Mon. Not. R. astr. Soc. (1983) **203**, 945–954

Limits from the timing of pulsars on the cosmic gravitational wave background

B. Bertotti *Istituto di Fisica Teorica, Universita degli Studi di Pavia, Via Bassi, 4, 27100 Pavia, Italy and Institute of Astronomy, Cambridge*

B. J. Carr and M. J. Rees *Institute of Astronomy, Madingley Road, Cambridge CB3 0HA*

Received 1982 August 27

Summary. We calculate how a stochastic background of gravitational waves might contribute to the 'timing noise' of pulsars. The published timing data, extending over a timespan $T \lesssim 10$ yr, already provide a limit $\Omega_g \lesssim 10^{-3}$ on the fraction of the critical cosmological density contributed by waves with periods of a few years; less sensitive limits can also be set for waves of shorter periods. Further analysis of existing data could tighten these limits; they will also improve as the time-base T lengthens. A genuine contribution from background gravitational waves could be distinguished from irregularities intrinsic to the pulsars by searching for correlations between the 'timing noise' of different pulsars. The timing data set poor limits to backgrounds with periods $\gg T$, because such waves would merely give a contribution to $\dot{P}$ indistinguishable from the effect of intrinsic spin-down, which is much larger and cannot be predicted *a priori.* However, the *orbital* period of the binary pulsar provides a 'clock' whose intrinsic secular behaviour can already be *predicted,* on the basis of general relativity, with an accuracy $\sim 10^{-11}$ yr^{-1}. Searches for disparities between the predicted and measured changes in the orbit could, within a few years, probe the gravitational wave background at periods up to $\sim 10^4$ yr with a sensitivity corresponding to $\Omega_g \simeq 10^{-4}$.

1 Introduction

A background flux of long-wavelength gravitational waves would induce small fluctuations in the measured periods (P) of pulsars. The 'timing noise' – i.e. the small residuals when a polynomial fit has been made to P and its derivatives – could be intrinsic to the pulsar (Cordes & Helfand 1980), but the timing is so precise, and the 'noise' so low, that the same data can be used to place limits to non-intrinsic influences on P and $\dot{P}$ due to gravitational waves (Detweiler 1979; Mashhoon 1982). In this paper we attempt to quantify these limits. If $\rho_g(\omega)$ is defined as the spectral energy density of the waves (*cf.* Isaacson 1968), then the limits on a broad-band background become interesting if they imply $\omega\rho_g(\omega) \lesssim \rho_c$, where $\rho_c = 3H_0^2/8\pi G$ is the critical cosmological density (H_0 being Hubble's constant). Following the notation of Bertotti & Carr (1980), we define $\Omega_g(\omega) \equiv \omega \rho_g(\omega)/\rho_c$. Interesting limits can already be set: we find $\Omega_g(\omega) \lesssim 10^{-3}$ for $\omega \simeq 10^{-8}\,s^{-1}$ and there are prospects of improving and extending the limits when longer time-spans of pulsar data have been accumulated.

Searches for gravitational waves by precise Doppler tracking of spacecraft involve analogous physics (Mashhoon & Grishchuk 1980; Bertotti & Carr 1980), though the pulsar data probe a lower range of wave frequencies. The fact that pulsar observations stretch over a time-base T of $\lesssim 10$ yr renders the limits much less sensitive for $\omega/2\pi \lesssim 0.1\,\mathrm{yr}^{-1}$, because the influence of very long waves would be absorbed in the polynomial fit to P and its derivatives. Mashhoon's (1982) claim to have derived exceedingly strong upper limits, $\Omega_g(\omega) \lesssim 10^{-10}$ for $\omega \lesssim 10^{-4}\,\mathrm{yr}^{-1}$, is incorrect because he did not allow for this (see Section 3).

If a 'clock' were available whose intrinsic long-term behaviour could be predicted (rather than inferred from a fit to a stretch of timing data), one could in principle set better limits to $\rho_g(\omega)$ for $\omega \ll T^{-1}$. We suggest in Section 4 that the binary pulsar's *orbital motion* provides such a clock. It is changing on a time-scale $\sim 3 \times 10^8$ yr, consistent with the Landau–Lifshitz formula for gravitational radiation. If one *accepts* the Landau–Lifshitz formula (rather than using the data to test it), then the binary pulsar provides a clock whose intrinsic properties are well enough known to test already for $\lesssim 10^{-2}$ of the closure density of gravitational waves at all frequencies $1\,\mathrm{yr}^{-1} \gtrsim \omega \gtrsim 10^{-4}\,\mathrm{yr}^{-1}$; again, the sensitivity of the limit will improve considerably when the timing data have accumulated for a few more years.

