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LIMITING DISTRIBUTIONS FOR CONTINUOUS STATE MARKOV VOTING MODELS

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SOCIAL SCIENCE WORKING PAPER 394

July 1981

ABSTRACT

This paper proves the existence of a stationary distribution for a class of Markov voting models. We assume that alternatives to replace the current status quo arise probabilistically, with the probability distribution at time t+1 having support set equal to the set of alternatives that defeat, according to some voting rule, the current status quo at time t. When preferences are based on Euclidean distance, it is shown that for a wide class of voting rules, a limiting distribution exists. For the special case of majority rule, not only does a limiting distribution always exist, but we obtain bounds for the concentration of the limiting distribution around a centrally located set. The implications are that under Markov voting models, small deviations from the conditions for a core point will still leave the limiting distribution quite concentrated around a generalized median point. Even though the majority relation is totally cyclic in such situations, our results show that such chaos is not probabilistically significant.

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1. INTRODUCTION

It is becoming increasingly evident that nondeterministic models of individual and group behavior have an important role to play in the social sciences. Such models can provide relief from impossibility results, and may yield equilibria not generally present in deterministic formulations. In the realm of game-theoretic models of committee voting there are an abundance of solution concepts, but none are fully adequate in that they can often fail to make a prediction (equilibria do not exist) or they predict indiscriminately (the set of solutions is unworkably large). Nondeterministic solutions, on the other hand, afford the possibility of always existing while discriminating probabilistically over any set of possible outcomes. In addition, probabilistic solutions may reduce to nondeterministic solutions when the latter do exist and are appropriate.

A natural and widely applicable means of generating probabilistic predictions is provided by Markov processes. For voters making a choice from a finite set of alternatives, a finite state discrete time Markov chain model of sequential voting is developed by Ferejohn, Fiorina, and Packel [1980]. When the alternatives form a compact subset of $\mathbb{R}^m$, a continuous state Markov process can be used to generate a probabilistic solution concept (Packel [1981]). Each of these models assumes that alternatives to replace the current status quo arise probabilistically, with the probability distribution of time t + 1 having support equal to the set of alternatives that defeat (by some voting rule) the current status quo at time t. Once this probability distribution is obtained, one can then try to determine the limiting distribution (i.e., the steady state probabilities) for the Markov process. For a finite set of alternatives, this procedure is straightforward. When alternatives are a general subset of $\mathbb{R}^m$, a number of interesting and important questions arise about the existence and structure of the limiting distribution. In this paper we formulate and address these questions.

When the alternative space is compact and voter preferences are ''reasonable,'' it is shown in Packel [1981] that a limiting probability distribution must exist and will be concentrated at the strong equilibrium if such an equilibrium exists. In the absence of an equilibrium, however, one would like to know to what extent the distribution is concentrated in the Pareto set or some other centrally located region in the alternative space.

If we allow the alternative space to be all of $\mathbb{R}^m$, additional subtleties arise. A result by McKelvey [1976, 1979] shows that, for almost all distributions of voter preferences, any point in $\mathbb{R}^m$ can be reached from any other point by a finite sequence of majority rule votes. This result suggests that any of the following disjoint conclusions might plausibly hold:

(a) The limiting distribution may fail to exist.

(b) The limiting distribution will exist, but fail to concentrate in the Pareto set, or in any centrally located region.

(c) The limiting distribution will exist and have most of its probability near some centrally located region in the Pareto set.

As we show in the final section, both (a) and (b) may occur if some contrived assumptions are made about the support sets for the transition probabilities. However, under more reasonable assumptions about the transition probabilities, and assuming circular preferences for the voters, we show that a limiting distribution must always exist. This is true regardless of the structure of the decisive coalitions which generate the social preference relation -- even, for example, if the underlying game is not proper. We can also obtain some weak bounds on the limiting probability distribution of such a process and its relation to the Pareto set.

In the special case of majority rule, we get much stronger results. Here we not only get existence of the limiting distribution, but we can bound the limiting distribution in terms of its concentrations around a more centrally located set. It follows from the results here that if the distribution of voter ideal points is ''close'' to symmetric, i.e. if it is close the situation when a core exists, then the limiting distribution will be quite concentrated near a ''generalized median.'' Even when there is less symmetry, the distribution will be concentrated around a centrally located region which for large numbers of voters will be contained in the Pareto set and will also contain Kramer's minimax set. Thus, the class of probabilistic solution concepts we consider are consistent with the chaos of total cyclicity suggested by McKelvey's theorem, but our results show that such chaos is not probabilistically significant.

2. DEFINITIONS AND NOTATION

We assume a set $N = \{1,2,\dots,n\}$ of <u>voters,</u> a set $X \subseteq \mathbb{R}^m$ of <u>alternatives,</u> and, for each $i \in N$, a complete binary relation $R_i \subseteq X \times X$ representing voter i's <u>preferences.</u> We interpret R_i as weak preference, denoting its asymmetric part (strict preference) by P_i. Any nonempty subset C of N is called a <u>coalition,</u> with $|C|$ denoting the number of members in C. For any $x, y \in X$, preferences are defined for coalitions by,

$$xP_Cy \Leftrightarrow xP_iy (\forall\ i \in C). \tag{2.1}$$

We impose the structure of a <u>simple game</u> on N. Thus we are given a collection $\underline{W}$ of subsets of N with the property that

$$C \in \underline{W},\ C \subseteq C' \Rightarrow C' \in \underline{W} \tag{2.2}$$

The collection $\underline{W}$ can be thought of as the set of ''winning,'' or ''decisive'' coalitions. (Note that we do not necessarily assume that the set $\underline{W}$ is generated by a proper game.)

We can then define the group preference relation $P \subseteq X \times X$ by

$$xPy \Leftrightarrow xP_Cy \text{ for some } C \in \underline{W}. \tag{2.3}$$

Define for any $x \in X$, and $i \in N$,

$$P_i(x) = \{y \in X | yP_ix\}. \tag{2.4}$$

For any $C \subseteq N$, define

$$P_C(x) = \{y \in X | yP_Cx\} \tag{2.5}$$

and

$$P(x) = \{y \in X | yPx\}. \tag{2.6}$$

A core point for the group preference relation, P, is any $x \in X$ for which $P(x) = \emptyset$. Some of the results we obtain will require absolute majority rule for the group preference. In this case we define $\underline{W}$ by

$$\underline{W} = \{C \subseteq N: |C| > \frac{n}{2}\}, \tag{2.7}$$

and we denote the induced majority relation by P^M.

3. THE MARKOV MODEL

Let $\underline{M}$ denote the Lebesgue measurable subsets of X and let μ

denote Lebesgue measure on $\mathbb{R}^m$. We start with a function $p : \underline{M} \times X \to \mathbb{R}$ satisfying

(i) $p(\cdot,x)$ is a probability measure on $\underline{M}$ for each $x \in X$.

(ii) $p(A,\cdot)$ is a Lebesgue measurable function for each $A \in \underline{M}$.

(iii) $x \notin A$ and $p(A,x) > 0 \Rightarrow \exists y \in A$ such that yPx.

Conditions (i) and (ii) are standard assumptions for generating a Markov process with stationary transition probabilities. Thus the following interpretation is placed on p. Given that alternative (state) x has occurred at time t and given $A \in \underline{M}$, $p(A,x)$ gives the probability that an alternative will be selected from A at time $t + 1$ $(t = 0,1,2,\ldots)$. Condition (iii) requires that a set A of alternatives excluding x, none of which is preferred to x, cannot have a positive transition probability from x. Intuitively, any new alternative selected at time $t + 1$ must defeat the status quo alternative of time t. Throughout, we will use the following notation. For each $x \in X$,

$$p_x : \underline{M} \to \mathbb{R} \text{ is the measure } p(\cdot,x)$$

For each $A \in \underline{M}$, (3.1)

$$p_A : X \to \mathbb{R} \text{ is the function } p(A,\cdot)$$

We are concerned in this paper with the existence and properties of the limiting (or steady state) probability $p^* : \underline{M} \to \mathbb{R}$ of the Markov process determined by p. To define p^*, we first obtain the k-step density functions inductively as follows:

$$p_x^1(A) = p_x(A) \tag{3.2}$$

$$p_x^{k+1}(A) = \int p_A dp_x^k$$

We now define p_x^* by

$$p_x^*(A) = \lim_{k \to \infty} p_x^k(A) \tag{3.3}$$

provided this limit exists and that p_x^* is a probability measure over X for all $x \in X$. In situations we shall be considering, p_x^* will frequently turn out to be independent of x. We can therefore regard p^* as a function from $\underline{M}$ to R and for any measurable $A \subseteq X$, $p^*(A)$ denotes the limiting probability of the process ''ending up'' in A.

