Random Oper. and Stoch. Equ., Vol. 14, No. 4, pp. 385–392 (2006)

Limiting distribution of random motion in a n-dimensional parallelepiped

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Received for *ROSE* March 2, 2006

Abstract—In this paper we study a continuous time random walk in a n-dimensional parallelepiped with pairs of boundaries $[a_i, b_i]$. In a pair of boundaries the particle can move in any of two directions with different velocities $v_i^{(1)}$ and $v_i^{(2)}$. We consider a special type of boundary which can trap the particle for a random time, and we found the limiting distribution of this random motion for the position of the particle. Our formulation allows us to find the limiting distribution for a broad class of alternating semi-Markov processes.

Key words and phrases: Random evolutions; semi-Markov processes; delaying boundaries; random walk

1. INTRODUCTION

In 1951 S. Goldstein introduced the telegrapher's stochastic process in his seminal paper [1], which is a random motion driven by a homogeneous Poisson process. This basic telegrapher process has been extended in many manners since then [2], [3], [4], [5], and references therein. In this work we investigate the random motion driven by the superposition of an alternating semi-Markov process in n-dimensional space. We study the limiting distribution of a n-dimensional random motion performed with two velocities, where the random times separating consecutive velocity changes perform an alternating semi-Markov process.

We assume that the particle moves inside the parallelepiped

$$\prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n,\ n \geq 1,$$

with a vector of velocities

$$\vec{v}(t) = (v_1(t), v_2(t), \ldots, v_n(t)) = \sum_{i=1}^{n} v_i(t)\vec{e}_i,$$

where every $v_i(t)$ can take one of two values $v_i^{(1)} > 0$ or $v_i^{(2)} < 0$, and $\vec{e}_i$, $i = 1, 2, \ldots, n$, is a Cartesian basis. The motion is performed in the following manner: At each instant the particle moves according to one of two velocities, namely $v_i^{(1)}$ or $v_i^{(2)}$. Starting at the position

$$\vec{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \prod_{i=1}^{n} [a_i, b_i]$$

the particle continues its motion with velocity $v_i^{(1)}$ during random time $\tau_i^{(1)}$, where $\tau_i^{(1)}$ is a random variable with distribution function $G_i^{(1)}(t)$. Afterwards the particle moves with velocity $v_i^{(2)}$ during random time $\tau_i^{(2)}$, where $\tau_i^{(2)}$ is a random variable with distribution function $G_i^{(2)}(t)$. Furthermore, the particle moves with velocity $v_i^{(1)}$ and so on. When the particle reaches boundary a_i or b_i it will stay at that boundary a random time given by the time the particle remains in the same direction up to the time such a particle changes direction. So, a_i and b_i are two delaying boundaries of direction $\vec{e}_i$. We assume that the random variables $\tau_i^{(1)}$ and $\tau_i^{(2)}$ are independent. Similar partly reflecting (or trapping) boundaries have been considered in [6] for the case of one-dimensional motion, and they may be found in optical photon propagation in a turbid medium or chemical processes with sticky layers or boundaries. We believe that our model is more realistic for these applications.

Our main contribution is to find the limiting stationary distribution of this continuous time random walk when the sojourn times are generally distributed.

2. MATHEMATICAL MODEL

On the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ consider a collection of independent alternating semi-Markov process $\{\kappa_i(t),\, t \geq 0,\, i = 1, 2, \ldots, n\}$, on the phase space $\mathbb{E} = \{1, 2\}$, having the sojourn time $\tau_i^{(k)}$ corresponding to the state $k \in \mathbb{E}$, where

$$\kappa_i(t) = \begin{cases} 1, & \text{if} \quad v_i(t) = v_i^{(1)} \\ 2, & \text{if} \quad v_i(t) = v_i^{(2)}. \end{cases} \tag{1}$$

So, a sojourn time of $\kappa_i(t)$ at state k has the distribution $G_i^{(k)}(t)$, and transition probability matrix of the embedded Markov chain

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

Denote by $x(t)$, $t \geq 0$ the position of the particle at time t. Consider the function $C_i(x, k)$ on the space $[a_i, b_i] \times \mathbb{E}$ which is defined as

$$C_i(x, k) := \begin{cases} v_i^{(1)}, & \text{if} \quad a_i < x < b_i,\ k = 1; \\ v_i^{(2)}, & \text{if} \quad a_i < x < b_i,\ k = 2; \\ 0, & \text{if} \quad (x = a_i,\ k = 2) \text{ or } (x = b_i,\ k = 1). \end{cases} \tag{3}$$

The position of the particle at any time t is denoted as $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, and it is easily verified that it can be expressed by

$$x_i(t) = x_i^0 + \int_0^t C(x_i(s), \kappa_i(s))\, ds, \tag{4}$$

where the starting point $x_i^0 \in [a_i, b_i]$.