2 The spectrum of timing fluctuations induced by the gravitational wave background

Following Mashhoon (1982) and Mashhoon & Grishchuk (1980), we choose a coordinate gauge in which the only metric component we need reads

$$h_{zz}(t, \mathbf{x}) = -\tfrac{1}{2}\Sigma_{\mathbf{k}} \sin^2\theta \, Re\,[H(\mathbf{k}) \exp i(\omega t + \mathbf{k}\cdot\mathbf{x})]. \tag{1}$$

Here, θ is the angle between $\mathbf{k} = \omega\hat{\mathbf{k}}$ and the pulsar, which is placed on the z-axis at distance L. $H = H_+ + H_\times$ is the sum of the complex amplitudes of the two polarization modes; the experiment does not distinguish between them. The factor $\sin^2\theta$ arises from the transverse character of the waves. The energy flux in each mode is

$$\mathscr{F}(\mathbf{k}) = \frac{\omega^2}{64}\,[|H_+|^2 + |H_\times|^2] \tag{2}$$

(taking units with $G = c = 1$); the spectral energy density is defined by

$$\rho_g(\omega)\,d\omega = \sum_{\omega < k < \omega + d\omega} \mathscr{F}(\mathbf{k}) = \Omega_g(\omega)\rho_c\,d\omega. \tag{3}$$

Integrating $-\tfrac{1}{2}\partial h_{zz}/\partial t$ along the electromagnetic path, one readily obtains the redshift

$$z_g(t) = \tfrac{1}{2}\Sigma_{\mathbf{k}}\; Re\left\{H(\mathbf{k})\sin^2\theta\left[\frac{1-\exp(-i\omega u L)}{u}\right]\exp(i\omega t)\right\}, \tag{4}$$

where $u \equiv 1 + \cos\theta$.

The gravitational background has a random phase distribution; we also assume an isotropic spectrum. The spectrum of z_g is the Fourier transform of the correlation function:

$$\langle z_g(t) z_g(t+\tau)\rangle = 2\int_0^\infty d\omega \cos\omega\tau\, S_{z_g}(\omega). \tag{5}$$

One obtains

$$S_{z_g}(\omega) = \frac{8\pi}{3}\frac{\rho_g(\omega)}{\omega^2} B(\omega L), \tag{6}$$

where the distance-dependence is contained in the function

$$B(x) = \frac{3}{4}\int_0^\pi d\theta \frac{\sin^5\theta}{u^2}\sin^2\left(\frac{xu}{2}\right) = 1 - \frac{3}{4}\left(\frac{2x - \sin 2x}{x^3}\right) \tag{7}$$

[proportional to Mashhoon's $R(x)$]. For $x \ll 1, B(x) = O(x^2)$; for $x \gg 1, B(x) \simeq 1$.

The isotropic background we are looking for is determined statistically by its energy spectrum $\rho_g(\omega)$, the phases and polarization of each mode being random. Consequently the theory enables us to calculate only the expectation value of any physical quantity over the random phase ensemble. Mashhoon (1982) used this averaging procedure to evaluate mean-square residuals, and compared these with the actual time-averaged square residual over an observational time T of only a few years. The ergodic theorem, however, ensures equality of the ensemble and time averages only if T exceeds the characteristic time-scale of the phenomenon (which is $\sim L/c$ because the convergence of the spectrum (6) is ensured by $\omega L \ll 1$). We prefer to proceed by deducing upper limits to $\rho_g(\omega)$ for each accessible frequency (i.e. $\omega \gtrsim T^{-1}$) and only afterwards draw conclusions about the total energy density Ω_g.

A few remarks are in order about the distance-dependence. On intuitive grounds one would expect the integral along the ray path that yields the frequency shift to be (for $\omega L \gg 1$) the sum of many contributions, corresponding to gravitational wave packets crossing the ray; this would lead to a spectrum proportional to L. Equation (6) shows, contrariwise, that there is no such secular effect ($B \to 1$ when $L \to \infty$). This is a consequence of the transverse character of the gravitational waves. A secular term can arise only from those waves with $\omega u L \ll 1$, from (4), which requires $|\theta - \pi| < (\omega L)^{1/2}$. On the other hand, these waves have a negligible effect because of the transversality factor $\sin^2\theta$ appearing in equation (4). Were purely longitudinal modes present, the function $B(x)$ would be proportional to L, as naively expected. Secular terms in the electromagnetic effects of gravitational waves are also absent, for essentially the same reason, in the scintillation of a distant source (Zipoy & Bertotti 1968; Bertotti & Trevese 1972).