When such a limiting distribution $p^* : \underline{M} \to \mathbb{R}$ exists, the Markov process determined by p is called <u>stationary</u> and p^* is any probability measure defined recursively by the equation

$$p^*(A) = \int p_A dp^* \tag{3.4}$$

We refer the reader to Doob [1953] for expository details of these Markov process ideas.

Under specific assumptions on p, on the voting rule, and on the types of individual preferences, the probability measure determined by p is the end result of the voting models developed in Ferejohn, Fiorina and Packel [1980], and Packel [1981]. The solution concept obtained has been called the <u>stochastic solution.</u> In what follows, we examine the qualitative behavior of the stochastic solution and some of its natural extensions.

4. SPECIAL ASSUMPTIONS

All of our results will require added structure on the Markov process p, as well as on individual preferences. For any $A \subseteq X$, let $\chi_A : X \to \{0,1\}$ denote the <u>characteristic function</u> of A:

$$\chi_A(y) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases} \tag{4.1}$$

We make the following assumption for the transition function p.

<u>Assumption 1 (Preferred set Assumption):</u> For all $x \in X$ and $A \in \underline{M}$, p(A,x) can be written in the form

$$p(A,x) = p_A(x) = p_x(A) = \begin{cases} \chi_A(x) & \text{if } P(x) = \emptyset \\ \sum_{j=1}^{k} a_j(x)\mu(A \cap D_j(x)) & \text{if } P(x) \neq \emptyset \end{cases}$$

where $a_j : X \to \mathbb{R}$ is continuous with $a_j(x) > 0$ for all x, and $D_j(x) = \bigcup_{C \in \underline{B}_j} P_C(x)$ for all $j = 1,2,\ldots,J$, and $\underline{B}_j \subseteq \underline{W}$.

This assumption on p says that the transition probabilities from an $x \varepsilon X$ are determined by selecting various collections of winning coalitions, identifying the set of alternatives preferred to x by each such collection, and then taking some weighted combination of uniform distributions over these sets. It should be noted that many natural conditions imposed on group voting behavior in the literature can be captured under this assumption. We give three examples:

PSA1: Let $\underline{B}_j \equiv \{N\}$ with $J = 1$, so that $D_j(x)$ becomes the set $P_N(x)$ of alternatives unanimously preferred to x. Then the form of p requires that, for an x outside the Pareto set, its successor will be chosen by means of a uniform distribution on $P_N(x)$. For x in the Pareto set, the process stops (points in the Pareto set are absorbing states). In the limit one would expect this process to choose points from the Pareto set with probability one.

PSA2: Let $J = 1$, and $\underline{B}_j \equiv \{C_1,\ldots,C_m\}$ be the collection of minimal winning coalitions. Then $D_j(x)$ is the set $P(x)$ of points which are preferred by some winning coalitions to x. The resulting transition function is a uniform distribution over $P(x)$.

PSA3: Let $\{C_1,C_2,\ldots,C_m\}$ be the collection of minimal winning coalitions $J = m$, and let $\underline{B}_j \equiv \{C_j\}$. Then $D_j(x)$ is the set $P_{C_j}(x)$. For each x with $P(x) \neq \phi$ and any $j = 1,2,\ldots,J$ set $a_j(x) = 1/(\sum_{i=1}^{J} \mu(D_i(x)))$. The resulting transition function p assigns probabilities weighted in proportion to the number of minimal winning

coalitions preferring points to x. This is the form of the models developed in Ferejohn, Fiorina and Packel [1980] and Packel [1981].

Also, throughout the remainder of the paper, we require the following assumption on individual preferences.

Assumption 2: $X = \mathbb{R}^m$, and each voter has Type 1 preferences. I.e., for each $i \varepsilon N$, $\exists\, x^i \varepsilon X$ such that $\forall\, x,y \varepsilon X$

$$xR_iy \iff ||x - x^i|| \leq ||y - x^i||$$

The vector x^i is called voter i's ideal point.

Assumption 2 is common in the voting literature, and says that each voter's utility is a monotone decreasing function of Euclidean distance from some ideal point. In other words indifference surfaces are spheres.

The following lemma establishes continuity of the transition probability functions for any Markov voting process satisfying Assumptions 1 and 2.

Lemma 4.1. Under Assumptions 1 and 2, for each $A \varepsilon \underline{M}$, $p_A(x)$ is a continuous function of x on the set $\{x \varepsilon X \mid P(x) \neq \phi\}$.

Proof. Let $\underline{M}(X)$ denote the measurable functions on X. It is easy to verify that, for each i, the mapping $f_i : X \to \underline{M}(X)$ defined by

$$f_i(x) = \chi_{P_i(x)} \tag{4.2}$$

is continuous in the ℓ_1 norm. Thus, for each j, the mapping

$g_j : X \to \underline{M}(X)$ defined by

$$g_j(x) = \chi_{D_j(x)} \tag{4.3}$$

is also continuous in the ℓ_1 norm. This follows because g_j can be written in the form

$$g_j(x) = \max_{C \in \underline{B}_j} f_C(x), \tag{4.4}$$

where

$$f_C(x) = \prod_{i \in C} f_i(x) \tag{4.5}$$

But then we can write, for any $A \in \underline{M}$,

$$\mu(A \cap D_j(x)) = \int_A \chi_{D_j(x)} d\mu \tag{4.6}$$

and it follows that $\mu(A \cap D_j(x))$ is continuous in x, because since $g_j(x)$ is continuous, for any $\varepsilon > 0$ and $x_0 \in X$, we can find a neighborhood $N(x_0)$ of x_0 for which $x \in N(x_0)$ implies

$$\int | \chi_{D_j(x)} - \chi_{D_j(x_0)} | \, d\mu < \varepsilon.$$

But then

$$|\mu(A \cap D_j(x)) - \mu(A \cap D_j(x_0))|$$

$$= \left| \int_A (\chi_{D_j(x)} - \chi_{D_j(x_0)}) d\mu \right|$$

$$\leq \int_A | \chi_{D_j(x)} - \chi_{D_j(x_0)} | \, d\mu < \varepsilon. \tag{4.7}$$

So $\mu(A \cap D_j(x))$ is continuous.

Now

$$p_A(x) = \sum_{j=1}^{k} a_j(x) \mu(A \cap D_j(x)) \tag{4.8}$$

which is the sum of the product of continuous functions, so the result follows.

Q.E.D.

5. EXISTENCE OF A LIMITING PROBABILITY DISTRIBUTION.

This section shows existence of a stationary distribution for any Markov voting process satisfying assumptions 1 and 2. In addition, we get some weak bounds on the concentration of the limiting probability distributions around the set of Pareto optimals. To do this, we define B_0 to be a closed ball of minimal diameter containing all ideal points. We then find bounds on the limiting probability distribution in terms of the proportion of the distribution which lies within a given distance from the center of B_0.

More specifically, we define

$$B_0 = \{x \in \mathbb{R}^m \mid \|x - y^*\| \leq \max_{i\in N} \|x_i - y^*\|\} \tag{5.1}$$

where $y^* \in \mathbb{R}^m$ is chosen to minimize $\max_{i\in N} \|x_i - y^*\|$. Set $t^* = 2 \max_{i\in N} \|x_i - y^*\|$ to be the diameter of B_0, and define, for $j = 0,1,2,...$

$$B_j = \{x \in \mathbb{R}^m \mid 2\|x - y^*\| \leq (2j + 1)t^*\}. \tag{5.2}$$

Let A_j denote the annulus in $\mathbb{R}^m$ determined by B_{j-1} and B_j, so that $A_j \equiv B_j - B_{j-1}$ for $j = 1,2,...$(and set $A_0 = B_0$). See Figure 1 for an example of this construction for a particular configuration of 7 voters in two dimensions.

For the following lemma, we define $C^m(t,t_0)$ to be the m-dimensional cardioid whose boundary, in m-dimensional spherical coordinates is defined by the equation

$$\rho = t \sin \theta_1 + t_0. \tag{5.3}$$

See Appendix A for a more rigorous definition of $C^m(t,t_0)$.