Eq. (4) determines the random evolution of the particle in the alternating semi-Markov medium $\{\kappa_i(t),\, t \geq 0\}$ [7], [8]. So, our purpose is to calculate the stationary measure of the process $\vec{x}(t)$.

Since $x_1(t), x_2(t), \ldots, x_n(t)$ are independent we have

$$\mathbf{P}\{(x_1(t), x_2(t), \ldots, x_n(t)) \in d\vec{x}\} = \prod_{i=1}^{n} \mathbf{P}\{x_i(t) \in dx_i\}.$$

Now, to compute the stationary distribution of $\vec{x}(t)$ we must compute the stationary distribution of $x_i(t)$ for all $i = 1, 2, \ldots, n$.

We suppose that the distributions $G_i^{(1)}(t)$ and $G_i^{(2)}(t)$ are not degenerated. In addition, there exist the densities

$$g_i^{(k)}(t) = \frac{dG_i^{(k)}(t)}{dt},$$

and the first moments

$$m_i^{(k)} = \int_0^\infty t g_i^{(k)}(t) dt,$$

for all $k \in \mathbb{E}$. We also assume the existence of the hazard rate

$$r_i^{(k)}(t) = \frac{g_i^{(k)}(t)}{1 - G_i^{(k)}(t)}.$$

Let us define

$$\zeta_i(t) = t - \sup_{0 \leq u \leq t} \{\kappa_i(u) \neq \kappa_i(t)\}.$$

Now, consider the three-component stochastic process $\{\chi_i(t) = (\zeta_i(t), x_i(t), \kappa_i(t))\}$ on the phase space $\mathbb{Z}_i = [0, \infty) \times [a_i, b_i] \times \{1, 2\}$, $i = 1, 2, \ldots, n$. It is well-known that for every $i = 1, 2, \ldots, n$, the process $\chi_i(t)$ is a Markov process with the following infinitesimal operator [7], [8], [9]:

$$A_i\, \phi(\zeta, x, s) = \frac{\partial}{\partial \zeta} \phi(\zeta, x, s) + r_i^{(s)}(\zeta)[P\, \phi(0, x, s) - \phi(\zeta, x, s)] + C_i(x, s) \frac{\partial}{\partial x} \phi(\zeta, x, s), \quad (5)$$

where $s = 1, 2$. The boundary conditions are given by

$$\phi_\zeta(\zeta, a_i, 1) = \phi_\zeta(\zeta, b_i, 2) = 0,$$

where $\phi(\zeta, x, s)$ is a continuously differentiable function with respect to ζ and x. It is easy to see that

$$\mathbf{P}\phi(0, x, 1) = \phi(0, x, 2) \text{ and } \mathbf{P}\phi(0, x, 2) = \phi(0, x, 1).$$

In the rest of the paper we will calculate the stationary distribution $\rho_i(\cdot)$ of $\chi_i(t)$. The analysis of the properties of the process $\chi_i(t)$ leads up to the conclusion that the stationary distribution ρ_i has atoms or singularities at points $(\zeta, a_i, 2)$ and $(\zeta, b_i, 1)$, and we denote them as $\rho_i[\zeta, a_i, 2]$ and $\rho_i[\zeta, b_i, 1]$, respectively. The continuous part of ρ_i is denoted as $\rho_i(\zeta, x, k)$, $k \in \mathbb{E}$.