One wonders whether the refractive index due to interstellar plasma, tampering with the resonance between electromagnetic and gravitational waves, might perhaps restore the secular behaviour. This is indeed the case, under some conditions, for scintillation (Bertotti & Catenacci 1975). To estimate the magnitude of this effect, note that the only change induced by a propagation velocity different from c for the radio waves is that the quantity u becomes $u = 1 + (1 + \delta v)\cos\theta$. In equation (7) the resonance corresponds to the interval in u for which $\omega L u \lesssim O(1)$ and hence the function $u^{-2}\sin^2 \omega L u/2$ is $\sim \omega^2 L^2/4$. If $\delta v < 0$, no such interval exists and there is no resonance; if $\delta v > 0$, we need $\omega L \delta v \gg 1$ so that there is a resonance peak at $\theta = \pi - (2\delta v)^{1/2}$ with width $\Delta\theta \simeq (2\delta v)^{1/2}/\omega L$. This interval contributes to B (equation 7) a secular term of order

$$\Delta\theta(\delta v)^{5/2}\omega^2 L^2 = (\delta v)^3 \omega L, \tag{8}$$

interesting only when $\omega L > (\delta v)^{-3}$. For an electrostatic plasma, $\delta v = \omega_p^2/2\omega_{em}^2$ has the right sign but its value in interstellar space is far too small. (We expect a plasma frequency ω_p of only $\sim 10^3$ Hz, far below the electromagnetic frequency ω_{em}.)

3 Comparison with the timing noise of pulsars

The observed pulsar phase $\phi(t)$ is affected by measurement noise $\phi_M(t)$ which can be assumed to have a white spectrum. Its average over a time Δt has a rms value proportional to $(\Delta t)^{-1/2}$:

$$\frac{1}{\Delta t}\left\langle\left[\int_t^{t+\Delta t} dt'\,\phi_M(t')\right]^2\right\rangle = T_M = \text{constant}. \tag{9}$$

Observations suggest that this is the dominant noise for the binary pulsar up to $t \simeq 3.8 \times 10^4$ s (Taylor & Weisberg 1982). The spectrum of ϕ_M is simply T_M; if P is the pulsar period, then

$$z(t) = \frac{P}{2\pi}\frac{d\phi}{dt} \tag{10}$$

and the corresponding spectrum for the frequency residuals is

$$S_{ZM}(\omega) = \left(\frac{P}{2\pi}\right)^2 \omega^2 T_M. \tag{11}$$

This simulates a gravitational spectral density $\propto \omega^4$ (*cf.* equation 6).

In all other pulsars the phase residuals $\phi_R(t)$ are apparently dominated by the effects of intrinsic fluctuations in the rotation rate. It is found that $\phi_R(t)$ is not a stationary stochastic variable, but it can be modelled as a random walk in the phase (PN), the frequency (FN), or the time derivative of the frequency (SN) (Helfand *et al.* 1980; Cordes 1980). For these three cases, the quantity

$$\sigma^2(T) \equiv \frac{1}{T}\int_0^T dt\,\phi_R(t)^2 \tag{12}$$

grows with integration time (on the average) like T, T^3 and T^5 respectively:

$$\langle\sigma^2(T)\rangle = \begin{cases} \frac{1}{2} S_{PN} T & \text{(PN)} \\ \frac{1}{12} S_{FN} T^3 & \text{(FN)} \\ \frac{1}{120} S_{SN} T^5 & \text{(SN)} \end{cases} \tag{13}$$

(Cordes & Greenstein 1981).

Each process is entirely determined by its strength, S_{PN}, S_{FN} or S_{SN}. When the time-origin is displaced, a random walk in phase is changed by the addition of a constant, the sum of all the random phase jumps that have been produced in the added time interval. If we have a random walk in frequency, the added frequency jumps change the phase by a term linear with time; for a random walk in $\dot{P}$, a shift of time-origin adds a quadratic term to the phase. Consequently, a combination of these three processes can be defined consistently in a manner independent of the time-origin only to within an arbitrary quadratic form in time. To make the process definite one chooses, for each time interval of the data analysis, ϕ_R

to satisfy the conditions

$$0=\int_0^T dt\,\phi_R=\int_0^T dt\,t\phi_R=\int_0^T dt\,t^2\phi_R. \tag{14}$$

They are fulfilled by fitting to the phase in each interval a quadratic polynomial $a+bt+ct^2$.

