Limits on classical communication from quantum entropy power inequalities

Graeme Smith, IBM Research
(joint work with Robert Koenig)

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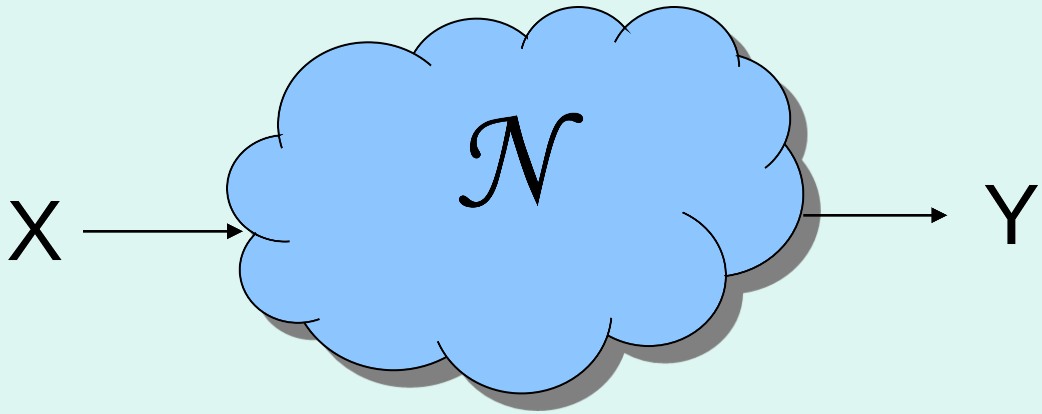
Channel Capacity



