal distribution. In the high frequency setting the limit distribution factorizes in a random factor independent of the MCARMA parameters and a deterministic factor, which is determined by the model parameters (the integral over the kernel function). This is the same pattern as for multivariate ARMA models. However, the normalization differs in the continuous-time and the discrete-time case. The grid distance h_n has an influence on the convergence rate and hence, determines the normalization in the continuous-time model. Furthermore, we study the asymptotic behavior of the sample autocovariance. The results show that in the finite second moment case the sample autocovariance is a consistent estimator for the autocovariance. In the infinite second moment case it converges to an $\alpha/2$-stable random matrix. Again the convergence rate depends on the sampling distance h_n.

Another issue of this paper is the estimation of a cointegrated model in continuous time, where the MCARMA process is embedded. Co-integration plays an important role in financial econometrics, see, e.g., Engle and Granger [14] and is well understood in discrete time if second moments exist (cf. the monographs [24, 27]). Most of the literature on cointegrated models in continuous time is restricted to Gaussian processes as, e.g, [3, 10, 36, 44]. First approaches to drop the Gaussian assumption go back to Phillips [33]; see also Fasen [16] and references therein. Let $\mathbf{L}_1 = (\mathbf{L}_1(t))_{t\in\mathbb{R}}$ be the $\mathbb{R}^m$-valued driving Lévy process of the $\mathbb{R}^d$-valued MCARMA process $\mathbf{V}$ and $\mathbf{L}_2 = (\mathbf{L}_2(t))_{t\in\mathbb{R}}$ be an $\mathbb{R}^\nu$-valued Lévy process independent of $\mathbf{L}_1$. Then we investigate for $\mathbf{A} \in \mathbb{R}^{d\times\nu}$ the *multivariate cointegrated model*

$$\begin{aligned} \mathbf{Y}(t) &= \mathbf{A}\mathbf{X}(t) + \mathbf{V}(t), & t \geq 0, \quad \text{in } \mathbb{R}^d, \\ \mathbf{X}(t) &= \mathbf{L}_2(t), & t \geq 0, \quad \text{in } \mathbb{R}^\nu. \end{aligned} \tag{1.1}$$

The observation scheme is

$$\mathbb{Y}_n' = (\mathbf{Y}(h_n), \ldots, \mathbf{Y}(nh_n)) \in \mathbb{R}^{d\times n}, \qquad \mathbb{X}_n' = (\mathbf{X}(h_n), \ldots, \mathbf{X}(nh_n)) \in \mathbb{R}^{\nu\times n}. \tag{1.2}$$

However, the paper investigates a more general model. Let $(\xi_{n,k})_{k\in\mathbb{Z}}$ and $(\varepsilon_{n,k})_{k\in\mathbb{Z}}$ be independent sequences of iid (independently and identically distributed) random vectors in $\mathbb{R}^m$ and $\mathbb{R}^\nu$, respectively, for any $n \in \mathbb{N}$, and $(\mathbf{C}_{n,k})_{k\in\mathbb{N}}$ be a sequence of deterministic matrices in $\mathbb{R}^{d\times m}$ satisfying some general constraints. Then we may define for any $n \in \mathbb{N}$ the $\mathbb{R}^d$-valued stationary moving average process

$$\mathbf{Z}_{n,k} = \sum_{j=0}^{\infty} \mathbf{C}_{n,j} \xi_{n,k-j} \quad \text{for } k \in \mathbb{N}_0, \tag{1.3}$$

and the cointegrated model as

$$\begin{aligned} \mathbf{Y}_{n,k} &= \mathbf{A}\mathbf{X}_{n,k} + \mathbf{Z}_{n,k} \qquad \text{for } n,k \in \mathbb{N}, \quad \text{in } \mathbb{R}^d, \\ \mathbf{X}_{n,k} &= \mathbf{X}_{n,k-1} + \varepsilon_{n,k} \qquad \text{for } n,k \in \mathbb{N}, \quad \text{in } \mathbb{R}^v. \end{aligned} \tag{1.4}$$

In this case the observation scheme is

$$\mathbb{Y}_n' = (\mathbf{Y}_{n,1},\ldots,\mathbf{Y}_{n,n}) \in \mathbb{R}^{d\times n}, \qquad \mathbb{X}_n' = (\mathbf{X}_{n,1},\ldots,\mathbf{X}_{n,n}) \in \mathbb{R}^{v\times n}. \tag{1.5}$$

Since the high frequency sampled MCARMA process $(\mathbf{V}(kh_n))_{k\in\mathbb{Z}}$ has a representation as in (1.3) and

$$\mathbf{L}_2(kh_n) = \mathbf{L}_2((k-1)h_n) + [\mathbf{L}_2(kh_n) - \mathbf{L}_2((k-1)h_n)],$$

where $(\mathbf{L}_2(kh_n) - \mathbf{L}_2((k-1)h_n))_{k\in\mathbb{N}}$ is an iid sequence by the independent and stationary increment property of a Lévy process, (1.2) can be interpreted as a special case of (1.5). As estimator for $\mathbf{A}$ we use the least squares estimator

$$\widehat{\mathbf{A}}_n = \mathbb{Y}_n'\mathbb{X}_n(\mathbb{X}_n'\mathbb{X}_n)^{-1}. \tag{1.6}$$

The paper is structured in the following way. First, in Section 2 we present some preliminaries on MCARMA processes, regular variation and model assumptions. The main results of this paper on limit theory for high-frequency sampled MCARMA processes but also for equidistant sampled MCARMA processes are topic of Section 3. We show that the properly normalized partial sum $\sum_{k=1}^n \mathbf{V}(kh_n)$ and the sample autocovariance $\sum_{k=1}^n \mathbf{V}(kh_n)\mathbf{V}(kh_n)'$ of the MCARMA process with either $h_n \downarrow 0$ and $nh_n \to \infty$ as $n \to \infty$, or $h_n = h$ (but with different normalization) converge weakly, and we completely characterize their limit distributions. Moreover, we investigate the cointegrated model (1.1)-(1.2). All results are compared to multivariate ARMA models in discrete time. The proofs of this section are based on some general limit theorems as constituted in Section 4. There we present under some general assumptions the asymptotic behavior of $\widehat{\mathbf{A}}_n$ for the multivariate cointegrated model (1.4)-(1.5). Finally, Section 5 contains the proofs of the stated results and the Appendix A involves the asymptotic behavior of stochastic integrals where the driving Lévy process has either a finite second moment or is multivariate regularly varying. On the one hand, these results are interesting for their own but on the other hand, they act as preliminaries to the results in this paper.

We use the notation $\Longrightarrow$ for weak convergence, $\xrightarrow{\mathbb{P}}$ for convergence in probability, and $\stackrel{v}{\Longrightarrow}$ for vague convergence. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\infty\}$ be the compactification of $\mathbb{R}$ and let $\mathscr{B}(\cdot)$ be the Borel-σ-algebra. For two random vectors $\mathbf{X},\mathbf{Y}$ the notation $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$ means equality in distribution. We use as norms the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^d$ and the corresponding operator norm $\|\cdot\|$ for matrices, which is submultiplicative. Recall that two norms on a finite-dimensional linear space are always equivalent and hence, our results remain true if we replace the Euclidean norm by any other norm. For a measurable function $f:(0,\infty)\to(0,\infty)$ and $\alpha \in \mathbb{R}$ we say that f is regularly varying of index $-\alpha$, if $\lim_{x\to\infty} f(xu)/f(x) = u^{-\alpha}$ for any $u > 0$, and we write $f \in \mathscr{R}_{-\alpha}$. The set of $d\times m$ matrices over $\mathbb{R}$ is denoted by $M_{d\times m}(\mathbb{R})$. The matrix $\mathbf{0}_{d\times m}$ is the zero matrix in $M_{d\times m}(\mathbb{R})$ and $\mathbf{I}_{d\times d}$ is the identity matrix in $M_{d\times d}(\mathbb{R})$. For a vector $\mathbf{x} \in \mathbb{R}^d$ we write $\mathbf{x}'$ for its transpose and for $x \in \mathbb{R}$ we write $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \le x\}$. The space $(\mathbb{D}[0,1],\mathbb{R}^d)$ denotes the space of all càdlàg (continue à droite et limitée à gauche = right continuous, with left limits) functions on $[0,1]$ with values in $\mathbb{R}^d$ equipped with the Skorokhod J_1 topology. Finally, for a semimartingale $\mathbf{W} = (\mathbf{W}_1(t),\ldots,\mathbf{W}_d(t))_{t\ge0}$ in $\mathbb{R}^d$ we denote by $[\mathbf{W},\mathbf{W}]_t = ([\mathbf{W}_i,\mathbf{W}_j]_t)_{i,j=1,\ldots,d}$ for $t \ge 0$ the quadratic variation process.

2. Preliminaries

2.1. MCARMA process

Let $\mathbf{L}_1 = (\mathbf{L}_1(t))_{t\in\mathbb{R}}$ be a two-sided $\mathbb{R}^m$-valued Lévy process and $p > q$ are positive integers. Then the d-dimensional MCARMA(p,q) model can be interpreted as the solution to the p-th-order d-dimensional stochastic differential equation

$$\mathbf{P}(D)\mathbf{V}(t) = \mathbf{Q}(D)D\mathbf{L}_1(t) \quad \text{for } t \in \mathbb{R},$$

where D is the differential operator,

$$\mathbf{P}(z) := \mathbf{I}_{d\times d}z^p + \mathbf{P}_1 z^{p-1} + \ldots + \mathbf{P}_{p-1}z + \mathbf{P}_p \tag{2.1}$$

with $\mathbf{P}_1, \ldots, \mathbf{P}_p \in M_{d\times d}(\mathbb{R})$ is the auto-regressive polynomial and

$$\mathbf{Q}(z) := \mathbf{Q}_0 z^q + \mathbf{Q}_1 z^{q-1} + \ldots + \mathbf{Q}_{q-1}z + \mathbf{Q}_q \tag{2.2}$$

with $\mathbf{Q}_0, \ldots, \mathbf{Q}_q \in M_{d\times m}(\mathbb{R})$ is the moving-average polynomial. Since a Lévy process is not differentiable, this definition can not be used, however, it can be interpreted to be equivalent to the following.

Definition 2.1. *Let $(\mathbf{L}_1(t))_{t\in\mathbb{R}}$ be an $\mathbb{R}^m$-valued Lévy process and let the polynomials $\mathbf{P}(z), \mathbf{Q}(z)$ be defined as in* (2.1) *and* (2.2) *with $p, q \in \mathbb{N}_0$, $q < p$, and $\mathbf{Q}_0 \neq \mathbf{0}_{d\times m}$. Moreover, define*

$$\Lambda = -\begin{pmatrix} \mathbf{0}_{d\times d} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times d} & \cdots & \mathbf{0}_{d\times d} \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times d} & \mathbf{I}_{d\times d} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0}_{d\times d} \\ \mathbf{0}_{d\times d} & \cdots & \cdots & \mathbf{0}_{d\times d} & \mathbf{I}_{d\times d} \\ -\mathbf{P}_p & -\mathbf{P}_{p-1} & \cdots & \cdots & -\mathbf{P}_1 \end{pmatrix} \in M_{pd\times pd}(\mathbb{R}),$$

$\mathbf{E} = (\mathbf{I}_{d\times d}, \mathbf{0}_{d\times d}, \ldots, \mathbf{0}_{d\times d}) \in M_{d\times pd}(\mathbb{R})$ *and* $\mathbf{B} = (\mathbf{B}_1' \cdots \mathbf{B}_p')' \in M_{pd\times m}(\mathbb{R})$ *with*

$$\mathbf{B}_1 := \ldots := \mathbf{B}_{p-q-1} := \mathbf{0}_{d\times m} \quad \text{and} \quad \mathbf{B}_{p-j} := -\sum_{i=1}^{p-j-1} \mathbf{P}_i \mathbf{B}_{p-j-i} + \mathbf{Q}_{q-j} \quad \text{for } j = 0, \ldots, q.$$

Assume $\mathscr{N}(\mathbf{P}) = \{z \in \mathbb{C} : \det(\mathbf{P}(z)) = 0\} \subseteq (-\infty, 0) + i\mathbb{R}$. Furthermore, the Lévy measure $\nu_{\mathbf{L}_1}$ of $\mathbf{L}_1$ satisfies

$$\int_{\|\mathbf{x}\|>1} \log\|\mathbf{x}\|\, \nu_{\mathbf{L}_1}(d\mathbf{x}) < \infty.$$

Then the $\mathbb{R}^d$-valued causal MCARMA(p,q) process $(\mathbf{V}(t))_{t\in\mathbb{R}}$ *is defined by the state-space equation*

$$\mathbf{V}(t) = \mathbf{E}\mathbf{Z}(t) \quad \text{for } t \in \mathbb{R}, \tag{2.3}$$

where

$$\mathbf{Z}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\Lambda(t-s)}\mathbf{B}\,\mathrm{d}\mathbf{L}_1(s) \quad \text{for } t \in \mathbb{R} \tag{2.4}$$

is the unique solution to the pd-dimensional stochastic differential equation $\mathrm{d}\mathbf{Z}(t) = -\Lambda\mathbf{Z}(t)\,\mathrm{d}t + \mathbf{B}\,\mathrm{d}\mathbf{L}(t)$. *The function $\mathbf{f}(t) = \mathbf{E}\mathrm{e}^{-\Lambda t}\mathbf{B}\mathbb{1}_{(0,\infty)}(t)$ for $t \in \mathbb{R}$ is called kernel function.*

In particular, the MCARMA$(1,0)$ process and $\mathbf{Z}$ in (2.4) are multivariate Ornstein-Uhlenbeck processes. To see that the MCARMA(p,q) process is well-defined compare Marquardt and Stelzer [28]. Moreover, Lemma 3.8 of Marquardt and Stelzer [28] says that the set $\mathscr{N}(\mathbf{P})$ is equal to the set of eigenvalues of $-\Lambda$, which means that for a MCARMA(p,q) process the eigenvalues of Λ have strictly positive real parts. The class of MCARMA processes is huge. Schlemm and Stelzer [43], Corollary 3.4, showed that the class of state-space models of the form

$$\begin{aligned} \mathrm{d}\widetilde{\mathbf{Z}}(t) &= -\widetilde{\Lambda}\widetilde{\mathbf{Z}}(t)\,\mathrm{d}t + \widetilde{\mathbf{B}}\,\mathrm{d}\mathbf{L}(t), \\ \widetilde{\mathbf{V}}(t) &= \widetilde{\mathbf{C}}\widetilde{\mathbf{Z}}(t), \end{aligned}$$

where $\widetilde{\Lambda} \in \mathbb{R}^{N\times N}$ has only eigenvalues with strictly positive real parts, $\widetilde{\mathbf{B}} \in \mathbb{R}^{N\times m}$ and $\widetilde{\mathbf{C}} \in \mathbb{R}^{d\times N}$ and the class of causal MCARMA processes are equivalent if $\mathbb{E}\|\mathbf{L}(1)\|^2 < \infty$ and $\mathbb{E}(\mathbf{L}(1)) = \mathbf{0}_m$.

2.2. Multivariate regular variation and assumptions

Multivariate regular variation plays a basic part in our model assumption. First, we recall the definition.

Definition 2.2. *A random vector $\mathbf{U} \in \mathbb{R}^d$ is multivariate regularly varying with index $-\alpha < 0$ if and only if there exists a non-zero Radon measure μ on $(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}_d\}, \mathscr{B}(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}_d\}))$ with $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers increasing to ∞ such that*

$$n\mathbb{P}(a_n^{-1}\mathbf{U} \in \cdot) \overset{v}{\Longrightarrow} \mu(\cdot) \quad \text{as } n \to \infty \quad \text{on } \mathscr{B}(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}_d\}),$$

where the limit measure μ is homogenous of order $-\alpha$, i.e., $\mu(uB) = u^{-\alpha}\mu(B)$ for $u > 0$, $B \in \mathscr{B}(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}_d\})$. We write $\mathbf{U} \in \mathscr{R}_{-\alpha}(a_n, \mu)$.

If the representation of the limit measure μ or the norming sequence $(a_n)_{n\in\mathbb{N}}$ does not matter we also write $\mathscr{R}_{-\alpha}(a_n)$ and $\mathscr{R}_{-\alpha}$, respectively. For further information regarding multivariate regular variation of random vectors we refer to Resnick [40].

Definition 2.3. *Let $\mathbf{U}$ be an $\mathbb{R}^d$-valued random vector, $\alpha \in (0,2]$, $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive constants tending to ∞, μ be a Radon measure on $(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}_d\}, \mathscr{B}(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}_d\}))$ with $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ and $\Sigma \in M_{d\times d}(\mathbb{R})$ be a positive semi-definite matrix. We write $U \in DA(\alpha, a_n, \Sigma, \mu)$ if either*

(a) $\alpha < 2$, $\Sigma = \mathbf{0}_{d\times d}$, μ is non-zero and $\mathbf{U} \in \mathscr{R}_{-\alpha}(a_n, \mu)$, or

(b) $\alpha = 2$, $a_n = n^{1/2}$, $\mu = 0$ and $\mathbb{E}\|\mathbf{U}\|^2 < \infty$ with $\mathbb{E}(\mathbf{U}\mathbf{U}') = \Sigma$.

The abbreviation DA stands for *domain of attraction* because of the following argument. Let $(\mathbf{U}_k)_{k\in\mathbb{N}}$ be a sequence of iid $\mathbb{R}^d$-valued random vectors with $\mathbf{U}_1 \in \text{DA}(\alpha, a_n, \mu, \Sigma)$, $\alpha \neq 1$, and $\mathbf{S} = (\mathbf{S}(t))_{t\geq 0}$ be an $\mathbb{R}^d$-valued α-stable Lévy process with characteristic triplet $(\int_{\|\mathbf{x}\|\leq 1} \mathbf{x}\mu(\mathrm{d}\mathbf{x}), \Sigma, \mu)$ if $\alpha \in (0,1)$ and $(-\int_{\|\mathbf{x}\|>1} \mathbf{x}\mu(\mathrm{d}\mathbf{x}), \Sigma, \mu)$ if $\alpha > 1$. In particular if $\alpha = 2$, $\mathbf{S}$ is a Brownian motion with covariance matrix Σ. Assume $\mathbb{E}(\mathbf{U}_1) = \mathbf{0}_d$ if $\alpha > 1$. Then

$$a_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{U}_k \Longrightarrow \mathbf{S} \quad \text{as } n \to \infty \text{ in } \mathbb{D}([0,1], \mathbb{R}^d).$$

This means that the triplet (α, μ, Σ) characterizes completely the limit distribution and $(a_n)_{n\in\mathbb{N}}$ the convergence rate. For $\alpha = 1$ we need additionally a deterministic shift factor to obtain the convergence, which we can neglect if $\mathbf{U}_1$ is symmetric. In general the only possible limit of a normalized partial sum of iid random vectors is an α-stable distribution with $\alpha \in (0,2]$ (cf. Rvačeva [41]). The limit distribution is an α-stable random vector with $\alpha < 2$ if and only if $\mathbf{U}_1$ is multivariate regularly varying of index $-\alpha$. Then also $\mathbb{E}\|\mathbf{U}_1\|^2 = \infty$. On the other hand, $\mathbb{E}\|\mathbf{U}_1\|^2 < \infty$ is only a sufficient assumption to be in the domain of attraction of a normal distribution.