The actual phase residuals $\phi_R(t)$ for a given pulsar are obtained from the observed phase by applying the appropriate astrometric correction due to the motion of the receiver and the source, and by subtracting the systematic component due to the spin-down. The polynomial fitting method described above then leads, for each interval, to three numbers a, b and c. These are each the sum of: (1) a constant average component corresponding to the initial phase, the period P and its time-derivative; and (2) a component which varies from interval to interval and should be ascribed to the residual random phase. The separation between these two parts is a delicate procedure, depending on the total length of the record and on the size and number of single data blocks that one can usefully construct. This, however, is a problem of no concern to us. It is essential to our purpose only to point out that the actual observables which we can get from the timing residuals are the quantities $\sigma(T)$ (equation 12), where the phase has been normalized to fulfil (14) so as to take away an additive constant and any linear or quadratic trend. This is, of course, equivalent to cutting off the Fourier transform of the phase at a frequency α/T (where $\alpha\simeq 1$). The actual value of the numerical constant α (or, more precisely, the shape of the cut-off) would need to be computed separately for different assumptions about the low-frequency spectrum. Such precision does not seem worthwhile at present, so we take $\alpha=2\pi$: we know that a full sine-wave cannot be fitted by a quadratic form, whereas up to a half-period of a sine-wave can.

We can now compute the average of the quantity (12) for an arbitrary gravitational wave spectrum:

$$\langle\sigma^2(T)\rangle=\left(\frac{2\pi}{P}\right)^2\frac{1}{T}\int_0^T dt\int_0^t dt_1\int_0^t dt_2\langle z_g(t_1)z_g(t_2)\rangle. \tag{15}$$

Using the definition (5), we get

$$\langle\sigma^2(T)\rangle=\frac{2}{T}\left(\frac{2\pi}{P}\right)^2\int_{\alpha/T}^{\infty} d\omega\,S_z(\omega)\int_0^T dt\int_0^t dt_1\int_0^t dt_2\cos\omega(t_1-t_2)$$

$$=\frac{32\pi}{3}\left(\frac{2\pi}{P}\right)^2\int_{\alpha/T}^{\infty} d\omega\,\frac{\rho_g(\omega)}{\omega^4}\left(1-\frac{\sin\omega T}{\omega T}\right). \tag{16}$$

Mashhoon (1982) has a similar expression, but his frequency integral starts from zero; this is the formal error which invalidates his conclusions. This also explains why $z_g(t)$, a stationary random variable, can appear as a random walk of some sort if sampled over a limited time interval much shorter than L/c. If the sampling procedure is not extended over times $\gg L/c$, and therefore does not include frequencies $\ll (L/c)^{-1}$, the convergence of the spectrum is not assured and the mean-square-value of the frequency shift diverges. The stationary character and ergodicity of $z_g(t)$ are recovered only if it is sampled over an interval $\gg L/c$ (Papoulis 1965).

It is easy to see that an appropriate power spectrum for the gravitational waves can lead to $\sigma(T)$ growing with any positive power of T. If $\rho_g(\omega)=\rho_n\omega^n$, then from equation (16) we have (writing $x=\omega T$),

$$\langle\sigma^2(T)\rangle=\frac{32\pi}{3}\left(\frac{2\pi}{P}\right)^2\rho_n T^{3-n}\int_{\alpha}^{\infty} dx\,x^{n-4}\left(1-\frac{\sin x}{x}\right). \tag{17}$$

Denoting the integral as

$$f_n(\alpha) \equiv \int_\alpha^\infty dx\, x^{n-4}\left(1-\frac{\sin x}{x}\right), \tag{18}$$

we get phase-noise for $n = 2$,

$$S_{PN} = 64\pi\left(\frac{2\pi}{P}\right)^2 \rho_2 f_2(\alpha); \tag{19}$$

frequency noise for $n = 0$,

$$S_{FN} = 128\pi\left(\frac{2\pi}{P}\right)^2 \rho_0 f_0(\alpha); \tag{20}$$

slowing-down (torque) noise for $n = -2$,

$$S_{SN} = 1280\pi\left(\frac{2\pi}{P}\right)^2 \rho_{-2} f_{-2}(\alpha). \tag{21}$$

The number $f_2(2\pi)$ is easily evaluated in terms of the sine integral and gives $64\pi f_2(2\pi)/3 \simeq 10$. However, $f_0(2\pi) \simeq 1.2 \times 10^{-4}$ and $f_{-2}(2\pi) \simeq 1.5 \times 10^{-5}$ are much smaller, making the FN and SN spectra less powerful.