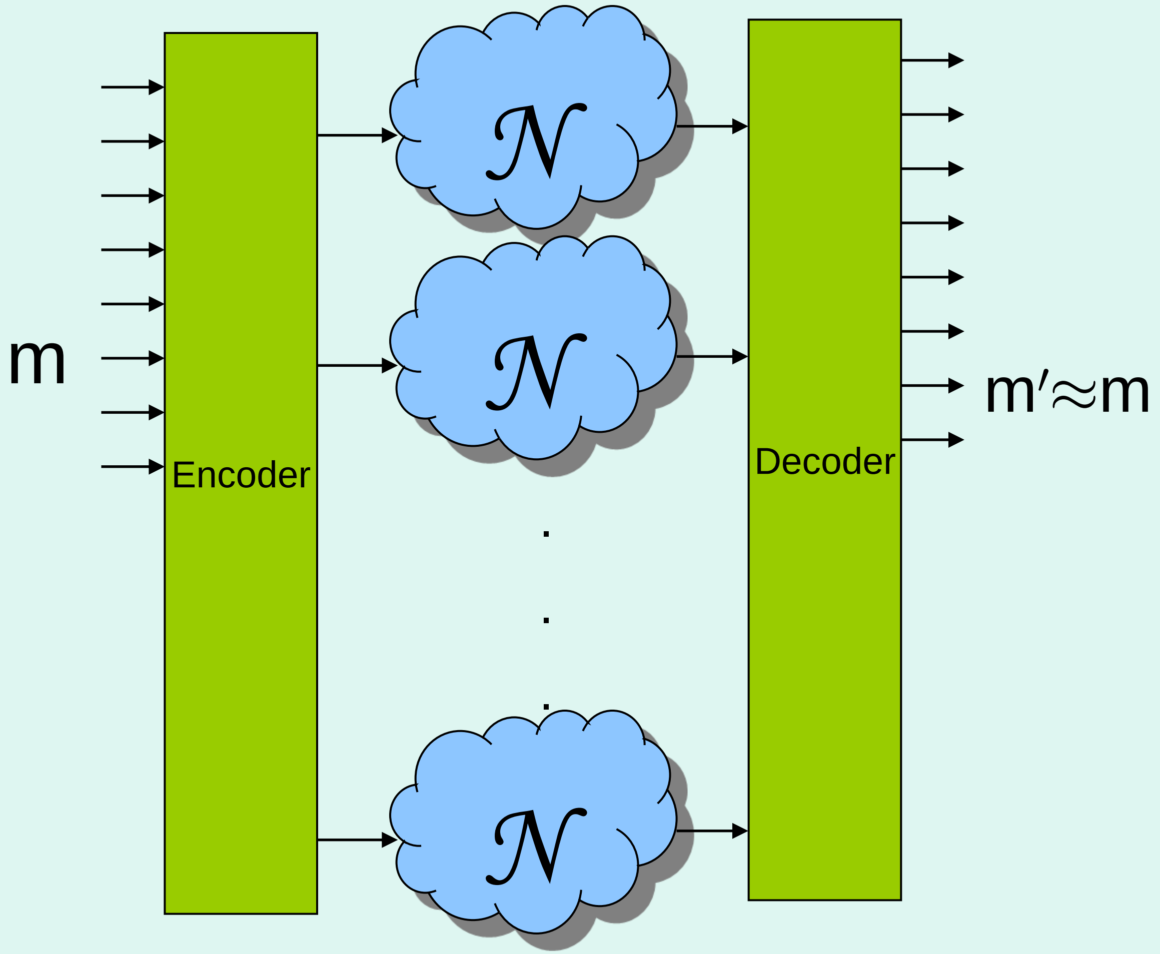
$p(y|x)$

Capacity: bits per channel use in the limit of many channels

$$C = \max_X I(X;Y)$$

$I(X;Y) = H(X)+H(Y)-H(XY)$ is the mutual information

Classical Capacity of Quantum Channel



Send a classical message over a quantum message using a code

$m \to \rho_m$

such that all ρ_m can be distinguished at the channel output.

$C(\mathcal{N})$ is the capacity

Classical Capacity of Quantum Channel

We can understand coding schemes for classical information in terms of the Holevo Information:

$$\chi(\mathcal{N}) = \max_{\{p_x,\ \rho_x\}} I(X;B) = \max_{\{p_x,\ \rho_x\}} H(\rho_{av}) - \sum_x p_x H(\rho_x)$$

where $I(X;B) = H(X) + H(B) - H(XB)$ uses von Neumann entropy and is evaluated on the state $\sum_x p_x |x\rangle\langle x| \, \mathcal{N}(\rho_x)$

Random coding arguments show that $\chi(\mathcal{N})$ is an achievable rate, so $C(\mathcal{N}) \geq \chi(\mathcal{N})$. Furthermore,

n uses

$$C(\mathcal{N}) = \lim_{n \to \infty} (1/n)\, \chi(\mathcal{N} \ldots \mathcal{N})$$

(see Holevo 98, Schumacher-Westmoreland 97)

χ isn't additive

- $C(\mathcal{N}) = \lim_{n \to \infty} (1/n)\, \chi(\mathcal{N} \ldots \mathcal{N})$
- Hastings 2009: $\exists \mathcal{N}$ with $\chi(\mathcal{N}\mathcal{N}) > 2\chi(\mathcal{N})$

Attempts at salvage / Denial:

- Surely this won't happen for "natural" channels
- Anyway, the effect is small and therefore not relevant, at least for natural channels

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Attempts at salvage / Denial:

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What is "natural"? What's "small"?

Outline

- Bosonic thermal noise channel
- Bounds on capacity (old and new)
- Entropy Power Inequalities (Quantum and Classical)
- Proof Ideas
- Outlook

Bosonic Modes

- Hilbert space spanned by $|n\rangle$, n = 0…∞
- Raising and lower operators:

$$a|n\rangle = \sqrt{n}|n{-}1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n{+}1\rangle$$

$$[a, a^\dagger] = 1$$

- Quadratures:

$$Q = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad P = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

$$[Q, P] = i$$

- Covariance matrix

$$°_{ij} = \mathrm{Tr}\,[(R_i R_j + R_j R_i)½]$$

$$\vec{R} = (P_1; Q_1; :::; P_n; Q_n)$$

Gaussian Quantum Channels

- Classical Additive White Gaussian Noise:

$$X \;!\; aX + N$$

- Quantum Generalization:

$$\gamma \rightarrow A\gamma A^T + N$$

- Generated by quadratic interactions between input signal and vacuum environment

Additive White Gaussian Noise

Input X is a real variable (eg, component of EM field)