3. Main results

We start with a central limit theorem for MCARMA processes in

Theorem 3.1. *Let $(\mathbf{V}(t))_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued causal MCARMA(p,q) process as given in Definition 2.1 driven by the $\mathbb{R}^m$-valued Lévy process $(\mathbf{L}_1(t))_{t\in\mathbb{R}}$ with $\mathbf{L}_1(1) \in DA(\alpha, a_n, \mu_1, \Sigma_1)$ and $\mathbb{E}(\mathbf{L}_1(1)) = \mathbf{0}_m$ if $\alpha > 1$. Set $a_t := a_{\lfloor t \rfloor}$ for $t \geq 0$. If $\alpha = 1$ we assume additionally that $\mathbf{L}_1(1)$ is symmetric.*
(a) Let $(\mathbf{S}_1(t))_{t\geq 0}$ be an $\mathbb{R}^m$-valued α-stable Lévy process with characteristic triplet $(\int_{\|\mathbf{x}\|\leq 1} \mathbf{x}\mu_1(\mathrm{d}\mathbf{x}), \Sigma_1, \mu_1)$

if $\alpha \in (0,1]$ *and* $(-\int_{\|\mathbf{x}\|>1} \mathbf{x}\mu_1(\mathrm{d}\mathbf{x}), \Sigma_1, \mu_1)$ *if* $\alpha > 1$. *Suppose the sequence of positive constants* $(h_n)_{n\in\mathbb{N}}$ *satisfies* $h_n \downarrow 0$ *as* $n \to \infty$ *and* $\lim_{n\to\infty} nh_n = \infty$. *Then as* $n \to \infty$,

$$h_n a_{nh_n}^{-1} \sum_{k=1}^{n} \mathbf{V}(kh_n) \Longrightarrow \left(\int_0^\infty \mathbf{f}(s)\,\mathrm{d}s\right) \mathbf{S}_1(1).$$

(b) Let $h > 0$ *and let* $(\mathbf{S}_{\mathbf{f},h}(t))_{t\geq 0}$ *be an* $\mathbb{R}^d$*-valued* α*-stable Lévy process with characteristic triplet* $(\int_{\|\mathbf{x}\|\leq 1} \mathbf{x}\mu_{\mathbf{f},h}(\mathrm{d}\mathbf{x}), \Sigma_{\mathbf{f},h}, \mu_{\mathbf{f},h})$ *if* $\alpha \in (0,1]$ *and* $(-\int_{\|\mathbf{x}\|>1} \mathbf{x}\mu_{\mathbf{f},h}(\mathrm{d}\mathbf{x}), \Sigma_{\mathbf{f},h}, \mu_{\mathbf{f},h})$ *if* $\alpha > 1$, *where*

$$\mu_{\mathbf{f},h}(B) = \int_0^h \int_{\mathbb{R}^m} \mathbb{1}_B\left(\sum_{k=0}^\infty \mathbf{f}(kh+s)\mathbf{x}\right) \mu_1(\mathrm{d}\mathbf{x})\mathrm{d}s \qquad \text{for } B \in \mathscr{B}(\mathbb{R}^d\backslash\{\mathbf{0}_d\}), \tag{3.1}$$

$$\Sigma_{\mathbf{f},h} = \int_0^h \left(\sum_{k=0}^\infty \mathbf{f}(kh+s)\right) \Sigma_1 \left(\sum_{k=0}^\infty \mathbf{f}(kh+s)\right)' \mathrm{d}s. \tag{3.2}$$

Suppose $\mathbb{E}\|\mathbf{L}_1(1)\|^r < \infty$ *for some* $r > 2$ *if* $\alpha = 2$. *Then as* $n \to \infty$,

$$a_n^{-1} \sum_{k=1}^{n} \mathbf{V}(kh) \Longrightarrow \mathbf{S}_{\mathbf{f},h}(1).$$

We shall compare this result to the limit results for ARMA models and present a motivation for the normalization.

Remark 3.2.
(a) Let $(\xi_k)_{k\in\mathbb{Z}}$ *be a sequence of iid random vectors in* $\mathbb{R}^m$ *with* $\xi_1 \in \mathscr{R}_{-\alpha}(a_n, \mu_1)$ *for some* $0 < \alpha < 2$. *If* $\alpha > 1$ *then suppose* $\mathbb{E}(\xi_1) = \mathbf{0}_m$, *and if* $\alpha = 1$ *then suppose* ξ_1 *is symmetric. Furthermore, let* $(\mathbf{C}_k)_{k\in\mathbb{N}}$ *be a sequence of matrices in* $M_{d\times m}(\mathbb{R})$ *with* $\sum_{k=0}^\infty k\|\mathbf{C}_k\|^\theta < \infty$ *for some* $0 < \theta < \alpha$, $\theta \leq 1$. *The* $\mathbb{R}^d$*-valued stationary MA process* $(\mathbf{X}_k)_{k\in\mathbb{Z}}$ *is defined as*

$$\mathbf{X}_k = \sum_{j=0}^\infty \mathbf{C}_j \xi_{k-j} \quad \text{for } k \in \mathbb{Z}. \tag{3.3}$$

Then a special case of Theorem 4.2 (from below) is that as $n \to \infty$,

$$a_n^{-1} \sum_{k=1}^{n} \mathbf{X}_k \Longrightarrow \left(\sum_{k=0}^\infty \mathbf{C}_k\right) \mathbf{S}_1(1).$$

On the one hand, we observe the similar structure of the limit distribution $(\int_0^\infty \mathbf{f}(s)\,\mathrm{d}s)\,\mathbf{S}_1(1)$ *and* $(\sum_{k=0}^\infty \mathbf{C}_k)\,\mathbf{S}_1(1)$ *in the continuous-time high frequency and the discrete-time model. On the other hand, the normings are different. To explain the different normings we consider an* α*-stable Lévy process* $(\mathbf{L}_1(t))_{t\geq 0}$ *and an* α*-stable random variable* ξ_1. *Then the idea in the continuous-time model is that as* $n \to \infty$,

$$\begin{aligned} h_n a_{nh_n}^{-1} \sum_{k=1}^{n} \mathbf{V}(kh_n) &= \left(\sum_{j=0}^\infty \mathbf{f}(jh_n)h_n\right) \left(a_{nh_n}^{-1} \sum_{k=1}^{n} [\mathbf{L}_1(kh_n) - \mathbf{L}_1((k-1)h_n)]\right) + o_P(1) \qquad (3.4)\\ &\stackrel{d}{=} \left(\sum_{j=0}^\infty \mathbf{f}(jh_n)h_n\right) \left((nh_n)^{-\frac{1}{\alpha}} h_n^{\frac{1}{\alpha}} \sum_{k=1}^{n} [\mathbf{L}_1(k) - \mathbf{L}_1(k-1)]\right) + o_P(1)\\ &\stackrel{d}{=} \left(\sum_{j=0}^\infty \mathbf{f}(jh_n)h_n\right) \mathbf{S}_1(1) + o_p(1) = \left(\int_0^\infty \mathbf{f}(s)\,\mathrm{d}s\right) \mathbf{S}_1(1) + o_P(1), \end{aligned}$$

and in the discrete-time model that as $n\to\infty$,

$$a_n^{-1}\sum_{k=1}^{n}\mathbf{X}_k=\left(\sum_{j=0}^{\infty}\mathbf{C}_j\right)\left(a_n^{-1}\sum_{k=1}^{n}\xi_k\right)+o_p(1)\stackrel{d}{=}\left(\sum_{j=0}^{\infty}\mathbf{C}_j\right)\xi_1+o_p(1). \tag{3.5}$$

In (3.4) *and* (3.5) *we see where the different normings have their origin. In the continuous-time model, the h_n of the norming $h_n a_{nh_n}^{-1}$ goes into the first factor of (3.4), which converges to $(\int_0^\infty \mathbf{f}(s)\,\mathrm{d}s)$ and the norming $a_{nh_n}^{-1}$ goes into the second, the random factor.*
(b) Representation (3.4) *gives also a motivation for the fact that the classical techniques of Davis and Resnick [12] to prove the asymptotic behavior of one-dimensional MA processes by using truncated MA processes will not work for the high-frequency case, because $\lim_{n\to\infty}\sum_{j=0}^{M}\mathbf{f}(jh_n)h_n=\mathbf{0}_{d\times m}$ for $M>0$.* □

Remark 3.3. *A straightforward extension is the convergence of the finite dimensional distribution for any $l\in\mathbb{N}$, as $n\to\infty$,*

$$h_n a_{nh_n}^{-1}\left(\sum_{k=1}^{n}\mathbf{V}(kh_n),\ldots,\sum_{k=1}^{n}\mathbf{V}((k+l)h_n)\right)\Longrightarrow\left(\int_0^\infty\mathbf{f}(s)\,\mathrm{d}s\right)(\mathbf{S}_1(1),\ldots,\mathbf{S}_1(1))$$

since for any $l\in\mathbb{N}_0$,

$$h_n a_{nh_n}^{-1}\sum_{k=1}^{n}\mathbf{V}((k+l)h_n)=\left(\sum_{j=0}^{\infty}\mathbf{f}(jh_n)h_n\right)\left(a_{nh_n}^{-1}\sum_{k=1}^{n}[\mathbf{L}_1(kh_n)-\mathbf{L}_1((k-1)h_n)]\right)+o_P(1)$$

as in (3.4)*.* □

Next we investigate the co-integrated model (1.1)-(1.2).

Theorem 3.4. *Let model (1.1)-(1.2) be given where $\mathbb{X}_n$ has full rank and let the assumptions of Theorem 3.1 hold. Furthermore, let $(\mathbf{L}_2(t))_{t\in\mathbb{R}}$ be an $\mathbb{R}^v$-valued Lévy process independent of $(\mathbf{L}_1(t))_{t\in\mathbb{R}}$, where $\mathbf{L}_2(1)\in DA(\beta,b_n,\mu_2,\Sigma_2)$ and $\mathbb{E}(\mathbf{L}_2(1))=\mathbf{0}_v$ if $\beta>1$. If $\beta=1$ assume additionally that $\mathbf{L}_2(1)$ is symmetric. Set $a_t:=a_{\lfloor t\rfloor}$ and $b_t=b_{\lfloor t\rfloor}$ for $t\geq 0$. Moreover, let $(\mathbf{S}_2(t))_{t\geq 0}$ be an $\mathbb{R}^v$-valued β-stable Lévy process independent of $(\mathbf{S}_1(t))_{t\geq 0}$ with characteristic triplet $(\int_{\|\mathbf{x}\|\leq 1}\mathbf{x}\mu_2(\mathrm{d}\mathbf{x}),\Sigma_2,\mu_2)$ if $\beta\in(0,1]$ and $(-\int_{\|\mathbf{x}\|>1}\mathbf{x}\mu_2(\mathrm{d}\mathbf{x}),\Sigma_2,\mu_2)$ if $\beta>1$, and suppose*

$$\mathbb{P}\left(\det\left(\int_0^1\mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s\right)=0\right)=0.$$

(a) Suppose the sequence of positive constants $(h_n)_{n\in\mathbb{N}}$ satisfies $h_n\downarrow 0$ as $n\to\infty$ and $\lim_{n\to\infty}nh_n=\infty$. If $\min(\alpha,\beta)<2$ and either $\nu_{\mathbf{L}_2}(\mathbb{R}^v)=\infty$ or $\Sigma_{\mathbf{L}_2}\neq\mathbf{0}_{v\times v}$ we additionally assume that for some $\varepsilon>0$,

$$\begin{aligned}&\lim_{n\to\infty}n^{\frac{1}{\min(\alpha,\beta)}+\varepsilon}h_n^{\frac{1}{2}}a_{nh_n}^{-1}b_{nh_n}^{-1}=0 && \text{if } \min(\alpha,\beta)\leq 1, \text{ and moreover,}\\ &\lim_{n\to\infty}nh_n^{\frac{1}{2}}a_{nh_n}^{-1}b_{nh_n}^{-1}=0 && \text{if } 1<\min(\alpha,\beta)<2.\end{aligned} \tag{3.6}$$

Then $\widehat{\mathbf{A}}_n$ as given in (1.6) *satisfies as $n\to\infty$,*

$$nh_n a_{nh_n}^{-1}b_{nh_n}(\widehat{\mathbf{A}}_n-\mathbf{A})\Longrightarrow\left(\int_0^\infty\mathbf{f}(s)\,\mathrm{d}s\right)\left(\mathbf{S}_1(1)\mathbf{S}_2(1)'-\int_0^1\mathbf{S}_1(s-)\mathrm{d}\mathbf{S}_2(s)'\right)\left(\int_0^1\mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s\right)^{-1}.$$

In particular, $\widehat{\mathbf{A}}_n\xrightarrow{\mathbb{P}}\mathbf{A}$ as $n\to\infty$ if $\alpha>\beta/(\beta+1)$, i.e. $\widehat{\mathbf{A}}_n$ is a consistent estimator.
(b) Let $h>0$ and $h_n=h$ for any $n\in\mathbb{N}$. Suppose $\mathbb{E}\|\mathbf{L}_1(1)\|^r<\infty$ for some $r>2$ if $\alpha=2$. Then $\widehat{\mathbf{A}}_n$ as given in (1.6) *satisfies as $n\to\infty$,*

$$na_n^{-1}b_n(\widehat{\mathbf{A}}_n-\mathbf{A})\Longrightarrow\left(\mathbf{S}_{\mathbf{f},h}(1)\mathbf{S}_2(1)'-\int_0^1\mathbf{S}_{\mathbf{f},h}(s-)\mathrm{d}\mathbf{S}_2(s)'\right)\left(\int_0^1\mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s\right)^{-1}.$$

In particular, $\widehat{\mathbf{A}}_n \xrightarrow{\mathbb{P}} \mathbf{A}$ *as* $n \to \infty$ *if* $\alpha > \beta/(\beta+1)$*, i.e.* $\widehat{\mathbf{A}}_n$ *is a consistent estimator.*

Remark 3.5.

(a) Assumption (3.6) *can be relaxed, which goes beyond this paper because it uses a completely different approach, and can be found in Fasen [15].*

(b) If $\alpha = \beta < 2$*, sufficient conditions for* (3.6) *are that for some* $\varepsilon > 0$*,*

$$\begin{aligned} &\lim_{n\to\infty} nh_n^{2-\frac{\alpha}{2}+\varepsilon} = \infty \qquad \text{if} \quad \alpha \le 1, \\ &\lim_{n\to\infty} nh_n^{\frac{1}{2}+\frac{1}{2-\alpha}+\varepsilon} = \infty \quad \text{if} \quad 1 < \alpha < 2 \end{aligned}$$

holds. □

Finally, we investigate the asymptotic behavior of the sample autocovariance. Both Theorem 3.1 and Theorem 3.6 are used in Fasen and Fuchs [17, 18] to derive the asymptotic behavior of the normalized, the self-normalized and the smoothed periodogram as well as for statistical inference of CARMA processes.

Theorem 3.6. *Let* $(\mathbf{V}(t))_{t\ge 0}$ *be an* $\mathbb{R}^d$*-valued MCARMA*(p,q) *process as given in Definition 2.1 driven by the* $\mathbb{R}^m$*-valued Lévy process* $(\mathbf{L}_1(t))_{t\in\mathbb{R}}$ *with* $\mathbf{L}_1(1) \in DA(\alpha, a_n, \mu_1, \Sigma_1)$*. Set* $a_t := a_{\lfloor t \rfloor}$ *for* $t \ge 0$*.*
(a) Let $(\mathbf{S}_1(t))_{t\ge 0}$ *be an* $\mathbb{R}^m$*-valued* α*-stable Lévy process with characteristic triplet* $(\mathbf{0}_m, \Sigma_1, \mu_1)$*. Suppose the sequence of positive constants* $(h_n)_{n\in\mathbb{N}}$ *satisfies* $h_n \downarrow 0$ *as* $n\to\infty$ *and* $\lim_{n\to\infty} nh_n = \infty$*. Then as* $n \to \infty$*,*

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \mathbf{V}(kh_n)\mathbf{V}(kh_n)' \Longrightarrow \int_0^\infty \mathbf{f}(s)[\mathbf{S}_1,\mathbf{S}_1]_1 \mathbf{f}(s)' \,\mathrm{d}s,$$

which is equal to $\mathbb{E}(\mathbf{V}(0)\mathbf{V}(0)')$ *if* $\alpha = 2$*. In particular, this means for a one-dimensional CARMA process* $(V(t))_{t\ge 0}$ *with* $f = \mathbf{f}$*,* $L_1 = \mathbf{L}_1$ *and* $S_1 = \mathbf{S}_1$ *that as* $n\to\infty$*,*

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} V(kh_n)^2 \Longrightarrow \left(\int_0^\infty f^2(s)\,\mathrm{d}s\right)[S_1,S_1]_1.$$

(b) Let $h > 0$ *and let* $(\mathbf{S}_{\mathbf{f},h}(t))_{t\ge 0}$ *be an* $\mathbb{R}^d$*-valued* α*-stable Lévy process with characteristic triplet* $(\mathbf{0}_d, \Sigma_{\mathbf{f},h}, \mu_{\mathbf{f},h})$ *where* $\mu_{\mathbf{f},h}$ *and* $\Sigma_{\mathbf{f},h}$ *are given as in (3.1) and (3.2), respectively. Then as* $n\to\infty$*,*

$$a_n^{-2} \sum_{k=1}^{n} \mathbf{V}(kh)\mathbf{V}(kh)' \Longrightarrow [\mathbf{S}_{\mathbf{f},h}, \mathbf{S}_{\mathbf{f},h}]_1,$$

which is equal to $\Sigma_{\mathbf{f},h}$ *if* $\alpha = 2$*.*

Thus, if $\mathbb{E}\|\mathbf{L}_1(1)\|^2 < \infty$, the sample autocovariance is a consistent estimator. Further, we want to point out that in contrast to Theorem 3.1, Theorem 3.6 does not require $\mathbb{E}(\mathbf{L}_1(1)) = \mathbf{0}_d$ if $1 < \alpha < 2$ and the symmetry of $\mathbf{L}_1(1)$ if $\alpha = 1$. Also the drift term of $\mathbf{S}_1$ can be chosen arbitrary since it doesn't has an influence on $[\mathbf{S}_1, \mathbf{S}_1]_1$.

As in Remark 3.2 we shall make a comparison to the discrete-time case.

Remark 3.7.

Let a discrete-time MA process as in Remark 3.2 be given. Then by Davis et al. [11], Theorem 2.1, for the 2-dimensional case (see also Meerschaert and Scheffler [29], (4.7)) as $n\to\infty$*,*

$$a_n^{-2} \sum_{k=1}^{n} \mathbf{X}_k \mathbf{X}_k' \Longrightarrow \sum_{k=0}^{\infty} \mathbf{C}_k [\mathbf{S}_1, \mathbf{S}_1]_1 \mathbf{C}_k'.$$

Again we see the similarity between the continuous-time high frequency and the discrete-time model. Considering an α*-stable Lévy process* $(\mathbf{L}_1(t))_{t\ge 0}$ *and an* α*-stable random variable* ξ_1*, the normings can be* *understood in the continuous-time high-frequency model by*

$$\begin{aligned}
&h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \mathbf{V}(kh_n)\mathbf{V}(kh_n)' \\
&= \sum_{j=0}^{\infty} \mathbf{f}(jh_n) \left(a_{nh_n}^{-2} \sum_{k=1}^{n} [\mathbf{L}_1(kh_n) - \mathbf{L}_1((k-1)h_n)][\mathbf{L}_1(kh_n) - \mathbf{L}_1((k-1)h_n)]' \right) \mathbf{f}(jh_n)' h_n + o_P(1) \\
&\stackrel{d}{=} \sum_{j=0}^{\infty} \mathbf{f}(jh_n)[\mathbf{L}_1, \mathbf{L}_1]_1' \mathbf{f}(jh_n)' h_n + o_P(1) \\
&= \int_0^{\infty} \mathbf{f}(s)[\mathbf{S}_1, \mathbf{S}_1]_1 \mathbf{f}(s)' \, \mathrm{d}s + o_P(1).
\end{aligned}$$

The first factor h_n of $h_n a_{nh_n}^{-2}$ is required for the convergence of the integral and $a_{nh_n}^{-2}$ for the random part. In the discrete-time model we have

$$a_n^{-2} \sum_{k=1}^{n} \mathbf{X}_k \mathbf{X}_k' = \sum_{j=0}^{\infty} \mathbf{C}_j \left(a_n^{-2} \sum_{k=1}^{n} \xi_k \xi_k' \right) \mathbf{C}_j' + o_P(1) \stackrel{d}{=} \sum_{j=0}^{\infty} \mathbf{C}_j [\mathbf{S}_1, \mathbf{S}_1]_1 \mathbf{C}_j' + o_P(1).$$

□

Remark 3.8. *The finite dimensional distribution of the sample autocovariance function has for any $l \in \mathbb{N}$ the asymptotic behavior as $n \to \infty$,*

$$\begin{aligned}
&h_n a_{nh_n}^{-2} \left(\sum_{k=1}^{n} \mathbf{V}(kh_n)\mathbf{V}(kh_n)', \ldots, \sum_{k=1}^{n} \mathbf{V}(kh_n)\mathbf{V}((k+l)h_n)' \right) \\
&\Longrightarrow \left(\int_0^{\infty} \mathbf{f}(s)[\mathbf{S}_1, \mathbf{S}_1]_1 \mathbf{f}(s)' \, \mathrm{d}s, \ldots, \int_0^{\infty} \mathbf{f}(s)[\mathbf{S}_1, \mathbf{S}_1]_1 \mathbf{f}(s)' \, \mathrm{d}s \right).
\end{aligned}$$

□

4. Multivariate high frequency model

Under the following general assumption we derive the properties of the least squares estimator given in (1.6) for model (1.4)-(1.5). As mentioned in the introduction and used in the proof of Theorem 3.1, the cointegrated MCARMA model can been seen as a special case of this more general model.

