Erasmus University Rotterdam

Econometric Institute

Netherlands school of economics.

Report 7603/S

LIMIT THEORY FOR MULTIVARIATE SAMPLE EXTREMES

Laurens de Haan and Sidney I. Resnick

february, 1976

Limit Theory for Multivariate Sample Extremes

by Laurens de Haan* and Sidney I. Resnick**

Abstract

Let $\{(X_n^{(1)}, \ldots, X_n^{(k)}), n \geqslant 1\}$ be k-dimensional iid random vectors. Necessary and sufficient conditions are found for the weak convergence of the maxima $\{\vee_{j=1}^n X_j^{(1)}, \ldots, \vee_{j=1}^n X_j^{(k)}\}$ suitably normed to a non-degenerate limit df. The class of such limits is specified and conditions stated for the limit joint df to be a product of marginal df's. Some results are presented concerning extremal processes generated by multivariate df's.

Key Words: maxima, weak convergence, extrema, extremal processes, Poisson process

1970 AMS classification: Primary 60F05, Secondary 60G99

* Research supported by the Netherlands Organisation for the Advancement of Pure Research (Z.W.O.) and by CSIRO, Division of Mathematics and Statistics, Canberra while on leave from Erasmus University, Rotterdam.

** Support provided by National Science Foundation Grant OIP75-14513 while on leave from Stanford University. The hospitality of CSIRO, Division of Mathematics and Statistics, Canberra and the Department of Statistics, SGS, Australian National University is gratefully acknowledged.

1. Introduction

Suppose $\{\underset{\sim}{X}_n, n \geq 1\} = \{(X_n^{(1)}, \ldots, X_n^{(k)}), n \geq 1\}$ are independent, identically distributed (iid) random vectors with k-dimensional distribution function (df) F. Define the sample maxima as $\underset{\sim}{Y}_n = (Y_n^{(1)}, \ldots, Y_n^{(k)}) = (\bigvee_{j=1}^n X_j^{(1)}, \ldots, \bigvee_{j=1}^n X_j^{(k)})$. We seek conditions under which $\exists$ normalizing constants $a_n^{(j)} > 0$, $b_n^{(j)}$, $n \geq 1$, $1 \leq j \leq k$ such that

$$(1) \qquad \left(\frac{Y_n^{(1)} - b_n^{(1)}}{a_n^{(1)}}, \ldots, \frac{Y_n^{(k)} - b_n^{(k)}}{a_n^{(k)}} \right)$$

converges weakly to a non-degenerate limit df and we seek specifications of the class of such limits. To avoid trivialities we assume each marginal sequence $(Y_n^{(j)} - b_n^{(j)})/a_n^{(j)}$ in (1) converges weakly to a non-degenerate limit. This problem has been considered previously by Geffroy (1958), Tiago de Oliveira (1959) and Sibuya (1960). Their results are for k = 2 and do not extend in an obvious manner to higher dimensions.

A multivariate convergence to types argument (see Geffroy (1958)) quickly shows that the class of limit df's for (1) is the class of <u>max-stable</u> distributions where we define a df G in R^k to be max-stable iff for every n, $\exists$ $\alpha_n^{(j)} > 0$, $\beta_n^{(j)}$, $1 \leq j \leq k$ such that

$$(2) \qquad G^n(\alpha_n^{(1)} x_1 + \beta_n^{(1)}, \ldots, \alpha_n^{(k)} x_k + \beta_n^{(k)}) = G(x_1, \ldots, x_k) \quad .$$

Note that each marginal of G must be one of the three classical extreme value df's studied by Gnedenko (1943) and de Haan (1970, 1971). Max-stable df's form a subclass of the max-infinitely divisible (max-id) df's introduced and characterized in Balkema and Resnick (1975).

We begin in section 1 by deriving the form of max-stable df's in R^k which have specified marginals. Several representations are given. The restriction on the marginals is next removed after which we take up domain of attraction and asymptotic independence questions. Finally we close with some observations about the extremal processes generated by the max-stable and max id df's.

The max id df's as discussed in Balkema and Resnick (1975) are a proper subclass of the df's on R^k which can be defined as follows: $F(x_1, \ldots, x_k)$ is max id iff for every $t > 0$, $F^t(x_1, \ldots, x_k)$ is a df or equivalently iff $\forall n$ $F^{1/n}$ is a df. It is then immediate from (2) that max-stable df's are max id.

The following is a criterion for F to be max id: Let $A(x_1, \ldots, x_k) = (-\infty, x_1] \times \ldots \times (-\infty, x_k]$. Then there must exist a measure ν on $[-\infty,\infty)^k$, called the <u>exponent measure</u>, such that $\nu([-\infty,\infty)^k) = \infty$, $\nu(A^c(x_1, \ldots, x_k) < \infty$ for some $(x_1, \ldots, x_k)$ and

$$F(x_1, \ldots, x_k) = \exp\{-\nu(A^c(x_1, \ldots, x_k))\}.$$

From a process point of view the max id df's are precisely the class of df's F which can be used to define a multivariate extremal process $\underset{\sim}{Y}(t) = (Y_1(t), \ldots, Y_k(t))$. Such a process is defined to have marginals: $\forall\, n, \forall\, 0 < t_1 < \ldots < t_n$

$$\text{(3)} \qquad P\left[Y_1(t_i) \leqslant x_1^{(i)}, \ldots, Y_k(t_i) \leqslant x_k^{(i)}, i = 1,\ldots,n\right]$$

$$= F^{t_1}\left(\bigwedge_{i=1}^{n} x_1^{(i)}, \bigwedge_{i=1}^{n} x_2^{(i)}, \ldots, \bigwedge_{i=1}^{n} x_k^{(i)}\right)$$

$$F^{t_2-t_1}\left(\bigwedge_{i=2}^{n} x_1^{(i)}, \ldots, \bigwedge_{i=2}^{n} x_k^{(i)}\right) \cdots$$

$$F^{t_n-t_{n-1}}(x_1^{(n)}, \ldots, x_k^{(n)}) \; .$$

A related viewpoint is that F is max id iff there exists a measure ν on $[-\infty,\infty)^k$ such that if we construct a Poisson random measure on $R_+ \times [-\infty,\infty)^k$ with points $\{T_n; J_n^{(1)}, \ldots, J_n^{(k)}\}$ and mean measure $dt \times \nu(dx_1, \ldots, dx_k)$ then defining the extremal process $\underset{\sim}{Y}(t)$ by

(4) $$Y_i(t) = \sup \{J_k^{(i)} | T_k \leqslant t\}$$

we have $F^t(x_1, \ldots, x_k) = P[Y_i(t) \leqslant x_i, i=1, \ldots, k] = \exp\{-t\nu(A^C(x_1, \ldots, x_k))\}$.

Our methods differ from those of previous authors because of our reliance on the concept of max infinite divisibility and judicious use of polor coordinates. Also insight is gained by comparing the multivariate stable Lévy processes with certain of our extremal provesses $\underset{\sim}{Y}$ which satisfy $\{\underset{\sim}{Y}(at), t > 0\} = \{a^\alpha \underset{\sim}{Y}(t), t > 0\}$ $\forall$ $a > 0$ where α is a positive parameter.

2. Max-stable df's with prescribed marginals

Call a max-stable df G in R^k simple if each marginal is equal to the extreme value df $\Phi_1(x) = e^{-x^{-1}}$, $x > 0$; i.e.

$$G(\infty, \ldots \infty, x_i, \infty, \ldots, \infty) = e^{-x_i^{-1}}, \; x_i > 0 .$$

We begin by deriving the form of a simple G. The reason why it is sensible to start with a simple G becomes clear in section 4 where we remove this restriction on the marginals.