$$X \to X + bN = Y$$

N is normally distributed with variance 1, and mean zero, so

$$\Pr(y|x) = \frac{1}{\sqrt{2\pi}\,b} e^{-(x-y)^2/2b^2}$$

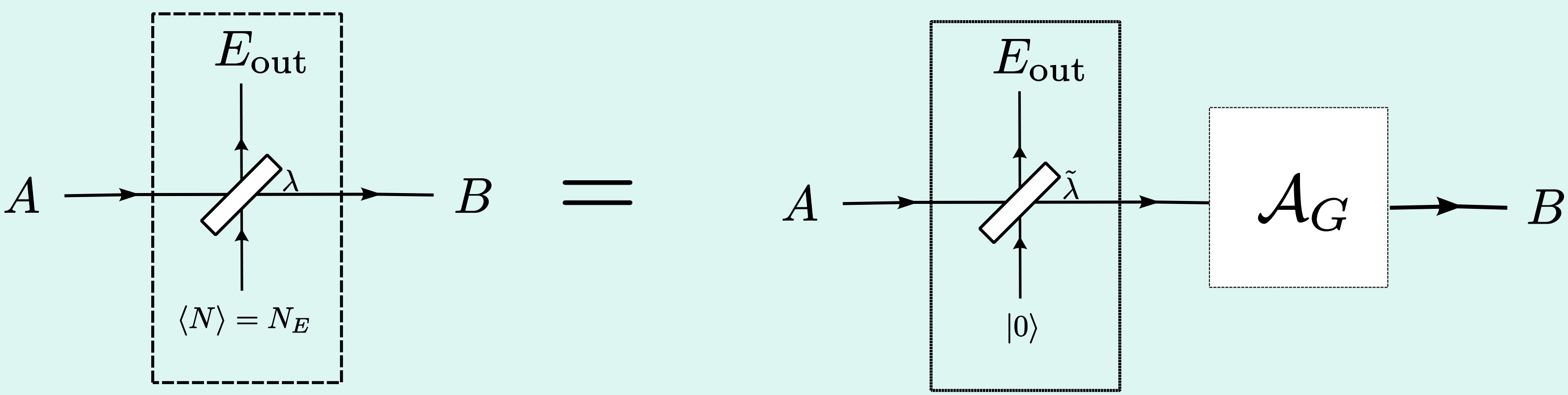
Capacity of this channel is infinite, but makes sense if we introduce a power constraint: $E[X^2] \leq P$. Then the capacity becomes

$$C = \frac{1}{2}\log(1 + SNR)$$

Where $SNR = P/b^2$ is the ratio of max signal power to noise power

Gaussian Thermal Noise Channel

- Evolution: ° ! (1 ¡ ¸)° + ¸ N_E I
- Models combination of attenuation and amplification present in optical fiber



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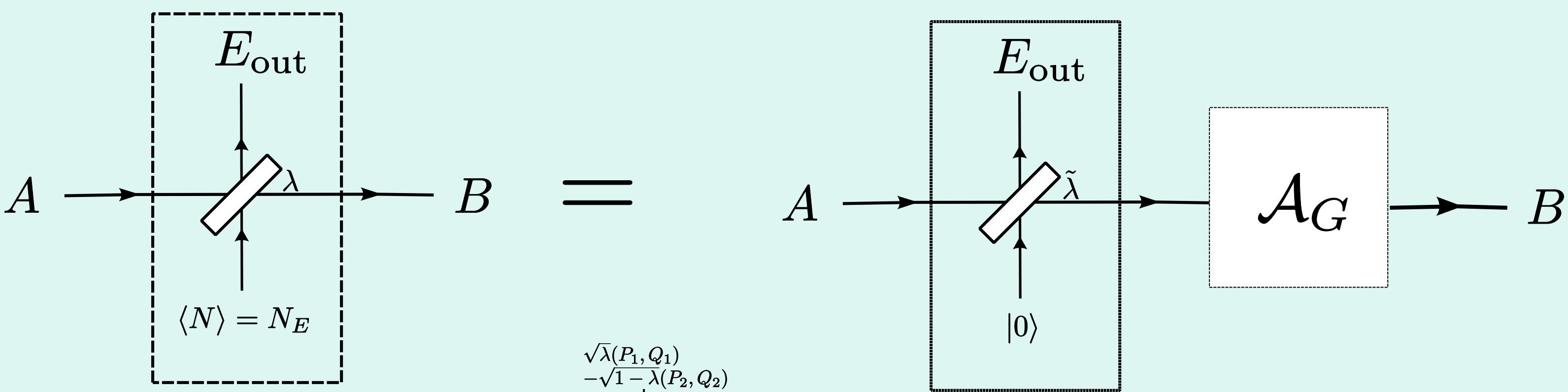

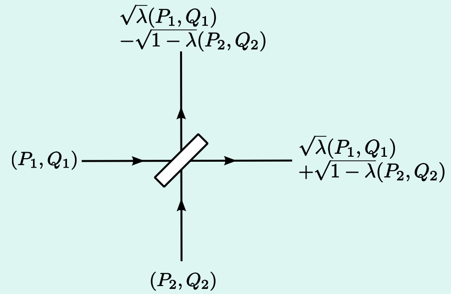

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Lower Bound

Achievable rate: $\chi(\mathcal{N}) = \max_{\{p_x, \rho_x\}} H(\mathcal{N}(\rho_{av})) - \sum_x p_x H(\mathcal{N}(\rho_x))$

To get a lower bound, just exhibit a particular ensemble.