Assumption 4.1. *Let model* (1.4)-*(1.5) be given.*

(a) Suppose that there exist sequences of positive constants $\widetilde{a}_n, \widetilde{b}_n \uparrow \infty$ as $n \to \infty$ such that

$$\left(\widetilde{a}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \xi_{n,k}', \widetilde{b}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \varepsilon_{n,k}' \right)_{t \geq 0} \Longrightarrow (\mathbf{S}_1(t)', \mathbf{S}_2(t)')_{t \geq 0} \quad \text{as } n \to \infty \text{ in } \mathbb{D}([0,1], \mathbb{R}^{m+v}), \tag{4.1}$$

where $\mathbf{S}_1 = (\mathbf{S}_1(t))_{t \geq 0}$ is a càdlàg stochastic process in $\mathbb{R}^m$ and $\mathbf{S}_2 = (\mathbf{S}_2(t))_{t \geq 0}$ is a càdlàg stochastic process in $\mathbb{R}^v$, respectively. Furthermore, suppose that

$$\mathbb{P}\left(\det\left(\int_0^1 \mathbf{S}_2(s)\mathbf{S}_2(s)' \mathrm{d}s \right) = 0 \right) = 0. \tag{4.2}$$

(b) Define

$$\widetilde{\mathbf{Z}}_{n,k} := \sum_{j=0}^{\infty} \left(\sum_{l=j+1}^{\infty} \mathbf{C}_{n,l} \right) \xi_{n,k-j} \quad \text{for } k \in \mathbb{N}_0, n \in \mathbb{N}.$$

Suppose that there exist a sequence of positive constants $(h_n)_{n\in\mathbb{N}}$ and a positive bounded decreasing function g with either $g \in \mathscr{R}_{-\alpha}$, $\alpha \in (0,2)$, or $\int_0^\infty xg(x)\,\mathrm{d}x < \infty$ and $\alpha := 2$, such that

$$\mathbb{P}(h_n\|\widetilde{\mathbf{Z}}_{n,0}\| > x) \le g(x) \quad \text{for } x \ge 0,\, n \in \mathbb{N}.$$

(c) *Let for some $0 < \theta < \alpha$ and $\theta \le 1$,*

$$\sum_{k=0}^{\infty} k\|\mathbf{C}_{n,k}\|^{\theta} < \infty.$$

Furthermore, there exists a matrix $\mathbf{C} \in M_{d\times m}(\mathbb{R})$ for $(h_n)_{n\in\mathbb{N}}$ in (b) such that

$$\lim_{n\to\infty} h_n \sum_{k=0}^{\infty} \mathbf{C}_{n,k} = \mathbf{C}.$$

(d) *There exist constants $K_1, K_2, K_3 < \infty$ and some $0 < \delta < \alpha$ with $\delta \le 1$ such that the following holds:*

(i) $n\widetilde{b}_n^{-2}\mathbb{E}(\|\varepsilon_{n,1}\|^2 \mathbb{1}_{\{\|\varepsilon_{n,1}\|\le \widetilde{b}_n\}}) \le K_1 \;\forall n \in \mathbb{N}$.

(ii) $n\widetilde{b}_n^{-1}\|\mathbb{E}(\varepsilon_{n,1}\mathbb{1}_{\{\|\varepsilon_{n,1}\|\le \widetilde{b}_n\}})\| \le K_2 \;\forall n \in \mathbb{N}$.

(iii) $n\widetilde{b}_n^{-\delta}\mathbb{E}(\|\varepsilon_{n,1}\|^{\delta} \mathbb{1}_{\{\|\varepsilon_{n,1}\| > \widetilde{b}_n\}}) \le K_3 \;\forall n \in \mathbb{N}$.

Furthermore, one of the following conditions is satisfied for g in (b):

(iv1) $g \in \mathscr{R}_{-\alpha}$ *with* $\alpha \in (0,2)$ *and* $\lim_{n\to\infty} n\widetilde{a}_n^{-\delta}\widetilde{b}_n^{-\delta}\mathbb{E}\|\varepsilon_{n,1}\|^{\delta} = 0$.

(iv2) $\int_0^\infty xg(x)\,\mathrm{d}x < \infty$ *and* $\lim_{n\to\infty} n\widetilde{a}_n^{-2}\widetilde{b}_n^{-2}\mathbb{E}\|\varepsilon_{n,1}\|^2 = 0$.

Note that if g is a positive bounded decreasing function with $g \in \mathscr{R}_{-\alpha}$, $\alpha \in (0,2)$ then $\int_0^\infty x^{\gamma-1}g(x)\,\mathrm{d}x < \infty$ for any $0 < \gamma < \alpha$ (apply Karamata's Theorem (cf. Resnick [40], Theorem 2.1)). Moreover, $\lim_{n\to\infty} g(\widetilde{a}_n) = 0$.

We start with the first limit result.

Theorem 4.2. *Let model (1.4)-(1.5) be given where $\mathbb{X}_n$ has full rank and let Assumption 4.1 hold. Define*

$$\mathbf{S}_{1,n}(t) := h_n\widetilde{a}_n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}\mathbf{Z}_{n,k} \qquad \text{and} \qquad \mathbf{S}_{2,n}(t) := \widetilde{b}_n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}\varepsilon_{n,k} \quad \text{for } t \ge 0,\, n \in \mathbb{N}.$$

Then as $n \to \infty$,

$$\left(\mathbf{S}_{1,n}(1), \mathbf{S}_{2,n}(1), \int_0^1 \mathbf{S}_{2,n}(s)\mathbf{S}_{2,n}(s)'\mathrm{d}s, \int_0^1 \mathbf{S}_{1,n}(s-)\,\mathrm{d}\mathbf{S}_{2,n}(s)'\right) \Longrightarrow \left(\mathbf{C}\mathbf{S}_1(1), \mathbf{S}_2(1), \int_0^1 \mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s, \mathbf{C}\int_0^1 \mathbf{S}_1(s-)\,\mathrm{d}\mathbf{S}_2(s)'\right)$$

in $\mathbb{R}^d \times \mathbb{R}^v \times \mathbb{R}^{v\times v} \times \mathbb{R}^{d\times v}$.

Based on this theorem we are able to derive the asymptotic behavior of the least squares estimator in the cointegrated model.

Theorem 4.3. *Let model (1.4)-(1.5) be given and let Assumption 4.1 hold. Then $\widehat{\mathbf{A}}_n$ as given in* (1.6) *satisfies as $n \to \infty$,*

$$nh_n\widetilde{a}_n^{-1}\widetilde{b}_n(\widehat{\mathbf{A}}_n - \mathbf{A}) \Longrightarrow \mathbf{C}\left(\mathbf{S}_1(1)\mathbf{S}_2(1)' - \int_0^1 \mathbf{S}_1(s-)\mathrm{d}\mathbf{S}_2(s)'\right)\left(\int_0^1 \mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s\right)^{-1}.$$

In particular, $\widehat{\mathbf{A}}_n \xrightarrow{\mathbb{P}} \mathbf{A}$ as $n \to \infty$ if $\lim_{n\to\infty} nh_n\widetilde{a}_n^{-1}\widetilde{b}_n = \infty$, i.e. $\widehat{\mathbf{A}}_n$ is a consistent estimator.

5. Proofs

5.1. Proofs of Section 4

The proofs of this section are very similar to Fasen [16]. However, we mimic them to show where the different assumptions are going in. An essential piece of the proof will be that as $n \to \infty$,

$$h_n \widetilde{a}_n^{-1} \sum_{k=1}^{n} \mathbf{Z}_{n,k} = \left(h_n \sum_{j=0}^{\infty} \mathbf{C}_{n,j} \right) \left(\widetilde{a}_n^{-1} \sum_{k=1}^{n} \xi_{n,k} \right) + o_P(1). \tag{5.1}$$

As Lemma 5.6 in Fasen [16] we can prove the next lemma. This lemma we require for the proof of Theorem 3.6 and Theorem 4.2.

Lemma 5.1. *Let $(\varepsilon_{n,k})_{k\in\mathbb{N}}$ be an iid sequence of random vectors in $\mathbb{R}^v$ for any $n \in \mathbb{N}$, and let $(\mathbf{W}_{n,k})_{k\in\mathbb{N}}$ be a sequence of random vectors in $\mathbb{R}^d$ for any $n \in \mathbb{N}$, where $(\mathbf{W}_{n,k-j})_{j=1,\dots,k-1}$ is independent of $(\varepsilon_{n,k+j})_{j\in\mathbb{N}}$ for any $n,k \in \mathbb{N}$. Suppose that there exists a positive, bounded, decreasing function g such that*

$$\mathbb{P}(\|\mathbf{W}_{n,k}\| > x) \le g(x) \quad \text{for any } x \ge 0, n \in \mathbb{N}, k \in \mathbb{N}.$$

Assume that one of the following conditions is satisfied:

(1) $g \in \mathscr{R}_{-\alpha}$, $0 < \alpha < 2$, and for some $0 < \delta \le 1$, $\delta < \alpha$, the condition $\lim_{n\to\infty} n \widetilde{a}_n^{-\delta} \widetilde{b}_n^{-\delta} \mathbb{E}\|\varepsilon_{n,1}\|^\delta = 0$ holds.

(2) $\int_0^\infty x g(x)\,\mathrm{d}x < \infty$, $\mathbb{E}(\varepsilon_{n,1}) = \mathbf{0}_v$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} n \widetilde{a}_n^{-2} \widetilde{b}_n^{-2} \mathbb{E}\|\varepsilon_{n,1}\|^2 = 0$.

Then as $n \to \infty$,

$$\widetilde{a}_n^{-1} \widetilde{b}_n^{-1} \sum_{k=1}^{n} \mathbf{W}_{n,k-1} \varepsilon'_{n,k} \xrightarrow{\mathbb{P}} \mathbf{0}_{d\times v}.$$

Proof. *Case (1).* Taking $\delta \le 1$ into account we have

$$\begin{aligned}
\widetilde{a}_n^{-\delta} \widetilde{b}_n^{-\delta} \mathbb{E} \left\| \sum_{k=1}^{n} \mathbf{W}_{n,k-1} \varepsilon'_{n,k} \right\|^\delta &\le \widetilde{a}_n^{-\delta} \widetilde{b}_n^{-\delta} \sum_{k=1}^{n} \mathbb{E} \left\| \mathbf{W}_{n,k-1} \right\|^\delta \mathbb{E} \left\| \varepsilon_{n,k} \right\|^\delta \\
&\le n \widetilde{a}_n^{-\delta} \widetilde{b}_n^{-\delta} \left(\delta \int_0^\infty x^{\delta-1} g(x)\,\mathrm{d}x \right) \mathbb{E} \left\| \varepsilon_{n,1} \right\|^\delta \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Case (2). We investigate the sequence of random matrices componentwise and denote by (l,m) the component in the l-th row and m-th column. Since $((\mathbf{W}_{n,k-1}\varepsilon'_{n,k})_{(l,m)})_{k\in\mathbb{N}}$ are uncorrelated,

$$\begin{aligned}
\widetilde{a}_n^{-2} \widetilde{b}_n^{-2} \mathbb{E} \left(\left(\sum_{k=1}^{n} \mathbf{W}_{n,k-1} \varepsilon'_{n,k} \right)^2_{(l,m)} \right) &= \widetilde{a}_n^{-2} \widetilde{b}_n^{-2} \sum_{k=1}^{n} \mathbb{E} \left(\left(\mathbf{W}_{n,k-1} \varepsilon'_{n,k} \right)^2_{(l,m)} \right) \\
&\le C_1 \widetilde{a}_n^{-2} \widetilde{b}_n^{-2} \sum_{k=1}^{n} \mathbb{E}\|\mathbf{W}_{n,k-1}\|^2 \mathbb{E}\|\varepsilon_{n,k}\|^2 \\
&\le C_2 n \widetilde{a}_n^{-2} \widetilde{b}_n^{-2} \mathbb{E}\|\varepsilon_{n,1}\|^2.
\end{aligned}$$

The last expression tends to 0 as $n \to \infty$ by assumption. □

We will prove Theorem 4.2 by an application of Jacod and Shiryaev [23], Theorem VI.6.22. Therefore, we need some definition.

Definition 5.2. *Let $\mathbf{S}^n = (\mathbf{S}^n(t))_{t\ge0} = (S_1^n(t), \dots, S_v^n(t))_{t\ge0}$ for any $n \in \mathbb{N}$ be an $\mathbb{R}^{m\times v}$-valued adapted càdlàg stochastic process on $(\Omega, \mathscr{F}, ((\mathscr{F}_t^n)_{t\ge0})_{n\in\mathbb{N}}, \mathbb{P})$ and $\mathscr{H}^n$ be the set of all $(\mathscr{F}_t^n)_{t\ge0}$ predictable pro-*

cesses $\mathbf{H}^n$ *in* $\mathbb{R}^{d\times m}$ *having the form*

$$\mathbf{H}^n_t = \mathbf{Y}^n_0 \mathbb{1}_{\{0\}} + \sum_{k=1}^{m(\mathbf{H}^n)} \mathbf{Y}^n_k \mathbb{1}_{(t^n_k, t^n_{k+1}]}(t) \quad \text{for } t \geq 0$$

with $m(\mathbf{H}^n) \in \mathbb{N}$, $0 = t^n_0 < \ldots < t^n_{m(\mathbf{H}^n)+1} < \infty$, *and* $\mathbf{Y}^n_k$ *in* $\mathbb{R}^{d\times m}$ *is* $\mathscr{F}^n_{t^n_k}$*-measurable with* $\|\mathbf{Y}^n_k\| \leq 1$. *Then the sequence of stochastic processes* $(\mathbf{S}^n)_{n\in\mathbb{N}}$ *is said to be* predictably uniformly tight (P-UT) *if for any* $t > 0$*:*

$$\lim_{x\uparrow\infty} \sup_{\mathbf{H}^n\in\mathscr{H}^n, n\in\mathbb{N}} \mathbb{P}\left(\left\| \sum_{k=1}^{m(\mathbf{H}^n)} \mathbf{Y}^n_k (\mathbf{S}^n(t^n_{k+1}\wedge t) - \mathbf{S}^n(t^n_k \wedge t)) \right\| > x \right) = 0.$$

Similarly to Lemma 5.5 in Fasen [16] we derive the next Lemma.

Lemma 5.3. *Let Assumptions 4.1 (d) hold. Then the sequence of stochastic processes* $(\mathbf{S}_{2,n})_{n\in\mathbb{N}}$ *as given in Theorem 4.2 is P-UT on* $(\Omega, \mathscr{F}, ((\mathscr{F}^n_t)_{t\geq 0})_{n\in\mathbb{N}}, \mathbb{P})$ *with* $\mathscr{F}^n_t = \sigma(\varepsilon_{n,k} : k \leq \lfloor nt \rfloor)$, $t \geq 0$, $n \in \mathbb{N}$.

Proof. We define for $t \geq 0$, $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbf{M}_n(t) &:= \widetilde{b}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \left(\varepsilon_{n,k} \mathbb{1}_{\{\|\varepsilon_{n,k}\| \leq \widetilde{b}_n\}} - \mathbb{E}(\varepsilon_{n,1} \mathbb{1}_{\{\|\varepsilon_{n,1}\| \leq \widetilde{b}_n\}}) \right), \\
\mathbf{D}^{(1)}_n(t) &:= \lfloor nt \rfloor \widetilde{b}_n^{-1} \mathbb{E}\left(\varepsilon_{n,1} \mathbb{1}_{\{\|\varepsilon_{n,1}\| \leq \widetilde{b}_n\}} \right), \\
\mathbf{D}^{(2)}_n(t) &:= \widetilde{b}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \varepsilon_{n,k} \mathbb{1}_{\{\|\varepsilon_{n,k}\| > \widetilde{b}_n\}}.
\end{aligned}$$

It is obvious that $(\mathbf{M}_n(t))_{t\geq 0}$ is a martingale with respect to $(\mathscr{F}^n_t)_{t\geq 0}$ and in particular, a local martingale. All three processes are adapted with respect to $(\mathscr{F}^n_t)_{t\geq 0}$ and we have the semimartingale decomposition

$$\mathbf{S}_{2,n}(t) = \mathbf{M}_n(t) + \mathbf{D}^{(1)}_n(t) + \mathbf{D}^{(2)}_n(t).$$

If $(\mathbf{M}_n)_{n\in\mathbb{N}}$, $(\mathbf{D}^{(1)}_n)_{n\in\mathbb{N}}$ and $(\mathbf{D}^{(2)}_n)_{n\in\mathbb{N}}$ are *P-UT* then VI.6.4 in Jacod and Shiryaev [23] gives that the sum $(\mathbf{S}_{2,n})_{n\in\mathbb{N}}$ is *P-UT* as well.

Let $\mathrm{VT}_s(\mathbf{W}) = \sup_{i=1,\ldots,v} \mathrm{VT}_s(\mathbf{W}_i)$ for $s \geq 0$ denote the variation process of the càdlàg stochastic process $(\mathbf{W}(s))_{s\geq 0} = (\mathbf{W}_1(s), \ldots, \mathbf{W}_v(s))_{s\geq 0}$. To prove the *P-UT*ness of $(\mathbf{D}^{(1)}_n)_{n\in\mathbb{N}}$ and $(\mathbf{D}^{(2)}_n)_{n\in\mathbb{N}}$ it is sufficient to show that $(\mathrm{VT}_t(\mathbf{D}^{(1)}_n))_{n\in\mathbb{N}}$ and $(\mathrm{VT}_t(\mathbf{D}^{(2)}_n))_{n\in\mathbb{N}}$ are tight for any $t \geq 0$; see Jacod and Shiryaev [23], VI.6.6. Let $t \geq 0$ be fixed. We start with the verification of the tightness of $(\mathrm{VT}_t(\mathbf{D}^{(1)}_n))_{n\in\mathbb{N}}$ by showing that it is uniformly bounded. Assumption 4.1 *(d) (ii)* gives the uniform bound

$$\sup_{n\in\mathbb{N}} \mathrm{VT}_t(\mathbf{D}^{(1)}_n) \leq C_1 \sup_{n\in\mathbb{N}} nt \widetilde{b}_n^{-1} \left\| \mathbb{E}(\varepsilon_{n,1} \mathbb{1}_{\{\|\varepsilon_{n,1}\| \leq \widetilde{b}_n\}}) \right\| \leq C_2 t, \tag{5.2}$$

which results in the tightness of $(\mathrm{VT}_t(\mathbf{D}^{(1)}_n))_{n\in\mathbb{N}}$.

For the proof of the tightness of $(\mathrm{VT}_t(\mathbf{D}^{(2)}_n))_{n\in\mathbb{N}}$ we use that for $\delta \leq 1$,

$$(\mathrm{VT}_t(\mathbf{D}^{(2)}_n))^\delta \leq C_3 \widetilde{b}_n^{-\delta} \sum_{k=1}^{\lfloor nt \rfloor} \|\varepsilon_{n,k}\|^\delta \mathbb{1}_{\{\|\varepsilon_{n,k}\| > \widetilde{b}_n\}}.$$

Then a conclusion of Assumption 4.1 *(d)(iii)* and Markov's inequality is

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\mathrm{VT}_t(\mathbf{D}^{(2)}_n) > \eta) \leq C_4 \eta^{-\delta} \sup_{n\in\mathbb{N}} \widetilde{b}_n^{-\delta} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(\|\varepsilon_{n,k}\|^\delta \mathbb{1}_{\{\|\varepsilon_{n,k}\| > \widetilde{b}_n\}}) \leq C_5 \eta^{-\delta} t \overset{\eta\to\infty}{\longrightarrow} 0. \tag{5.3}$$

Hence, $(\mathrm{VT}_t(\mathbf{D}_n^{(2)}))_{n\in\mathbb{N}}$ is also tight.