Consideration of properties exhibited by $\Phi_1(x)$ shows that (2) can be written as

$$G^n(nx_1, \ldots, nx_k) = G(x_1, \ldots, x_k)$$

$\forall n$ and it is easy to switch to a continuous variable s in place of n so that $\forall$ $s > 0$

(5) $$G^s(sx_1, \ldots, sx_k) = G(x_1, \ldots, x_k) .$$

Letting ν be the exponent measure of G (5) becomes

(6) $$s\nu(A^C(sx_1, \ldots, sx_k)) = \nu(A^C(x_1, \ldots, x_k))$$

where recall $A(x_1, \ldots, x_k) = (-\infty, x_1] \times \ldots \times (-\infty, x_k]$ so that (6) entails

$$s\nu(sA^c(x_1, \ldots, x_k)) = \nu(A^c(x_1, \ldots, x_k)).$$

For fixed s the measure $s\nu(s\cdot)$ agrees with ν on a generating class closed under intersections and hence we conclude $\forall B \in \mathcal{B}(R^k)$

(7) $$s\nu(sB) = \nu(B) .$$

Let $\Xi := [0, \pi/2]^{k-1}$ and let $T : R^k \to R_+ \times \Xi$ be the transformation to polar coordinates: $T(x_1, \ldots, x_k) = (r, \underset{\sim}{\theta})$ where $r^2 = \Sigma_{i=1}^k x_i^2$, $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ and $\sin^2 \theta_i = (x_{i+1}^2 + \ldots + x_k^2)/x_i^2 + \ldots + x_k^2)$ for $i = 1, \ldots, k-1$. Fix a Borel set $C \subset \Xi$ and set $D(r,C) = \{(s, \underset{\sim}{\theta}) | s > r, \underset{\sim}{\theta} \in C\}$. Note that for $r > 0$, $\nu(T^{-1}(D(r,C))) < \infty$ because for some $x_1, \ldots, x_k$, $x_i \geqslant 0$, $i = 1 \ldots k$ we have $T^{-1}(D(r,C)) \subset A^c(x_1, \ldots, x_k)$ and $\nu(A^c(x_1, \ldots, x_k)) < \infty$. Referring back to (7) we have

$$s\nu(sT^{-1}(D(r,C))) = s\nu(T^{-1}(D(rs,C))) = \nu(T^{-1}(D(r,C))),$$

i.e. if $M(r) = \nu(T^{-1}(D(r,C)))$ we have

$$M(r) = sM(rs).$$

Setting $s = r^{-1}$ and $S(C) = M(1)$ gives $M(r) = r^{-1}S(C)$ where S is a finite measure on Ξ. Thus we have

<u>Theorem 1</u>: G is simple stable with exponent measure ν iff there exists a finite measure S on Ξ such that

$$\nu \circ T^{-1}(dr, d\underset{\sim}{\theta}) = r^{-2}dr\, S(d\underset{\sim}{\theta})$$

and $$\int_{\Xi} \sin \theta_1, \ldots, \sin \theta_{i-1} \cos \theta_i \, S(d\underset{\sim}{\theta}) = 1$$

for $i = 1, \ldots, k-1$ with the convention that $\theta_k = 0$ and for $i = 1$ the integrand is just $\cos \theta_1$. Recall T is the transformation to polar coordinates.

The integral condition in Theorem 1 arises because of the requirement that G be simple (cf. Theorem 2) and disappears when this requirement is waived. To check that the integral must equal 1 note that for $i = 1, \ldots, k$

$$x_i^{-1} = \nu(A^c(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty))$$
$$= \int_{TA^c} r^{-2} dr\, S(d\underset{\sim}{\theta})$$

where $TA^c = \{(r,\underset{\sim}{\theta}) \mid r \sin\theta_1 \ \ldots \ \sin\theta_{i-1}\cos\theta_i > x_i\}$. Integrating on r gives the result.

<u>Remark</u>: For $r > 0$, $0 \leqslant \theta \leqslant \pi/2$ we have $\nu\circ T^{-1}(A^c(r,\theta)) = +\infty$. Thus $\nu\circ T^{-1}$ cannot serve as the exponent measure of a max id df.

The product measure appearing in Theorem 1 has the following interpretation: Suppose G is the df of $\underset{\sim}{Y}(1)$ where $\underset{\sim}{Y}$ is related to the Poisson random measure (as described in (4)) with points $\{T_n; J_n^{(1)}, \ldots, J_n^{(k)}\}$ and mean measures $dt \times \nu(dx_1, \ldots, dx_k)$. Transform these points into $\{T_n; T(J_n^{(1)}, \ldots, J_n^{(k)})\} = \{T_n, r_n, \underset{\sim}{\theta}_n\}$. The resulting set of points constitute a Poisson random measure on $R_+ \times R_+ \times \Xi$ with mean measure $dt \times r^{-2} dr \times S(d\underset{\sim}{\theta})$. Therefore $\{T_k\}$, $\{r_k\}$, $\{\underset{\sim}{\theta}_k\}$ are independent sequences.

Further understanding of the meaning of Theorem 1 is obtained from the following considerations: For a function x(t) which is right continuous with finite left limits $\forall t > 0$ define the functional h via

$$(hx)(t) = \sup_{0 < s \leqslant t} ((x(s) - x(s-)) \vee 0) .$$

<u>Corollary 1</u>: Let $\underset{\sim}{X}(t) = (X_1(t), \ldots, X_k(t))$ be a k-variate stable Lévy process of index 1; i.e. a process with stationary independent increments and the property $\forall\, a > 0$ $\{\underset{\sim}{X}(at), t \geqslant 0\} = \{a\underset{\sim}{X}(t) + \underset{\sim}{C}(a), t \geqslant 0\}$ where $\underset{\sim}{C}(a)$ is a non-random vector. Suppose further that for $i = 1, \ldots, k$ the Lévy measure ν_i of X_i has the property that $\nu_i(x,\infty) = x^{-1}$ for $x > 0$. The class of extremal processes generated by the simple stable df's described in

Theorem 1 is precisely the class of extremal processes realized through the scheme $\underset{\sim}{Y}(t) = \{Y_1(t), \ldots, Y_k(t)\} = \{(hX_1)(t), \ldots, (hX_k)(t)\}$.

Proof: That $\underset{\sim}{Y}$ is an extremal process follows (as in the 1-dimensional case - cf. Dwass 1964, Resnick and Rubinovitch 1973) from the fact that $\underset{\sim}{X}$ induces Poisson random measure with points $\{T_n; J_n^{(1)}, \ldots, J_n^{(k)}\}$ where T_n is the time of a jump and $(J_n^{(1)}, \ldots, J_n^{(k)}) = \underset{\sim}{X}(T_n) - \underset{\sim}{X}(T_n-)$. The mean measure is $dt \times \nu(dx_1, \ldots, dx_k)$ where ν is the Lévy measure of $\underset{\sim}{X}$. However, if $\underset{\sim}{X}$ is stable with index 1, it is well known (Lévy 1937) that $\nu \circ T^{-1}(dr, d\underset{\sim}{\theta}) = r^{-2} dr\, S(d\underset{\sim}{\theta})$ where S is a finite measure on Ξ.

Remark: Corollary 1 was deduced as a consequence of Theorem 1. The order can be reversed. One can first show that $(hX_1, \ldots, hX_k)$ gives the totality of max stable extremal processes with prescribed marginals and use this to deduce the criterion on the exponent measure.