By letting ρ_x be displaced coherent states and taking a gaussian mixture, ρ_{av} is thermal and we get

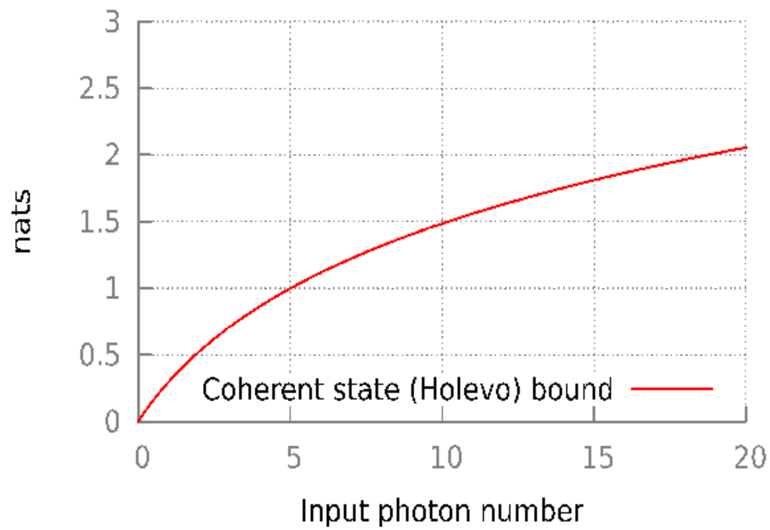
$$C(\mathcal{N}_{\lambda, N_e}; N) \geq g(\lambda N + (1-\lambda)N_E) - g((1-\lambda)N_E)$$

where $g(x) = (x+1)\log(x+1) - x\log x$ is the entropy of a thermal state with average photon number x

- Holevo 1998 (see also Gordon 1964)

Known bounds on classical capacity

transmissivity $\lambda = 1/2$, $N_E = 2$



Maximum output entropy

- $\chi(\mathcal{N}) = \max_{\{p_x, \rho_x\}} H(\mathcal{N}(\rho_{av})) - \sum_x p_x H(\mathcal{N}(\rho_x)) \leq \max_\rho H(\mathcal{N}(\rho)) = H_{max}(\mathcal{N})$
- $H_{max}(\mathcal{N}^n) = n\, H_{max}(\mathcal{N})$
- $\chi(\mathcal{N}^n) \leq n\, H_{max}(\mathcal{N})$
- $C(\mathcal{N}) = \lim_{n \to \infty} 1/n\, \chi(\mathcal{N}^n) \leq H_{max}(\mathcal{N})$

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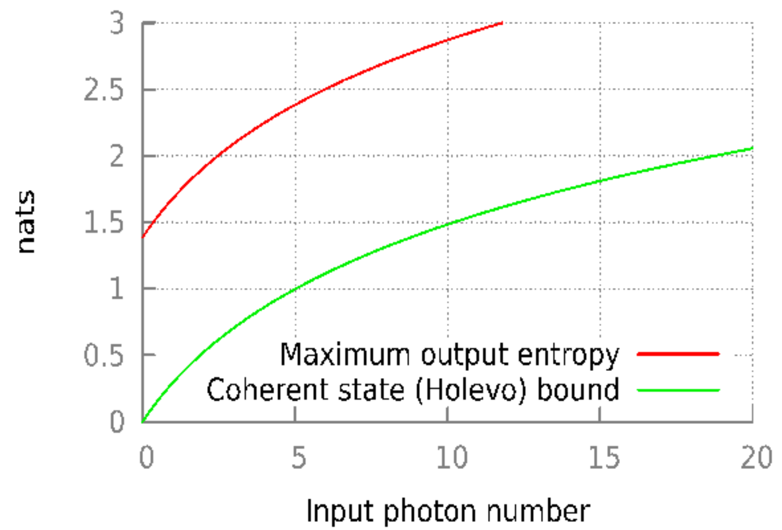
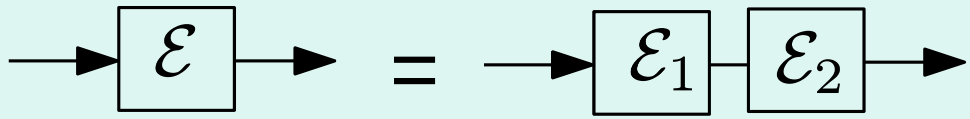