Setting $\alpha = \sin^{-1}\left(\frac{t_0}{t}\right)$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, it is shown in Appendix A that the m dimensional Lebesgue content of $C^m(t,t_0)$ is given by:

$$\mu[C^m(t,t_0)] = \frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})} \int_\alpha^{\frac{\pi}{2}} (t \sin \theta_1 + t_0)^m \cos^{m-2} \theta_1 d\theta_1. \tag{5.4}$$

When $t_0 = 0$, (5.3) becomes the formula for a sphere, and $\mu[C^m(t,t_0)]$ reduces to the m-dimensional Lebesgue content of a sphere of diameter t. In this case we write

$$S^m(t) = C^m(t, 0) \tag{5.5}$$

We now seek lower bounds on the limiting probability that the process will end up within each ball B_j(i.e., bounds on $p^*(B_j)$) for a transition density function satisfying Assumptions 1 and 2. This will tell us, for a general configuration of ideal points within B_0, to what extent the limiting distribution is concentrated near the ''centrally located'' region determined by the set B_0. For each $j,k = 0,1,2,...,$ define nonnegative real numbers $q_{j,k}$ by

$$q_{j,k} = \inf_{y\in A_j} p(B_k,y) \tag{5.6}$$

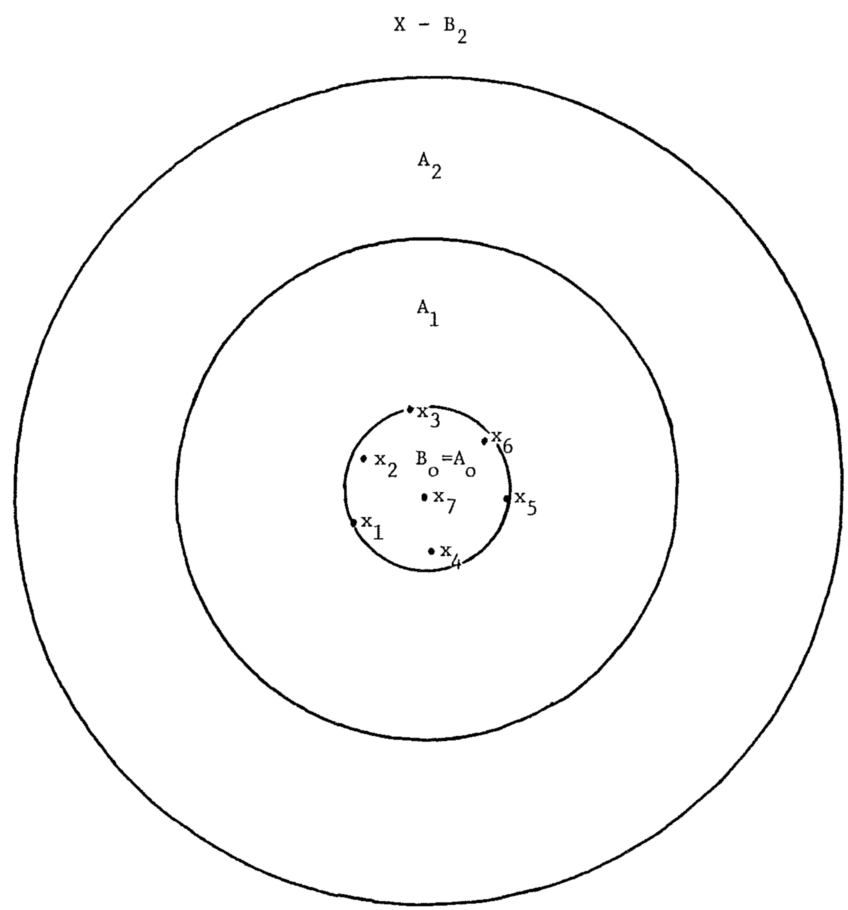
The following lemma obtains bounds for the $q_{j,k}$ based on the above expressions for $C^m(t, t_0)$ and $S^m(t)$.

<u>Lemma 5.1</u> Given Assumptions 1 and 2, the transition function p satisfies:

(a) $q_{j,k} \geq \dfrac{\mu[S^m(2k+1)]}{\mu[C^m(2j+1,1)]}$ for $0 \leq k \leq j-2$, $j \geq 2$ or $k = j$, $j \geq 1$

(b) $q_{j,k} \geq \dfrac{\mu[C^m(2j-1,-1)]}{\mu[C^m(2j+1,1)]}$ for $k = j - 1$, $j \geq 2$

(c) $q_{j,k} = 1$ for $k > j$



ANNULUS CONSTRUCTION FOR A
PARTICULAR SET OF IDEAL POINTS

Figure 1

<u>Proof:</u> Follows from Lemma A1 and A2 of Appendix A.

By using the $q_{j,k}$ estimates, we now bound the original continuous state Markov process by a countably infinite state Markov chain. The states correspond to the annuli $\{A_j\}_{j=0}^{\infty}$. In the transition matrix $S = [s_{j,k}]_{j,k=o}^{\infty}$ for the Markov chain, $s_{j,k}$ represents the 1 step probability of reaching state k, given that j is the current state. We define S as follows:

First, set

$$\tilde{r}_{j,k} = \begin{cases} \dfrac{\mu[S^m(2k+1)]}{\mu[C^m(2j+1,1)]} & \text{for } 0 \leq k \leq j-2,\ j \geq 2 \\ & \quad \text{or } k = j,\ j \geq 1 \\ \dfrac{\mu[C^m(2j-1,-1)]}{\mu[C^m(2j+1,1)]} & \text{for } k = j-1 \\ & \quad j \geq 2 \\ 1 & \text{for } k > j \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

Then define $r_{j,k}$ recursively such that

$$r_{j,k} = \min(\tilde{r}_{j,k}, r_{j-1,k}, r_{j,k+1}) \tag{5.8}$$

and then, set

$$s_{j,k} = \begin{cases} r_{j,k} & \text{if } k = 0 \\ r_{j,k} - r_{j,k-1} & \text{otherwise} \end{cases} \tag{5.9}$$

The $r_{j,k}$ give the cumulative densities for the transition process defined by $s_{j,k}$. We must modify the natural bounds $\tilde{r}_{j,k}$ given in

(5.7) to those given by $r_{j,k}$ in (5.8) for technical reasons, which will become apparent in the proof of Theorem 1. The definition of S is motivated by the fact that its transition probabilities give <u>less</u> pull towards the center A_0 and <u>more</u> pull outward than is present in the original Markov process. In other words, for all $x \in X$, if $x \in A_j$, then

$$p(B_k, x) \geq r_{j,k} \tag{5.10}$$

We first find steady state probabilities for S and then prove that a stationary limiting distribution p* must exist for the original Markov process p, and further that p* is at least as concentrated as the limiting distribution generated by S.

<u>Lemma 5.2:</u> The Markov chain S defined by (5.9) has a convergent limiting distribution.

<u>Proof:</u> We use Theorem 7 of Kushner ([1971], p. 211, see also the Corollary on the same page). Since all states communicate, we need only show that for all but a finite number of states, j, the expected state after one transition is less than or equal to j − 1. I.e., we must show, for all but a finite number of j,

$$\sum_{k=0}^{\infty} k s_{jk} \leq j - 1. \tag{5.11}$$

But

$$\sum_{k=0}^{\infty} ks_{jk} \leq (j+1)\left[\sum_{k=j-1}^{j+1} s_{jk}\right] + (j-2)\left[\sum_{k=0}^{j-2} s_{jk}\right] \tag{5.12}$$

$$= (j+1)\left[1 - \sum_{k=0}^{j-2} s_{jk}\right] + (j-2)\left[\sum_{k=0}^{j-2} s_{jk}\right]$$

$$= (j+1) - 3r_{j,j-2}$$

Now

$$r_{j,j-2} \geq \min_{\substack{j' \leq j \\ k \geq j-2}} \tilde{r}_{j',k} \tag{5.13}$$

Assume the minimum is achieved at $\tilde{r}_{j^*,k^*}$. Then there are four possible oases.