In case $k = 2$, the criteria obtained in terms of $\nu \circ T^{-1}$ for G to be max stable can be rephrased in terms of ν:

Corollary 2: $G(x,y)$ is simple stable with exponent measure ν iff

$$\nu(A^c(x,y)) = x^{-1} \int_0^{\arctan y/x} \cos\theta\, S(d\theta) + y^{-1} \int_{\arctan y/x}^{\pi/2} \sin\theta\, S(d\theta)$$

where $S(\cdot)$ is a finite measure on $[0, \frac{\pi}{2}]$ such that

$$\int_0^{\pi/2} \cos\theta\, S(d\theta) = \int_0^{\pi/2} \sin\theta\, S(d\theta) = 1 .$$

Proof: The last two conditions arise because we require $G(x,\infty) = \exp\{-x^{-1}\} = G(\infty,x)$. For the rest note that by Theorem 1 $\nu \circ T^{-1}(dr,d\theta) = r^{-2} dr\, S(d\theta)$ so that

$$\begin{aligned}\nu(A^c(x,y)) &= \iint_{T(A^c(x,y))} r^{-2}\, dr\, S(d\theta) \\ &= \iint_{\{(r,\theta)\,|\, r\cos\theta \leqslant x,\ r\sin\theta \leqslant y\}^c} r^{-2}\, dr\, S(d\theta) \\ &= \iint_{\{(r,\theta)\,|\, r > \frac{x}{\cos\theta} \wedge \frac{y}{\sin\theta}\}} r^{-2}\, dr\, S(d\theta)\end{aligned}$$

and evaluating the integral on r for fixed θ gives

$$x^{-1}\int_0^{\arctan y/x} \cos\theta\, S(d\theta) + y^{-1}\int_{\arctan y/x}^{\pi/2} \sin\theta\, S(d\theta)$$

as asserted.

Corollary 3: If G is as in Corollary 2 and $P(X \leqslant x,\ Y \leqslant y) = G(x,y)$ then

(i) X,Y are independent iff $S\{0\} = S\{\pi/2\} = 1$ and S places no mass elsewhere. This can be seen either from Corollary 2 or by checking directly from $G(x,y) = \exp\{-(\frac{1}{x}+\frac{1}{y})\}$ that $\nu\{(t,s)\,|\,t > x,\ s > y\} = 0$ for all $x,y > 0$.

(ii) $P(X = Y) = 1$ iff $S\{\pi/4\} = \sqrt{2}$ and S places no mass elsewhere.

Remark: If the measure S concentrates on some point $\theta_0 \in [0, \pi/2]$ with $\theta_0 \neq \pi/4$ we have $Y = (\tan\theta_0)X$ a.s. and hence the marginals are both of type $\Phi_1(x)$, but are not equal. This means that G is not simple according to our definition.

Remark: We can connect our results with those of Sibuya (1960) (see also Geffroy (1958)) as follows: In Corollary 2 when k = 2 set

$$(8)\qquad W(t) = \int_0^{\arctan t} \cos\theta\, S(d\theta) = \int_0^t \cos(\arctan y)\, d\, S(\arctan y)$$

and

$$(9)\qquad G(x,y) = \exp\{-(\frac{1}{x}+\frac{1}{y}+\frac{1}{y}\chi(\frac{y}{x}))\},$$

so that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{y}\chi(\frac{y}{x}) = \frac{1}{x}\int_0^{\arctan y/x} \cos\theta\, S(d\theta) + \frac{1}{y}\int_{\arctan y/x}^{\pi/2} \sin\theta\, S(d\theta)$$

i.e.

$$\begin{aligned}
t + 1 + \chi(t) &= t\int_0^{\arctan t} \cos\theta\, S(d\theta) + \int_{\arctan t}^{\pi/2} \sin\theta\, S(d\theta) \\
&= tW(t) + \int_{\arctan t}^{\pi/2-} \tan\theta\cos\theta\, S(d\theta) + S(\{\pi/2\}) \\
&= tW(t) + \int_t^{\infty} y\, W(dy) + S(\{\pi/2\}) \\
&= \int_t^{\infty} (y-t)\, W(dy) + t(1 - W(t)) + tW(t) + S(\{\pi/2\}) \\
&= t + \int_t^{\infty} (1 - W(s))\,ds + S(\{\pi/2\}).
\end{aligned}$$

Therefore we conclude

$$\chi(t) = t\,W(t) + \int_t^\infty y\,W(dy) + S(\{\tfrac{\pi}{2}\}) - 1 - t$$

$$= \int_t^\infty (1 - W(s))ds + S(\{\pi/2\}) - 1.$$

Note χ has the properties specified by Sibuya:

(10) χ is continuous and convex since it is the integral of a monotone function

(11) $\max(-t, -1) \leqslant \chi(t) \leqslant 0, \quad \forall\, t \geqslant 0.$

Conversely if G is of form (9) where χ satisfies (10) and (11) then one checks directly that G is max stable.

<u>Example 1:</u> (Cf. Geffroy 1958, p.71): Let $S[0, \theta] = \theta$ for $0 \leqslant \theta \leqslant \pi/2$ so that $\int_0^{\pi/2} \cos\theta\, S(d\theta) = \int_0^{\pi/2} \sin\theta\, S(d\theta) = 1$. Then $\chi(t) = (1+t^2)^{\frac{1}{2}} - (1+t)$ and $G(x,y) = \exp\{-(x^{-2} + y^{-2})^{\frac{1}{2}}\}$ for $x \geqslant 0$, $y \geqslant 0$.

<u>Example 2:</u> Take $S[0, \theta] = 3\int_0^\theta \cos t \sin t\, dt$, $0 \leqslant \theta \leqslant \pi/2$. Then $\chi(t) = -t(1 + t^2)^{-\frac{1}{2}}$ and for $x \geqslant 0$, $y \geqslant 0$ $G(x,y) = \exp\{-(x^{-1} + y^{-1} - (x^2 + y^2)^{-\frac{1}{2}})\}$.

<u>Example 3:</u> (Sibuya 1960, p.208): $\chi(t) = -kt(1+t)^{-1}$ for $0 < k \leqslant 1$ corresponds to $S(t) = \int_0^t 2k(\cos y + \sin y)^{-3}dy$ and $G(x,y) = \exp\{-(x^{-1} + y^{-1} - k(x+y)^{-1})\}$.

<u>A constructive approach</u>:

Next we follow a constructive approach which leads to a representation of the simple stable df's in Cartesian coordinates. Recalling that the required marginals are $\Phi_1(x) = e^{-x^{-1}}$, $x > 0$ observe that in R^2 the Frechet df $G(x,y) = \Phi_1(x) \wedge \Phi_1(y) = \exp\{-x^{-1} \vee y^{-1}\}$ for $x, y > 0$ is a simple stable df. This df is concentrated on the line $x = y$. The df

$$G(x,y) = \exp\{-(1+a)^{-1}((ax^{-1}) \vee y^{-1} + x^{-1} \vee (ay^{-1}))\}$$

is simple stable for $a > 0$ and concentrates on the lines $x = ay$ and $x = a^{-1}y$. (G is the product of two distribution functions each of which concentrating on one of the lines.) Generalizing this procedure we get the most general simple stable df in R^k. Let $\Omega = \{(x_1, \ldots, x_k) \mid x_i \geqslant 0,\ i = 1, \ldots, k,\ \sum_1^k x_i^2 = 1\}$.

Theorem 2: $G(x_1, \ldots, x_k)$ is simple stable iff there exists a finite measure U on Ω with

$$\int_\Omega a_i \, U(da_1, \ldots, da_k) = 1 \quad \text{for } i = 1, \ldots, k$$

and such that

$$G(x_1, \ldots, x_k) = \exp\{-\int_\Omega \max(a_1 x_1^{-1}, \ldots, a_k x_k^{-1}) \, U(da_1, \ldots, da_k)\} .$$

Proof: That any G of the given form is simple can be verified easily. To prove the converse we use Theorem 1. We have

$$- \log G(x_1, \ldots, x_k) = \int_{\{(r,\underset{\sim}{\theta}) | r \sin \theta_1 \ldots \sin \theta_{i-1} \cos \theta_i \leq x_i, \, i=1, \ldots, k\}^c} r^{-2} \, dr \, S(d\underset{\sim}{\theta})$$

and integrating on r gives

$$= \int_{\underset{\sim}{\theta} \in \Xi} \max \left\{\frac{\sin \theta_1 \ldots \sin \theta_{i-1} \cos \theta_i}{x_i}, \, i=1, \ldots, k\right\} S(d\underset{\sim}{\theta})$$

which completes the proof.