We have used the data given by Cordes & Helfand (1980) to set limits to $\rho_g(\omega)$. These authors have studied 11 pulsars, determining the form of random walk and the corresponding strength for each of them (*cf.* equation 13). PSR 2217 + 47 and 1133 + 16 are particularly interesting because their PN spectra penetrate deeply into the 'closure line' defined by $\Omega_g(\omega) = 1$, yielding a limit on $\Omega_g(\omega)$ at $\omega = 2.6 \times 10^{-8}\ s^{-1}$ almost three orders of magnitude below the closure line (see Fig. 1). In the figure we also plot the limits from PSR 2016 + 28, found by Cordes & Helfand to have an FN spectrum. Cordes & Greenstein (1981) quote limits to the contributions of the other (non-dominant) forms of noise; we have checked that they do not modify the gravitational wave limits drawn in Fig. 1. With the lengthening of the time record, the limits will improve, not only because of the higher accuracy, but also because the frequency cut-off $2\pi/T$ will become lower.

For most pulsars, we have no definite information about the spectrum of the timing residuals, and their amplitude is measured by a single parameter, the 'activity' of the pulsar (Cordes & Helfand 1980). This is essentially the quantity $\sigma^2(T)$ for a time-span of length from a third to the full available record, and it provides a relevant limit to the dimensionless integral (*cf.* equation 16)

$$K \equiv \frac{1}{T^4}\int_{\alpha/T}^\infty d\omega\, \frac{\Omega_g(\omega)}{\omega^5}\left(1-\frac{\sin \omega T}{\omega T}\right). \tag{22}$$

(The upper limit is really the reciprocal of the resolution time.) The 'activity parameter' then provides an upper limit $\propto \omega^4$ to the value of $\Omega_g(\omega)$ in any waveband with $\omega \gg 1/T$:

$$\Omega_g(\omega) < K(\omega T)^4. \tag{23}$$

This is essentially the argument first given by Detweiler (1979). Values of K can be calculated from the data in table I of Cordes & Helfand (1980); the values obtained for some specific pulsars are given in Table 1 of this paper.

The time-span T is typically ~ 1000 day. Inserting the values of Table 1 into (23) seems to yield impressively low limits. However, we can only apply (23) with confidence when $\omega T \gg \alpha \simeq 2\pi$, and the sensitivity diminishes rapidly towards higher frequencies. A represen-

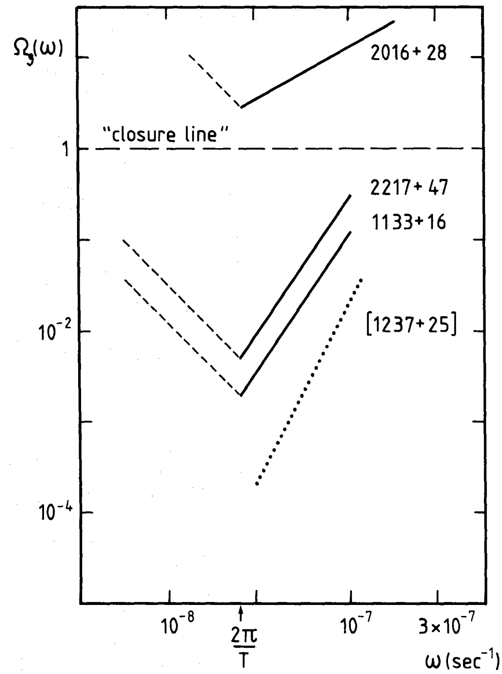

Figure 1. This diagram shows, on a logarithmic plot, the upper limits on the energy density in a stochastic gravitational wave background, in terms of the density parameter $\Omega_g(\omega)$: this is defined as the fraction of the critical density in gravitational waves in the waveband ω to 2ω. The limits are drawn for $h = 0.5$ (i.e. a Hubble constant of $50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$) and are proportional to h^{-2}. The two pulsars 1133+16 and 2217+47 display 'phase noise' (PN); 2016+28 displays 'frequency noise' (FN). The limits are strongest for frequencies of 2π/(length of available record) $= 2\pi/T \simeq 2.6\times10^{-8}\,\mathrm{s^{-1}}$. We also plot as a dotted line the limit ($\Omega_g \propto \omega^4$) that can be set from the 'activity parameter' of a typical quiet pulsar, 1237 + 25; this limit is, however, less reliable than the others except when $\omega \gg 2\pi/T$ (see text). The limits can be extended to $\omega < 2\pi/T$ (i.e. to wave periods exceeding the time-span of the data) because intense slow waves would contribute a large second derivative to the observed period. These limits, shown by the dashed lines, weaken as ω^{-2} at lower frequencies. Potentially much better limits down to $\omega \simeq 10^{-4}\,\mathrm{yr^{-1}}$ come from a different argument involving the *orbital* behaviour of the binary pulsar, discussed in Section 4 of the text.

tative limit derived from the 'activity parameter' of 1237 + 25 is plotted in Fig. 1. These limits are comparable to those for the more carefully-studied pulsars for periods ~ 1 yr; but the limits on $\Omega_g(\omega)$ worsen more steeply at higher frequency. This is because the activity parameter does not tell us how $\sigma(T)$ diminishes as we consider shorter spans of data, whereas for PN we get a $T^{1/2}$ dependence.