Case I. $k^* = j^* - 2$. (Note here, $j = j^*$.) Then

$$\tilde{r}_{j^*,k^*} = \frac{\mu[S^m(2j^*-3)]}{\mu[C^m(2j^*+1,\ 1)]} \geq \frac{\mu[S^m(2j^*-3)]}{\mu[S^m(2j^*+3)]}$$

$$= \frac{(2j^*-3)^m}{(2j^*+3)^m} = \frac{(2j-3)^m}{(2j+3)^m} \tag{5.14}$$

Case II. $k^* = j^* - 1$. (Note here, $j \geq j^* \geq j - 1$.) Then

$$\tilde{r}_{j^*,k^*} = \frac{\mu[C^m(2j^*-1,\ -1)]}{\mu[C^m(2j^*+1,\ 1)]} \geq \frac{\mu[S^m(2j^*-3)]}{\mu[S^m(2j^*+3)]}$$

$$= \frac{(2j^*-3)^m}{(2j^*+3)^m} \geq \frac{(2j-3)^m}{(2j+3)^m} \tag{5.15}$$

Case III. $k^* = j^*$. (Note here $j \geq j^* \geq j - 2$.) Then

$$\tilde{r}_{j^*,k^*} = \frac{\mu[S^m(2j^*+1)]}{\mu[C^m(2j^*+1,\ 1)]} \geq \frac{\mu[S^m(2j^*+1)]}{\mu[S^m(2j^*+3)]}$$

$$= \frac{(2j^*+1)^m}{(2j^*+3)^m} \geq \frac{(2j^*-3)^m}{(2j^*+3)^m} \geq \frac{(2j-3)^m}{(2j+3)^m} \tag{5.16}$$

Case IV. $k^* > j^*$. Here

$$\tilde{r}_{j^*,k^*} = 1 \geq \frac{(2j-3)^m}{(2j+3)^m}. \tag{5.17}$$

It follows that

$$r_{j,j-2} \geq \frac{(2j-3)^m}{(2j+3)^m} \tag{5.18}$$

so

$$r_{j,j-2} \to 1 \text{ as } j \to \infty \tag{5.19}$$

so, for large j,

$$\sum_{k=0}^{j-2} ks_{jk} \leq (j+1) - 3r_{j,j-2} \leq j - 1 \tag{5.20}$$

as we wished to show.

Q.E.D.

We let $s^* = \{s_i^*\}_{i=0}^{\infty}$ denote the limiting distribution for the Markov chain S. It follows that for all k

$$s_k^* = \sum_{j=0}^{\infty} s_j^* s_{jk} \tag{5.21}$$

Further, let $r^* = \{r_i^*\}_{i=0}^{\infty}$ be the cumulative density of s^* i.e., for all k,

$$r_k^* = \sum_{i=0}^{k} s_i^*$$

We can now present an existence theorem for the limiting distribution of the Markov voting models we are studying.

<u>Theorem 1:</u> Given a Markov process p satisfying Assumptions 1 - 2, a limiting distribution p^* exists. Furthermore, any limiting distribution p^* is more concentrated on the sets B_k than the process S. I.e., for all k,

$$p^*(B_k) \geq r_k^* \tag{5.22}$$

<u>Proof:</u> If $P(x_0) = \emptyset$ for some $x_0 \in \mathbb{R}^m$, then set $p^*(A) = \chi_A(x_0)$. It follows easily that this is a stationary distribution for p. Further $x_0 \in B_0$. Otherwise, taking $x \in B_0$ to minimize $||x_0 - x||$, we have $xP_i x_0$ for all $i \in N$, implying xPx_0. But then $p^*(B_k) = 1$ for all k, and (5.22) is satisfied. So we assume $P(x) \neq \emptyset$ for all $x \in \mathbb{R}^m$. Then,

the theorem follows by application of Corollary 1 of Green, McKelvey and Packel [1981]. Thus S defines a Markov process $s : \underline{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ on the Borel sets $\underline{\mathbb{Z}}$ of $\mathbb{Z}$, (the natural numbers), where, for any $A \in \underline{\mathbb{Z}}$, $j \in \mathbb{Z}$,

$$s(A,j) = s_j(A) \equiv \sum_{k \in A} s_{jk} \tag{5.23}$$

Let $K : X \to \mathbb{Z}$ be the mapping

$$K(x) = \sum_{k=1}^{\infty} k \chi_{A_k}(x) \tag{5.24}$$

By construction, under this mapping, for all $x \in X$, if $K(x) = j$ (i.e., $x \in A_j$), then by (5.10), $p(B_k,x) \geq r_{j,k}$. In other words, letting $T_k = \{t \in \mathbb{Z} \mid t \leq k\}$,

$$\begin{aligned} p_x \circ K^{-1}(T_k) &= p_x(B_k) = p(B_k,x) \\ &\geq r_{j,k} = s_{K(x)}(T_k) \end{aligned} \tag{5.25}$$

Thus, the Markov process s stochastically dominates p with respect to K. Further, by construction, if $i \geq j$, then $r_{ik} \leq r_{jk}$ for all k. So the process s is stochastically increasing. Finally, by Lemma 4.1, since $P(x) \neq \emptyset$ for all $x \in \mathbb{R}^m$, $p(A,x)$ is a continuous function of x for each fixed A. Therefore, the theorem follows as a direct application of the cited corollary.

Q.E.D.

6. LIMITING DISTRIBUTIONS FOR MAJORITY PROCESSES.

The previous section placed no restriction on the underlying coalition structure generating the Markov process. Here we assume majority rule, and using methods very similar to the previous section, get considerably tighter bounds on the concentration of the limiting distribution.

Assumption 3: $P = P^M$, $\{x \in X : P(x) = \emptyset\} = \emptyset$ and the transition function p has the form $p(A,x) = a(x)\cdot\mu(A \cap D(x))$ where $D(x) = \bigcup_{C\in\underline{W}} P_C(x)$ and $a(x) = 1/\mu(D(x))$.

Assumption 3 says that the voting rule is majority rule, and no majority rule core exists, which of course is the generic state of affairs. Also, when Assumption 2 is combined with Assumption 3 it follows from Theorem 1 of McKelvey [1976] that any point in X can be reached by a finite sequence of majority rule votes from any other point. Assumption 3 imposes a uniform distribution over the set of points that defeat a given point, as in example PSA2.

Now, for any $\alpha \in \mathbb{R}^m$ and $c \in \mathbb{R}$ write

$$\begin{aligned} H(\alpha,c) &= \{x \in \mathbb{R}^m : x\cdot\alpha = c\} \\ H^+(\alpha,c) &= \{x \in \mathbb{R}^m : x\cdot\alpha > c\} \qquad (6.1) \\ H^-(\alpha,c) &= \{x \in \mathbb{R}^m : x\cdot\alpha < c\} \end{aligned}$$

The hyperplane $H(\alpha,c)$ is called a median hyperplane iff

$$|\{i : x^i \in H^+(\alpha,c)\}| \leq \frac{n}{2} \text{ and } |\{i : x^i \in H^-(\alpha,c)\}| \leq \frac{n}{2} \qquad (6.2)$$

We now let $\overline{B}_0$ be a closed ball of minimum diameter such that every median hyperplane has a nonempty intersection with $\overline{B}_0$. Assume $\overline{B}_0$ can be written as follows.

$$\overline{B}_0 = \{x \in \mathbb{R}^m \mid 2||x - \overline{y}|| \leq \overline{t}\} \qquad (6.3)$$

As before, we define a series of concentric spheres around $\overline{y}$ as follows: For j = 0,1,... define

$$\overline{B}_j = \{x \in \mathbb{R}^m \mid 2||x - \overline{y}|| \leq (2j + 1)\overline{t}\}, \qquad (6.4)$$

and define

$$\overline{A}_j = \overline{B}_j - \overline{B}_{j-1}. \qquad (6.5)$$

See Figure 2 for an example of this construction for a particular configuration of 7 voters in 2 dimensions.

Now define, for each j,k = 0,1,2,... nonnegative real numbers $\overline{q}_{j,k}$ by

$$\overline{q}_{j,k} = \inf_{y\in\overline{A}_j} p(\overline{B}_k, y) \qquad (6.6)$$

We then obtain the following analogue of Lemma 5.1:

Lemma 6.1: Given Assumptions 1 - 3, the transition function p

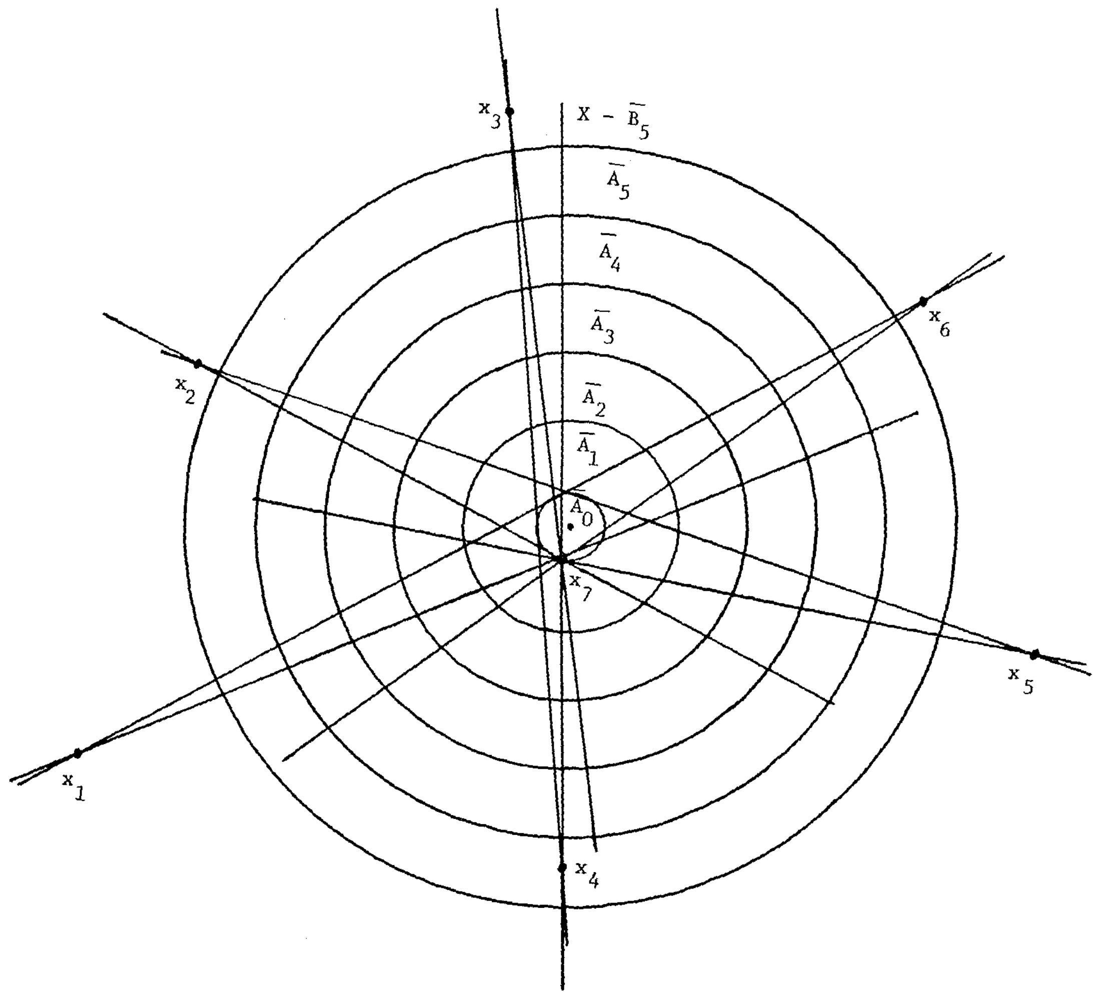


Figure 2 : ANNULUS CONSTRUCTION
FOR A PARTICULAR SET OF IDEAL POINTS
UNDER MAJORITY RULE (ASSUMPTION 3)

satisfies:

(a) $\bar{q}_{j,k} \geq \dfrac{\mu[S^m(2k+1)]}{\mu[C^m(2j+1,1)]}$ for $0 \leq k \leq j-2$, $j \geq 2$ or $k = j$, $j \geq 1$

(b) $\bar{q}_{j,k} \geq \dfrac{\mu[C^m(2j-1,-1)]}{\mu[C^m(2j+1,1)]}$ for $k = j-1$, $j \geq 2$

(c) $\bar{q}_{j,k} = 1$ for $k > j$

Proof: The proof follows from Lemma A1 and A3 of Appendix A.

With the above Lemma in hand, we can use exactly the same methods as in the previous section. Specifically, the countable state Markov chain defined by equations (5.7) - (5.9) can be used to bound the majority process as well. I.e. for all $x \in X$, if $x \in A_j$,

$$p(\bar{B}_k, x) \geq r_{j,k} \tag{6.7}$$

We then get the following Theorem for the majority process:

Theorem 2. Given a Markov process p satisfying Assumptions 1 - 3, a limiting distribution, p^*, exists. Furthermore, the limiting distribution is more concentrated on the sets $\bar{B}_k$ than the process S. I.e., for all k

$$p^*(\bar{B}_k) \geq r_k^*$$

Proof: If $P(x_0) = \phi$ for some $x_0 \in \mathbb{R}^m$, then $p^*(A) = \chi_A(x_0)$ is a stationary distribution. In this case, from Davis, Degroot and Hinich [1972], it follows that x_0 is a total median. Therefore $\bar{y}$, the center of $\bar{B}_0$ is at x_0, and $\bar{t}$, the diameter of B_0 is zero. So $p^*(\bar{B}_k) = 1$ for all k, satisfying the inequality required. If $P(x) \neq \phi$ for all $x \in \mathbb{R}^m$, then replacing the sets B_k by $\bar{B}_k$ and A_k by $\bar{A}_k$, the proof is exactly the same as the proof of Theorem 1.

Q.E.D.

Since McKelvey's theorem ensures that $\mathbb{R}^m$ itself will be the unique ergodic set for p (all states communicate), a direct consequence of Theorem 2 is that any subset of $\mathbb{R}^m$ with positive measure will be assigned positive probability under the stochastic solution. The final section looks at the concentration of this distribution.

7. CONCENTRATION OF THE LIMITING DISTRIBUTION

We have shown that the limiting distribution, p^*, of the Markov voting processes can be bounded by the cumulative density of the limiting distribution of the countable state Markov chain, S, defined by (5.7) - (5.9). For any process satisfying Assumptions 1 and 2, we have, for all k,

$$p^*(B_k) \geq r_k^* \tag{7.1}$$

and for majority processes, we have, for all k

$$p^*(\overline{B}_k) \geq r_k^* \qquad (7.2)$$

In this section, we compute numerical values for the r^* to obtain bounds on the concentration of p^*.

For actual computation we approximate the countably infinite process S by a finite state process and we perform a numerical integration to approximate the quantities $\mu[C^m(j,k)]$ of (5.4). The limiting distribution of S can be approximated to any degree of accuracy by starting with a large enough finite ''truncation'' of S (with all ''excess'' probability thrown into the largest available state). The computational results of Table I indicate that, at least for Euclidean dimensions between m = 2 and m = 8, a finite approximation with 30 states (inside the ball B_{30} in $\mathbb{R}^m$ of radius 30.5 t_0) is more than adequate for three decimal place accuracy. We therefore take the values in Table I as accurate estimates for the $r_k^* = \sum_{j \leq k} s_j^*$, and hence, according to Theorems 1 and 2, as lower bounds of $p^*(B_k)$ or $p^*(\overline{B}_k)$.

In the two dimensional case, note that over 72 percent of the probability is within B_3 (or $\overline{B}_3$ if Assumption 3 is met) and that essentially all the probability is within B_8. The situation for higher dimensions appears to be analogous. As might be expected, the distributions become less concentrated as m increases from 2 to 8; nevertheless, the results indicate that the entire distribution in R^m is essentially contained within the ball $B_{2(m+2)}$ having radius $[2(m+2) + 1/2]t_0$. It seems unlikely that this exact pattern would continue in still higher dimensions, but we leave such investigation

for a later time and a larger computer budget.

The above results give bounds on the percentage of the limiting distribution which must be within a given distance of the set B_0 (or $\overline{B}_0$). It therefore is important to discuss the properties of the sets B_0 and $\overline{B}_0$ as a function of the distribution of preferences.

For general processes, satisfying Assumptions 1 and 2, the limits are on the probabilities of lying in B_k. Here we recall that B_0 is the smallest closed ball containing all ideal points. Of course the diameter of this set will always be large in relation to the distribution of ideal points, and the diameter of B_0 will not decrease with the symmetry of the ideal points or with larger numbers of voters. Hence, the bounds on $p^*(B_k)$ are weak. The important fact here, however is that these bounds hold regardless of the form of p(A,x), as long as Assumption 1 is met, even for processes generated by coalition structures that are not proper.

For majority processes, the results are much stronger. Recall that the set $\overline{B}_0$ is a minimal sphere which intersects with every median hyperplane. We can think of $\overline{B}_0$ as a ''generalized median''. When the distribution of ideal points is symmetric, then there will be a unique total median (See Davis, Degroot and Hinich [1972]), implying that $\overline{B}_0$ will be a singleton set, consisting of that point. In that case, of course, there is a core, and the limiting distribution will be the distribution assigning the core probability one and all other points probability zero.