Since ρ_i is the stationary distribution of $\chi_i(t)$ then for any function $\phi(\cdot)$ from the domain of the operator A_i we have

$$\int_{\mathbb{Z}_i} A_i\phi(z)\,\rho_i(dz) = 0. \tag{6}$$

Now, let A_i^* be the conjugate or adjoint operator of A_i. Then by changing the order of integration in Eq. (6) (integrating by parts), we can obtain the following expressions for the continuous part of $A_i^*\rho_i = 0$

$$\frac{\partial}{\partial\zeta}\rho_i(\zeta,x,k) + r_i^{(k)}(\zeta)\rho_i(\zeta,x,k) + v_i^{(k)}\frac{\partial}{\partial x}\rho_i(\zeta,x,k) = 0, \quad k = 1,2. \tag{7}$$

and

$$\int_0^\infty r_i^{(k)}(\zeta)\rho_i(\zeta,x,k)d\zeta = \rho_i(0,x,j);\ j \neq k, \quad j,k = 1,2, \tag{8}$$

with the limiting behavior $\rho_i(+\infty,x,1) = \rho_i(+\infty,x,2) = 0$, for all $x \in [a_i,b_i]$.

For the atoms we have

$$\frac{\partial}{\partial\zeta}\rho_i[\zeta,a_i,2] + r_i^{(2)}(\zeta)\rho_i[\zeta,a_i,2] + v_i^{(2)}\rho_i(\zeta,a_i+,2) = 0 \tag{9}$$

$$\frac{\partial}{\partial\zeta}\rho_i[\zeta,b_i,1] + r_i^{(1)}(\zeta)\rho_i[\zeta,b_i,1] - v_i^{(1)}\rho_i(\zeta,b_i-,1) = 0 \tag{10}$$

where

$$\rho_i(\zeta,b_i-,k) := \lim_{x\uparrow b_i}\rho_i(\zeta,x,k)$$

and

$$\rho_i(\zeta,a_i+,k) := \lim_{x\downarrow a_i}\rho_i(\zeta,x,k),$$

for $k = 1,2$. We also have

$$\rho_i[+\infty,a_i,2] = \rho_i[0,a_i,2] = \rho_i[+\infty,b_i,1] = \rho_i[0,b_i,1] = 0.$$

Now, by taking into account boundary conditions we have

$$\int_0^\infty r_i^{(1)}(\zeta)\rho_i[\zeta,b_i,1]d\zeta = -v_i^{(2)}\int_0^\infty \rho_i(\zeta,b_i-,2)d\zeta \tag{11}$$

and

$$\int_0^\infty r_i^{(2)}(\zeta)\rho_i[\zeta,a_i,2]d\zeta = v_i^{(1)}\int_0^\infty \rho_i(\zeta,a_i+,1)d\zeta. \tag{12}$$

By solving Eqs. (7) we obtain

$$\rho_i(\zeta,x,k) = f_i^{(k)}(x - v_i^{(k)}\zeta)\exp\left(-\int_0^\zeta r_i^{(k)}(t)dt\right), \quad k = 1,2, \tag{13}$$

where $f_i^{(k)} \in \mathcal{C}^1$.

By substituting Eqs. (13) into Eqs. (8) and by noting that

$$\exp\left(-\int_0^\zeta r_i^{(k)}(t)dt\right) = 1 - G_i^{(k)}(\zeta)$$

we obtain

$$\int_0^\infty f_i^{(k)}(x - v_i^{(k)}\zeta) g_i^{(k)}(\zeta) d\zeta = f_i^{(j)}(x), \quad j \neq k;\; j,k = 1,2. \tag{14}$$

It follows from the two Eqs. (14) that

$$\int_0^\infty \int_0^\infty f_i^{(k)}(x - v_i^{(1)}\zeta - v_i^{(2)}t) g_i^{(1)}(\zeta) g_i^{(2)}(t) d\zeta dt = f_i^{(k)}(x), \quad k = 1,2. \tag{15}$$

From Eqs. (14) and (15) we can assume that the functions $f_i^{(k)}(x)$ are of the form

$$f_i^{(k)}(x) = c_i^{(k)} e^{\lambda_i x}, \quad k = 1,2. \tag{16}$$

Now, by substituting Eq. (16) into Eq. (15), we obtain

$$\hat{g}_i^{(1)}(\lambda_i v_i^{(1)})\, \hat{g}_i^{(2)}(\lambda_i v_i^{(2)}) = 1, \tag{17}$$

where

$$\hat{g}_i^{(k)}(s) = \int_0^\infty g_i^{(k)}(t) e^{-st} dt$$

is the Laplace transform of $g_i^{(k)}(t)$, $k = 1,2$. The set of *pdf*s for which Eq. (17) exists is similar to the set of functions that satisfies the Cramér condition. Furthermore, we need to answer the question: Is there a nonzero solution of Eq. (17) for the unknowns λ_i?