If we show that $([\mathbf{M}_n,\mathbf{M}_n]_t)_{n\in\mathbb{N}}$ is tight for any $t\geq 0$, then the *P-UT*ness of $(\mathbf{M}_n)_{n\in\mathbb{N}}$ follows by Jacod and Shiryaev [23], Proposition VI.6.13. Here, we use Assumption 4.1 *(d)(i)* for

$$\sup_{n\in\mathbb{N}}\mathbb{P}(\|[\mathbf{M}_n,\mathbf{M}_n]_t\|>\eta)\leq\eta^{-1}\sup_{n\in\mathbb{N}}n\widetilde{b}_n^{-2}\mathbb{E}(\|\varepsilon_{n,1}\|^2\mathbb{1}_{\{\|\varepsilon_{n,1}\|\leq\widetilde{b}_n\}})\leq C_6\eta^{-1}\longrightarrow 0\quad\text{ as }\eta\to\infty.$$

Finally, $([\mathbf{M}_n,\mathbf{M}_n]_t)_{n\in\mathbb{N}}$ is tight as well. □

Proof of Theorem 4.2. The Beveridge-Nelson decomposition (cf. [4]) has the representation

$$\mathbf{Z}_{n,k}=\left(\sum_{j=0}^{\infty}\mathbf{C}_{n,j}\right)\xi_{n,k}+\widetilde{\mathbf{Z}}_{n,k-1}-\widetilde{\mathbf{Z}}_{n,k}\quad\text{ for }k,n\in\mathbb{N}.$$

Thus,

$$\mathbf{S}_{1,n}(t)=h_n\widetilde{a}_n^{-1}\left(\sum_{j=0}^{\infty}\mathbf{C}_{n,j}\right)\sum_{k=1}^{\lfloor nt\rfloor}\xi_{n,k}+h_n\widetilde{a}_n^{-1}\left[\widetilde{\mathbf{Z}}_{n,0}-\widetilde{\mathbf{Z}}_{n,\lfloor nt\rfloor}\right]\quad\text{ for }t\geq 0. \tag{5.4}$$

Therefore we define

$$\widetilde{\mathbf{S}}_{1,n}(t):=\left(h_n\sum_{j=0}^{\infty}\mathbf{C}_{n,j}\right)\widetilde{a}_n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}\xi_{n,k}\quad\text{ for }t\geq 0. \tag{5.5}$$

By Assumption 4.1 *(a)* and *(c)* we have as $n\to\infty$,

$$\left(\widetilde{\mathbf{S}}_{1,n}(t)',\mathbf{S}_{2,n}(t)'\right)'_{t\geq 0}\Longrightarrow\left((\mathbf{C}\mathbf{S}_1(t))',\mathbf{S}_2(t)'\right)'_{t\geq 0}\text{ in }\mathbb{D}([0,1],\mathbb{R}^{d+v}).$$

A straightforward conclusion of the continuous mapping theorem is then as $n\to\infty$,

$$\begin{aligned}&\left(\widetilde{\mathbf{S}}_{1,n}(1),\mathbf{S}_{2,n}(1),\int_0^1\mathbf{S}_{2,n}(s)\mathbf{S}_{2,n}(s)'\mathrm{d}s,(\widetilde{\mathbf{S}}_{1,n}(t)',\mathbf{S}_{2,n}(t)')'_{t\geq 0}\right)\\&\qquad\Longrightarrow\quad\left(\mathbf{C}\mathbf{S}_1(1),\mathbf{S}_2(1),\int_0^1\mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s,((\mathbf{C}\mathbf{S}_1(t))',\mathbf{S}_2(t)')'_{t\geq 0}\right)\end{aligned}$$

in $\mathbb{R}^d\times\mathbb{R}^v\times\mathbb{R}^{v\times v}\times(\mathbb{D}[0,1],\mathbb{R}^{d+v})$. Since $(\mathbf{S}_{2,n})_{n\in\mathbb{N}}$ is *P-UT* by Lemma 5.3, a result of Jacod and Shiryaev [23], Theorem VI.6.22, is that as $n\to\infty$,

$$\begin{aligned}&\left(\widetilde{\mathbf{S}}_{1,n}(1),\mathbf{S}_{2,n}(1),\int_0^1\mathbf{S}_{2,n}(s)\mathbf{S}_{2,n}(s)'\mathrm{d}s,\int_0^1\widetilde{\mathbf{S}}_{1,n}(s-)\,\mathrm{d}\mathbf{S}_{2,n}(s)'\right)\\&\qquad\Longrightarrow\quad\left(\mathbf{C}\mathbf{S}_1(1),\mathbf{S}_2(1),\int_0^1\mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s,\mathbf{C}\int_0^1\mathbf{S}_1(s-)\,\mathrm{d}\mathbf{S}_2(s)'\right)\end{aligned} \tag{5.6}$$

in $\mathbb{R}^d\times\mathbb{R}^v\times\mathbb{R}^{v\times v}\times\mathbb{R}^{d\times v}$.

On the one hand, by (5.4) we have

$$\int_0^1\mathbf{S}_{1,n}(s-)\,\mathrm{d}\mathbf{S}_{2,n}(s)'=\int_0^1\widetilde{\mathbf{S}}_{1,n}(s-)\,\mathrm{d}\mathbf{S}_{2,n}(s)'+\left[h_n\widetilde{a}_n^{-1}\widetilde{\mathbf{Z}}_{n,0}\mathbf{S}_{2,n}(1)-h_n\widetilde{a}_n^{-1}\widetilde{b}_n^{-1}\sum_{k=1}^{n}\widetilde{\mathbf{Z}}_{n,k-1}\varepsilon'_{n,k}\right]. \tag{5.7}$$

Applying Lemma 5.1, $h_n\widetilde{a}_n^{-1}\widetilde{\mathbf{Z}}_{n,0}\xrightarrow{\mathbb{P}}\mathbf{0}_d$ as $n\to\infty$ (by Assumption 4.1 *(b)*), and $\mathbf{S}_{2,n}(1)\Longrightarrow\mathbf{S}_2(1)$ as $n\to\infty$ gives on the other hand,

$$h_n\widetilde{a}_n^{-1}\widetilde{b}_n^{-1}\widetilde{\mathbf{Z}}_{n,0}\mathbf{S}_{2,n}(1)-h_n\widetilde{a}_n^{-1}\widetilde{b}_n^{-1}\sum_{k=1}^{n}\widetilde{\mathbf{Z}}_{n,k-1}\varepsilon'_{n,k}\xrightarrow{\mathbb{P}}\mathbf{0}_{d\times v}\quad\text{ as }n\to\infty. \tag{5.8}$$

Finally, from (5.6)-(5.8) the statement follows. □

Proof of Theorem 4.3.
(a) Since $\mathbb{Y}_n' = \mathbf{A}\mathbb{X}_n' + \mathbb{Z}_n'$ with $\mathbb{X}_n, \mathbb{Y}_n$ as given in (1.5) and $\mathbb{Z}_n' = (\mathbf{Z}_{n,1}, \ldots, \mathbf{Z}_{n,n})$, we have

$$\widehat{\mathbf{A}}_n - \mathbf{A} = \mathbf{A}\mathbb{X}_n'\mathbb{X}_n(\mathbb{X}_n'\mathbb{X}_n)^{-1} + \mathbb{Z}_n'\mathbb{X}_n(\mathbb{X}_n'\mathbb{X}_n)^{-1} - \mathbf{A} = \mathbb{Z}_n'\mathbb{X}_n(\mathbb{X}_n'\mathbb{X}_n)^{-1}. \tag{5.9}$$

This gives

$$nh_n\widetilde{a}_n^{-1}\widetilde{b}_n\left(\widehat{\mathbf{A}}_n - \mathbf{A}\right) = nh_n\widetilde{a}_n^{-1}\widetilde{b}_n(\mathbb{Z}_n'\mathbb{X}_n)(\mathbb{X}_n'\mathbb{X}_n)^{-1} = \left(h_n\widetilde{a}_n^{-1}\mathbb{Z}_n'\mathbb{X}_n\widetilde{b}_n^{-1}\right)\left(n^{-1}\widetilde{b}_n^{-1}\mathbb{X}_n'\mathbb{X}_n\widetilde{b}_n^{-1}\right)^{-1}. \tag{5.10}$$

Now we will prove the convergence

$$\left(h_n\widetilde{a}_n^{-1}\mathbb{Z}_n'\mathbb{X}_n\widetilde{b}_n^{-1}, n^{-1}(\widetilde{b}_n^{-1}\mathbb{X}_n'\mathbb{X}_n\widetilde{b}_n^{-1})\right) \Longrightarrow \left(\mathbf{C}\mathbf{S}_1(1)\mathbf{S}_2(1)' - \mathbf{C}\int_0^1 \mathbf{S}_1(s-)\mathrm{d}\mathbf{S}_2(s)', \int_0^1 \mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s\right) \tag{5.11}$$

in $\mathbb{R}^{d\times v} \times \mathbb{R}^{v\times v}$ as $n \to \infty$, giving us the claim by a continuous mapping theorem, since (4.2) holds. We get for the left-hand side of (5.11),

$$h_n\widetilde{a}_n^{-1}\widetilde{b}_n^{-1}\mathbb{Z}_n'\mathbb{X}_n = \mathbf{S}_{1,n}(1)\mathbf{S}_{2,n}(1)' - \int_0^1 \mathbf{S}_{1,n}(s-)\,\mathrm{d}\mathbf{S}_{2,n}(s)', \tag{5.12}$$

$$n^{-1}\widetilde{b}_n^{-2}\mathbb{X}_n'\mathbb{X}_n = \int_0^1 \mathbf{S}_{2,n}(s)\mathbf{S}_{2,n}(s)'\mathrm{d}s. \tag{5.13}$$

The result follows then from Theorem 4.2 and (5.10)-(5.13). □

5.2. Proof of Theorem 3.1

It is well known that the stationary Ornstein-Uhlenbeck process $\mathbf{Z}$ given in (2.4) observed at the time-grid $h_n\mathbb{Z}$ has the representation as a MA process

$$\mathbf{Z}(kh_n) = \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j}\xi_{n,k-j} \quad \text{for } k \in \mathbb{Z},$$

where

$$\xi_{n,k} = \int_{(k-1)h_n}^{kh_n} \mathrm{e}^{-\Lambda(kh_n - s)}\,\mathbf{B}\,\mathrm{d}\mathbf{L}_1(s) \quad \text{for } k \in \mathbb{Z}, n \in \mathbb{N}.$$

As (5.1) suggests as $n \to \infty$,

$$h_n a_{nh_n}^{-1}\sum_{k=1}^{n}\mathbf{V}(kh_n) = \left(h_n\sum_{j=0}^{\infty}\mathbf{E}\mathrm{e}^{-\Lambda h_n j}\right)\left(a_{nh_n}^{-1}\sum_{k=1}^{\lfloor n\rfloor}\xi_{n,k}\right) + o_P(1).$$

The convergence of $a_{nh_n}^{-1}\sum_{k=1}^{n}\xi_{n,k}$ is based on central limit results for arrays and the properties of the sequence of iid random vectors $(\xi_{n,k})_{k\in\mathbb{Z}}$ as presented in Appendix A.

Before we state the proof of Theorem 3.1, we present the analogous result for the state process $\mathbf{Z}$ which is essential for the proof of Theorem 3.1.

Lemma 5.4. *Let the assumptions of Theorem 3.1 hold. Then as $n \to \infty$,*

$$h_n a_{nh_n}^{-1}\sum_{k=1}^{n}\mathbf{Z}(kh_n) \Longrightarrow \Lambda^{-1}\mathbf{B}\mathbf{S}_1(1).$$

Proof. First, we define $\widetilde{a}_n := a_{nh_n}$, $\mathbf{C}_{n,k} := \mathrm{e}^{-\Lambda h_n k}$ and

$$\xi_{n,k} := \int_{(k-1)h_n}^{kh_n} \mathrm{e}^{-\Lambda(kh_n - s)} \mathbf{B} \mathrm{d}\mathbf{L}_1(s) \quad \text{for } k \in \mathbb{Z}, n \in \mathbb{N}.$$

Then

$$\mathbf{Z}_{n,k} := \mathbf{Z}(kh_n) = \sum_{j=0}^{\infty} \mathbf{C}_{n,j} \xi_{n,k-j} \quad \text{for } k \in \mathbb{Z}, n \in \mathbb{N}.$$

We will show that Assumption 4.1 *(a)-(d)* with $\varepsilon_{n,k} := 0$ are satisfied because then the result follows by Theorem 4.2 (it does not matter that (4.2) is not satisfied for $\varepsilon_{n,k} = 0$).

(a) Consider the case $0 < \alpha < 2$. By Proposition A.2 *(a,c,d)*, $\mathbb{E}(\xi_{n,0}) = \mathbf{0}_{pd}$ if $\alpha > 1$, $\xi_{n,0}$ symmetric for $\alpha = 1$, and Resnick [40], Theorem 7.1, we have

$$\left(\widetilde{a}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \xi_{n,k} \right)_{t \geq 0} \Longrightarrow (\mathbf{B}\mathbf{S}_1(t))_{t \geq 0} \quad \text{as } n \to \infty \text{ in } \mathbb{D}([0,1], \mathbb{R}^{pd}). \tag{5.14}$$

Consider $\alpha = 2$. Then Proposition A.1 *(c,e,f,g)* and Kallenberg [25], Corollary 15.16 give (5.14).

(b) Since

$$\widetilde{\mathbf{Z}}_{n,k} = \sum_{j=0}^{\infty} \left(\sum_{l=j+1}^{\infty} \mathrm{e}^{-\Lambda h_n l} \right) \xi_{n,k-j} = (\mathbf{I}_{d \times d} - \mathrm{e}^{-\Lambda h_n})^{-1} \mathrm{e}^{-\Lambda h_n} \mathbf{Z}(kh_n),$$

the inequality

$$\mathbb{P}(h_n \| \widetilde{\mathbf{Z}}_{n,0} \| > x) \leq \mathbb{P}(2 \| \Lambda^{-1} \| \| \mathbf{Z}(0) \| > x) =: g(x) \quad \text{for } x \geq 0$$

holds, where for $\alpha < 2$ the function $g \in \mathscr{R}_{-\alpha}$ due to Moser and Stelzer [30], Theorem 3.2, such that by Karamata's Theorem $\int_0^\infty x^{\gamma-1} g(x)\,\mathrm{d}x < \infty$ for any $0 < \gamma < \alpha$, and for $\alpha = 2$ we have $2\int_0^\infty x g(x)\,\mathrm{d}x = 8\|\Lambda^{-1}\|^2 \mathbb{E}\|\mathbf{Z}(0)\|^2 < \infty$.

(c) We have $\sum_{k=0}^\infty k \| \mathrm{e}^{-\Lambda h_n k} \|^\theta \leq \sum_{k=0}^\infty k \mathrm{e}^{-\lambda \theta h_n k} < \infty$ for any $\theta > 0$, $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} h_n \sum_{k=0}^{\infty} \mathbf{C}_{n,k} = \lim_{n \to \infty} h_n (\mathbf{I}_{d \times d} - \mathrm{e}^{-\Lambda h_n})^{-1} = \Lambda^{-1}.$$

(d) is obviously satisfied since $\varepsilon_{n,k} = 0$. □

Proof of Theorem 3.1.

(a) Due to Lemma 5.4,

$$h_n a_{nh_n}^{-1} \sum_{k=1}^{n} \mathbf{Z}(kh_n) \Longrightarrow \Lambda^{-1} \mathbf{B}\mathbf{S}_1(1) \quad \text{as } n \to \infty,$$

and by (2.3)

$$h_n a_{nh_n}^{-1} \sum_{k=1}^{n} \mathbf{V}(kh_n) = h_n a_{nh_n}^{-1} \sum_{k=1}^{n} \mathbf{E}\mathbf{Z}(kh_n) \Longrightarrow \mathbf{E}\Lambda^{-1}\mathbf{B}\mathbf{S}_1(1) = \left(\int_0^\infty \mathbf{f}(s) \mathrm{d}s \right) \mathbf{S}_1(1) \quad \text{as } n \to \infty,$$

such that we receive the statement.

(b) Define $\mathbf{g}(s) := \mathrm{e}^{-\Lambda s} \mathbf{B} \mathbb{1}_{(0,\infty)}(s)$. A conclusion of Fasen [16], Proposition 2.1, is that as $n \to \infty$,

$$a_n^{-1} \sum_{k=1}^{n} \mathbf{Z}(kh) \Longrightarrow \mathbf{S}_{\mathbf{g},h}(1).$$

Thus, as $n \to \infty$,

$$a_n^{-1} \sum_{k=1}^{n} \mathbf{V}(kh) \Longrightarrow \mathbf{E}\mathbf{S}_{\mathbf{g},h} \stackrel{d}{=} \mathbf{S}_{\mathbf{f},h}(1)$$

completes the proof. □

5.3. Proof of Theorem 3.4

Again we use for the proof of Theorem 3.4 the similar result for the state process $\mathbf{Z}$ as stated in

Lemma 5.5. *Let model (1.1)-(1.2) be given with $\mathbf{V} = \mathbf{Z}$ and $\mathbf{A} \in \mathbb{R}^{pd\times v}$, and let the assumptions of Theorem 3.4 hold. Then $\widehat{\mathbf{A}}_n$ as given in* (1.6) *satisfies as $n \to \infty$,*

$$nh_n a_{nh_n}^{-1} b_{nh_n} (\widehat{\mathbf{A}}_n - \mathbf{A}) \quad \Longrightarrow \quad \Lambda^{-1}\mathbf{B}\left(\mathbf{S}_1(1)\mathbf{S}_2(1)' - \int_0^1 \mathbf{S}_1(s-)\mathrm{d}\mathbf{S}_2(s)'\right)\left(\int_0^1 \mathbf{S}_2(s)\mathbf{S}_2(s)'\mathrm{d}s\right)^{-1}.$$

In particular, $\widehat{\mathbf{A}}_n \xrightarrow{\mathbb{P}} \mathbf{A}$ as $n \to \infty$ if $\alpha > \beta/(\beta+1)$, i.e. $\widehat{\mathbf{A}}_n$ is a consistent estimator.

Proof. We use the same notation as in the proof of Lemma 5.4 only that we define $\widetilde{b}_n := b_{nh_n}$, and

$$\varepsilon_{n,k} := \mathbf{L}_2(kh_n) - \mathbf{L}_2((k-1)h_n).$$

Again we will show that Assumption 4.1 *(a)-(d)* are satisfied following then the statement by Theorem 4.3.

(a) If $\alpha < 2$ due to the independence of $(\xi_{n,k})$ and $(\varepsilon_{n,k})$, Proposition A.2 and Resnick [40], Theorem 7.1, the limit result

$$\left(\widetilde{a}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \xi_{n,k}', \widetilde{b}_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \varepsilon_{n,k}'\right)_{t\geq 0} \Longrightarrow (\mathbf{S}_1(t)', \mathbf{S}_2(t)')_{t\geq 0} \quad \text{as } n \to \infty \text{ in } \mathbb{D}([0,1], \mathbb{R}^{pd+v}) \tag{5.15}$$

holds; see also Paulauskas and Rachev [31]. If $\alpha = 2$, (5.15) is a conclusion of Proposition A.1 and Kallenberg [25], Corollary 15.15.
(b,c) is satisfied by the proof of Lemma 5.4.
(d) *(i)* is a conclusion from Proposition A.2 *(c)* and Proposition A.1 *(e)*, respectively. *(ii)* follows from Proposition A.2 *(e)* and Proposition A.1 *(f)*, respectively. Only for $\alpha = 1$ it follows by symmetry. Moreover, we obtain *(iii)* by Proposition A.2 *(d)* and Proposition A.1 *(d)*.
Let $\min(\alpha, \beta) < 2$, then using $\mathbb{E}\|\mathbf{L}_2(h_n)\|^\delta \leq C_1 h_n^{\delta/2}$ and (3.6) gives *(iv1)*. In the case of a compound Poisson process, Lemma A.4 says that $\mathbb{E}\|\mathbf{L}_2(h_n)\|^\delta \leq C_2 h_n$, such that no additional assumption is necessary.
Finally, if $\alpha = \beta = 2$, then $\lim_{n\to\infty} n(nh_n)^{-2}\mathbb{E}\|\mathbf{L}_2(h_n)\|^2 = \lim_{n\to\infty} n(nh_n)^{-2} h_n \mathbb{E}\|\mathbf{L}_2(1)\|^2 = 0$, such that *(iv2)* holds. □

Proof of Theorem 3.4. The proof goes as the proof of Theorem 3.1 using only Lemma 5.5 and Fasen [16], Theorem 3.4. □

5.4. Proof of Theorem 3.6

The main idea of the proof is to show that as $n \to \infty$,

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \mathbf{V}(kh_n)\mathbf{V}(kh_n)' = \mathbf{E} \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j} \left(a_{nh_n}^{-2} \sum_{k=1}^{n} \xi_{n,k}\xi_{n,k}'\right) \mathrm{e}^{-\Lambda' h_n j} \mathbf{E}' h_n + o_P(1).$$

The convergence of $a_{nh_n}^{-2} \sum_{k=1}^{n} \xi_{n,k}\xi_{n,k}'$ follows by the limit results of Resnick [40], Theorem 7.1 as well, respectively by the law of large numbers for arrays of independent random vectors and the properties of

$(\xi_{n,k})_{k\in\mathbb{Z}}$ as given in Appendix A.

In the same spirit as before we start with the result for $\mathbf{Z}$.