In the more usual case, when there is no total median, all

Lower Bounds on General Process	Lower Bounds on Majority Process	Dimensions 2	3	4	5	6	7	8
$p^*(B_0)$	$p^*(\overline{B}_0)$	.000	.000	.000	.000	.000	.000	.000
$p^*(B_1)$	$p^*(\overline{B}_1)$	.131	.024	.003	.001	.000	.000	.000
$p^*(B_2)$	$p^*(\overline{B}_2)$	.432	.141	.037	.008	.002	.000	.000
$p^*(B_3)$	$p^*(\overline{B}_3)$	.723	.364	.139	.043	.011	.003	.001
$p^*(B_4)$	$p^*(\overline{B}_4)$	.896	.613	.319	.133	.046	.014	.003
$p^*(B_5)$	$p^*(\overline{B}_5)$	.969	.805	.534	.284	.125	.046	.015
$p^*(B_6)$	$p^*(\overline{B}_6)$	.992	.917	.724	.470	.253	.115	.045
$p^*(B_7)$	$p^*(\overline{B}_7)$	.998	.970	.858	.652	.417	.226	.106
$p^*(B_8)$	$p^*(\overline{B}_8)$	.999	.991	.936	.796	.587	.372	.203
$p^*(B_9)$	$p^*(\overline{B}_9)$	1.0	.997	.975	.894	.736	.530	.333
$p^*(B_{10})$	$p^*(\overline{B}_{10})$		.999	.991	.950	.847	.678	.479
$p^*(B_{11})$	$p^*(\overline{B}_{11})$		1.0	.997	.979	.919	.798	.623
$p^*(B_{12})$	$p^*(\overline{B}_{12})$			.999	.992	.961	.883	.748
$p^*(B_{13})$	$p^*(\overline{B}_{13})$			1.0	.997	.983	.938	.844
$p^*(B_{14})$	$p^*(\overline{B}_{14})$				.999	.993	.970	.910
$p^*(B_{15})$	$p^*(\overline{B}_{15})$				1.0	.997	.986	.952
$p^*(B_{16})$	$p^*(\overline{B}_{16})$					.999	.994	.976
$p^*(B_{17})$	$p^*(\overline{B}_{17})$					1.0	.998	.989
$p^*(B_{18})$	$p^*(\overline{B}_{18})$						.999	.995
$p^*(B_{19})$	$p^*(\overline{B}_{19})$						1.0	.998
$p^*(B_{20})$	$p^*(\overline{B}_{20})$							.999
$p^*(B_{21})$	$p^*(\overline{B}_{21})$							1.0

TABLE 1*

Lower Bounds on the Limiting Distribution, p^*, of the Markov Process

*Entries in kth row of each column are $r_k^* = \sum_{j \leq k} s_j^*$ of the dominating process S.

points communicate, and the results of the previous section apply. In this case, the set $\overline{B}_0$ will be a sphere whose diameter is a measure of the degree of nonsymmetry present in the distribution of ideal points. Thus, the more symmetric the distribution, the smaller will be the diameter of $\overline{B}_0$ and the more concentrated will be the distribution of p^*.

A second question concerns the location of the set $\overline{B}_0$ in the relation to the ideal points of the voters. Kramer [1977], in a recent article defines a set, called the minimax set, consisting of those points which can be beaten by the fewest number of votes. Kramer shows that the minimax set is a centrally located subset of the Pareto optimals, and shrinks to a point as the number of voters increases. It also follows, from results of his, that $\overline{B}_0$ has a nonempty intersection with the minimax set (see [Kramer 1981]), and that as $n \to \infty$ the minimax set will be a strict subset of B_0. It follows that the limiting distribution of the Markov process defined by $p(\cdot|x)$ will be centered around the minimax set, being more or less concentrated as the symmetry of the distribtion of ideal points increases.

Finally, we note that all the above conclusions depend on the particular transition process assumed in Assumption 1. Changing these assumptions could drastically alter the above results. For example, it is possible to imagine alternative assumptions on the transition probability function $p(\cdot,x)$ which would lead to situations where a limiting probability fails to exist. Let $X = \mathbb{R}^m$ and assume that ideal

points are situated so that no majority rule equilibrium exists (this is ''almost always'' the case, as follows from Plott [1967]). Given $X \in \mathbb{R}^m$, find a nonnegative integer j such that $x \in A_j$. The theorem in McKelvey [1976] then ensures that there will be a subset $A_x \subseteq A_{j+1}$ of positive Lebesgue measure such that $yP^{M}x$ ($\forall\, y \in A_x$). If we define $p(\cdot,x)$ to have support set A_x, for all $x \in \mathbb{R}^m$, it is clear that $p(\cdot,x)$ fails to have a limiting distribution and its Markov process will be <u>transient</u>. By modifying the definition so that $p(\cdot|x)$ has a larger support set intersecting A_{k-1}, A_k and A_{k+1}, we could arrange things so that $p(\cdot,x)$ was either null recurrent ($p^* = 0$) or so it was stationary with a limiting distribution requiring arbitrarily large balls B_k to contain most of the probability.

We do not formalize the above ideas; but they do suggest, in line with McKelvey's result, that under sufficiently contrived assumptions about the transition process, literally anything can happen. Under more natural assumptions such as Assumption 1, however, a limiting probability must exist, and it will tend to concentrate near the Pareto set. While there are other reasonable assumptions about group preference besides that of Assumption 1 it seems they would generally lead to transition probabilities which are even more concentrated towards the minimax set. Accordingly our existence results still apply and even better lower bounds would be expected for limiting probabilities.

APPENDIX A

This appendix computes bounds for the transition probability $p_x(B_k)$. To do this, we need to first develop limits on the set P(x) of points that beat a given point. It turns out that the set P(x) can be bounded, in a wide class of situations, by a pair of cardioids. So we first present general formuli for a cardioid in $\mathbb{R}^m$ and for its volume.

Let $\theta(x) = (\theta_1(x),\theta_2(x),\ldots,\theta_{m-1}(x),\rho(x))$ denote the m dimensional, spherical coordinates of the vector $x \in \mathbb{R}^m$. Thus

$$\rho(x) = ||x||$$

and, for $1 \leq i \leq m-1$,

$$\theta_i(x) = \sin^{-1}\left[\frac{x_i}{\rho(x)\prod_{j<i}\cos\theta_j(x)}\right]$$

Here, the θ_j range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, except θ_{n-1}, which ranges between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Now pick $x^*, y^* \in \mathbb{R}^m$, $t_0 \in \mathbb{R}$, and set $t = 2||x^* - y^*||$. Let Q be an m x m orthonormal rotation matrix such that

$$Q(y^* - x^*) = (\tfrac{t}{2},0,\ldots,0)$$

write $Q(x - x^*) = (z_1,\ldots,z_m) = z$. Then define $\xi_{x^*,y^*}(x) = \theta(Q(y^* - x^*)) = \theta(z)$. So $\xi_{x^*,y^*}(x)$ are m dimensional spherical coordinates of x which are centered at x^* and have one axis orthogonal to the vector $y^* - x^*$. Now set

$$\alpha = \sin^{-1}(-\frac{t_0}{t}) \qquad \text{where } -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}.$$

So that the above is also well defined for the case when $|t_0| \geq t$, we use the convention that, for $r \in \mathbb{R}$, $|r| \geq 1$, $\sin^{-1} r = \frac{\pi}{2}\,\mathrm{sgn}\, r$. We then set

$$\beta = \begin{cases} \frac{\pi}{2} & \text{if } m > 2 \\ \pi - \alpha & \text{if } m = 2 \end{cases}$$

Then, define, for $t_0 \in \mathbb{R}$,

$$C^m(x^*,y^*,t_0) = \{x \in X \mid 0 \leq \rho(z) \leq t\sin\theta_1(z) + t_0,$$

and

$$\alpha \leq \theta_1(z) \leq \beta(\text{where } z = Q(x - x^*))\}.$$

Thus, $C^m(x^*,y^*,t_0)$ is the <u>m-dimensional cardioid</u> which has <u>cusp</u> at x^*, center at y^*, <u>eccentricity</u> of t_0, and <u>radius</u> of $\frac{t}{2}$. See Figure A1. Note that if $t_0 = 0$, then $C^m(x^*,y^*,t_0)$ becomes a sphere, with center at y^* and diameter t. If $t_0 < 0$, then the resulting cardioid is contained in this sphere, otherwise it contains the sphere. Also note that if $t_0 < -t$, then $C^m(x^*,y^*,t_0) = \phi$. We adopt the following shorthand notation: If $x^0 = (0,0,\ldots,0)$, and $y^0 = (t,0,\ldots,0)$, then we write

$$C^m(x^0,y^0,t_0) = C^m(t,t_0)$$

Note that for arbitrary $x^*,y^* \in \mathbb{R}^m$, if $t = ||x^* - y^*||$, then

$$\mu[C^m(x^*,y^*,t_0)] = \mu[C^m(t,t_0)].$$

This follows because the transformation $Q(x - x^*)$ is just a translation and rotation. So we only need to compute $\mu[C^m(t,t_0)]$. Now, using the fact that the Jacobian of the transformation to spherical coordinates is given by

$$J = \rho^{n-1} \sum_{i=1}^{n-2} \cos^{n-i+1} \theta_i \tag{A6}$$

(See Kendall [1961], p. 17), we get:

$$\mu[C^m(t,t_0)] = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_\alpha^\beta \int_0^{t \sin \theta_1 + t_0} \rho^{m-1} \cos^{m-2} \theta_1$$

$$\cos^{m-3} \theta_2 \ldots \cos \theta_{m-2} d\rho d\theta_1 \ldots d\theta_{m-1}$$

$$= \left[\int_\alpha^\beta \int_0^{t \sin \theta_1 + t_0} \rho^{m-1} \cos^{m-2}\theta_1 d\rho d\theta_1 \right].$$

$$\left[\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{m-3} \theta_2 \ldots \cos \theta_{n-2} d\theta_2 \ldots d\theta_{n-2} \right]$$

$$= \frac{2\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} \int_\alpha^{\frac{\pi}{2}} \frac{(t \sin \theta_1 + t_0)^m}{m} \cos^{m-2}\theta_1 d\theta_1 \tag{A7}$$

Here, the evaluation of the m-2 fold integral in the last step follows from results of Kendall ([1961],p. 35).