Lemma. *If*

$$v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} \neq 0,$$

where

$$m_i^{(k)} = \int_0^\infty t g_i^{(k)}(t) dt,$$

and there exist $\ell_1 < \ell_2$*,* $p_1 < p_2$*,* $\sigma_1 > 0$ *and* $\sigma_2 > 0$ *such that* $g_i^{(1)}(t) \geq \sigma_1$*,* $t \in [\ell_1, \ell_2]$*,* $g_i^{(2)}(t) \geq \sigma_2$*,* $t \in [p_1, p_2]$ *and* $0 < v_i^{(1)}\ell_2 + v_i^{(2)}p_2$*,* $0 < -(v_i^{(1)}\ell_1 + v_i^{(2)}p_1)$*. Then, there exists* $\lambda_i^0 \neq 0$ *that satisfies Eq. (17).*

Proof. Let us define

$$p(\lambda_i) = \hat{g}_i^{(1)}(\lambda_i v_i^{(1)})\, \hat{g}_i^{(2)}(\lambda_i v_i^{(2)}),$$

so

$$p'(0) = -(v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)}) \neq 0.$$

Now, suppose

$$p'(0) = -(v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)}) < 0$$

then

$$p(\lambda_i) \geq \int_{\ell_1}^{\ell_2} e^{-v_i^{(1)}\lambda_i t} g_i^{(1)}(t) dt \int_{p_1}^{p_2} e^{-v_i^{(2)}\lambda_i t} g_i^{(2)}(t) dt,$$

hence

$$p(\lambda_i) \geq \frac{\sigma_1\sigma_2}{v_i^{(1)} v_i^{(2)}} \left(e^{-v_i^{(1)}\lambda_i\ell_1} - e^{-v_i^{(1)}\lambda_i\ell_2}\right)\left(e^{-v_i^{(2)}\lambda_i p_1} - e^{-v_i^{(2)}\lambda_i p_2}\right) \to +\infty \text{ as } \lambda_i \to +\infty.$$

The case

$$p'(0) = v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} > 0$$

can be reduced to the previous one by assuming $s = -t$ and using $0 < v_i^{(1)} \ell_i^{(2)} + v_i^{(2)} p_2$.

Theorem. A) *If*

$$v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} \neq 0 \quad \text{and} \quad \lambda_i^0 \neq 0$$

is the solution for Eq. (17), and

$$\int_0^\infty e^{-\lambda_i^0 v_i^{(1)} u} (1 - G_i^{(1)}(u)) du < +\infty,$$

then there exists a stationary distribution of $\chi_i(t)$ with the following continuous part:

$$\rho_i(\zeta, x, 1) = c_i e^{\lambda_i^0 (x - v_i^{(1)} \zeta)} (1 - G_i^{(1)}(\zeta)), \tag{18}$$

$$\rho_i(\zeta, x, 2) = c_i \hat{g}_i^{(1)}(\lambda_i^0 v_i^{(1)}) e^{\lambda_i^0 (x - v_i^{(2)} \zeta)} (1 - G_i^{(2)}(\zeta)) \tag{19}$$

and singular parts:

$$\rho_i[\zeta, b_i, 1] = c_i v_i^{(1)} e^{\lambda_i^0 b_i} (1 - G_i^{(1)}(\zeta)) \int_0^\zeta e^{-\lambda_i^0 v_i^{(1)}} dt, \tag{20}$$

$$\rho_i[\zeta, a_i, 2] = c_i v_i^{(2)} e^{\lambda_i^0 a_i} \hat{g}_i^{(1)}(\lambda_i^0 v_i^{(1)}) (G_i^{(2)}(\zeta) - 1) \int_0^\zeta e^{-\lambda_i^0 v_i^{(2)}} dt. \tag{21}$$