Lemma 5.6. *Let the assumptions of Theorem 3.6 hold. Then as $n\to\infty$,*

$$h_n a_{nh_n}^{-2}\sum_{k=1}^{n}\mathbf{Z}(kh_n)\mathbf{Z}(kh_n)'\Longrightarrow\int_0^\infty \mathrm{e}^{-\Lambda s}[\mathbf{BS}_1,\mathbf{BS}_1]_1\mathrm{e}^{-\Lambda' s}\,\mathrm{d}s. \tag{5.16}$$

Proof. A multivariate version of the second order Beveridge-Nelson decomposition given in Phillips and Solo [34], Equation (28), gives the representation

$$\begin{aligned}\mathbf{Z}(kh_n)\mathbf{Z}(kh_n)' &= \sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,k}\xi_{n,k}'\mathrm{e}^{-\Lambda' h_n j}+(\mathbf{F}_{n,k-1}^{(1)}-\mathbf{F}_{n,k}^{(1)})+\sum_{r=1}^{\infty}(\mathbf{F}_{n,k,r}^{(2)}+\mathbf{F}_{n,k,-r}^{(2)})\\ &\quad+\sum_{r=1}^{\infty}(\mathbf{F}_{n,k-1,r}^{(3)}+\mathbf{F}_{n,k-1,-r}^{(3)}-\mathbf{F}_{n,k,r}^{(3)}-\mathbf{F}_{n,k,-r}^{(3)}),\end{aligned}$$

where

$$\begin{aligned}\mathbf{F}_{n,k}^{(1)} &= \sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty}\mathrm{e}^{-\Lambda h_n s}\xi_{n,k-j}\xi_{n,k-j}'\mathrm{e}^{-\Lambda' h_n s},\\ \mathbf{F}_{n,k,r}^{(2)} &= \sum_{j=\max(0,-r)}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,k}\xi_{n,k-r}'\mathrm{e}^{-\Lambda' h_n (j+r)},\\ \mathbf{F}_{n,k,r}^{(3)} &= \sum_{j=0}^{\infty}\sum_{s=\max(j+1,-r)}^{\infty}\mathrm{e}^{-\Lambda h_n s}\xi_{n,k-j}\xi_{n,k-j-r}'\mathrm{e}^{-\Lambda' h_n (s+r)}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=1}^{n}\mathbf{Z}(kh_n)\mathbf{Z}(kh_n)' &= \sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\left(\sum_{k=1}^{n}\xi_{n,k}\xi_{n,k}'\right)\mathrm{e}^{-\Lambda' h_n j}+(\mathbf{F}_{n,0}^{(1)}-\mathbf{F}_{n,n}^{(1)})\\ &\quad+\sum_{k=1}^{n}\sum_{r=1}^{\infty}(\mathbf{F}_{n,k,r}^{(2)}+\mathbf{F}_{n,k,-r}^{(2)})+\sum_{r=1}^{\infty}(\mathbf{F}_{n,0,r}^{(3)}+\mathbf{F}_{n,0,-r}^{(3)}-\mathbf{F}_{n,n,r}^{(3)}-\mathbf{F}_{n,n,-r}^{(3)})\\ &=: J_{n,1}+J_{n,2}+J_{n,3}+J_{n,4}.\end{aligned} \tag{5.17}$$

Step 1. Let $\alpha\in(0,2)$ and assume that $\mathbf{L}_1$ is a compound Poisson process as given in (A.5) with characteristic triplet $(\mathbf{0}_m,\mathbf{0}_{m\times m},\nu_{\mathbf{L}_1})$. On the one hand, by Lemma 5.7 from below we have for $i=2,3,4$

$$h_n a_{nh_n}^{-2}J_{n,i}\xrightarrow{\mathbb{P}}\mathbf{0}_{pd\times pd}\qquad\text{as } n\to\infty. \tag{5.18}$$

On the other hand, by Proposition A.2 *(a,c)* and Resnick [40], Theorem 7.1, we have

$$\mathbf{S}_n:=a_{nh_n}^{-2}\sum_{k=1}^{n}\xi_{n,k}\xi_{n,k}'\Longrightarrow[\mathbf{BS}_1,\mathbf{BS}_1]_1\qquad\text{as } n\to\infty.$$

We denote by $\mathbf{g}_n$ and $\mathbf{g}$ maps from $M_{pd\times pd}(\mathbb{R})\to M_{pd\times pd}(\mathbb{R})$ with

$$\mathbf{g}_n(\mathbf{C})=h_n\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\mathbf{C}\mathrm{e}^{-\Lambda' h_n j}\quad\text{and}\quad\mathbf{g}(\mathbf{C})=\int_0^\infty\mathrm{e}^{-\Lambda s}\mathbf{C}\mathrm{e}^{-\Lambda' s}\,\mathrm{d}s. \tag{5.19}$$

Since $\mathbf{g}_n$ and $\mathbf{g}$ are continuous with $\lim_{n\to\infty}\mathbf{g}_n(\mathbf{C}_n)=\mathbf{g}(\mathbf{C})$ for any sequence $\mathbf{C}_n,\mathbf{C}\in M_{pd\times pd}(\mathbb{R})$ with $\lim_{n\to\infty}\mathbf{C}_n=\mathbf{C}$, we can apply a generalized version of the continuous mapping theorem (cf. Whitt [46], Theorem 3.4.4) to obtain $g_n(\mathbf{S}_n) \Longrightarrow g([\mathbf{S}_1,\mathbf{S}_1]_1)$ as $n\to\infty$, which means that as $n\to\infty$,

$$h_n a_{nh_n}^{-2} J_{n,1} = h_n a_{nh_n}^{-2} \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j} \left(\sum_{k=1}^{n} \xi_{n,k}\xi_{n,k}'\right) \mathrm{e}^{-\Lambda' h_n j} \Longrightarrow \int_0^\infty \mathrm{e}^{-\Lambda s}[\mathbf{BS}_1,\mathbf{BS}_1]_1 \mathrm{e}^{-\Lambda' s}\,\mathrm{d}s. \tag{5.20}$$

Then the result (5.16) follows by (5.17)-(5.20).

Step 2. Let $\alpha\in(0,2)$ and $\mathbf{L}_1$ be some Lévy process. We use the decomposition of $\mathbf{L}_1 = \mathbf{L}_1^{(1)} + \mathbf{L}_1^{(2)}$ and $\xi_{n,k} = \xi_{n,k}^{(1)} + \xi_{n,k}^{(2)}$ as given in (A.3) and (A.4), respectively, such that

$$\mathbf{Z}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\Lambda(t-s)}\,\mathbf{B}\,\mathrm{d}\mathbf{L}^{(1)}(s) + \int_{-\infty}^{t} \mathrm{e}^{-\Lambda(t-s)}\,\mathbf{B}\,\mathrm{d}\mathbf{L}^{(2)}(s) =: \mathbf{Z}_1(t) + \mathbf{Z}_2(t) \quad \text{for } t\geq 0,$$

and

$$\begin{aligned}
&\sum_{k=1}^{n} \mathbf{Z}(kh_n)\mathbf{Z}(kh_n)' \\
&\quad = \sum_{k=1}^{n} \mathbf{Z}_1(kh_n)\mathbf{Z}_1(kh_n)' + \sum_{k=1}^{n} \mathbf{Z}_1(kh_n)\mathbf{Z}_2(kh_n)' + \sum_{k=1}^{n} \mathbf{Z}_2(kh_n)\mathbf{Z}_1(kh_n)' + \sum_{k=1}^{n} \mathbf{Z}_2(kh_n)\mathbf{Z}_2(kh_n)' \\
&\quad =: I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned} \tag{5.21}$$

Applying Step 1 we obtain as $n\to\infty$,

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \mathbf{Z}_1(kh_n)\mathbf{Z}_1(kh_n)' \Longrightarrow \int_0^\infty \mathrm{e}^{-\Lambda s}[\mathbf{BS}_1,\mathbf{BS}_1]_1 \mathrm{e}^{-\Lambda' s}\,\mathrm{d}s. \tag{5.22}$$

Furthermore, Hölder inequality results in the decomposition

$$h_n a_{nh_n}^{-2} \max(\|I_{n,2}\|,\|I_{n,3}\|) \leq \left(h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \|\mathbf{Z}_1(kh_n)\|^2\right)^{\frac{1}{2}} \left(h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \|\mathbf{Z}_2(kh_n)\|^2\right)^{\frac{1}{2}} \tag{5.23}$$

of independent factors. Now we use that $\mathbf{L}_1^{(1)}$ has the representation (A.5) and we define

$$L^*(t) := \|\mathbf{B}\| \sum_{k=1}^{N(t)} \|\mathbf{J}_k\|, \quad \xi_{n,k}^* := \int_{(k-1)h_n}^{kh_n} \mathrm{e}^{-\lambda(kh_n-s)}\,\mathrm{d}L^*(s), \quad Z^*(t) := \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}L^*(s). \tag{5.24}$$

Hence,

$$\|\mathbf{BL}_1^{(1)}(t)\| \leq L^*(t), \quad \|\xi_{n,k}^{(1)}\| \leq \xi_{n,k}^* \quad \text{and} \quad \|\mathbf{Z}_1(t)\| \leq Z^*(t).$$

Then a conclusion of Step 1 is

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \|\mathbf{Z}_1(kh_n)\|^2 \leq h_n a_{nh_n}^{-2} \sum_{k=1}^{n} Z^*(kh_n)^2 \Longrightarrow \frac{1}{2\lambda}[S,S]_1 \quad \text{as } n\to\infty, \tag{5.25}$$

where $S=(S(t))_{t\geq 0}$ is an α-stable Lévy process. Since

$$\lim_{n\to\infty} h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \mathbb{E}\|\mathbf{Z}_2(kh_n)\|^2 = \lim_{n\to\infty} nh_n a_{nh_n}^{-2} \mathbb{E}\|\mathbf{Z}_2(1)\|^2 = 0,$$

we obtain

$$\left(h_n a_{nh_n}^{-2}\sum_{k=1}^{n}\|\mathbf{Z}_2(kh_n)\|^2\right)^{\frac{1}{2}} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty. \tag{5.26}$$

Hence, (5.23)-(5.26) give $h_n a_{nh_n}^{-2}\|I_{n,2}\| \xrightarrow{\mathbb{P}} 0$ and $h_n a_{nh_n}^{-2}\|I_{n,3}\| \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$. A conclusion of (5.26) is $h_n a_{nh_n}^{-2}\|I_{n,4}\| \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$ as well. Finally, the result follows by (5.21) and (5.22).
Step 3. Let $\alpha=2$. On the one hand, by Lemma 5.8 from below we have for $i=2,3,4$ as $n\to\infty$,

$$h_n a_{nh_n}^{-2} J_{n,i} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}\,. \tag{5.27}$$

On the other hand, by Proposition A.1 *(g)* as $n\to\infty$,

$$\mathbf{S}_n := a_{nh_n}^{-2}\sum_{k=1}^{n}\xi_{n,k}\xi_{n,k}' \xrightarrow{\mathbb{P}} \mathbf{B}\Sigma_1\mathbf{B}' = [\mathbf{B}\mathbf{S}_1,\mathbf{B}\mathbf{S}_1]_1.$$

The same arguments as in Step 1 complete the proof. □

First we present Lemma 5.7 and 5.8 and then give the proof of Theorem 3.6.

Lemma 5.7. *Let the assumptions of Lemma 5.6 hold with $\alpha\in(0,2)$ and suppose that $\mathbf{L}_1$ is a compound Poisson process as given in (A.5) with characteristic triplet $(\mathbf{0}_m,\mathbf{0}_{m\times m},\nu_{\mathbf{L}_1})$.*

(a) Then $\mathbf{F}_{n,0}^{(1)} \stackrel{d}{=} \mathbf{F}_{n,n}^{(1)}$ and as $n\to\infty$,

$$h_n a_{nh_n}^{-2}\mathbf{F}_{n,0}^{(1)} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}\,.$$

(b) Then as $n\to\infty$,

$$h_n a_{nh_n}^{-2}\sum_{k=1}^{n}\sum_{r=1}^{\infty}(\mathbf{F}_{n,k,r}^{(2)}+\mathbf{F}_{n,k,-r}^{(2)}) \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}\,.$$

(c) Then as $n\to\infty$,

$$h_n a_{nh_n}^{-2}\sum_{r=1}^{\infty}(\mathbf{F}_{n,0,r}^{(3)}+\mathbf{F}_{n,0,-r}^{(3)}-\mathbf{F}_{n,n,r}^{(3)}-\mathbf{F}_{n,n,-r}^{(3)}) \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}\,.$$

Proof.
(a) We use the notation given in (5.24). Then

$$\|\mathbf{F}_{n,0}^{(1)}\| \le (1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda h_n j}\xi_{n,-j}^{*2} \le (1-\mathrm{e}^{-2\lambda h_n})^{-1}Z^*(0)^2.$$

Hence,

$$\mathbb{P}(\|\mathbf{F}_{n,0}^{(1)}\| > a_{nh_n}^2 h_n^{-1}) \le \mathbb{P}(Z^*(0)^2 > C_1 a_{nh_n}^2) \xrightarrow{n\to\infty} 0.$$

(b) The upper bound

$$\begin{aligned}\left\|\sum_{k=1}^{n}\sum_{r=1}^{\infty}\mathbf{F}_{n,k,r}^{(2)}\right\| &\le \sum_{k=1}^{n}\sum_{j=0}^{\infty}\mathrm{e}^{-\lambda h_n(2j+1)}\xi_{n,k}^{*}\left(\sum_{r=0}^{\infty}\xi_{n,k-1-r}^{*}\mathrm{e}^{-\lambda h_n r}\right)\\ &\le (1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{k=1}^{n}\xi_{n,k}^{*}Z^*((k-1)h_n) \end{aligned} \tag{5.28}$$

holds. Applying Lemma 5.1 (here we require that for a compound Poisson process $\mathbb{E}\|\xi_{n,0}*\|^{\delta} \le C_2 h_n$ by

Lemma A.4, which is used to show *(1)* for some $0<\delta<1$, $\delta<\alpha$ and $2\delta>\alpha$) gives

$$h_n(1-\mathrm{e}^{-2\lambda h_n})^{-1}a_{nh_n}^{-2}\sum_{k=1}^{n}\xi_{n,k}^{*}Z^{*}((k-1)h_n)\xrightarrow{\mathbb{P}}0 \quad \text{as } n\to\infty. \tag{5.29}$$

On the other hand, if we define $W^{*}(kh_n):=\sum_{r=0}^{\infty}\mathrm{e}^{-\lambda h_n r}\xi_{n,k+r}^{*}$, then $W^{*}(kh_n)\stackrel{d}{=}Z^{*}(0)$ and

$$\begin{aligned}\left\|\sum_{k=1}^{n}\sum_{r=1}^{\infty}\mathbf{F}_{n,k,-r}^{(2)}\right\| &\leq \sum_{k=1}^{n}\sum_{r=1}^{\infty}\sum_{j=r}^{\infty}\mathrm{e}^{-2\lambda h_n j}\xi_{n,k}^{*}\xi_{n,k+r}^{*}\mathrm{e}^{\lambda h_n r}\\ &\leq (1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{k=1}^{n}\xi_{n,k}^{*}W^{*}((k+1)h_n).\end{aligned} \tag{5.30}$$

Using again Lemma 5.1 yields

$$h_n(1-\mathrm{e}^{-2\lambda h_n})^{-1}a_{nh_n}^{-2}\sum_{k=1}^{n}\xi_{n,k}^{*}W^{*}((k+1)h_n)\xrightarrow{\mathbb{P}}0 \quad \text{as } n\to\infty. \tag{5.31}$$

Hence, (5.28)-(5.31) give the statement.
(c) We will show that on the one hand,

$$h_n a_{nh_n}^{-2}\sum_{r=1}^{\infty}\mathbf{F}_{n,0,r}^{(3)}\xrightarrow{\mathbb{P}}\mathbf{0}_{pd\times pd} \quad \text{as } n\to\infty,$$

and on the other hand,

$$h_n a_{nh_n}^{-2}\sum_{r=1}^{\infty}\mathbf{F}_{n,0,-r}^{(3)}\xrightarrow{\mathbb{P}}\mathbf{0}_{pd\times pd} \quad \text{as } n\to\infty.$$

Since $\sum_{r=1}^{\infty}\mathbf{F}_{n,0,r}^{(3)}\stackrel{d}{=}\sum_{r=1}^{\infty}\mathbf{F}_{n,n,r}^{(3)}$ and $\sum_{r=1}^{\infty}\mathbf{F}_{n,0,-r}^{(3)}\stackrel{d}{=}\sum_{r=1}^{\infty}\mathbf{F}_{n,n,-r}^{(3)}$ the proof will then be finished. Again we use the notation given in (5.24). For the first term we derive the upper bound

$$\begin{aligned}\left\|\sum_{r=1}^{\infty}\mathbf{F}_{n,0,r}^{(3)}\right\| &\leq \sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty}\mathrm{e}^{-2\lambda h_n s}\xi_{n,-j}^{*}\left(\sum_{r=0}^{\infty}\xi_{n,-j-1-r}^{*}\mathrm{e}^{-\lambda h_n r}\right)\\ &\leq (1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda h_n j}\xi_{n,-j}^{*}Z^{*}((-j-1)h_n).\end{aligned}$$

Applying for $0<\delta<\alpha$, $\delta\leq 1$,

$$\mathbb{E}\left(\left(\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda h_n j}\xi_{n,-j}^{*}Z^{*}((-j-1)h_n)\right)^{\delta}\right)\leq\sum_{j=0}^{\infty}\mathrm{e}^{-2\delta\lambda h_n j}\mathbb{E}(\xi_{n,0}^{*\delta})\mathbb{E}(Z^{*}(0)^{\delta}),$$

where we used the independence of $\xi_{n,-j}^{*}$ and $Z^{*}((-j-1)h_n)$ in the first inequality, and Lemma A.4 results in

$$h_n^{\delta}a_{nh_n}^{-2\delta}\mathbb{E}\left\|\sum_{r=1}^{\infty}\mathbf{F}_{n,0,r}^{(3)}\right\|^{\delta}\leq C_3 a_{nh_n}^{-2\delta}\xrightarrow{n\to\infty}0.$$

For the second term we have the upper bound

$$\begin{aligned}\left\|\sum_{r=1}^{\infty}\mathbf{F}_{n,0,-r}^{(3)}\right\| &\leq \sum_{r=1}^{\infty}\sum_{j=0}^{\infty}\sum_{s=\max(j+1,r)}^{\infty} \mathrm{e}^{-\lambda h_n s}\xi_{n,-j}^{*}\xi_{n,-j+r}^{*}\mathrm{e}^{-\lambda h_n(s-r)} \\ &= (1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{j=0}^{\infty}\sum_{r=1}^{j}\mathrm{e}^{-2\lambda h_n(j+1)}\xi_{n,-j}^{*}\xi_{n,-j+r}^{*}\mathrm{e}^{\lambda h_n r} \\ &\quad +(1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty}\mathrm{e}^{-2\lambda h_n r}\xi_{n,-j}^{*}\xi_{n,-j+r}^{*}\mathrm{e}^{\lambda h_n r} \\ &=: I_{n,1}+I_{n,2}. \end{aligned} \tag{5.32}$$

Moreover,

$$I_{n,1} = (1-\mathrm{e}^{-2\lambda h_n})^{-1}\mathrm{e}^{-2\lambda h_n}\sum_{j=0}^{\infty}\mathrm{e}^{-\lambda h_n j}\xi_{n,-j}^{*}\sum_{r=1}^{j}\xi_{n,-j+r}^{*}\mathrm{e}^{-\lambda h_n(j-r)} \leq (1-\mathrm{e}^{-2\lambda h_n})^{-1}Z^{*}(0)^{2}, \tag{5.33}$$

and

$$\begin{aligned} I_{n,2} &= (1-\mathrm{e}^{-2\lambda h_n})^{-1}\sum_{j=0}^{\infty}\mathrm{e}^{-\lambda h_n j}\xi_{n,-j}^{*}\sum_{r=j+1}^{\infty}\xi_{n,-j+r}^{*}\mathrm{e}^{-\lambda h_n(r-j)} \\ &\stackrel{d}{=} (1-\mathrm{e}^{-2\lambda h_n})^{-1}\mathrm{e}^{-\lambda h_n}Z^{*}(0)\widetilde{Z}(0), \end{aligned} \tag{5.34}$$

where $\widetilde{Z}(0)$ is an independent copy of $Z^{*}(0)$. A conclusion of (5.32)-(5.34) is that for any $\varepsilon > 0$,

$$\mathbb{P}\left(h_n a_{nh_n}^{-2}\left\|\sum_{r=1}^{\infty}\mathbf{F}_{n,0,r}^{(3)}\right\| > \varepsilon\right) \leq \mathbb{P}(Z^{*}(0)^{2}+Z^{*}(0)\widetilde{Z}(0) > C_4 a_{nh_n}^{2}) \longrightarrow 0 \quad \text{as } n\to\infty,$$

what was the aim to show. □

Lemma 5.8. *Let the assumptions of Lemma 5.6 hold with $\alpha = 2$.*

(a) Then $\mathbf{F}_{n,0}^{(1)} \stackrel{d}{=} \mathbf{F}_{n,n}^{(1)}$ and as $n\to\infty$,

$$h_n a_{nh_n}^{-2}\mathbf{F}_{n,0}^{(1)} \stackrel{\mathbb{P}}{\longrightarrow} \mathbf{0}_{pd\times pd}\,.$$