Remark: Independence of the k-marginals of G corresponds to a measure U concentrated on the k extreme points of Ω. If U concentrates on a subset of Ω, then G concentrates on the straight lines through the origin and this subset of Ω.

Here are some examples in R^3:

Example 4: Suppose $U\{(1/\sqrt{2}, 1/\sqrt{2}, 0)\} = U\{(1/\sqrt{2}, 0, 1/\sqrt{2})\} = U\{(0, 1/\sqrt{2}, 1/\sqrt{2})\} = 1/\sqrt{2}$ with U placing no mass elsewhere. Then

$$G(x, y, z) = \exp\{-\tfrac{1}{2}(x^{-1} \vee y^{-1} + y^{-1} \vee z^{-1} + x^{-1} \vee z^{-1})\}$$

for $x, y, z \geq 0$.

Example 5: Let U concentrate on $\Omega \cap \{(x, y, z) | x=0 \text{ or } y=0 \text{ or } z=0\}$ and have density $\tfrac{1}{2}$ there. Then

$$G(x, y, z) = \exp -\tfrac{1}{2}\{(x^{-2} + y^{-2})^{\frac{1}{2}} + (x^{-2} + z^{-2})^{\frac{1}{2}} + (y^{-2} + z^{-2})^{\frac{1}{2}}\}.$$

<u>Example 6</u>: Let U have constant density $4/\pi$ on Ω. Then

$$G(x, y, z) = \exp \left\{-\tfrac{1}{2}\left(x^{-1} \arcsin \frac{yz}{(x^2+ y^2)^{\frac{1}{2}}(x^2+ z^2)^{\frac{1}{2}}} + y^{-1} \arcsin \frac{xz}{(x^2+y^2)^{\frac{1}{2}}(y^2+z^2)^{\frac{1}{2}}}\right.\right.$$

$$\left.\left. + z^{-1} \arcsin \frac{xy}{(x^2+ z^2)^{\frac{1}{2}}(y^2+ z^2)^{\frac{1}{2}}}\right)\right\}$$

for $x, y, z \geqslant 0$.

<u>Remark</u>: Examples 4 and 5 are based on the observation that if Ω is partitioned into n measurable sets $\Omega_1, \ldots, \Omega_n$, the stable df can be written as the product of n stable df's with angular measures concentrated on Ω_i $(i=1, \ldots, n)$.

3. <u>Domains of attraction of simple max-stable distributions.</u>

Here we characterize the domain of attraction of a simple stable df G and again we recall that each marginal of G equals $\Phi_1(x) = e^{-x^{-1}}$, $x > 0$.

Suppose F is in the domain of attraction of a simple stable df G; i.e. $\exists\, a_n^{(j)} > 0$, $b_n^{(j)}$, $n \geqslant 1$, $j = 1, \ldots, k$ such that

$$(12) \qquad F^n(a_n^{(1)}x_1 + b_n^{(1)}, \ldots, a_n^{(k)}x_k + b_n^{(k)}) \to G(x_1, \ldots, x_k)$$

for $(x_1, \ldots, x_k)$ a continuity point of G, $x_i \geqslant 0$, $i=1, \ldots, k$. Consideration of the marginals shows that (12) still holds if $b_n^{(j)} = 0$, $n \geqslant 1$, $j=1, \ldots, k$ (cf. Gnedenko (1943), de Haan (1970)). Suppose for the moment $a_n = a_n^{(1)} = \ldots = a_n^{(k)}$. When this is the case we say F is in the <u>domain of symmetric attraction</u> of G. Recall the notation $A(x_1, \ldots, x_k) = (-\infty, x_1] \times \ldots \times (-\infty, x_k]$. Note (12) holds iff

$$\lim_{n \to \infty} n(1-F(a_n x_1, \ldots, a_n x_k)) = - \log G(x_1, \ldots, x_k)$$

so that if ν is the exponent measure of G we have

$$\lim_{n\to\infty} nP\,[\underset{\sim}{X} \in a_n\, A^c(x_1, \ldots, x_k)] = \nu(A^c(x_1, \ldots, x_k))$$

for all A with $\nu(\partial A) = 0$, where we suppose $\underset{\sim}{X}$ is a random vector with df F. Hence $\forall\, B \in \mathcal{B}(R^k)$ with $\nu(\partial B) = 0$ we have

$$\lim_{n\to\infty} nP\,[\underset{\sim}{X} \in a_n B] = \nu(B).$$

Now we switch to polar coordinates. Let C be a Borel subset of Ξ and set for $r > 0$

$$B(r,C) = \{(x_1, \ldots, x_k) \mid \sum_1^k x_i^2 > r^2,\ \underset{\sim}{\theta} \in C\}.$$

Then

$$\lim_{n\to\infty} nP\,[\underset{\sim}{X} \in B\,(a_n r,\ C)] = \lim_{n\to\infty} nP\,[\underset{\sim}{X} \in a_n B(r,\ C)]$$

$$= \nu(B(r,C)) = r^{-1} S(C)$$

(by Theorem 1 provided $S(\partial C) = 0$), i.e.

(13) $$\lim_{n\to\infty} nP\,[\|\underset{\sim}{X}\| > a_n r,\ \Theta(\underset{\sim}{X}) \in C] = r^{-1}\, S(C)$$

where $\|\underset{\sim}{X}\|$, Θ are the polar coordinates of $\underset{\sim}{X}$. Setting $r = 1$ and $C = \Omega$ we obtain

(14) $$\lim_{n\to\infty} P\,[\|\underset{\sim}{X}\| > a_n r] / P\,[\|\underset{\sim}{X}\| > a_n] = r^{-1}$$

and furthermore it follows from (13) that

(15) $$\lim_{n\to\infty} P\,[\|\underset{\sim}{X}\| > a_n,\ \Theta(\underset{\sim}{X}) \in C] / P\,[\|\underset{\sim}{X}\| > a_n] =$$

$$\lim_{n\to\infty} P\,[\,\Theta(\underset{\sim}{X}) \in C \,|\, \|\underset{\sim}{X}\| > a_n] = S(C)/S(\Omega)\ .$$

It is not hard to see that a_n may be replaced by a continuous variable t and that in this form (14) and (15) imply (13). Thus we have proved

Theorem 3: The random vector $\underset{\sim}{X}$ with df F is in the domain of symmetric attraction of the simple stable df G with exponent measure ν and $\nu\circ T^{-1}(dr,\ d\underset{\sim}{\theta}) = r^{-2} dr\ S(d\underset{\sim}{\theta})$ iff

$$(16)\quad \lim_{t\to\infty} P[\|\underset{\sim}{X}\| > tr] / P[\|\underset{\sim}{X}\| > t] = r^{-1}$$

$$(17)\quad \lim_{t\to\infty} P[\Theta(\underset{\sim}{X}) \in C \mid \|\underset{\sim}{X}\| > t] = S(C)/S(\Omega).$$

Corollary 3: In the case of symmetric attraction to G, the partial maxima of $\|\underset{\sim}{X}_i\|$ where $\underset{\sim}{X}_i$, $i \geqslant 1$ are iid vectors from F, converge to Φ_1.

Remark: The criteria for convergence of sums of iid vectors are the same. See Rvačeva (1962 Theorem 4.2; set α=1).

The situation of non-symmetric attraction is discussed in the next section.

Sufficient conditions for convergence can be given in terms of the density of F when this density exists.