We can now prove the following key lemma. For this lemma, the transition process $p(A,x)$ and the sets $D_i(x)$, etc., are as defined in the text. (See Assumption 1.)

Lemma A1. Let Assumptions 1 and 2 be met, and assume there exists an $y^* \varepsilon \mathbb{R}^m$ and $t_0 \varepsilon \mathbb{R}$ such that for all $x \varepsilon \mathbb{R}^m$, and all $1 \leq i \leq J$,

$$C^m(x,y^*,-t_0) \subseteq D_i(x) \subseteq C^m(x,y^*,t_0).$$

Then, setting, for $j \geq 0$

$$B_j = \{y \varepsilon \mathbb{R}^m \mid ||y - y^*|| \leq jt_0 + \frac{t_0}{2}\}$$

and

$$A_j = B_j - B_{j-1}$$

(where $B_{-1} = \phi$), and defining, for $j,k \geq 0$,

$$q_{j,k} = \inf_{y \varepsilon A_j} p(B_k,y)$$

it follows that

(a) $\quad q_{j,k} \geq \dfrac{\mu[C^m(2k + 1,0)]}{\mu[C^m(2j + 1,1)]}$ for $0 \leq k \leq j - 2$, $j \geq 2$ or $k = j$, $j \geq 1$

(b) $\quad q_{j,k} \geq \dfrac{\mu[C^m(2j - 1,-1)]}{\mu[C^m(2j + 1,1)]}$ for $k = j - 1$, $j \geq 2$

(c) $\quad q_{j,k} = 1$ for $k > j$

Proof. Note that if $j \geq 1$, and $x \varepsilon A_j$, then setting $t = 2||x - y^*||$, we have $t > t_0$. By assumption of the lemma,

$$C^m(x,y^*,-t_0) \subseteq D_i(x)$$

So

$$0 < \mu[C^m(t,-t_0)] = \mu[C^m(x,y^*,-t_0)] \leq \mu[D_i(x)].$$

for all $1 \leq i \leq J$. But if $\mu[D_i(x)] \neq 0$ for all i, then we can write

$$p(B_k,x) = \sum_{i=1}^{k} b_i(x)p_i(B_k,x)$$

where $b_i(x) = a_i(x)\mu(D_i(x))$ and

$$p_i(B_k,x) = \frac{\mu(B_k \cap D_i(x))}{\mu(D_i(x))}.$$

Further $\sum b_i(x) = 1$, so $p(B_k,x)$ is a convex combinationn of the $p_i(B_k,x)$. Thus, defining

$$q^i_{j,k} = \inf_{y \varepsilon A_j} p_i(B_k,y),$$

when $j \geq 1$, it is sufficient to show that, for all i, the bounds in (a), (b) and (c) hold for the $q^i_{j,k}$.

Note that the only case in which we can have $j = 0$, arises in (c). This will be treated as a separate subcase. In all other cases, since we only need to show that the inequalities of (a), (b), and (c) hold for the $q^i_{j,k}$, we can drop the subscripts and superscripts on i, and assume, without loss of generality, that $p_i(B_k,x)$ is of the form

$$p_i(B_k,x) = p(B_k,x) = \frac{\mu(B_k \cap D(x))}{\mu(D(x))}$$

where

$$D(x) = \bigcup_{C \varepsilon \underline{B}} P_C(x).$$

We prove (a), (b) and (c) in turn.

(a) Let $x \varepsilon A_j$. There are 2 cases.

<u>Case I:</u> $0 \leq k \leq j - 2$, $j \geq 2$. Here,

$$B_k = C^m(x,y^*,-t_0) \subseteq D(x) \subseteq C^m(x,y^*,t_0)$$

but then

$$p(B_k,x) = \frac{\mu[D(x) \cap B_k]}{\mu[D(x)]} = \frac{\mu(B_k)}{\mu(P(x))} \geq \frac{\mu(B_k)}{\mu[C^m(x,y^*,t_0)]}$$

$$= \frac{\mu(B_k)}{\mu[C^m(2||x - y^*||,t_0)]}.$$

Since $\mu[C^m(2||x - y^*||,t_0)]$ is monotone increasing in $||x - y^*||$, it follows that this is maximized when $||x - y^*|| = jt_0 + \frac{t_0}{2}$. So

$$q_{j,k} = \inf_{x \varepsilon A_j} p(B_k,x) \geq \frac{\mu(B_k)}{\mu[C^m((2j + 1)t_0,t_0)]}$$

$$= \frac{\mu[C^m(2k+1,0)]}{\mu[C^m(2j+1,1)]}$$

Case II. $k = j,\ j \geq 2.$

Here, the worst case occurs when x is on the outside boundary of A_j, so $D(x) - B_k$ is as large as possible. Since $D(x)$ must be starlike from x, this worst case occurs when $||x - y^*|| = (j + \frac{1}{2})t_0$ and $D(x) = C^m(x,y^*,t_0)$. But then

$$p(B_k,x) = \frac{\mu[D(x) \cap B_k]}{\mu[D(x)]} \geq \frac{\mu[C^m(x,y^*,t_0) \cap B_k]}{\mu[C^m(x,y^*,t_0)]}$$

$$= \frac{\mu[B_k]}{\mu[C^m((2j+1)t_0,t_0)]} = \frac{\mu[C^m(2k+1,0)]}{\mu[C^m(2j+1,1)]}$$

Since the above holds for all $x \in A_j$, the result follows.

(b) $k = j - 1,\ j \geq 2.$

Let $x \in A_j$. Also pick $x^1 \in \bar{A}_j$ and $x^2 \in \bar{A}_j$ (here $\bar{A}$ denotes the closure of A) such that

$$||x^1 - y^*|| = (j - \tfrac{1}{2})t_0$$

$$||x^2 - y^*|| = (j + \tfrac{1}{2})t_0$$

and such that for some $r^1, r^2 \in \mathbb{R}^+$,

$$x^1 = y^* + r^1(x - y^*)$$

$$x^2 = y^* + r^2(x - y^*)$$

It follows that

$$C^m(x^1,y^*,-t_0) \subseteq C^m(x,y^*,-t_0) \subseteq D(x) \subseteq C^m(x,y^*,t_0) \subseteq C^m(x^2,y^*,t_0)$$

Also

$$C^m(x^1,y^*,-t_0) \subseteq B_k$$

Hence

$$p(B_k,x) = \frac{\mu[D(x) \cap B_k]}{\mu[D(x)]} \geq \frac{\mu[C^m(x^1,y^*,-t_0) \cap B_k]}{\mu[C^m(x^2,y^*,t_0)]}$$

$$= \frac{\mu[C^m((2j-1)t_0,t_0)]}{\mu[C^m((2j+1)t_0,t_0)]} = \frac{\mu[C^m(2j-1,1)]}{\mu[C^m(2j+1,1)]}$$

So the result follows.

(c) $k > j$. Here there are two subcases.

Case I. $j \neq 0$.

Pick $x \in A_j$. Then

$$D(x) \subseteq C^m(x,y^*,t_0) \subseteq B_{j+1}$$

So

$$D(x) \subseteq B_k$$

But then

$$p(B_k,x) = \frac{\mu[B_k \cap D(x)]}{\mu[D(x)]} = \frac{\mu[D(x)]}{\mu[D(x)]} = 1$$

Since this is true for all x, the result follows.

Case II. $j = 0$.