(b) Then as $n\to\infty$,

$$h_n a_{nh_n}^{-2}\sum_{k=1}^{n}\sum_{r=1}^{\infty}(\mathbf{F}_{n,k,r}^{(2)}+\mathbf{F}_{n,k,-r}^{(2)}) \stackrel{\mathbb{P}}{\longrightarrow} \mathbf{0}_{pd\times pd}\,.$$

(c) Then as $n\to\infty$,

$$h_n a_{nh_n}^{-2}\sum_{r=1}^{\infty}(\mathbf{F}_{n,0,r}^{(3)}+\mathbf{F}_{n,0,-r}^{(3)}-\mathbf{F}_{n,n,r}^{(3)}-\mathbf{F}_{n,n,-r}^{(3)}) \stackrel{\mathbb{P}}{\longrightarrow} \mathbf{0}_{pd\times pd}\,.$$

Proof.
(a) We rewrite

$$\mathbf{F}_{n,0}^{(1)} = \sum_{s=0}^{\infty}\mathrm{e}^{-\Lambda h_n(s+1)}\left(\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\xi_{n,-j}'\mathrm{e}^{-\Lambda' h_n j}\right)\mathrm{e}^{-\Lambda' h_n(s+1)}.$$

With $\mathbf{g}_n$ and $\mathbf{g}$ as defined in (5.19) and

$$\mathbf{S}_n := a_{nh_n}^{-2} \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j} \xi_{n,-j} \xi'_{n,-j} \mathrm{e}^{-\Lambda' h_n j},$$

the equality $h_n a_{nh_n}^{-2} \mathbf{F}_{n,0}^{(1)} = \mathrm{e}^{-\Lambda h_n} \mathbf{g}_n(\mathbf{S}_n) \mathrm{e}^{-\Lambda' h_n}$ is valid. If we are able to prove that $\mathbf{S}_n \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}$ as $n \to \infty$, then with a generalized continuous mapping theorem (the same arguments as in the proof of Lemma 5.6) we can conclude $h_n a_{nh_n}^{-2} \mathbf{F}_{n,0}^{(1)} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}$ as $n \to \infty$. Finally, due to Proposition A.1 *(a)*

$$\mathbb{E}\|\mathbf{S}_n\| \leq a_{nh_n}^{-2} \sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda h_n j} \mathbb{E}\|\xi_{n,0}\|^2 \leq C_1 a_{nh_n}^{-2} \xrightarrow{n\to\infty} 0,$$

and $\mathbf{S}_n \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}$ as $n \to \infty$.

(b) The representation

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathbf{F}_{n,k,r}^{(2)} = \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j} \left(a_{nh_n}^{-2} \sum_{k=1}^{n} \xi_{n,k} \mathbf{Z}((k-1)h_n)' \right) \mathrm{e}^{-\Lambda' h_n (j+1)} h_n$$

holds. Using the same arguments as in *(a)* it is sufficient to prove that as $n \to \infty$,

$$a_{nh_n}^{-2} \sum_{k=1}^{n} \xi_{n,k} \mathbf{Z}((k-1)h_n)' \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}.$$

However, this follows from Proposition A.1 and Lemma 5.1. Similarly,

$$h_n a_{nh_n}^{-2} \sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathbf{F}_{n,k,-r}^{(2)} = \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda' h_n j} \left(a_{nh_n}^{-2} \sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathrm{e}^{-\Lambda h_n r} \xi_{n,k} \xi'_{n,k+r} \right) \mathrm{e}^{-\Lambda' h_n j} h_n.$$

As in *(a)* it is sufficient to show that

$$a_{nh_n}^{-2} \sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathrm{e}^{-\Lambda h_n r} \xi_{n,k} \xi'_{n,k+r} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}. \tag{5.35}$$

We prove the convergence of (5.35) componentwise. The sequence of (l,m)-components $\left(\left(\mathrm{e}^{-\Lambda h_n r} \xi_{n,k} \xi'_{n,k+r} \right)_{(l,m)} \right)_{k,r\in\mathbb{N}}$ is uncorrelated such that

$$\begin{aligned} \mathbb{E}\left(\left(\sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathrm{e}^{-\Lambda h_n r} \xi_{n,k} \xi'_{n,k+r} \right)_{(l,m)}^2 \right) &= \sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathbb{E}\left(\left(\mathrm{e}^{-\Lambda h_n r} \xi_{n,k} \xi'_{n,k+r} \right)_{(l,m)}^2 \right) \\ &\leq C_2 \sum_{k=1}^{n} \sum_{r=1}^{\infty} \mathrm{e}^{-2\lambda h_n r} (\mathbb{E}\|\xi_{n,0}\|^2)^2 \leq C_3 n h_n. \end{aligned}$$

Thus, (5.35) holds.

(c) Let us start with

$$h_n a_{nh_n}^{-2} \sum_{r=1}^{\infty} \mathbf{F}_{n,0,r}^{(3)} = \sum_{s=0}^{\infty} \mathrm{e}^{-\Lambda h_n (s+1)} \left(a_{nh_n}^{-2} \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j} \xi_{n,-j} \mathbf{Z}((-j-1)h_n)' \mathrm{e}^{-\Lambda' h_n j} \right) \mathrm{e}^{-\Lambda' h_n (s+2)} h_n.$$

As before it is sufficient to show that

$$a_{nh_n}^{-2} \sum_{j=0}^{\infty} \mathrm{e}^{-\Lambda h_n j} \xi_{n,-j} \mathbf{Z}((-j-1)h_n)' \mathrm{e}^{-\Lambda' h_n j} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd} \qquad \text{as } n \to \infty.$$

We prove it componentwise using the uncorrelation of the sequence of the (l,m)-components $((\xi_{n,-j}\mathbf{Z}((-j-1)h_n)')_{(l,m)})_{j\in\mathbb{N}}$. For the (l,m)-component we have

$$\begin{aligned}
&a_{nh_n}^{-4}\mathbb{E}\left(\left(\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\mathbf{Z}((-j-1)h_n)'\mathrm{e}^{-\Lambda' h_n j}\right)_{(l,m)}^2\right)\\
&= a_{nh_n}^{-4}\sum_{j=0}^{\infty}\mathbb{E}\left(\left(\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\mathbf{Z}((-j-1)h_n)'\mathrm{e}^{-\Lambda' h_n j}\right)_{(l,m)}^2\right)\\
&\leq a_{nh_n}^{-4}C_4\sum_{j=0}^{\infty}\mathrm{e}^{-4\lambda h_n j}\mathbb{E}\|\xi_{n,0}\|^2\mathbb{E}\|\mathbf{Z}(0)\|^2\\
&\leq C_5 a_{nh_n}^{-4}\stackrel{n\to\infty}{\longrightarrow}0.
\end{aligned}$$

Now we investigate

$$\begin{aligned}
\sum_{r=1}^{\infty}\mathbf{F}_{n,0,-r}^{(3)} &= \sum_{j=0}^{\infty}\sum_{r=1}^{j}\sum_{s=j+1}^{\infty}\mathrm{e}^{-\Lambda h_n s}\xi_{n,-j}\xi_{n,-j+r}'\mathrm{e}^{-\Lambda' h_n(s-r)}+\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty}\sum_{s=r}^{\infty}\mathrm{e}^{-\Lambda h_n s}\xi_{n,-j}\xi_{n,-j+r}'\mathrm{e}^{-\Lambda' h_n(s-r)}\\
&=: I_{n,1}+I_{n,2}. \qquad (5.36)
\end{aligned}$$

Then

$$I_{n,1}=\sum_{s=0}^{\infty}\mathrm{e}^{-\Lambda h_n(s+1)}\left(\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\left(\sum_{r=1}^{j}\xi_{n,-j+r}'\mathrm{e}^{-\Lambda' h_n(j-r)}\right)\right)\mathrm{e}^{-\Lambda' h_n(s+1)}.$$

For the convergence $h_n a_{nh_n}^{-2}I_{n,1}\stackrel{\mathbb{P}}{\longrightarrow}\mathbf{0}_{pd\times pd}$ as $n\to\infty$ it is again sufficient to show that

$$a_{nh_n}^{-2}\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\left(\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u}\right)\stackrel{\mathbb{P}}{\longrightarrow}\mathbf{0}_{pd\times pd}\quad\text{as } n\to\infty, \qquad (5.37)$$

what we will prove componentwise. Since the (l,m)-components $((\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u})_{(l,m)})_{j\in\mathbb{N}}$ are uncorrelated

$$\mathbb{E}\left(\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\left(\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u}\right)_{(l,m)}\right)^2=\sum_{j=0}^{\infty}\mathbb{E}\left(\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\left(\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u}\right)_{(l,m)}\right)^2. \qquad (5.38)$$

Furthermore, by Proposition A.1 *(a)* we get

$$\mathbb{E}\left\|\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\right\|^2\leq C_6\mathrm{e}^{-2\lambda h_n j}\mathbb{E}\|\xi_{n,0}\|^2\leq C_7 h_n\mathrm{e}^{-2\lambda h_n j}, \qquad (5.39)$$

and

$$\mathbb{E}\left\|\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u}\right\|^2\leq C_8\sum_{u=0}^{j-1}\mathrm{e}^{-2\lambda h_n u}\mathbb{E}\|\xi_{n,0}\|^2\leq C_9. \qquad (5.40)$$

Hence, (5.38)-(5.40) and the independence of $\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}$ and $\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u}$ give

$$\mathbb{E}\left(\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\xi_{n,-j}\left(\sum_{u=0}^{j-1}\xi_{n,-u}'\mathrm{e}^{-\Lambda' h_n u}\right)_{(l,m)}\right)^2\leq C_{10}h_n\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda h_n j}\leq C_{11},$$

which results in (5.37).

Next we have to show that $h_n a_{nh_n}^{-2} I_{n,2} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}$ as $n\to\infty$. Therefore we use the representation

$$I_{n,2} = \sum_{s=0}^{\infty} \mathrm{e}^{-\Lambda h_n s}\left(\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty} \mathrm{e}^{-\Lambda h_n r}\xi_{n,-j}\xi'_{n,-j+r}\right)\mathrm{e}^{-\Lambda' h_n s}$$

and prove that as $n\to\infty$,

$$a_{nh_n}^{-2}\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty} \mathrm{e}^{-\Lambda h_n r}\xi_{n,-j}\xi'_{n,-j+r} = a_{nh_n}^{-2}\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\sum_{u=1}^{\infty}\mathrm{e}^{-\Lambda h_n u}\xi_{n,-j}\xi'_{n,u} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}\,. \tag{5.41}$$

By the uncorrelation of the components of $\left(\left(\mathrm{e}^{-\Lambda h_n j}\mathrm{e}^{-\Lambda h_n u}\xi_{n,-j}\xi'_{n,u}\right)_{(l,m)}\right)_{j,u\in\mathbb{N}}$ we obtain similarly as above

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{j=0}^{\infty}\mathrm{e}^{-\Lambda h_n j}\sum_{u=1}^{\infty}\mathrm{e}^{-\Lambda h_n u}\xi_{n,-j}\xi'_{n,u}\right)^2_{(l,m)}\right) &\le \sum_{j=0}^{\infty}\sum_{u=1}^{\infty}\mathbb{E}\left(\left(\mathrm{e}^{-\Lambda h_n j}\mathrm{e}^{-\Lambda h_n u}\xi_{n,-j}\xi'_{n,u}\right)^2_{(l,m)}\right)\\
&\le C_{12}\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda h_n j}\sum_{u=0}^{\infty}\mathrm{e}^{-2\lambda h_n u}\mathbb{E}\|\xi_{n,-j}\|^2\mathbb{E}\|\xi_{n,u}\|^2\\
&\le C_{13}.
\end{aligned}$$

After all this gives (5.41) and $h_n a_{nh_n}^{-2} I_{n,2} \xrightarrow{\mathbb{P}} \mathbf{0}_{pd\times pd}$ as $n\to\infty$. □

Finally, we are able to prove the main statement in Theorem 3.6.

Proof of Theorem 3.6.
(a) The observation equation (2.3) and Lemma 5.6 yield

$$\begin{aligned}
h_n a_{nh_n}^{-2}\sum_{k=1}^{n}\mathbf{V}(kh_n)\mathbf{V}(kh_n)' &= h_n a_{nh_n}^{-2}\sum_{k=1}^{n}\mathbf{E}\mathbf{Z}(kh_n)\mathbf{Z}(kh_n)'\mathbf{E}'\\
&\Longrightarrow \int_0^{\infty}\mathbf{E}\mathrm{e}^{-\Lambda s}\mathbf{B}[\mathbf{S}_1,\mathbf{S}_1]_1\mathbf{B}'\mathrm{e}^{-\Lambda' s}\mathbf{E}'\,\mathrm{d}s = \int_0^{\infty}\mathbf{f}(s)[\mathbf{S}_1,\mathbf{S}_1]_1\mathbf{f}(s)'\,\mathrm{d}s \quad \text{as } n\to\infty.
\end{aligned}$$

(b) An application of Fasen [16], Proposition 2.1, gives that with $\mathbf{g}(s)=\mathrm{e}^{-\Lambda s}\mathbf{B}\mathbb{1}_{(0,\infty)}(s)$

$$a_n^{-2}\sum_{k=1}^{n}\mathbf{Z}(kh)\mathbf{Z}(kh)' \Longrightarrow [\mathbf{S}_{\mathbf{g},h},\mathbf{S}_{\mathbf{g},h}]_1 \quad \text{as } n\to\infty,$$

such that

$$a_n^{-2}\sum_{k=1}^{n}\mathbf{V}(kh)\mathbf{V}(kh)' \Longrightarrow \mathbf{E}[\mathbf{S}_{\mathbf{g},h},\mathbf{S}_{\mathbf{g},h}]_1\mathbf{E}' = [\mathbf{E}\mathbf{S}_{\mathbf{g},h},\mathbf{E}\mathbf{S}_{\mathbf{g},h}]_1 \stackrel{d}{=} [\mathbf{S}_{\mathbf{f},h},\mathbf{S}_{\mathbf{f},h}]_1 \quad \text{as } n\to\infty,$$

is the result. □

A. Appendix: Asymptotic behavior of stochastic integrals

In the appendix we present the tail behavior and extensions of Karamata's Theorem to stochastic integrals of the form $\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}(s)$ where $h_n\downarrow 0$ as $n\to\infty$. First, we start with a driving Lévy process which has a finite second moment. In the subsequent subsection the driving Lévy process has a regularly varying tail.

A.1. Finite second moments

Proposition A.1. *Let $(\mathbf{L}(t))_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with $\mathbb{E}\|\mathbf{L}(1)\|^2<\infty$ and $\mathbb{E}(\mathbf{L}(1)\mathbf{L}(1)')=\boldsymbol{\Sigma}$. Suppose $(h_n)_{n\in\mathbb{N}}$ is a sequence of positive constants such that $h_n\downarrow 0$ and $\lim_{n\to\infty}nh_n=\infty$. Moreover, let $\mathbf{f}:\mathbb{R}\to\mathbb{R}^{m\times d}$ be a measurable and bounded function with $\lim_{x\to 0}\mathbf{f}(x)=\mathbf{f}(0)$. Define $\xi_n=\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}(s)$ for $n\in\mathbb{N}$. Finally, let $\delta\in(0,2]$ and let $x>0$.*

(a) *There exists a finite positive constant K such that*

$$h_n^{-1}\mathbb{E}\|\xi_n\|^2\leq K\quad\forall n\in\mathbb{N}.$$

If $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$, then $\lim_{n\to\infty}h_n^{-1}\mathbb{E}\|\xi_n\|^2=\mathbb{E}\|\mathbf{f}(0)\mathbf{L}(1)\|^2$.

(b) *If $\mathbb{E}\|\mathbf{L}(1)\|^4<\infty$, then there exists a finite positive constant K such that $\mathbb{E}\|\xi_n\|^4\leq Kh_n\ \forall n\in\mathbb{N}$.*

(c) *$n\mathbb{P}((nh_n)^{-1/2}\xi_n\in\cdot)\stackrel{v}{\Longrightarrow}0$ as $n\to\infty$ on $\mathscr{B}(\overline{\mathbb{R}}^m\backslash\{\mathbf{0}_m\})$.*

(d) *$\lim_{n\to\infty}n(nh_n)^{-\delta/2}\mathbb{E}(\|\xi_n\|^\delta\mathbb{1}_{\{\|\xi_n\|>\sqrt{nh_n}x\}})=0$.*

(e) *There exists a finite positive constant K such that*

$$h_n^{-1}\mathbb{E}(\|\xi_n\|^2\mathbb{1}_{\{\|\xi_n\|\leq\sqrt{nh_n}x\}})\leq K\quad\forall n\in\mathbb{N}.$$

If $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$, then $\lim_{n\to\infty}h_n^{-1}\mathbb{E}(\|\xi_n\|^2\mathbb{1}_{\{\|\xi_n\|\leq\sqrt{nh_n}x\}})=\mathbb{E}\|\mathbf{f}(0)\mathbf{L}(1)\|^2$.