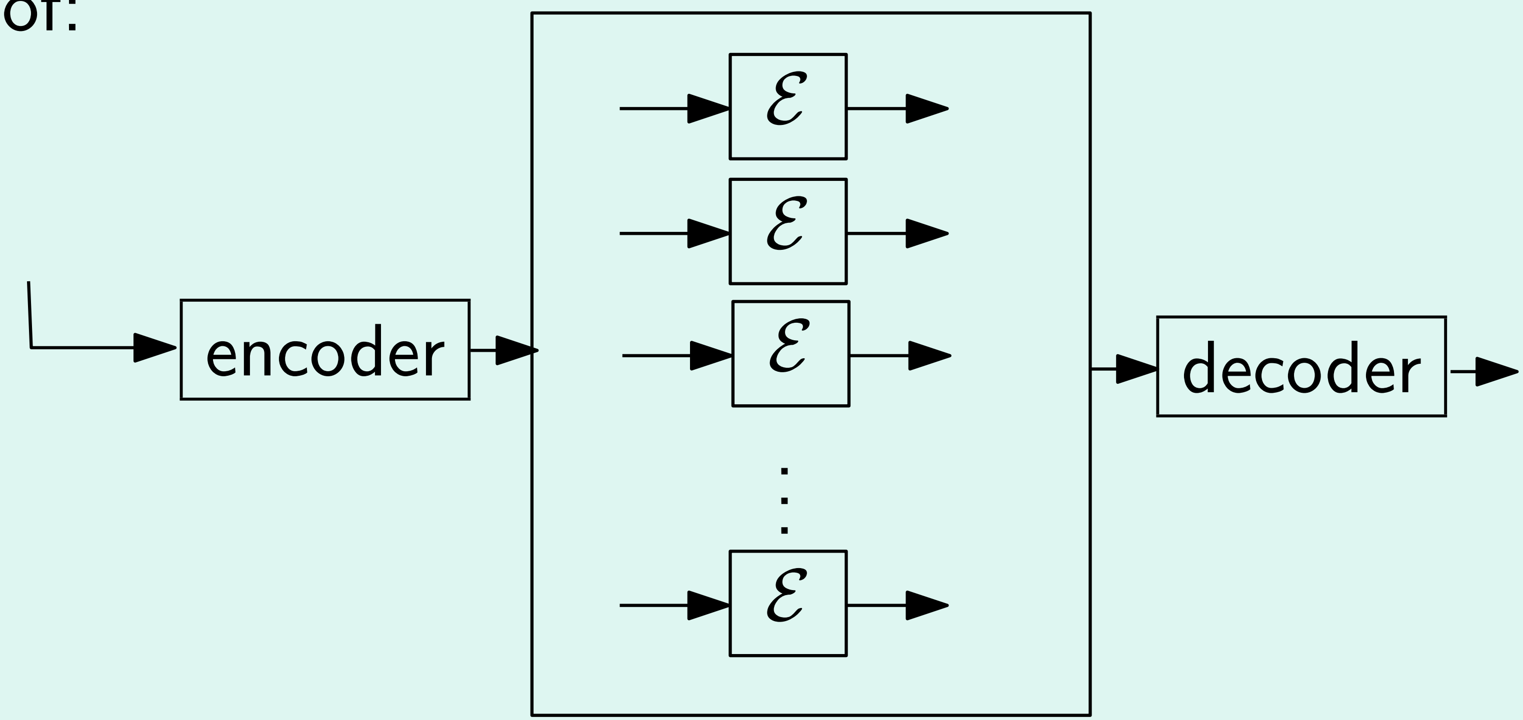
Bottleneck

Say



Then $C(E;N) \cdot C(E_1;N)$

Proof:



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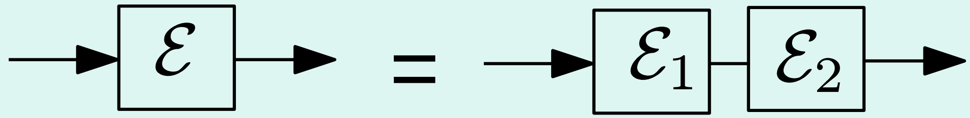