Corollary 4: Suppose G is simple stable and the measure S appearing in the representation of Theorem 1 has density $s(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k) \in \Xi$. Suppose F has density f. Then F is in the domain of symmetric attraction of G if for all $r > 0$

$$(18)\quad \lim_{t\to\infty} \frac{\int_\Xi f(tr\cos\theta_1, tr\sin\theta_1\cos\theta_2, \ldots, tr\sin\theta_1\ldots\sin\theta_{k-2}\cos\theta_{k-1}, tr\sin\theta_1\ldots\sin\theta_{k-1})d\underset{\sim}{\theta}}{\int_\Xi f(t\cos\theta_1, t\sin\theta_1\cos\theta_2, \ldots, t\sin\theta_1\ldots\sin\theta_{k-2}\cos\theta_{k-1}, t\sin\theta_1\ldots\sin\theta_k)d\underset{\sim}{\theta}}$$

$$= r^{-(k+1)}$$

$$(19)\quad \lim_{t\to\infty} \frac{f(t\cos\theta_1, t\sin\theta_1\cos\theta_2, \ldots, t\sin\theta_1\ldots\sin\theta_{k-2}\cos\theta_{k-1}, t\sin\theta_1\ldots\sin\theta_{k-1})}{f(t/\sqrt{2}, t/(\sqrt{2})^2, \ldots, t/(\sqrt{2})^k)}$$

$$= s(\underset{\sim}{\theta})/s(\tfrac{\pi}{4}) .$$

Proof: Let $f_*(r,\underset{\sim}{\theta})$ be the density of $\|\underset{\sim}{X}\| = (\sum_1^k X_i^2)^{\frac{1}{2}}$ and $\underset{\sim}{\Theta} = (\Theta_1, \ldots, \Theta_k)$ where $\Theta_i = \arcsin\left(\sum_{\ell=i+1}^k X_\ell^2 / \sum_{\ell=i}^k X_\ell^2\right)^{\frac{1}{2}}$ and suppose

$$(20)\quad \lim_{t\to\infty} \frac{\int_{\underset{\sim}{\theta}\in\Xi} f_*(rt,\underset{\sim}{\theta})\,d\underset{\sim}{\theta}}{\int_{\underset{\sim}{\theta}\in\Xi} f_*(t,\underset{\sim}{\theta})\,d\underset{\sim}{\theta}} = r^{-2}$$

and

$$(21)\quad \lim_{t\to\infty} f_*(t,\underset{\sim}{\theta})/\ f_*(t,\underset{\sim}{\tfrac{\pi}{4}}) = s(\underset{\sim}{\theta})/\ s(\underset{\sim}{\tfrac{\pi}{4}})\ .$$

Note that (21) and L'Hospital's rule give

$$\lim_{t\to\infty} \frac{\int_{s=t}^{\infty} f_*(s,\underset{\sim}{\theta})\,ds}{\int_{s=t}^{\infty} f_*(s,\underset{\sim}{\tfrac{\pi}{4}})\,ds} = \frac{s(\underset{\sim}{\theta})}{s(\underset{\sim}{\tfrac{\pi}{4}})}$$

and therefore

$$\lim_{t\to\infty} \frac{P\,[\|\underset{\sim}{X}\| > t,\ \underset{\sim}{\theta}\in C]}{\int_t^{\infty} f_*(r\ \underset{\sim}{\tfrac{\pi}{4}})\,dr}$$

$$= \lim_{t\to\infty} \frac{\int_t^{\infty}\int_{\underset{\sim}{\theta}\in C} f_*(r,\underset{\sim}{\theta})\,dr\ d\underset{\sim}{\theta}}{\int_t^{\infty} f_*(r,\ \underset{\sim}{\tfrac{\pi}{4}})\,dr}$$

$$= \frac{\int_{\underset{\sim}{\theta}\in C} s(\underset{\sim}{\theta})\,d\underset{\sim}{\theta}}{s(\underset{\sim}{\tfrac{\pi}{4}})} = \frac{S(C)}{s(\underset{\sim}{\tfrac{\pi}{4}})}$$

from which (17) follows and (16) follows directly from (20) thus implying F is symmetrically attracted to G. The conditions (20) and (21) readily translate into (18) and (19) and the proof is complete.

Example 7: On R^2 suppose $S[0,\theta] = \theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$ for G. Then (19) means

$$f(t\cos\theta_1, t\sin\theta_1) \sim f(t\cos\theta_2, t\sin\theta_2)$$

as $t \to \infty$ $\forall\, \theta_1, \theta_2 \in [0, \frac{\pi}{2}]$.

4. Stable df's that are not simple; domains of attraction.

We again suppose that (12) holds but now make no assumption about the marginals of the limit G except that they be non-degenerate. Denote the marginals of F by F_i, $i=1, \ldots, k$ and let $U_i(x)$ be an inverse of the monotone function $1/(1-F_i(x))$. Then U_i satisfies

$$\lim_{t \to \infty} \frac{U_i(tx) - U_i(t)}{U_i(te) - U_i(t)} = \Psi_i(x)$$

where

$$\text{(22)} \qquad \Psi_i(x) = \frac{x^\rho - 1}{e^\rho - 1}, \quad \frac{1-x^{-\rho}}{1-e^{-\rho}} \quad \text{or} \quad \log x$$

for $x > 0$ where ρ is a positive parameter (de Haan, 1970); in particular

$$\text{(23)} \qquad \lim_{n \to \infty} (U_i(nx) - b_n^{(i)})/ a_n^{(i)} = \Psi_i(x)$$

where $a_n^{(i)} > 0$, $b_n^{(j)}$, $j=1, \ldots, k$, $n \geqslant 1$ are the normalizing constants appearing in (12). Therefore

$$\lim_{n \to \infty} P\left[\frac{1}{1-F_i}(Y_n^{(i)}) \leqslant nx_i, \quad i=1, \ldots, k\right]$$

$$= \lim_{n \to \infty} P\left[(Y_n^{(i)} - b_n^{(i)})/ a_n^{(i)} \leqslant (U_i(nx_i) - b_n^{(i)})/ a_n^{(i)}, \ i=1, \ldots, k\right]$$

$$= G(\Psi_1(x_1), \ldots, \Psi_k(x_k))$$

from (12) and (23). If we suppose the marginals of $G(\Psi_1(x_1), \ldots, \Psi_k(x_k))$ are $e^{-x^{-1}}$, $x > 0$ (in any event, the marginals will be of this type), then we have symmetric convergence of the maxima of $\{\frac{1}{1-F_i}(X_n^{(1)}), \ldots, \frac{1}{1-F_k}(X_n^k);$ $n \geq 1\}$ to the simple stable df $G(\Psi_1(x_1), \ldots, \Psi_k(x_k))$. Using this we can generalize the results in the previous two sections to the general case: The type of the most general max-stable df with non-degenerate marginals is of the form $G_*(\Psi_1^{-1}(x_1), \ldots, \Psi_k^{-1}(x_k))$ with G_* a simple stable df and Ψ_i one of the functions given in (22), $i=1, \ldots, k$. A df F is in the domain of attraction of G iff $F(U_1(x_1), \ldots, U_k(x_k))$ is in the domain of symmetric attraction of $G(\Psi_1(x_1), \ldots, \Psi_k(x_k))$.

We end this section with a remark concerning our definition of the simple stable df's. We chose the approach used in section 2 because of the links described in Corollary 1 with the stable Lévy processes. However an alternative approach would be to start with df's whose marginals are double exponential df's. The transformation to polar coordinates is then replaced by the transformation

$$z_1 = x_1 + \ldots + x_k,\ z_2 = x_1 - x_2,\ \ldots,\ z_k = x_{k-1} - x_k$$

and all results can then be derived in an analogous fashion to the one given in section 2. For example in R^2 if

$$B(w,z) = \{(x,y) \mid x+y > 2w,\ x - y > 2z\}$$

then (6) is replaced by

$$s + \log \nu(B(w + s, z)) = \log \nu(B(w, z))$$

which entails

$$\nu(B(w, z)) = e^{-w}\rho(z)$$

where ρ is decreasing. There is a problem here however. In the previous case we had measures on the closed set $[0,\frac{\pi}{2}]^{k-1}$ so that here we have to consider measures on the closed set $[-\infty, \infty]^{k-1}$.

In R^2 the approach using marginals equal to $\exp\{-e^{-x}\}$ could be linked to the approach using marginals equal to $e^{-x^{-1}}$ directly if in the latter approach we had used the transformation $z = xy$, $w = \arctan y/x$ instead of the conventional transformation $(x,y) \to (r,\theta)$ to polar coordinates. Similar remarks hold in higher dimensions.