Here we cannot write $p(B_k,x)$ as a convex combination of the $p_i(B_k,x)$. We have

$$p(B_k,x) = \sum_{i=1} a_i(x)\mu(D_i(x) \cap B_k)$$

But now, for all $x \in A_0 = B_0$, we have, by the assumptions of the Lemma,

$$D_i(x) \subseteq C^m(x,y^*,t_0) \subseteq B_1 \subseteq B_k$$

So

$$p(B_k,x) = \sum_{i=1} a_i(x)\mu(D_1(x)) = p(\mathbb{R}^n,x) = 1$$

Q.E.D.

Lemma A2. Let Assumptions 1 and 2 be met. Let

$B_0 = \{x \in \mathbb{R}^m \mid 2||x - y^*|| \leq t_0\}$ for $y^* \in \mathbb{R}^m$ and $t_0 \in \mathbb{R}$ and assume $x_i \in B_0$ for all $i \in N$. Then for all $x \in \mathbb{R}^m$ and for all $1 \leq j \leq J$,

$$C^m(x,y^*,-t_0) \subseteq D_j(x) \subseteq C^m(x,y^*,t_0)$$

Proof: Pick $x \in \mathbb{R}^m$. Without loss of generality, we can assume the coordinate system is such that $x = (0,\ldots,0)$, $y^* = (\frac{||x - y^*||}{2},0,\ldots,0)$. Now, by assumption 2, for each $i \in N$,

$$P_i(x) = C^m(x,x_i,t_i)$$

where $t_i = 2||x - x_i||$. Using the fact that $x_i \in B_0$, it can be verified that

$$C^m(x,y^*,-t_0) \subseteq C^m(x,x_i,t_i) \subseteq C^m(x,y^*,t_0).$$

I.e. for all $i \in N$

$$C^m(x,y^*,-t_0) \subseteq P_i(x) \subseteq C^m(x,y^*,t_0).$$

But then, since for any $C \subseteq N$, $P_C(x) = \bigcap_{i \in C} P_i(x)$, it follows that, for any C,

$$C^m(x,y^*,-t_0) \subseteq P_C(x) \subseteq C^m(x,y^*,-t_0)$$

Finally, since $D_j(x) = \bigcup_{C \in \underline{B}_i} P_C(x)$

$$C^m(x,y^*,-t_0) \subseteq D_j(x) \subseteq C^m(x,y^*,-t_0)$$

Q.E.D.

Lemma A3. Let Assumptions 1 - 3 be met. Let $\overline{B}_0 = \{x \in \mathbb{R}^m \mid 2||x - \overline{y}|| \leq \overline{t}\}$, for $\overline{y} \in \mathbb{R}^m$, and $\overline{t} \in \mathbb{R}$. Assume that for every hyperplane $H(\alpha,c)$, with $\alpha \in \mathbb{R}^m$, $c \in \mathbb{R}$, if $H(\alpha,c)$ is a median hyperplane, then $H(\alpha,c) \cap \overline{B}_0 \neq \phi$. Then

$$C^m(x,\overline{y},-\overline{t}) \subseteq P(x) \subseteq C^m(x,\overline{y},\overline{t}).$$

Proof: Pick $x \in \mathbb{R}^m$. Let $t = ||x - \overline{y}||$. We choose coordinates so that B_0 is centered at $(\frac{t}{2},0,\dots,0)$, (which translates to $(\frac{\pi}{2},0,\dots,0,\frac{t}{2})$ in spherical coordinates), and so that x is at the origin. Now θ_1 (or actually $\frac{\pi}{2} - \theta_1$) measures the angle an arbitrary point $y = (\theta_1,\dots,\theta_{n-1},r)$ makes with the axis between the origin and the center of B_0. If we consider all points on the ray from the origin thru y, we can easily characterize those points which can be in P(x). As illustrated in Figure A1, any median hyperplane in the direction y must be between the two extreme possibilities illustrated by H_1 and H_2, which are tangent to B_0.

For example, in the illustration, H is a possible median. But the set of points on the ray, L, which beat x are the points up to but not including the point on L which is twice the distance from x to the median hyperplane. In fact this point, indicated by the point b in the diagram, divides the set of points on L which beat x from those which are beaten by x. We want to find limits on where this dividing

point occurs. This will clearly be the segment between $2b_1$ and $2b_2$. To obtain formulas for these points, note that the distance to c is given by $\frac{t}{2} \sin \theta_1$, so we have, adding the radius of B_0, that $b_2 = \frac{t}{2} \sin \theta_1 + \frac{t_0}{2}$, or

$$2b_2 = t \sin \theta_1 + t_0 \tag{A2}$$

Similarly

$$2b_1 = t \sin \theta_1 - t_0 \tag{A3}$$

Now $\rho(\theta_1) = t \sin \theta_1 + t_0$ is simply the formula for an n dimensional cardioid with cusp at the orgin with underlying diameter of t and eccentricity of t_0. It follows, from the above argument that in any other direction, at angle θ_1 from the origin, the same reasoning applies, so that the set of points that is majority preferred to x will be some set whose boundary lies between the inner and outer cardioids defined by equations (A2) and (A3), and whose boundary intersects each half ray through the origin at most once. Figure A2 illustates a possible set in the two dimensional case. Clearly the same exact argument applies in the m dimensional case. Further, it follows that the outer cardioid is an ''upper bound'' on how big this set can be, while the inner cardioid is a ''lower bound''. The volumes of these sets provide upper and lower bounds for the volume of the set P(x). Summarizing we have

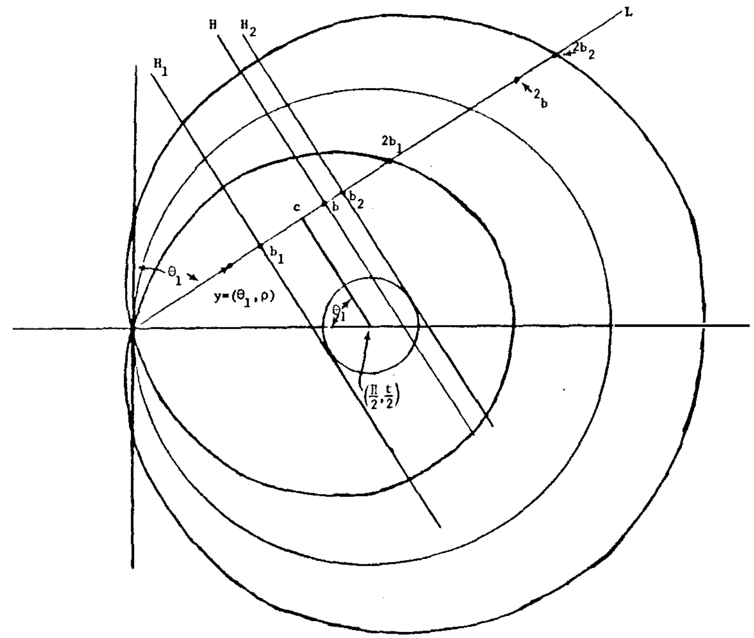


Figure A1

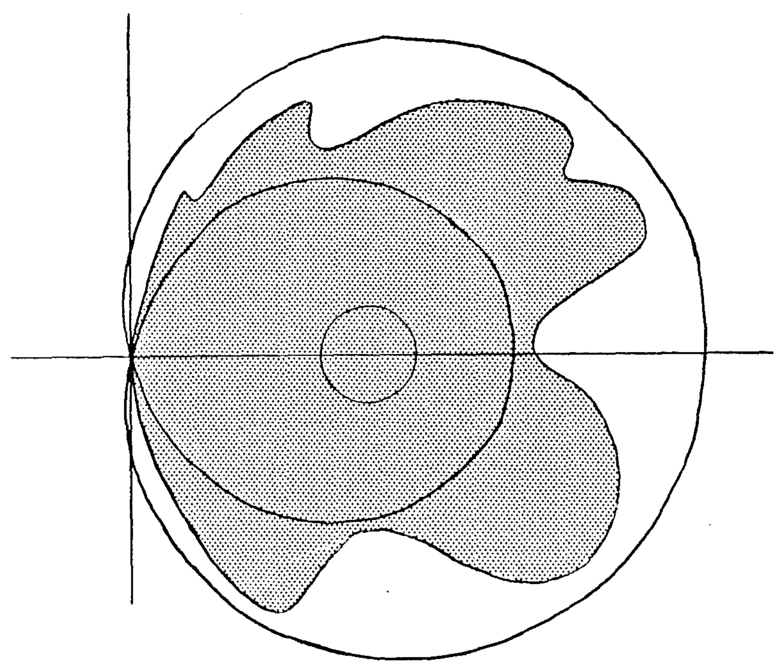


Figure A2

$$C^m(x,\overline{y},-\overline{t}) \subseteq P(x) \subseteq C^m(x,\overline{y},\overline{t}),$$

which is what we wanted to show.

Q.E.D.

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