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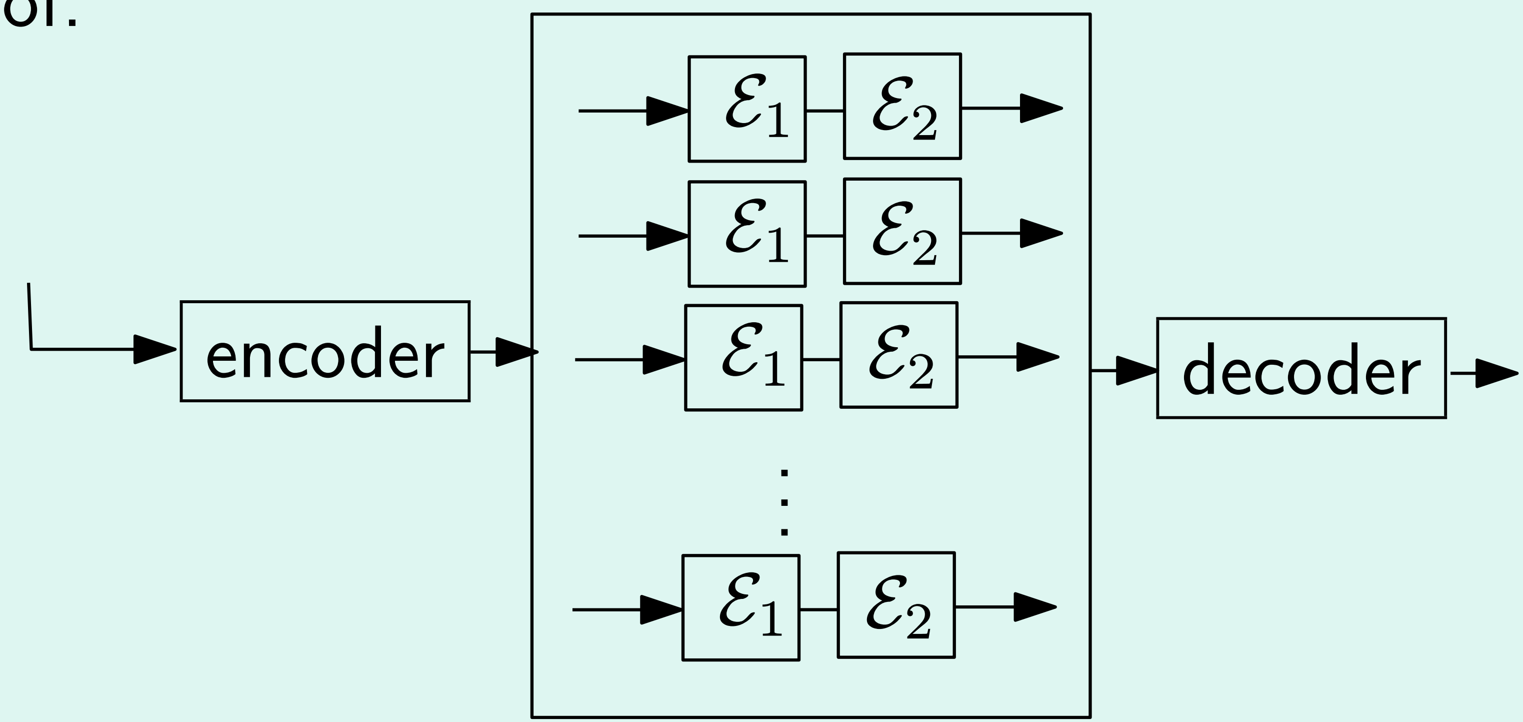

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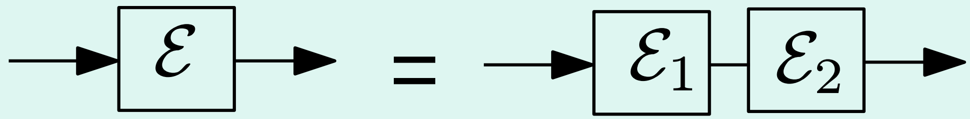

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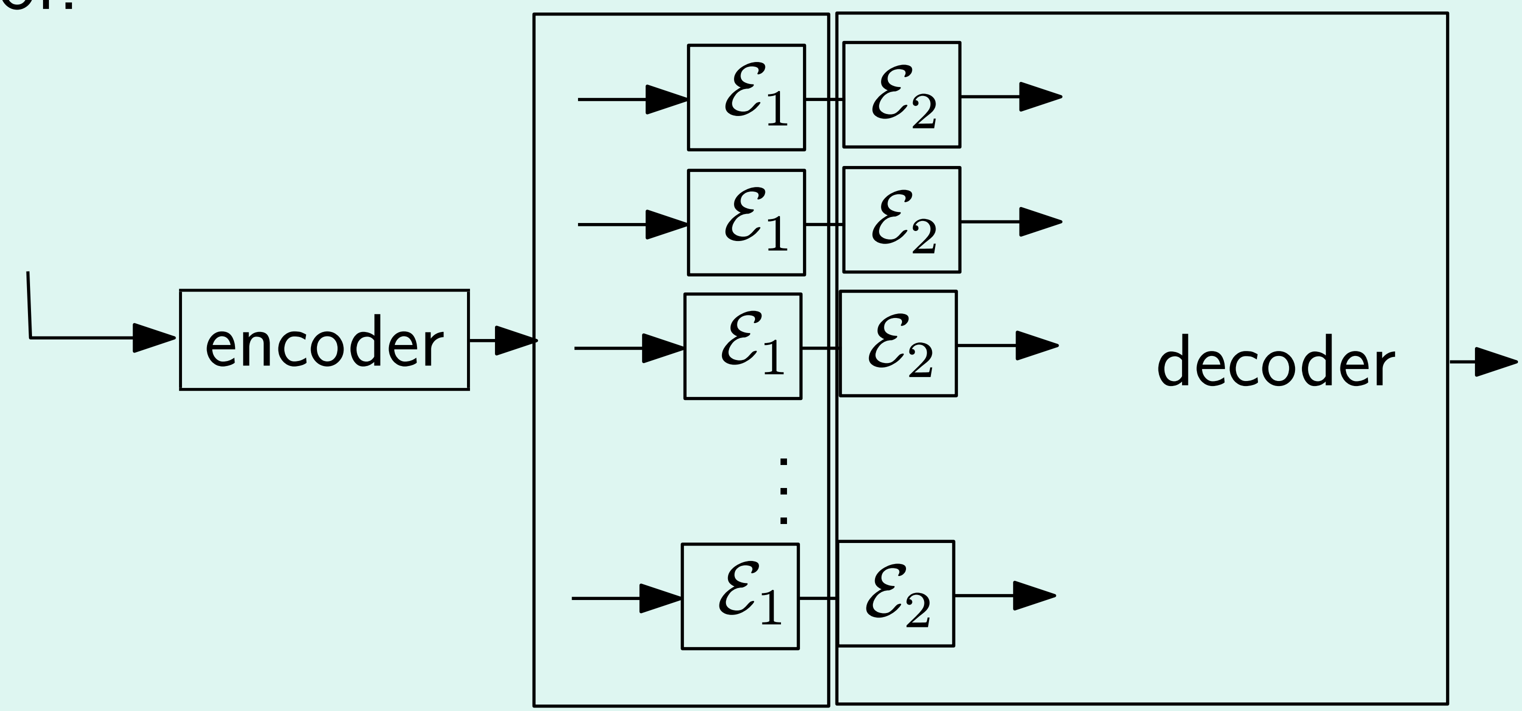