5. Asymptotic Independence

For completeness we derive by our methods two results of Sibuya (1960) concerning asymptotic independence and asymptotic full dependence of the components of the vector of maxima. We suppose that the vector of maxima converges to a limit df and for ease of writing we assume symmetric convergence to a simple stable df. Asymptotic independence then carries over to the general case. We confine ourselves to R^2 as the generalization to R^k is clear.

Theorem 4 (Sibuya): Suppose F is in the domain of symmetric attraction of the simple stable df G and (X,Y) has df F. Then asymptotic independence holds i.e.,

$$(24) \quad \lim_{n\to\infty} n(1-F(a_n x,\infty)F(\infty,a_n y)) = \lim_{n\to\infty} n(1-F(a_n x, a_n y))$$
$$= x^{-1} + y^{-1} = -\log G(x,y)$$

iff

$$(25) \quad \lim_{x\to\infty} P[Y > x | X > x) = \lim_{x\to\infty} \frac{P[Y > x, X > x]}{P[X > x]} = 0$$

Proof: Suppose asymptotic independence holds. Then from the marginal convergence

(26) $$\lim_{n \to \infty} n(1-F(a_n x, \infty)) = x^{-1}$$

together with (24) we obtain

$$\lim_{n \to \infty} (1-F(x, x))/(1-F(x, \infty)) = 2$$

and

$$\lim_{x \to \infty} (1-F(x, \infty))/(1-F(\infty, x)) = 1 .$$

From

$$P[X > x, Y > x] = (1-F(x, \infty)) + (1-F(\infty, x)) - (1-F(x, x))$$

we immediately get (25).

Conversely suppose (25) holds. From marginal convergence we have $P[X > ta] \sim a^{-1} P[X > t]$, $t \to \infty$, $\forall\, a > 0$ so that (25) entails

$$\lim_{t \to \infty} \frac{P[X > tx, Y > ty]}{P[X > t]} = 0$$

i.e. in view of (26)

$$\lim_{n \to \infty} nP[X > a_n x, Y > a_n y] = 0 .$$

Therefore

$$\lim_{n \to \infty} n(1-F(a_n x, a_n y))$$

$$= \lim_{n \to \infty} n((1-F(a_n x, \infty)) + (1-F(\infty, a_n y)) - P[X > a_n x, Y > a_n y])$$

$$= x^{-1} + y^{-1}$$

(using marginal convergence) and (24) ensues.

Example 8: Suppose F is the joint df of $(X, -X)$ and is symmetrically attracted to a simple stable df G. Then (25) holds because $P[X > x, -X > x] = 0$ for $x > 0$. Thus if $\{X_n, n \geq 1\}$ are iid copies of X

we have

$$P\left[\frac{\bigvee_{i=1}^{n} X_i}{a_n} \leqslant x, \; \frac{-\bigwedge_{i=1}^{n} X_i}{a_n} \leqslant y\right] \to \Phi_1(x)\,\Phi_1(y)$$

and consequently a limit law for the range ensues:

$$P\left[\frac{\bigvee_{i=1}^{n} X_i - \bigwedge_{i=1}^{n} X_i}{a_n} \leqslant x\right] \to \Phi_1 * \Phi_1(x) .$$

Cf. de Haan 1974.

We have the following counterpart of Theorem 4:

Theorem 5 (Sibuya): Suppose F is in the domain of symmetric attraction of the simple stable df G and (X,Y) has the df F. Then asymptotic full dependence holds, i.e.

$$(27) \quad \lim_{n\to\infty} n(1-F(a_n x,\, a_n y)) = x^{-1} \vee y^{-1} = -\log G(x,y)$$

for $x, y > 0$ iff

$$(28) \quad \lim_{x\to\infty} P[Y > x \mid X > x] = \lim_{x\to\infty} \frac{P[X > x,\, Y > x]}{P[X > x]} = 1.$$

Proof: To see (27) implies (28) proceed in a manner analogous to the previous proof. For the converse suppose (28) holds and note for $t, x, y > 0$ with $y > x$:

$$\frac{P[X > ty,\, Y > ty]}{P[X > t]} \leqslant \frac{P[X > tx,\, Y > ty]}{P[X > t]}$$

$$\leqslant \frac{P[X > ty,\, Y > ty]}{P[X > t]} + \frac{P[X \leqslant ty,\, Y > ty]}{P[X > ty,\, Y > ty]} \quad \frac{P[X > ty,\, Y > ty]}{P[X > t]} .$$

Now $\lim_{t\to\infty} P[X \leqslant ty,\, Y > ty] / P[X > ty,\, Y > ty] = 0$ from (28) and hence

$$\lim_{t \to \infty} P[X > tx,\ Y > ty] / P[X > t] = y^{-1}$$

and replacing t by a_n we see

$$\lim_{n \to \infty} n(1-F(a_n x,\ a_n y)) = \lim_{n \to \infty} n((1-F(a_n x,\ \infty)) + (1-F(\infty,\ a_n y))$$

$$- P[X > a_n x,\ Y > a_n y])$$

$$= x^{-1} + y^{-1} - y^{-1} = x^{-1} = x^{-1} \vee y^{-1}$$

as required.

6. Multidimensional Extremal Processes

Here we collect some results about multidimensional extremal processes in R^k. Let $\underset{\sim}{Y}(t) = (Y_1(t), \ldots, Y_k(t))$ be an extremal process generated by the max-id df F according to (3). From the form of the joint distribution of $\underset{\sim}{Y}(t_1), \ldots, \underset{\sim}{Y}(t_n)$ given by (3) it is clear that $\underset{\sim}{Y}$ is a Markov process in R^k with stationary transition probabilities. Again from (3) it follows that regular versions of the transition probabilities are

$$P_{x_1, \ldots, x_k}[Y_i(t) \leqslant y_i,\ i=1, \ldots, k]$$

$$:= P[Y_i(t+s) \leqslant y_i,\ i=1, \ldots, k \mid Y_i(s) = x_i,\ i=1, \ldots, k]$$

$$= F^t(y_1, \ldots, y_k)\ 1_{[y_i \geqslant x_i,\ i=1, \ldots, k]}\ .$$

The process $\underset{\sim}{Y}$ is in fact a Markov jump process and we will compute the parameters governing holding times and jumps. To facilitate this we compute the generator S. The computation is conducted for k=2. For f a bounded and continuous function $R^2 \to R$ we have for Sf:

$$Sf(x_1, x_2) = \lim_{t \downarrow 0} t^{-1} E_{x_1,x_2}(f(Y_1(t), Y_2(t)) - f(x_1, x_2))$$

$$= \lim_{t \downarrow 0} t^{-1} \int\int (f(y_1, y_2) - f(x_1, x_2)) P_{x_1,x_2} [Y_1(t) \in dy_1, Y_2(t) \in dy_2] .$$