Table 1. Limits to K (equation 22) from the 'activity parameter' of selected pulsars.

Pulsar	Limit on K
0818 – 13	7.5×10^{-7}
0834 + 06	1.1×10^{-7}
1237 + 25	5.7×10^{-8}
1913 + 16	1.35×10^{-5}
1919 + 21	1.2×10^{-7}
1946 + 35	1.1×10^{-6}
2303 + 30	2.1×10^{-6}

The contribution to $\sigma(T)$ from waves with $\omega T \ll 1$ is truncated owing to the $[1-(\sin\omega T)/\omega T]$ term in the integrals (16) and (22); it is essentially absorbed in the polynomial fit (*cf.* 14). Nevertheless, one can set *some* limits to $\Omega_g(\omega)$ at low frequencies – albeit with poorer sensitivity – from the nature of the polynomial fit itself. This is because a large-amplitude contribution to $\sigma(T)$ from a wave with $\omega \ll T^{-1}$ would yield a quadratic term (i.e. a second derivative) in the timing residuals. Cordes & Helfand (1980) show that there is no evidence for a significant second derivative except in the Crab pulsar. (They do this by comparing the size of the residuals after subtracting a second-order polynomial with those obtained by a third-order fit). This limit to the quadratic term yields a limit on $\Omega_g(\omega)$ proportional to ω^{-2} (for $T^{-1} \gg \omega > L^{-1}$). By matching this limit on to the results plotted in Fig. 1, we can rule out $\Omega_g(\omega) = 1$ for periods between ~ 1 month and ~ 100 yr.

These (already significant) limits have been derived from data collected and analysed for other purposes. Substantial improvements could be made by concentrating efforts on pulsars with the 'quietest' spin-rates (e.g. the binary pulsar). Furthermore, one can distinguish between 'noise' intrinsic to the pulsar and the effects of gravitational waves by *correlating* data from different pulsars. The gravitational wave contribution to z_g arises half from the phases of the metric variations at the pulsar and half from the phases at the Earth (equation 4); the latter would yield correlations between the timing data of different pulsars. If this were found, it would be positive evidence for a stochastic gravitational wave background.

4 A limit on the background at lower frequencies from the binary pulsar

The reason why the pulsar timing residuals do not provide useful information about the background at frequencies $\ll T^{-1}$ is that they are not predictable enough clocks. As we show below, we are looking for fractional period changes of order $10^{-10}\,\mathrm{yr}^{-1}$ (the Hubble constant); for typical pulsars, $\dot{P}/P$ is $10^{-8}\,\mathrm{yr}^{-1}$ – two orders of magnitude larger – and we have no independent way of calculating what $\dot{P}$ should be. Things are better if one considers the gravitational clock provided by the binary pulsar PSR 1913 + 16. The work by Taylor and co-workers (Taylor *et al.* 1976; McCulloch, Taylor & Weisberg 1979; Taylor, Fowler & McCulloch 1979; Boriakoff *et al.* 1982; and especially Taylor & Weisberg 1982) has pinned down the parameters of the system so well that the general relativistic secular speed-up of the *orbital* period due to gravitational radiation is predicted to be $|\dot{P}_K/P_K| \simeq (3\times10^8\,\mathrm{yr})^{-1}$ with an uncertainty of only 0.2 per cent (i.e. we have a 'clock' whose behaviour is known with precision better than $10^{-11}\,\mathrm{yr}^{-1}$). The observations have a precision of $(5\times10^9\,\mathrm{yr})^{-1}$ and agree with this prediction; moreover, there will be a rapid improvement in the measurement of $|\dot{P}_K/P_K|$ as the data accumulate over a longer time-base.