The normalization factor c_i can be calculated from

$$\int_{\mathbb{Z}_i} \rho_i(dz) = 1.$$

B) *If*

$$v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} = 0$$

and there exists the second moments

$$M_i^{(k)} = \int_0^\infty t^2 g_i^{(k)}(t) dt, \quad k \in \mathbb{E},$$

then the stationary measure of $\chi_i(t)$ is as follows

$$\rho_i(\zeta, x, 1) = d_i (1 - G_i^{(1)}(\zeta)), \qquad \rho_i(\zeta, x, 2) = d_i (1 - G_i^{(2)}(\zeta)) \tag{22}$$

with atoms

$$\rho_i[\zeta, b_i, 1] = d_i v_i^{(1)} \zeta (1 - G_i^{(1)}(\zeta)), \qquad \rho_i[\zeta, a_i, 2] = d_i v_i^{(2)} \zeta (G_i^{(2)}(\zeta) - 1), \tag{23}$$

where

$$d_i = \left[(m_i^{(1)} + m_i^{(2)})(b_i - a_i) + v_i^{(1)} \frac{M_i^{(1)}}{2} - v_i^{(2)} \frac{M_i^{(2)}}{2} \right]^{-1}.$$

Proof. A) It is easy to see that

$$f_i^{(1)}(x) = c_i e^{\lambda_i^0 x} \quad \text{and} \quad f_i^{(2)}(x) = c_i \hat{g}_i^{(1)}(\lambda_i^0 v_i^{(1)}) e^{\lambda_i^0 x}$$

satisfy Eq. (14). Substituting these functions $f_i^{(k)}(x)$ into Eqs. (13) we obtain Eqs. (18) and (19). Therefore we substitute Eqs. (18) and (19) into Eqs. (9) and (10), then by solving these equations we obtain Eqs. (20) and (21).

It can be easily verified that if $v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} \neq 0$ then the value $\lambda_i^0 \neq 0$, such that $\hat{g}_i^{(1)}(\lambda_i^0 v_i^{(1)}) \hat{g}_i^{(2)}(\lambda_i^0 v_i^{(2)}) = 1$, also satisfies Eqs. (11) and (12). If $v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} = 0$ then $\lambda_i^0 = 0$ satisfies Eqs. (11) and (12).

B) Similarly, for $v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} = 0$ we obtain Eqs. (22) and (23) in the same manner as for the case $v_i^{(1)} m_i^{(1)} + v_i^{(2)} m_i^{(2)} \neq 0$ when it is considered that $\lambda_i^0 = 0$. This concludes the proof.

We should notice that the stationary measure of the particle position $x_i(t)$ is determined by the following relations

$$\rho_i(x) = \int_0^\infty (\rho_i(\zeta, x, 1) + \rho_i(\zeta, x, 2))\, d\zeta, \quad \text{for } x \in (a_i, b_i), \tag{24}$$

$$\rho_i[a_i, 2] = \int_0^\infty \rho_i[\zeta, a_i, 2] d\zeta, \qquad \rho_i[b_i, 1] = \int_0^\infty \rho_i[\zeta, b_i, 1] d\zeta. \tag{25}$$

The limiting distribution $\rho(x_1, x_2, \ldots, x_n)$ of $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ can be written as follows

$$\rho(x_1, x_2, \ldots, x_n) = \prod_{i=0}^{n} \rho_i(x_i). \tag{26}$$

Therefore, by using Eqs. (18)-(23) we can compute $\rho(x_1, x_2, \ldots, x_n)$ for all

$$(x_1, x_2, \ldots, x_n) \in \prod_{i=0}^{n} [a_i, b_i].$$

3. CONCLUSIONS

The two-states continuous time random walk has been studied by many researchers for the Markov case and only a few have studied for non-Markovian processes [10]. This basic model has many applications in physics, biology, chemistry, and engineering. Most of the former models were oriented to solve the boundary-free particle motion. Recently this basic model has been extended in several directions, such as two and three dimensions, with reflecting and absorbing boundaries. Only a few of these works consider partly reflecting boundaries [6], [10], and references therein. However, in none of these previous works a stationary distribution for the particle position is presented, as we did in this paper. In this paper we have found the limiting distribution of a particle moving according to a semi-Markov evolution in a n-dimensional parallelepiped.

Acknowledgments

We thank ITESM, Campus Monterrey, through the Research Chair in Telecommunications for the support provided in the development of this work.

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