Since $P_{x_1,x_2} [Y_1(t) \in dy_1, Y_2(t) \in dy_2]$

$$= P[Y_1(t) \in dy_1, Y_2(t) \in dy_2] 1_{[y_1 > x_1, y_2 > x_2]}$$

$$+ P[Y_1(t) \leqslant x_1, Y_2(t) \in dy_2] 1_{[y_1 = x_1, y_2 > x_2]}$$

$$+ P[Y_1(t) \in dy_1, Y_2(t) \leqslant x_2] 1_{[y_1 > x_1, y_2 = x_2]}$$

and recalling $t^{-1} P[Y_1(t), Y_2(t) \in \cdot] \Rightarrow \nu(\cdot)$ as $t \to \infty$ where ν is the exponent measure of F (Balkema and Resnick, (1976)) we have:

$$Sf(x_1, x_2) = \int\int (f(y_1, y_2) - f(x_1, x_2)) \{1_{[y_1 > x_1, y_2 > x_2]} \nu(dy_1, dy_2)$$

$$+ 1_{[y_1 > x_1, y_2 = x_2]} \nu(dy_1, (-\infty, x_2])$$

$$+ 1_{[y_1 = x_1, y_2 > x_2]} \nu((-\infty, x_1], dy_2)\} .$$

Comparing the form just obtained with the canonical form of the generator for a Markov jump process (cf. Breiman p.331) we obtain the mean $\alpha^{-1}(x_1, x_2)$ of the holding time in state (x_1, x_2) and the conditional probability $\Pi((x_1, x_2); A)$ that starting from (x_1, x_2) the process jumps into A. For arbitrary k these quantities are given by

$$\alpha((x_1, \ldots, x_k)) = \nu(A^c(x_1, \ldots, x_k))$$

$$(29)\qquad \Pi((x_1, \ldots, x_k); A(y_1, \ldots, y_k)) = 1 - \frac{\nu(A^c(y_1, \ldots, y_k))}{\nu(A^c(x_1, \ldots, x_k))}$$

for $y_i \geqslant x_i$, i=1 ... k where as usual

$$A(y_1, \ldots, y_k) = \{ (t_1, \ldots, t_k) | t_i \leqslant y_i, \ i=1,\ldots,k\} .$$

For processes $\underset{\sim}{Y}$ generated by simple stable df's this result has the following interpretation: Let τ be the time of the first jump after t=1. Then

$$P\,[\underset{\sim}{Y}(\tau) \in A(y_1, \ldots, y_k) | Y_i(1) = x_i, \ i=1,\ldots,k]$$

$$=: P_{(x_1, \ldots, x_k)}[\underset{\sim}{Y}(\tau) \in A(y_1, \ldots, y_k)] = 1 - \frac{\nu(A^c(y_1, \ldots, y_k))}{\nu(A^c(x_1, \ldots, x_k))}$$

for $y_i \geqslant x_i$, i=1, ..., k. Therefore

$$P_{(x_1, \ldots, x_k)}[\underset{\sim}{Y}(\tau) \in A^c(y_1, \ldots, y_k)] = \frac{\nu(A^c(y_1, \ldots, y_k))}{\nu(A^c(x_1, \ldots, x_k)} .$$

If $B = \{ (t_1, \ldots, t_k) | t_i \geqslant x_i, \ i=1, \ldots, k\}$ then

$$P_{(x_1, \ldots, x_k)}[\underset{\sim}{Y}(\tau) \in A \cap B] = \frac{\nu(A \cap B)}{\nu(A^c(x_1, \ldots, x_k))}$$

for any $A \in \mathcal{B}(R^k)$. Supposing again that T is the transformation to polar coordinates and that $T\underset{\sim}{Y}(\tau) = (\|\underset{\sim}{Y}\|, \underset{\sim}{\Theta})$ we have on sets A' such that $T^{-1}A' \subset B$:

$$P_{(x_1, \ldots, x_k)}[(\|\underset{\sim}{Y}\|, \underset{\sim}{\Theta}) \in A'] = \iint_{A'} r^{-2}dr\; S(d\underset{\sim}{\theta})/\nu(A^c(x_1, \ldots, x_k)$$

so that with respect to $P_{(x_1, \ldots, x_k)}(\cdot)$ we have $\|\underset{\sim}{Y}\|$ and $\underset{\sim}{\Theta}$ independent.

Another independence result is given below which describes when the jumps of $\underset{\sim}{Y}$ are iid random vectors. Preparatory to this discussion we discuss the range $\mathcal{R}(\underset{\sim}{Y})$ which we define as

$$\mathcal{R}(\underset{\sim}{Y}) = \{ \underset{\sim}{x} \,|\, \forall \text{ open sets } O \ni \underset{\sim}{x},\ P[\underset{\sim}{Y}(t) \in O \text{ for some } t] > 0 \} .$$

For what follows we denote the support of a measure ν by supp ν.

To characterize $\mathcal{R}(Y)$ we need hitting probabilities for rectangles. This computation is done for k=2 and we seek $P[Y \text{ hits } (x_1, x_2] \times (y_1, y_2]]$ where $x_1 < x_2$, $y_1 < y_2$. Assume Y is related to a Poisson random measure as described in the introduction. Define $\sigma(A) = \inf\{ T_k | (J_k^{(1)}, J_k^{(2)}) \in A\}$ to be the first time there is a point in $A \in \mathcal{B}(R^2)$. Then

$$P[Y(t) \in (x_1, x_2] \times (y_1, y_2] \text{ for some } t]$$

$$= P[\sigma((-\infty, x_2] \times (y_1, y_2]) \vee \sigma((x_1, x_2] \times (-\infty, y_2])$$

$$< \sigma(A^c(x_2, y_2))] .$$

Note $\sigma((-\infty, x_2] \times (y_1, y_2]) = \sigma((-\infty, x_1] \times (y_1, y_2] \wedge \sigma((x_1, x_2] \times (y_1, y_2])$ $=: U \wedge V$ and

$\sigma((x_1, x_2] \times (-\infty, y_2]) = \sigma((x_1, x_2] \times (-\infty, y_1]) \wedge V =: W \wedge V.$

Set $Z = \sigma(A^c(x_2, y_2))$ and the required probability is

$$P[(U \wedge V) \vee (W \wedge V) < Z]$$

where U, V, W, Z are independent and for any $A \in \mathcal{B}(R^2)$ $P[\sigma(A) > t] = e^{-t\nu(A)}$. Set $\lambda_1 = \nu((x_1, x_2] \times (y_1, y_2])$, $\lambda_2 = \nu((-\infty, x_1] \times (y_1, y_2])$, $\lambda_3 = \nu((x_1, x_2] \times (-\infty, y_1])$, $\lambda_4 = \nu(A^c(x_2, y_2))$. Performing the required calculation by capitalizing on independence gives

$$P[Y \text{ hits } (x_1, x_2] \times (y_1, y_2]] = \lambda_4 \left\{ \frac{1}{\lambda_4} + \frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} - \frac{1}{\lambda_1+\lambda_3+\lambda_4} - \frac{1}{\lambda_1+\lambda_2+\lambda_4} \right\} .$$

Assuming that $\lambda_4 > 0$ we observe that the hitting probability is positive iff $\lambda_1 + \lambda_2 > 0$ and $\lambda_1 + \lambda_3 > 0$. This leads to:

Theorem 6: Let $\underset{\sim}{Y}$ be extremal in R^k with exponent measure ν. Then $(x_1, \ldots, x_k) \in \mathcal{R}(\underset{\sim}{Y})$ iff for all $\varepsilon > 0$

$$\nu\{(-\infty, x_1+\varepsilon] \times \ldots \times (-\infty, x_{i-1}+\varepsilon] \times (x_i-\varepsilon, x_i+\varepsilon] \times (-\infty, x_{i+1}+\varepsilon] \times \ldots \times (-\infty, x_k+\varepsilon]\} > 0 \quad \text{for } i=1, \ldots, k.$$

Equivalently we have

$$\mathcal{R}(\underset{\sim}{Y}) = \{(x_1, \ldots, x_k) \mid x_i = \sup\{y_i \mid \underset{\sim}{y} \in A\},\ i=1, \ldots, k \text{ for some } A \subset \operatorname{supp} \nu\}.$$

When $\underset{\sim}{Y}$ is generated by a simple stable df, the range has the following characterization. Recall the transformation to polar coordinates T: $(x_1, \ldots, x_k) \to (r, \underset{\sim}{\theta})$.

Corollary 5: If $\underset{\sim}{Y}$ is extremal generated by the simple stable df with exponent measure $\nu \circ T^{-1}(dr, d\underset{\sim}{\theta}) = r^{-2} dr\, S(d\underset{\sim}{\theta})$ then

$$\operatorname{supp} \nu = \{(x_1, \ldots, x_k) \mid \underset{\sim}{\theta} \in \operatorname{supp} S\}$$

and $\mathcal{R}(Y) = \{(x_1, \ldots, x_k) \mid \underset{\sim}{\theta} \in \text{closed convex hull of supp } S\}$.