To see the potentialities of these observations for probing the gravitational wave background, we must compute the expectation value of the square of the frequency change, $\Delta z_g(t) \equiv z_g(t) - z_g(0)$, which these waves would induce. Equations (5) and (6) give

$$\langle[\Delta z_g(T)]^2\rangle = 4\int_0^\infty d\omega\, S_{z_g}(\omega)\,(1-\cos\omega T) = \frac{64\pi}{3}\int_0^\infty \frac{d\omega}{\omega^2}\,\rho_g(\omega)\sin^2\frac{\omega T}{2}\,B(\omega L). \tag{24}$$

The integral is effectively cut off below $\omega \simeq L^{-1}$, unless $\rho_g(\omega)$ falls off faster than ω^{-1}, by the function $B(\omega L)$; the function $4/\omega^2 T^2 \sin^2 \omega T/2$ provides the upper cut-off at $\omega \sim T^{-1} \gg L^{-1}$, unless $\rho_g(\omega)$ rises faster than ω. Since these two functions are equal to unity far from their respective cut-offs, the integral (24) is approximated by

$$\langle[\Delta z_g(T)]^2\rangle = \frac{16\pi}{3}\,T^2\int_{L^{-1}}^{T^{-1}} d\omega\,\rho_g(\omega), \tag{25}$$

thus providing the precise upper limit $3/(16\pi T^2)\langle[\Delta z_g(T)]^2\rangle$ to the energy of the background between L^{-1} and T^{-1}. For wavelengths $>L$, equation (24) still yields a limit, but one which weakens as ω^{-2}. Therefore, assuming that relativity describes the orbital behaviour of the binary pulsar correctly, we get for this frequency interval

$$\Omega_g < \frac{1}{2}\left(\frac{\delta \dot{P}_K}{PH_0}\right)^2 = \frac{1}{2} h^{-2}\,[(\delta \dot{P}_K/P_K)/(10^{-10}\ \text{yr}^{-1})]^2, \tag{26}$$

where h is Hubble's constant H_0 in units of $100\ \text{km s}^{-1}\ \text{Mpc}^{-1}$, and $\delta\dot{P}_K$ is the part of $\dot{P}_K$ that could be due to gravitational waves (*cf.* equation 25). Contributions to $\delta\dot{P}_K$ come from (a) observational uncertainties and (b) uncertainties in the $\dot{P}_K$ given by the Landau–Lifshitz formula. The latter arise because the parameters of the system (masses, eccentricity, etc.) are imperfectly known. At present, the measurement uncertainties (a) are dominant, and (26) yields a limit no better than $\Omega_g \lesssim 2\,h^{-2}$. However, we now assess the various errors in order to show that there are excellent chances of pushing this limit down by several powers of 10.

4.1 OBSERVATIONAL ERRORS

If all observations were of similar quality, we would expect the measured uncertainty in $\dot{P}_K$ to decrease as the 5/2 – power of the time-base of the observations, since we are measuring a phase which increases quadratically with time. Since the present time-base is 6 yr, this would imply an improvement in the next 6 yr of $2^{5/2} \simeq 5.6$. Since the newer data have higher weight, the observational errors may fall still more rapidly.

4.2 THE PREDICTED $\dot{P}_K$ FROM THE LANDAU–LIFSHITZ FORMULA

Gravitational radiation causes the binary orbit to contract, and the orbital period P_K to decrease, at a rate

$$\dot{P}_K/P_K = (\text{constant}) \times P_K^{-8/3} f(e)\, m_1 m_2 (m_1+m_2)^{-1/3}, \tag{27}$$

where e is the orbital eccentricity and m_1 and m_2 are the component masses. The periastron precession, which is known with high accuracy, determines $(m_1+m_2)^{2/3}$; the main uncertainty in the mass function comes from the mass ratio, which has to be inferred from the gravitational redshift and second-order Doppler effect (Taylor & Weisberg 1982) and is 1 ± 0.04. However, equation (27) shows that $\dot{P}_K$ depends only quadratically on the error in the mass ratio when this ratio is close to unity; it is for this reason that the predicted $\dot{P}_K$ is only uncertain by 0.002. The determination of m_1/m_2 will improve as $T^{5/2}$ (since this is a secular observable, at least for time-spans smaller than the periastron precession period). The uncertainty in $\dot{P}_K$ will therefore reduce as T^{-5} (quadratically with the error in m_1/m_2) until m_1/m_2 is found to differ significantly from unity, and thereafter as $T^{-5/2}$. The uncertainty in eccentricity e will diminish only as $T^{-1/2}$; but since this is already known with 10^{-5} precision, it will be a long time before it becomes the dominant uncertainty in (27).