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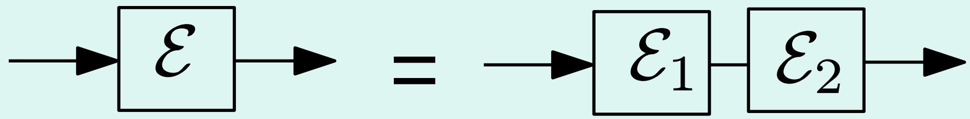
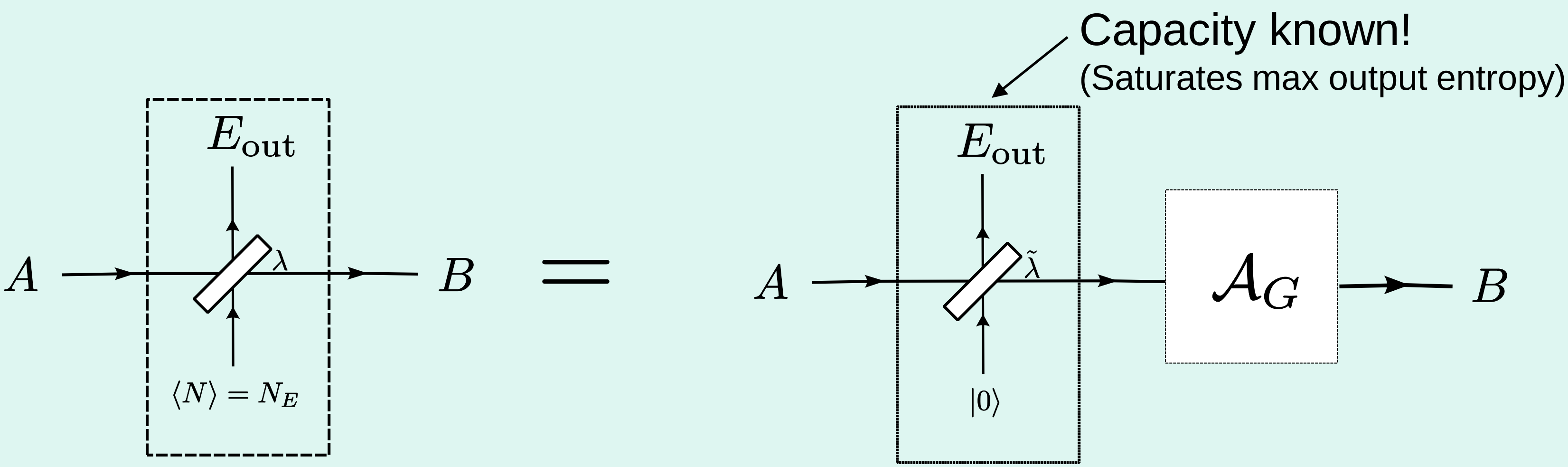
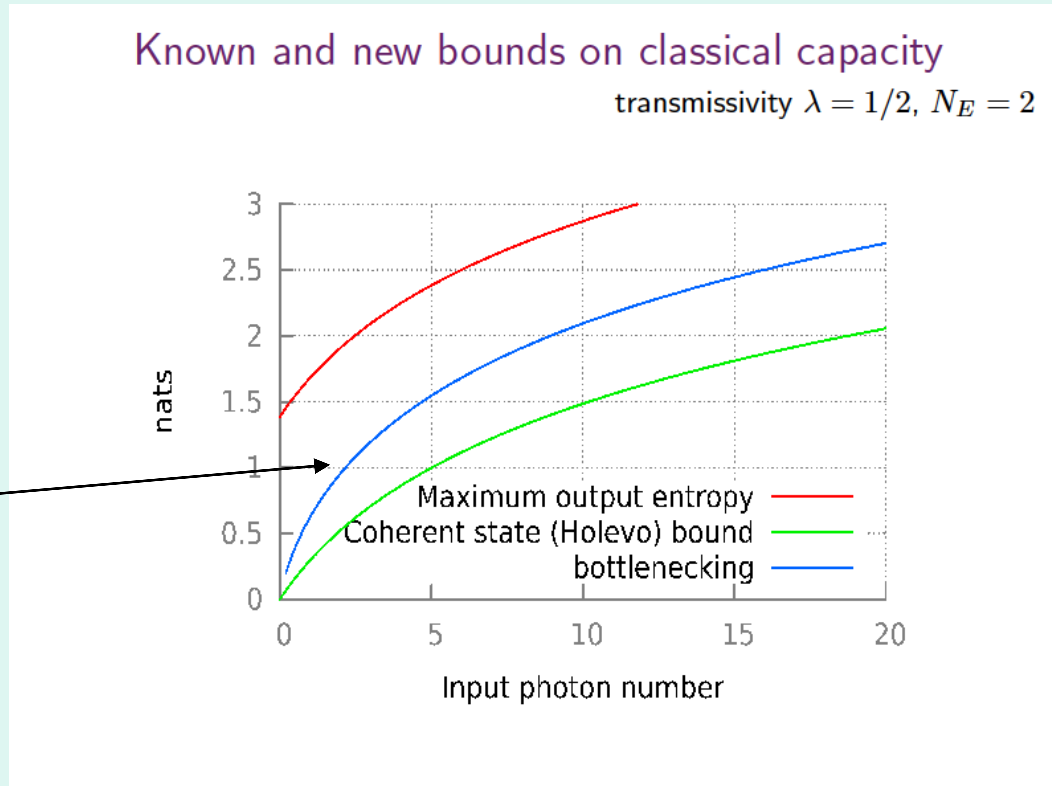