We now consider the following problem: Let $1 < T_1 < T_2 < \ldots$ be the times $\underset{\sim}{Y}$ jumps past $t=1$. For convenience set $T_0 = 1$. When is $\{\underset{\sim}{Y}(T_n) - \underset{\sim}{Y}(T_{n-1}), n \geq 1\}$ a sequence of iid random vectors? We begin by reviewing and completing the situation for $k=1$ (cf. Resnick and Rubinovitch, 1973).

If Y is extremal in one dimension generated by $F(x)$ set $Q(x) = -\log F(x) = \nu(x, \infty)$. Suppose $a = \inf\{x \mid F(x) > 0\}$. If the jumps of Y are iid then

$$(30) \qquad \{Y(T_n), n \geq 0\} \overset{d}{=} \{Z_0 + \sum_{1}^{n} Z_j, n \geq 0\}$$

where $\{Z_n, n \geq 1\}$ are iid rv's with common df $H(x)$. Note (30) holds iff $\forall x \in \mathcal{R}(Y)$

(31) $1 - Q(y)/Q(x) = H(y-x)$

for $y \geqslant x$ (cf. 29). The following facts are evident

(i) $\mathcal{R}(Y) = \text{supp}\ \nu$

(ii) $t \in \text{supp}\ H$ iff $t \geqslant 0$ and $t = x_2 - x_1$ where $x_1, x_2 \in \text{supp}\ \nu$.

This follows from (31).

(iii) If $x_1, x_2 \in \mathcal{R}(Y)$ and $x_1 < x_2$ then $\forall z \in \mathcal{R}(Y)$

$z + (x_2 - x_1) \in \mathcal{R}(y)$.

This is clear since $x_2 - x_1 \in \text{supp}\ H$.

(iv) Either $\mathcal{R}(y) = (a, \infty)$

or $\mathcal{R}(y) = \{x_0 + nd, -\infty < n < \infty \text{ and } x_0 + nd \geqslant a\}$, $d > 0$.

This is easily seen once one defines

$$d = \inf\{y - x | y > x,\ \ x, y \in \mathcal{R}(Y)\}\ .$$

Thus one is led to the possible structure of $\mathcal{R}(Y)$ when independent jumps are present. Analyzing (31) leads to functional equations which Q must satisfy. These equations are easily solved and the result is: Y has iid jumps iff

(i) $\mathcal{R}(Y) = (a, \infty)$, $-\infty \leqslant a$ and F is of type

$$F(x) = \begin{cases} e^{-e^{-x}}, & x \geqslant a \\ 0 & x < a \end{cases}$$

(ii) $\mathcal{R}(Y) = \{x_0 + nd, \forall n \text{ such that } x_0 + nd \geqslant a\}$

and F concentrates on $\{x_0 + nd\}$ and is of the form

$$F(x_0 + nd) = \begin{cases} e^{-p^n} & \text{for } x_0 + nd \geqslant a \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$.

We now consider the problem in R^k so suppose the jumps of $\underset{\sim}{Y}(\cdot) = (Y_1(\cdot), \ldots, Y_k(\cdot))$ are iid vectors. We are going to prove that the process is then one-dimensional; i.e. that $\mathcal{R}(\underset{\sim}{Y})$ is contained in a straight line. Pick two arbitrary components of Y. These components constitute an extremal process in R^2 and the jumps are iid pairs. The desired result will be proved if we prove the result for any two components of $\underset{\sim}{Y}$; i.e. it suffices to suppose $k = 2$.

Suppose in order to get a contradiction the process is not concentrated on a line. Then there are points (x_1, x_2), $(y_1, y_2) \in \mathcal{R}(\underset{\sim}{Y})$ with (say) $x_1 \leqslant y_1$, $x_2 > y_2$. It is evident that the following points must be in $\mathcal{R}(\underset{\sim}{Y})$:

$$\{ z(n, m)\} := \{ (y_1 + n(y_1 - x_1), x_2 + m(x_2 - y_2))\}$$

where $n \geqslant -1$, $m \geqslant -1$, n,m integers but we exclude $n = m = -1$. Define $g(n,m) = \nu\{ A^c(z(n,m))\}$. Referring to (29) and using the asumption of iid jumps we have that $g(n + r, m + s)/g(n, m)$ does not depend on n, $m(r,s = 0,1,\ldots)$. Call this ratio $f(r,s)$ so that

$$g(n + r, m) = g(n, m)f(r, 0).$$

From this we deduce

$$f(r + s, 0) = f(r, 0)f(s, 0)$$

and thus $f(r, 0) = e^{ar}$ for some constant a which entails

$$g(n, m) = e^{a(n-1)}g(1, m).$$

Similar analysis on the second variable shows

$$\begin{aligned} g(n, m) &= e^{a(n-1)}\, e^{b(m-1)}\, g(1, 1) \\ &= c\, e^{an}\, e^{bm} \end{aligned}$$

where c, a, b are constants and $c > 0$.

Since g must be non-increasing in n and m we must have $a < 0$, $b < 0$.

Define sets

$$\begin{aligned} B_{n,m} = \{ z_1, z_2) \mid & y_1 + (n-1)(y_1-x_1) < z_1 \leqslant y_1 + n(y_1-x_1), \\ & x_2 + (m-1)(x_2-y_2) < z_2 \leqslant x_2 + m(x_2-y_2)\} \end{aligned}$$

for $n, m = 1, 2, \ldots$ say and note

$$\nu(B_{n,m}) = -g(n-1,\ m-1) + g(n-1,\ m) + g(n,\ m-1) - g(n,\ m)$$
$$= -c\ e^{an}\ e^{bm}\ (1 - e^{-a})(a - e^{-b}) < 0$$

which gives the desired contradiction.

Thus if $\underset{\sim}{Y}$ has iid jumps then $\underset{\sim}{Y}$ is one-dimensional. The structure of $\mathcal{R}(\underset{\sim}{Y})$ and the possible distributions of the process are then obtained from the one-dimensional results.

References

1. Balkema, A. and Resnick, S.I. (1975). Max-infinite divisibility and a problem in the range. Tech. Report No.71, Stanford University, Dept. of Statistics.

2. Breiman, L. (1969). Probability, Addison Wesley.

3. de Haan, L. (1970). On Regular Variation and its Application to the Weak Convergence of Sample Extremes. MC Tract 32, Mathematisch Centrum, Amsterdam.

4. de Haan, L. (1971). A form of regular variation and its application to the domain of attraction of the double exponential distribution. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **17**, 241-258.

5. de Haan, L. (1974). Weak limits of sample range. *J. Appl. Probability.* **11**, 836-841.

6. Dwass, M. (1964). Extremal Processes. *Ann. Math. Statist.* **35**, 1718-1725.

7. Geffroy, J. (1958). Contributions à la theorie des valeurs extrèmes. *Publ. de l'Inst. Statist. de l'Un. de Paris*, **7**, **8**, 37-185.

8. Gnedenko, B.V. (1943). Sur la distribution limité du terme maximum d'une série aléatoire. *Ann. Math.* **44**, 423-453.

9. Lévy, P. (1937). Théorie de l'addition des variables aléatoires. Gauthier-Villars, Paris.

10. Resnick, S.I. and Rubinovitch, M. (1973). On the structure of extremal processes. *Adv. Appl. Probability* **5**, 287-307.

11. Rvačeva, E.L. (1962). On the domains of attraction of multi-dimensional distributions. Selected Translations in Mathematical Statistics and Probability, **2**, 183-207.

12. Sibuya, M. (1960) Bivariate extreme statistics. *Ann. Inst. Statist. Math. Tokyo,* **11**, 195-210.

13. Tiago de Oliveira, J. (1959). Extremal distributions. Faculdade de Ciencias de Lisboa, No.39.