(f) *Let $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$. Then $\lim_{n\to\infty}n(nh_n)^{-1/2}\mathbb{E}(\xi_n\mathbb{1}_{\{\|\xi_n\|\leq\sqrt{nh_n}x\}})=\mathbf{0}_{m\times m}$.*

(g) *Let $(\xi_{n,k})_{k\in\mathbb{N}}$ be an iid sequence with $\xi_{n,1}\stackrel{d}{=}\xi_n$ for any $n\in\mathbb{N}$ and $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$. Then*

$$(nh_n)^{-1}\sum_{k=1}^n\xi_{n,k}\xi_{n,k}'\stackrel{\mathbb{P}}{\longrightarrow}\mathbf{f}(0)\boldsymbol{\Sigma}\mathbf{f}(0)'\quad\text{as } n\to\infty.$$

Proof. *(a)* Suppose $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$. Due to (2.10) in Marquardt and Stelzer [28] the covariance matrix of ξ_n is $\int_0^{h_n}\mathbf{f}(s)\boldsymbol{\Sigma}\mathbf{f}(s)'\,\mathrm{d}s$. Hence, we obtain as $n\to\infty$,

$$\mathbb{E}\|\xi_n\|^2=\int_0^{h_n}\|\mathrm{diag}(\mathbf{f}(s)\boldsymbol{\Sigma}\mathbf{f}(s)')\|^2\mathrm{d}s\sim h_n\|\mathrm{diag}(\mathbf{f}(0)\boldsymbol{\Sigma}\mathbf{f}(0)')\|^2=h_n\mathbb{E}\|\mathbf{f}(0)\mathbf{L}(1)\|^2,\tag{A.1}$$

where $\mathrm{diag}(B)$ denotes the vector containing the diagonal elements of B.
Suppose $\mathbb{E}(\mathbf{L}(1))\neq\mathbf{0}_d$. Then define $\widetilde{\mathbf{L}}(t):=\mathbf{L}(t)-t\mathbb{E}(\mathbf{L}(1))$ for $t\geq 0$ and use the upper bound

$$\mathbb{E}\|\xi_n\|^2\leq 4\mathbb{E}\left\|\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\widetilde{\mathbf{L}}(s)\right\|^2+C_1h_n^2.$$

A conclusion of (A.1) is the statement.
(b) Suppose $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$. The characteristic function of $\xi(t)=\int_0^t\mathbf{f}(s)\,\mathrm{d}\mathbf{L}(s)=:(\xi_1(t),\ldots,\xi_m(t))'$ is $\mathbb{E}(\mathrm{e}^{i\Theta'\xi(t)})=\exp(-\boldsymbol{\Psi}_{\mathbf{f},t}(\Theta))$ for $\Theta\in\mathbb{R}^m$ where

$$\boldsymbol{\Psi}_{\mathbf{f},t}(\Theta)=\int_0^t\boldsymbol{\Psi}(\Theta'\mathbf{f}(s))\,\mathrm{d}s$$

(cf. Rajput and Rosinski [37], Proposition 2.6). Hence, for $k=1,\ldots,m$ and $\mathbf{e}_k=(0,\ldots,0,1,0,\ldots,0)'\in$

$\mathbb{R}^m$,

$$\begin{aligned}
\mathbb{E}|\xi_k(t)|^4 &= \frac{\mathrm{d}}{\mathrm{d}^4\theta}\mathbb{E}(\mathrm{e}^{i\theta\mathbf{e}_k'\xi(t)})\Big|_{\theta=0} = 3\left(\frac{\mathrm{d}}{\mathrm{d}^2\theta}\Psi_{\mathbf{f},t}(\theta\mathbf{e}_k)\Big|_{\theta=0}\right)^2 - \left(\frac{\mathrm{d}}{\mathrm{d}^4\theta}\Psi_{\mathbf{f},t}(\theta\mathbf{e}_k)\Big|_{\theta=0}\right)\\
&= 3\left(\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}^2\theta}\Psi(\theta\mathbf{e}_k'\mathbf{f}(s))\right)\Big|_{\theta=0}\mathrm{d}s\right)^2 - \int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}^4\theta}\Psi(\theta\mathbf{e}_k'\mathbf{f}(s))\right)\Big|_{\theta=0}\mathrm{d}s\\
&\sim 3t^2C_2 + tC_3 \quad \text{as } t\to 0.
\end{aligned}$$

Finally,

$$\mathbb{E}\|\xi_n\|^4 \le C_4\sum_{k=1}^m \mathbb{E}|\xi_k(h_n)|^4 \le C_5 h_n \quad \forall\, n\in\mathbb{N}. \tag{A.2}$$

Suppose $\mathbb{E}(\mathbf{L}(1)) \neq \mathbf{0}_d$. Then by (A.2)

$$\mathbb{E}\|\xi_n\|^4 \le 8\mathbb{E}\left\|\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\widetilde{\mathbf{L}}(s)\right\|^4 + C_6 h_n^4 \le C_7 h_n.$$

(c) In the following $f^* := \sup_{s\in\mathbb{R}}\|\mathbf{f}(s)\|$. Let $(\gamma_{\mathbf{L}},\Sigma_{\mathbf{L}},\nu_{\mathbf{L}})$ be the characteristic triplet of $(\mathbf{L}(t))_{t\ge 0}$ and $\mathbb{B}^{d-1} = \{\mathbf{x}\in\mathbb{R}^d : \|\mathbf{x}\|\le 1\}$ be the unit ball in $\mathbb{R}^d$. We factorize the Lévy measure $\nu_{\mathbf{L}}$ into two Lévy measures

$$\nu_{\mathbf{L}_1}(A) := \nu_{\mathbf{L}}(A\backslash\mathbb{B}^{d-1}) \qquad \text{and} \qquad \nu_{\mathbf{L}_2}(A) := \nu_{\mathbf{L}}(A\cap\mathbb{B}^{d-1}) \quad \text{for } A\in\mathscr{B}(\mathbb{R}^d\backslash\{\mathbf{0}_d\})$$

such that $\nu_{\mathbf{L}} = \nu_{\mathbf{L}_1} + \nu_{\mathbf{L}_2}$. Then we can decompose $(\mathbf{L}(t))_{t\ge 0}$ in two independent Lévy processes

$$\mathbf{L}(t) = \mathbf{L}^{(1)}(t) + \mathbf{L}^{(2)}(t) \quad \text{for } t\ge 0, \tag{A.3}$$

where $\mathbf{L}^{(1)} = (\mathbf{L}^{(1)}(t))_{t\ge 0}$ has the characteristic triplet $(\mathbf{0}_d,\mathbf{0}_{d\times d},\nu_{\mathbf{L}_1})$ and $\mathbf{L}^{(2)} = (\mathbf{L}^{(2)}(t))_{t\ge 0}$ has the characteristic triplet $(\gamma_{\mathbf{L}},\Sigma_{\mathbf{L}},\nu_{\mathbf{L}_2})$. Then

$$\xi_n = \int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}^{(1)}(s) + \int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}^{(2)}(s) =: \xi_n^{(1)} + \xi_n^{(2)}, \tag{A.4}$$

and $\xi_n^{(1)}$ and $\xi_n^{(2)}$ are independent. Since the Lévy measure of $\mathbf{L}^{(1)}$ is finite and $\mathbf{L}^{(1)}$ is without Gaussian part and drift, $\mathbf{L}^{(1)}$ has the representation as a compound Poisson process

$$\mathbf{L}^{(1)}(t) = \sum_{k=1}^{N(t)}\mathbf{J}_k, \quad t\ge 0, \quad \text{and} \quad \xi_n^{(1)} = \sum_{k=1}^{N(h_n)}\mathbf{f}(\Gamma_k)\mathbf{J}_k, \tag{A.5}$$

where $(\mathbf{J}_k)_{k\in\mathbb{N}}$ is a sequence of iid random vectors independent of the Poisson process $(N(t))_{t\ge 0}$ with intensity $\lambda = \nu_{\mathbf{L}_1}(\mathbb{R}^d)$ and jump times $(\Gamma_k)_{k\in\mathbb{N}}$. Now, let B be a relatively compact set in $\mathscr{B}(\overline{\mathbb{R}}^m\backslash\{\mathbf{0}_m\})$ with $\mu(\partial B) = 0$ and $\gamma_B = \inf_{x\in B}\|\mathbf{x}\|$, which is larger than 0. Then

$$n\mathbb{P}((nh_n)^{-1/2}\xi_n\in B) \le n\mathbb{P}(\|\xi_n^{(1)}\| > \gamma_B\sqrt{nh_n}/2) + n\mathbb{P}(\|\xi_n^{(2)}\| > \gamma_B\sqrt{nh_n}/2).$$

First, we will show that the first summand with $\xi_n^{(1)}$ converges to 0. Therefore, we will use the next conclusions. On the one hand, for $l\ge 1$,

$$\frac{\mathbb{P}(N(h_n)=l)}{h_n} = \mathrm{e}^{-\lambda h_n}\frac{(\lambda h_n)^l}{h_n l!} \le C_8\mathbb{P}(N(1)=l). \tag{A.6}$$

On the other hand, for $l \geq 2$,

$$\lim_{n\to\infty} \frac{\mathbb{P}(N(h_n)=l)}{h_n} = \lim_{n\to\infty} \mathrm{e}^{-\lambda h_n} \frac{\lambda^l h_n^{l-1}}{l!} = 0. \tag{A.7}$$

Finally,

$$\lim_{n\to\infty} \frac{\mathbb{P}(N(h_n)=1)}{h_n} = \lim_{n\to\infty} \mathrm{e}^{-\lambda h_n} \lambda = \lambda. \tag{A.8}$$

If $U_{l,1} < U_{l,2} < \ldots < U_{l,l}$ denotes the order statistic of l iid uniform random variables on $(0,1)$ then

$$n\mathbb{P}(\|\xi_n^{(1)}\| > \gamma_B\sqrt{nh_n}/2) = n\sum_{l=1}^{\infty} \mathbb{P}\left(\left\|\sum_{k=1}^{l} \mathbf{f}(h_n U_{l,k})\mathbf{J}_k\right\| > \gamma_B\sqrt{nh_n}/2\right)\mathbb{P}(N(h_n)=l)$$

(see Resnick [38], Theorem 4.5.2). On the one hand, by (A.6)

$$\begin{aligned} n\mathbb{P}(\|\xi_n^{(1)}\| > \gamma_B\sqrt{nh_n}/2) &\leq nh_n C_9 \mathbb{P}\left(f^* \sum_{k=1}^{N(1)} \|\mathbf{J}_k\| > \gamma_B\sqrt{nh_n}/2\right) \\ &\leq C_{10}\int_{\gamma_B\sqrt{nh_n}/2} x^2\,\mathbb{P}\left(f^* \sum_{k=1}^{N(1)} \|\mathbf{J}_k\| \in \mathrm{d}x\right) \stackrel{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

since $\mathbb{E}((\sum_{k=1}^{N(1)} \|\mathbf{J}_k\|)^2) < \infty$ by Sato [42], Corollary 25.8. On the other hand, since the Lévy measure of $\mathbf{L}^{(2)}$ has compact support, all moments of $\mathbf{L}^{(2)}(1)$ are finite (cf. Sato [42], Corollary 25.8), such that a conclusion of *(b)* is

$$n\mathbb{P}(\|\xi_n^{(2)}\| > \gamma_B\sqrt{nh_n}/2) \leq n(\gamma_B\sqrt{nh_n}/2)^{-4}\mathbb{E}\|\xi_n^{(2)}\|^4 \leq C_{11} n(nh_n)^{-2}h_n \stackrel{n\to\infty}{\longrightarrow} 0.$$

(d) Note that for any random variable X with $\mathbb{E}|X|^2 < \infty$ the limit $\lim_{y\to\infty} y^2\mathbb{P}(|X|>y) = 0$ and $\lim_{y\to\infty} y^{2-\delta}\mathbb{E}(|X|^\delta \mathbb{1}_{\{|X|>y\}}) = 0$ (apply Hölder inequality) holds. Then

$$\begin{aligned} n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}(\|\xi_n^{(1)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(1)}\|>\sqrt{nh_n}x\}}) &\leq C_{12}(nh_n)^{\frac{2-\delta}{2}}\mathbb{E}\left(\left(f^*\sum_{k=1}^{N(1)}\|\mathbf{J}_k\|\right)^\delta \mathbb{1}_{\{f^*\sum_{k=1}^{N(1)}\|\mathbf{J}_k\|>\sqrt{nh_n}x\}}\right) \\ &\stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{A.9}$$

Moreover, by Markov's inequality

$$\begin{aligned} &n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}(\|\xi_n^{(2)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(2)}\|>\sqrt{nh_n}x\}}) \\ &\leq n(nh_n)^{-\frac{\delta}{2}}\left((nh_n)^{\frac{\delta}{2}}x^\delta\mathbb{P}(\|\xi_n^{(2)}\|>\sqrt{nh_n}x) + \delta\int_{\sqrt{nh_n}x}^{\infty} x^{\delta-1}\mathbb{E}\|\xi_n^{(2)}\|^4 x^{-4}\,\mathrm{d}x\right) \\ &\leq C_{13}(nh_n)^{-1} \stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{A.10}$$

Taking $\mathbb{E}\|\xi_n\|^\delta \leq (\mathbb{E}\|\xi_n\|^2)^{\frac{\delta}{2}} \leq C_{14}h_n^{\frac{\delta}{2}}$ into account, the inequality

$$\begin{aligned} &n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}\|\xi_n^{(2)}\|^\delta\mathbb{P}(\|\xi_n^{(1)}\|>\sqrt{nh_n}x/2) + n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}\|\xi_n^{(1)}\|^\delta\mathbb{P}(\|\xi_n^{(2)}\|>\sqrt{nh_n}x/2) \\ &\leq C_{15}n(nh_n)^{-\frac{\delta}{2}}h_n^{\frac{\delta}{2}}h_n(nh_n)^{-1} \stackrel{n\to\infty}{\longrightarrow} 0 \end{aligned} \tag{A.11}$$

is valid. Finally, applying (A.9)-(A.11) yields

$$
\begin{aligned}
& n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}(\|\xi_n\|^\delta \mathbb{1}_{\{\|\xi_n\|>\sqrt{nh_n}x\}}) \\
\leq \quad & 2^\delta n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}(\|\xi_n^{(1)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(1)}\|>\sqrt{nh_n}x/2\}}) + 2^\delta n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}(\|\xi_n^{(2)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(2)}\|>\sqrt{nh_n}x/2\}}) \\
& +2^\delta n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}\|\xi_n^{(2)}\|^\delta \mathbb{P}(\|\xi_n^{(1)}\|>\sqrt{nh_n}x/2) + 2^\delta n(nh_n)^{-\frac{\delta}{2}}\mathbb{E}\|\xi_n^{(1)}\|^\delta \mathbb{P}(\|\xi_n^{(2)}\|>\sqrt{nh_n}x/2) \\
\overset{n\to\infty}{\longrightarrow} \quad & 0.
\end{aligned}
$$

(f) Since $\mathbb{E}(\xi_n) = \mathbf{0}_m$, an application of *(d)* results in

$$\lim_{n\to\infty} n(nh_n)^{-1/2}\|\mathbb{E}(\xi_n \mathbb{1}_{\{\|\xi_n\|\leq\sqrt{nh_n}x\}})\| = \lim_{n\to\infty} n(nh_n)^{-1/2}\|\mathbb{E}(\xi_n \mathbb{1}_{\{\|\xi_n\|>\sqrt{nh_n}x\}})\| = 0.$$

(g) Gut [21], Theorem 3.1, and

$$\lim_{n\to\infty} h_n^{-1}\mathbb{E}(\xi_n\xi_n') = \lim_{n\to\infty} h_n^{-1}\int_0^{h_n} \mathbf{f}(s)\mathbf{\Sigma}\mathbf{f}(s)'\,\mathrm{d}s = \mathbf{f}(0)\mathbf{\Sigma}\mathbf{f}(0)'$$

gives $(nh_n)^{-1}\sum_{k=1}^n \xi_{n,k}\xi_{n,k}' \xrightarrow{\mathbb{P}} \mathbf{f}(0)\mathbf{\Sigma}\mathbf{f}(0)'$ as $n\to\infty$. □

A.2. Infinite second moments

Moreover, we present some asymptotic results for $\mathbf{L}(1) \in \mathscr{R}_{-\alpha}(a_n,\mu)$, $\alpha\in(0,2)$.

Proposition A.2. *Let $(\mathbf{L}(t))_{t\geq0}$ be an $\mathbb{R}^d$-valued Lévy process with $\mathbf{L}(1)\in\mathscr{R}_{-\alpha}(a_n,\mu)$, $0<\alpha<2$. Suppose $(h_n)_{n\in\mathbb{N}}$ is a sequence of positive constants such that $h_n\downarrow 0$ and $\lim_{n\to\infty} nh_n=\infty$. Set $a_t := a_{\lfloor t\rfloor}$ for $t\geq 0$. Let $\mathbf{f}:\mathbb{R}\to\mathbb{R}^{m\times d}$ be a measurable and bounded function with $\lim_{x\to0}\mathbf{f}(x)=\mathbf{f}(0)$. Define $\xi_n=\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}(s)$ for $n\in\mathbb{N}$.*

(a) Then

$$n\mathbb{P}(a_{nh_n}^{-1}\xi_n\in\cdot) \overset{v}{\Longrightarrow} \mu\circ\mathbf{f}(0)^{-1}(\cdot) \quad \text{on } \mathscr{B}(\overline{\mathbb{R}}^m\backslash\{\mathbf{0}_m\}).$$

(b) There exists a finite positive constant K such that

$$\lim_{n\to\infty} n\mathbb{P}(\|\xi_n\|>a_{nh_n}x) = Kx^{-\alpha} \quad \text{for } x>0.$$

(c) Let either $\delta\geq2$, or $\delta>\alpha$ and $(\mathbf{L}(t))_{t\geq0}$ be a compound Poisson process. Then there exists for any $x>0$ a finite positive constant K_δ such that

$$na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta\mathbb{1}_{\{\|\xi_n\|\leq a_{nh_n}x\}}) \leq K_\delta x^{\delta-\alpha} \quad \forall n\in\mathbb{N}.$$

(d) Let $\delta\in(0,\alpha)$. Then there exists for any $x>0$ a finite positive constant K_δ such that

$$na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta\mathbb{1}_{\{\|\xi_n\|>a_{nh_n}x\}}) \leq K_\delta x^{\delta-\alpha} \quad \forall n\in\mathbb{N}.$$

(e) Suppose that $\alpha\neq1$ and $\mathbb{E}(\mathbf{L}(1))=\mathbf{0}_d$ if $1<\alpha<2$. Then there exists for any $x>0$ a finite positive constant K such that

$$na_{nh_n}^{-1}\|\mathbb{E}(\xi_n\mathbb{1}_{\{\|\xi_n\|\leq a_{nh_n}x\}})\| \leq Kx^{|1-\alpha|} \quad \forall n\in\mathbb{N}.$$

The proof of Proposition A.2 uses the next two Lemmatas.

Lemma A.3. *Let $(\mathbf{L}(t))_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with $\mathbb{E}\|\mathbf{L}(1)\|^2<\infty$, $(a_t)_{t\geq 0}$ be an increasing sequence of positive constants in $\mathscr{R}_{1/\alpha}$, $0<\alpha<2$, and $(h_n)_{n\in\mathbb{N}}$ be a sequence of positive constants such that $h_n\downarrow 0$ as $n\to\infty$ and $\lim_{n\to\infty} nh_n=\infty$. Moreover, let $\mathbf{f}:\mathbb{R}\to\mathbb{R}^{m\times d}$ be a measurable and bounded function with $\lim_{x\to 0}\mathbf{f}(x)=\mathbf{f}(0)$. Define $\xi_n=\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}(s)$ for $n\in\mathbb{N}$. Finally, let $(\alpha-1)_+<\delta<2$.*

(a) Then $\lim_{n\to\infty} n\mathbb{P}(a_{nh_n}^{-1}\xi_n\in B)=0$ for any relatively compact set $B\in\mathscr{B}(\overline{\mathbb{R}}^m\backslash\{\mathbf{0}_m\})$.
(b) $\lim_{n\to\infty} na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta\mathbb{1}_{\{\|\xi_n\|>a_{nh_n}x\}})=0\quad$ for $x>0$.

Proof.
(a) Let $\gamma_B:=\inf_{\mathbf{x}\in B}\|\mathbf{x}\|$, which is larger than 0, and $0<\varepsilon<2/\alpha-1$. Markov's inequality, Proposition A.1 *(a)* and Potter's Theorem result in

$$n\mathbb{P}(a_{nh_n}^{-1}\xi_n\in B)\leq n\mathbb{P}(\|\xi_n\|>a_{nh_n}\gamma_B)\leq \frac{n}{a_{nh_n}^2}\frac{1}{\gamma_B^2}\mathbb{E}\|\xi_n\|^2\leq \frac{C_1}{\gamma_B^2}\frac{n}{(nh_n)^{\frac{2}{\alpha}-\varepsilon}}h_n\overset{n\to\infty}{\longrightarrow}0,$$

which we had to show.
(b) Moreover,

$$\begin{aligned}
na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta\mathbb{1}_{\{\|\xi_n\|>a_{nh_n}x\}}) &= na_{nh_n}^{-\delta}(a_{nh_n}x)^\delta\mathbb{P}(\|\xi_n\|>a_{nh_n}x)+na_{nh_n}^{-\delta}\int_{a_{nh_n}x}^{\infty}\mathbb{P}(\|\xi_n\|>z)\delta z^{\delta-1}\,\mathrm{d}z\\
&\leq nx^\delta\mathbb{E}\|\xi_n\|^2a_{nh_n}^{-2}x^{-2}+na_{nh_n}^{-\delta}\int_{a_{nh_n}x}^{\infty}\mathbb{E}\|\xi_n\|^2z^{-2}\delta z^{\delta-1}\,\mathrm{d}z\\
&\leq C_2nh_na_{nh_n}^{-2}\overset{n\to\infty}{\longrightarrow}0,
\end{aligned}$$

where we also used Markov's inequality. □

Lemma A.4. *Let $\mathbf{L}_1=(\sum_{k=1}^{N(t)}\mathbf{J}_k)_{t\geq 0}$ be an $\mathbb{R}^d$-valued compound Poisson process and $\mathbf{f}:\mathbb{R}\to\mathbb{R}^{m\times d}$ be a measurable and bounded function with $\lim_{x\to 0}\mathbf{f}(x)=\mathbf{f}(0)$. Define $\xi_n=\int_0^{h_n}\mathbf{f}(s)\,\mathrm{d}\mathbf{L}_1(s)$ for $n\in\mathbb{N}$. Then for any $0<\delta\leq 1$ with $\mathbb{E}\|\mathbf{L}(1)\|^\delta<\infty$ there exists a finite positive constant K such that*

$$\mathbb{E}\|\xi_n\|^\delta\leq Kh_n.$$

Proof. We define the Lévy process $L^\delta(t):=\sum_{k=1}^{N(t)}\|\mathbf{J}_k\|^\delta$ for $t\geq 0$, which satisfies $\mathbb{E}(L^\delta(1))<\infty$ by Sato [42], Corollary 2.5.8. Let the increasing sequence $(\Gamma_k)_{k\in\mathbb{N}}$ denote the jump times of $(N(t))_{t\geq 0}$. Then

$$\|\xi_n\|^\delta\leq\sum_{k=1}^{N(h_n)}\|\mathbf{f}(\Gamma_k)\|^\delta\|\mathbf{J}_k\|^\delta=\int_0^{h_n}\|\mathbf{f}(s)\|^\delta L^\delta(\mathrm{d}s).$$

Since $\mathbb{E}\left(\int_0^{h_n}\|\mathbf{f}(s)\|^\delta L^\delta(\mathrm{d}s)\right)=\mathbb{E}(L^\delta(1))\int_0^{h_n}\|\mathbf{f}(s)\|^\delta\,\mathrm{d}s\leq Ch_n$, we get also $\mathbb{E}\|\xi_n\|^\delta\leq Ch_n$. □

Note, for an arbitrary driving Lévy process the result is not valid, e.g., Brownian motion. In general we only have $\mathbb{E}\|\xi_n\|^\delta\leq Ch_n^{\frac{\delta}{2}}$.