4.3 MOTION OF THE BINARY PULSAR SYSTEM IN THE GALAXY

Another limiting factor is the contribution to $\dot{P}_K$ as the pulsar's Doppler shift changes due to its motion in the Galaxy. The acceleration due to the gravitational field of the Galaxy contributes only $2 \times 10^{-13}\ \text{yr}^{-1}$. But if the binary pulsar system has a large peculiar velocity V, there is a changing Doppler effect as it moves (even with constant velocity) transverse to the line-of-sight. This yields $|\dot{P}_K/P_K| = (V^2/cD)\sin^2\theta$ (where D is the distance and θ the angle between the velocity and the line-of-sight). For $D = 5$ kpc, this contribution is at the level of $10^{-11}\ \mathrm{yr}^{-1}$ if V has the high value $\sim 200\ \mathrm{km\,s^{-1}}$ typical of ordinary pulsars. The proper motion $V \sin\theta/D$ may eventually be measurable. If this proves to be anomalously small, the associated $\dot{P}_K$ may be negligible. However, if the proper motion were (say) ~ 5 milliarcsec per year (corresponding to $V \simeq 200\ \mathrm{km\,s^{-1}}$, $\theta = 45°$), then, even if it were known exactly, there would still be an uncertainty $\sim 10^{-11}\ \delta D/D\ \mathrm{yr}^{-1}$ in the contribution to $\dot{P}_K$ resulting from the changing Doppler effect. It seems most unlikely that the fractional error $\delta D/D$ in the distance can be reduced below 0.1, so it may never be possible to discuss effects at the level below $10^{-12}\ \mathrm{yr}^{-1}$ (unless other binary pulsars are discovered).

In summary, the precision with which we know the orbital parameters and masses is improving so fast that we will, within a few years, be able to *calculate* $\dot{P}_K$ with a precision better than $10^{-12}\ \mathrm{yr}^{-1}$. Within 10 years the observations could also have achieved this same precision; we may then (if no discrepancy appears) be able to use (26) to set limits $\Omega_g \lesssim 10^{-4}$ to any wave background at wavelengths 10–10^4 light-years. Improvements beyond this level may be bedevilled by our ignorance of how the binary pulsar is moving through the Galaxy.

Acknowledgments

We have benefited from helpful discussions with B. Mashhoon and R. Hellings and are particularly grateful to D. Eardley for clarifying our ideas on the errors analysis of the binary pulsar. One of us (BB) acknowledges partial support from the Consiglio Nationale delle Recerche.

References

Bertotti, B. & Carr, B. J., 1980. *Astrophys. J.*, **236**, 1000.
Bertotti, B. & Catenacci, R., 1975. *Gen. Rel. Grav.*, **6**, 329.
Bertotti, B. & Trevese, D., 1972. *Nuovo Cim.*, **2B**, 223.
Boriakoff, V., Ferguson, D. C., Haugan, M. H., Terzian, Y. & Teukolsky, S., 1982. *Astrophys. J.*, **261**, L97.
Cordes, J. M., 1980. *Astrophys. J.*, **237**, 216.
Cordes, J. M. & Greenstein, G., 1981. *Astrophys. J.*, **245**, 1060.
Cordes, J. M. & Helfand, D. J., 1980. *Astrophys. J.*, **234**, 640.
Detweiler, S., 1979. *Astrophys. J.*, **234**, 1100.
Helfand, D. J., Taylor, J. H., Backus, P. R. & Cordes, J. M., 1980. *Astrophys. J.*, **237**, 206.
Isaacson, R. A., 1968. *Phys. Rev.*, **166**, 1263.
McCulloch, P. M., Taylor, J. H. & Weisberg, J. M., 1979. *Astrophys. J.*, **227**, L133.
Mashhoon, B., 1982. *Mon. Not. R. astr. Soc.*, **199**, 659.
Mashhoon, B. & Grishchuk, L. P., 1980. *Astrophys. J.*, **236**, 990.
Papoulis, A., 1965. *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, New York.
Taylor, J. H. & Weisberg, J. M., 1982. *Astrophys. J.*, **253**, 908.
Taylor, J. H., Fowler, L. A. & McCulloch, P. M., 1979. *Nature*, **277**, 437.
Taylor, J. H., Hulse, R. A., Fowler, L. A., Gullahorn, G. E. & Rankin, J. M., 1976. *Astrophys. J.*, **206**, L53.
Zipoy, D. M. & Bertotti, B., 1968. *Nuovo Cim.*, **56B**, 195.

Note added in proof

The newly-discovered pulsar in 4C21.53 (D. Backer, S. Kulkarni, C. Heiles, M. Davis & M. Goss: *IAU Circ. 3743*) may be much quieter than 1237+25. If the timing stability of 1.5 μs, reported by Backer *et al.* (*Nature*, **301**, 314) persists for several years, the sensitivity in terms of Ω_g may be better by as much as 3000 than that obtainable from 1237+25 (plotted in Fig. 1). By 1985 pulsar timing may therefore be able to detect a background of $\Omega_g \approx 10^{-7}$ with periods of order a year.