Proof of Proposition A.2. *(a)* We use the decomposition of $\xi_n=\xi_n^{(1)}+\xi_n^{(2)}$ as given in the proof of Proposition A.1 and the notation there. Moreover, $\mathbf{f}(0)\mathbf{J}_1\in\mathscr{R}_{-\alpha}(a_n,\lambda^{-1}\mu\circ\mathbf{f}(0)^{-1}(\cdot))$ due to Hult and Lindskog [22], Lemma 2.1 and $\|\mathbf{J}_1\|\in\mathscr{R}_{-\alpha}(a_n)$ as well. First, we will show that $\xi_n^{(1)}$ satisfies the statement. Now, let B be a relatively compact set in $\mathscr{B}(\overline{\mathbb{R}}^m\backslash\{\mathbf{0}_m\})$ with $\mu(\partial B)=0$ and $\gamma_B=\inf_{x\in B}\|\mathbf{x}\|$, which is larger than 0. We define

$$n\mathbb{P}(a_{nh_n}^{-1}\xi_n^{(1)}\in B)=\sum_{l=1}^{\infty}n\mathbb{P}\left(a_{nh_n}^{-1}\sum_{k=1}^{l}\mathbf{f}(h_nU_{l,k})\mathbf{J}_k\in B\right)\mathbb{P}(N(h_n)=l)=:\sum_{l=1}^{\infty}a_{n,l}^*. \tag{A.12}$$

Furthermore, (A.6) gives for any $l \geq 1$,

$$0 \leq a^*_{n,l} \leq C_1 nh_n \mathbb{P}\left(a^{-1}_{nh_n} f^* \sum_{k=1}^{l} \|\mathbf{J}_k\| > \gamma_B\right) \mathbb{P}(N(1)=l) =: b^*_{n,l},$$

and for some finite constants $C_2, C_3, C_4 > 0$,

$$\begin{aligned}
\lim_{n\to\infty} b^*_{n,l} &= C_2 l f^{*\alpha} \gamma_B^{-\alpha} \mathbb{P}(N(1)=l), \\
\lim_{n\to\infty} \sum_{l=1}^{\infty} b^*_{n,l} &= C_3 \lim_{n\to\infty} nh_n \mathbb{P}\left(a^{-1}_{nh_n} \sum_{k=1}^{N(1)} \|\mathbf{J}_k\| > f^{*-1}\gamma_B\right) = C_4 f^{*\alpha} \gamma_B^{-\alpha},
\end{aligned}$$

where we used that $\sum_{k=1}^{l} \|\mathbf{J}_k\|$ and $\sum_{k=1}^{N(1)} \|\mathbf{J}_k\|$ are in $\mathscr{R}_{-\alpha}(a_n)$ by Resnick [40], Theorem 6.1 and Proposition 7.4, and by Hult and Lindskog [22], Lemma 2.1, respectively. Since (A.8), (A.12) and $\lim_{n\to\infty} \mathbf{f}(h_n U_{1,1}) = \mathbf{f}(0)$ $\mathbb{P}$-a.s. yield

$$\lim_{n\to\infty} a^*_{n,1} = \lim_{n\to\infty} nh_n \mathbb{P}\left(a^{-1}_{nh_n} \mathbf{f}(0)\mathbf{J}_1 \in B\right) \lambda = \mu \circ \mathbf{f}(0)^{-1}(B),$$

and moreover (A.7) results in

$$\lim_{n\to\infty} a^*_{n,l} = 0 \quad \text{for } l \geq 2,$$

a conclusion of Pratt's Theorem (see Pratt [35]) is

$$\lim_{n\to\infty} n\mathbb{P}\left(a^{-1}_{nh_n} \xi^{(1)}_n \in B\right) = \sum_{l=1}^{\infty} \lim_{n\to\infty} a^*_{n,l} = \mu \circ \mathbf{f}(0)^{-1}(B). \tag{A.13}$$

Furthermore, the Lévy measure of $\mathbf{L}^{(2)}$ has compact support. Thus, Sato [42], Corollary 25.8, gives that all moments of $\|\mathbf{L}^{(2)}(1)\|$ exist. The statement follows then from Lemma A.3 *(a)*, (A.4) and (A.13).
(b) is a conclusion of *(a)* and Resnick [39], Proposition 3.12.
(c) *Step 1.* Let $(\mathbf{L}(t))_{t\geq 0}$ be a compound Poisson process as given in (A.5), $\mathbf{f}(s) = \mathbf{I}_{d\times d}$ and $\delta > \alpha$ (if $\delta \geq 2$ then in particularly $\delta > \alpha$). Keep in mind that $\mathbf{L}(1) \in \mathscr{R}_{-\alpha}(a_n,\mu)$ and $\mathbf{J}_1 \in \mathscr{R}_{-\alpha}(a_n,\mu/\lambda)$ by Hult and Lindskog [22], Lemma 2.1. Then

$$\begin{aligned}
\mathbb{E}(\|\mathbf{L}(h_n)\|^\delta \mathbb{1}_{\{\|\mathbf{L}(h_n)\|\leq a_{nh_n}x\}}) &= \mathbb{E}(\|\mathbf{J}_1\|^\delta \mathbb{1}_{\{\|\mathbf{J}_1\|\leq a_{nh_n}x\}}) \frac{\mathbb{P}(N(h_n)=1)}{h_n} \\
&\quad + \sum_{l=2}^{\infty} \mathbb{E}\left(\left\|\sum_{k=1}^{l} \mathbf{J}_k\right\|^\delta \mathbb{1}_{\{\|\sum_{k=1}^{l}\mathbf{J}_k\|\leq a_{nh_n}x\}}\right) \frac{\mathbb{P}(N(h_n)=l)}{h_n}.
\end{aligned} \tag{A.14}$$

By Resnick [40], Theorem 6.1 and Proposition 7.4, $\|\sum_{k=1}^{l} \mathbf{J}_k\| \in \mathscr{R}_{-\alpha}(a_n)$, a conclusion of Karamata's Theorem is for any $l \geq 1$,

$$\lim_{n\to\infty} nh_n a^{-\delta}_{nh_n} \mathbb{E}\left(\left\|\sum_{k=1}^{l} \mathbf{J}_k\right\|^\delta \mathbb{1}_{\{\|\sum_{k=1}^{l}\mathbf{J}_k\|\leq a_{nh_n}x\}}\right) = lC_5 x^{\delta-\alpha}. \tag{A.15}$$

As in *(a)* we are allowed to apply Pratt's Theorem, such that (A.7), (A.8), (A.14) and (A.15) result in

$$\lim_{n\to\infty} na^{-\delta}_{nh_n} \mathbb{E}(\|\mathbf{L}(h_n)\|^\delta \mathbb{1}_{\{\|\mathbf{L}(h_n)\|\leq a_{nh_n}x\}}) = \lambda C_5 x^{\delta-\alpha}. \tag{A.16}$$

Step 2. Let $(\mathbf{L}(t))_{t\geq 0}$ be a compound Poisson process as given in (A.5), $\mathbf{f}$ be arbitrary and $\delta > \alpha$. Since

$$\mathbb{P}(\|\xi_n\| > y) \leq \mathbb{P}\left(f^* \sum_{k=1}^{N(h_n)} \|\mathbf{J}_k\| > y\right) \quad \text{for any } y > 0$$

and $L^*(t) := f^* \sum_{k=1}^{N(t)} \|\mathbf{J}_k\|$ for $t \geq 0$ is a compound Poisson process with $L^*(1) \in \mathscr{R}_{-\alpha}(a_n)$, we have

$$na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta \mathbb{1}_{\{\|\xi_n\|\leq a_{nh_n}x\}}) \leq nx^\delta \mathbb{P}(L^*(h_n) > a_{nh_n}x) + na_{nh_n}^{-\delta}\mathbb{E}\left(L^*(h_n)^\delta \mathbb{1}_{\{L^*(h_n)\leq a_{nh_n}x\}}\right),$$

which converges to $C_6 x^{\delta-\alpha}$ due to *(b)* and Step 1.
Step 3. Let $(\mathbf{L}(t))_{t\geq 0}$ be a Lévy process, $\mathbf{f}$ be arbitrary, $\delta \geq 2$ and $\xi_n = \xi_n^{(1)} + \xi_n^{(2)}$ as given in (A.4). Further, let $\varepsilon > 0$. Then the decomposition

$$\begin{aligned} na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta \mathbb{1}_{\{\|\xi_n\|\leq a_{nh_n}x\}}) &= na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta \mathbb{1}_{\{\|\xi_n\|\leq a_{nh_n}x\}}\mathbb{1}_{\{\|\xi_n^{(1)}\|\leq a_{nh_n}(x+\varepsilon)\}}) \\ &\quad + na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta \mathbb{1}_{\{\|\xi_n\|\leq a_{nh_n}x\}}\mathbb{1}_{\{\|\xi_n^{(1)}\|> a_{nh_n}(x+\varepsilon)\}}) \\ &=: I_{n,1} + I_{n,2} \end{aligned}$$

holds. Further,

$$\begin{aligned} I_{n,1} &\leq na_{nh_n}^{-\delta}2^\delta \mathbb{E}(\|\xi_n^{(1)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(1)}\|\leq a_{nh_n}(x+\varepsilon)\}}) + n2^\delta(2x+\varepsilon)^\delta \mathbb{E}\left(\left\|\frac{\xi_n^{(2)}}{a_n(2x+\varepsilon)}\right\|^\delta \mathbb{1}_{\{\|\xi_n^{(2)}\|\leq a_n(2x+\varepsilon)\}}\right) \\ &\leq na_{nh_n}^{-\delta}2^\delta \mathbb{E}(\|\xi_n^{(1)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(1)}\|\leq a_{nh_n}(x+\varepsilon)\}}) + nC_7 a_{nh_n}^{-2}\mathbb{E}\|\xi_n^{(2)}\|^2 \xrightarrow{n\to\infty} C_8(x+\varepsilon)^{\delta-\alpha} \end{aligned}$$

by Step 2 and Proposition A.1 *(a)*. In the last inequality we required $\delta \geq 2$. Moreover, applying *(b)* and Proposition A.1 *(a)* results in

$$I_{n,2} \leq n\mathbb{P}(\|\xi_n^{(2)}\| > a_{nh_n}\varepsilon)\mathbb{P}(\|\xi_n^{(1)}\| > a_{nh_n}(x+\varepsilon)) \leq C_9\varepsilon^{-2}h_n a_{nh_n}^{-2} n\mathbb{P}(\|\xi_n^{(1)}\| > a_{nh_n}(x+\varepsilon)) \xrightarrow{n\to\infty} 0.$$

Thus, *(c)* follows.
(d) Let $\varepsilon \in (0,1)$. We use the upper bound

$$\begin{aligned} &na_{nh_n}^{-\delta}\mathbb{E}(\|\xi_n\|^\delta \mathbb{1}_{\{\|\xi_n\|> a_{nh_n}x\}}) \\ &\leq na_{nh_n}^{-\delta}2^\delta \mathbb{E}(\|\xi_n^{(1)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(1)}\|> a_{nh_n}x(1-\varepsilon)\}}) + na_{nh_n}^{-\delta}2^\delta \mathbb{E}(\|\xi_n^{(2)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(1)}\|> a_{nh_n}x(1-\varepsilon)\}}) \\ &\quad + na_{nh_n}^{-\delta}2^\delta \mathbb{E}(\|\xi_n^{(1)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(2)}\|> a_{nh_n}x\varepsilon\}}) + na_{nh_n}^{-\delta}2^\delta \mathbb{E}(\|\xi_n^{(2)}\|^\delta \mathbb{1}_{\{\|\xi_n^{(2)}\|> a_{nh_n}x\varepsilon\}}) \\ &=: I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}. \end{aligned} \tag{A.17}$$

As in *(c)* we can show that by Karamata's and Pratt's Theorem the compound Poisson process $(L^*(t))_{t\geq 0}$ satisfies

$$\lim_{n\to\infty} na_{nh_n}^{-\delta}\mathbb{E}(\|L^*(h_n)\|^\delta \mathbb{1}_{\{\|L^*(h_n)\|> a_{nh_n}x\}}) = \lambda C_{10}x^{\delta-\alpha}, \tag{A.18}$$

and

$$I_{n,1} \leq 2^\delta na_{nh_n}^{-\delta}\mathbb{E}(\|L^*(h_n)\|^\delta \mathbb{1}_{\{\|L^*(h_n)\|> a_{nh_n}x\}}) \leq C_{11}x^{\delta-\alpha} \quad \forall n \in \mathbb{N}.$$

Further, by *(b)* and Proposition A.1 *(a)*

$$I_{n,2} \leq 2^\delta a_{nh_n}^{-\delta}(\mathbb{E}\|\xi_n^{(2)}\|^2)^{\frac{\delta}{2}} n\mathbb{P}(\|\xi_n^{(1)}\| > a_{nh_n}x(1-\varepsilon)) \xrightarrow{n\to\infty} 0 \tag{A.19}$$

holds. Moreover, by Lemma A.4

$$I_{n,3} = a_{nh_n}^{-\delta} 2^{\delta} \mathbb{E}\|\xi_n^{(1)}\|^{\delta} n\mathbb{P}(\|\xi_n^{(2)}\| > a_{nh_n} x\varepsilon) \leq C_{12} a_{nh_n}^{-\delta} h_n^{\delta/2} nh_n a_{nh_n}^{-2} x^{-2}\varepsilon^{-2} \overset{n\to\infty}{\longrightarrow} 0. \tag{A.20}$$

Finally, by Lemma A.3 *(b)*, $\lim_{n\to\infty} I_{n,4} = 0$. Statement *(d)* is then a consequence from (A.17)-(A.20).
(e) *Step 1.* Let $1 < \alpha < 2$. Then $\mathbb{E}(\xi_n) = \mathbf{0}_m$. Hence,

$$na_{nh_n}^{-1} \|\mathbb{E}(\xi_n \mathbb{1}_{\{\|\xi_n\| \leq a_{nh_n} x\}})\| = na_{nh_n}^{-1} \|\mathbb{E}(\xi_n \mathbb{1}_{\{\|\xi_n\| > a_{nh_n} x\}})\| \leq na_{nh_n}^{-1} \mathbb{E}(\|\xi_n\| \mathbb{1}_{\{\|\xi_n\| > a_{nh_n} x\}})$$

such that we can apply *(d)*.
Step 2. Let $\alpha \in (0,1)$. Again we use the decomposition of $\xi_n = \xi_n^{(1)} + \xi_n^{(2)}$ as given in (A.4). Thus,

$$\mathbb{E}(\xi_n \mathbb{1}_{\{\|\xi_n\| \leq a_{nh_n} x\}}) = \mathbb{E}(\xi_n^{(1)} \mathbb{1}_{\{\|\xi_n\| \leq a_{nh_n} x\}}) + \mathbb{E}(\xi_n^{(2)} \mathbb{1}_{\{\|\xi_n\| \leq a_{nh_n} x\}}) =: I_{n,1} + I_{n,2}.$$

On the one hand, let for some $\varepsilon > 0$,

$$\begin{aligned} \|I_{n,1}\| &\leq \int_0^{a_{nh_n}(x+\varepsilon)} \mathbb{P}(\|\xi_n^{(1)}\| > y, \|\xi_n\| \leq a_{nh_n} x)\,\mathrm{d}y + \int_{a_{nh_n}(x+\varepsilon)}^{\infty} \mathbb{P}(\|\xi_n^{(1)}\| > y, \|\xi_n\| \leq a_{nh_n} x)\,\mathrm{d}y \\ &=: I_{n,1,1} + I_{n,1,2}. \end{aligned}$$

Then

$$\begin{aligned} I_{n,1,1} &\leq \mathbb{E}(\|\xi_n^{(1)}\| \mathbb{1}_{\{\|\xi_n^{(1)}\| \leq a_{nh_n}(x+\varepsilon)\}}) + \int_0^{a_{nh_n}(x+\varepsilon)} \mathbb{P}(\|\xi_n^{(1)}\| > y, \|\xi_n^{(2)}\| > a_{nh_n}\varepsilon)\,\mathrm{d}y \\ &\leq \mathbb{E}(\|\xi_n^{(1)}\| \mathbb{1}_{\{\|\xi_n^{(1)}\| \leq a_{nh_n}(x+\varepsilon)\}}) + a_{nh_n}(x+\varepsilon)\mathbb{P}(\|\xi_n^{(2)}\| > a_{nh_n}\varepsilon). \end{aligned}$$

Hence, by *(c)* and Proposition A.1 *(a)*

$$\limsup_{n\to\infty} na_{nh_n}^{-1} I_{n,1,1} \leq C_{13} x^{1-\alpha} + C_{14} \limsup_{n\to\infty} n\mathbb{E}\|\xi_n^{(2)}\|^2 a_{nh_n}^{-1} = C_{13} x^{1-\alpha}.$$

Furthermore,

$$I_{n,1,2} \leq \int_{a_{nh_n}(1+\varepsilon)}^{\infty} \mathbb{P}(\|\xi_n^{(1)}\| > y, \|\xi_n^{(2)}\| > y - a_{nh_n} x)\,\mathrm{d}y \leq \mathbb{P}(\|\xi_n^{(1)}\| > a_{nh_n}(x+\varepsilon))\mathbb{E}\|\xi_n^{(2)}\|,$$

such that by *(b)* and Proposition A.1 *(a)*,

$$\limsup_{n\to\infty} na_{nh_n}^{-1} I_{n,1,2} = 0.$$

To conclude, $na_{nh_n}^{-1} I_{n,1} \leq C_{14} x^{1-\alpha}\ \forall\, n \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned} \|I_{n,2}\| &\leq \|\mathbb{E}((\xi_n^{(2)} - \mathbb{E}(\xi_n^{(2)}))\mathbb{1}_{\{\|\xi_n\| > a_{nh_n} x\}})\| + \|\mathbb{E}(\mathbb{E}(\xi_n^{(2)})\mathbb{1}_{\{\|\xi_n\| \leq a_{nh_n} x\}})\| \\ &\leq \mathbb{E}(\|\xi_n^{(2)}\| \mathbb{1}_{\{\|\xi_n^{(1)}\| > a_{nh_n} x/2\}}) + \mathbb{E}(\|\xi_n^{(2)}\| \mathbb{1}_{\{\|\xi_n^{(2)}\| > a_{nh_n} x/2\}}) + 2\|\mathbb{E}(\xi_n^{(2)})\| \\ &=: I_{n,2,1} + I_{n,2,2} + I_{n,2,3}. \end{aligned}$$

Then by *(b)*, Proposition A.1 *(a)*, $\|\mathbb{E}(\xi_n^{(2)})\| \leq C_{15} h_n$ and $\alpha \in (0,1)$,

$$\begin{aligned} na_{nh_n}^{-1} I_{n,2,1} &= a_{nh_n}^{-1} \mathbb{E}\|\xi_n^{(2)}\| n\mathbb{P}(\|\xi_n^{(1)}\| > a_{nh_n} x/2) \overset{n\to\infty}{\longrightarrow} 0, \\ na_{nh_n}^{-1} I_{n,2,3} &\leq C_{16} nh_n a_{nh_n}^{-1} \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Finally, by Markov's inequality

$$na_{nh_n}^{-1} I_{n,2,2} = nx/2\mathbb{P}(\|\xi_n^{(2)}\| > a_{nh_n}x/2) + na_{nh_n}^{-1}\int_{a_{nh_n}x/2}^{\infty} \mathbb{P}(\|\xi_n^{(2)}\| > y)\,\mathrm{d}y \le C_{17} nh_n a_{nh_n}^{-2} \overset{n\to\infty}{\longrightarrow} 0,$$

and thus, $\lim_{n\to\infty} na_{nh_n}^{-1} I_{n,2} = 0$. □

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