Then C(E; N) · C(E$_1$; N)

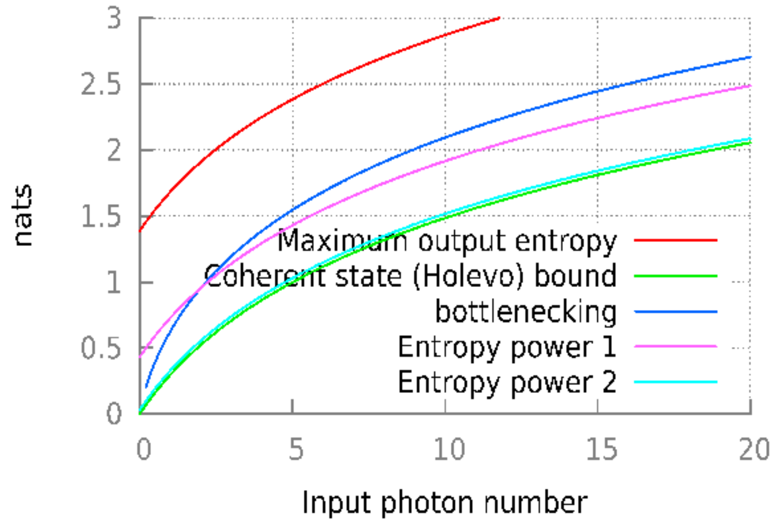




Always within
a nat

New bounds from entropy power inequalities

alternative (often better) bounds and new proof technique!



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- ~~Bosonic thermal noise channel~~
- ~~Bounds on capacity (old and new)~~
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Additive bounds on minimum output entropy and capacity

If $f(\lambda, N_E) \le \frac{1}{n} H(N_{\lambda, N_E}^{\otimes n}(\rho))$ for all ρ then

$$C(N_{\lambda, N_E}, N) \le g(\lambda N + (1 - \lambda) N_E) - f(\lambda, N_E)$$

Additive bounds on minimum output entropy and capacity

If $f(\lambda; N_E) \le \frac{1}{n} H(N_{\lambda;N_E}^{\otimes n}(\rho))$ for all ρ then

$$C(N_{\lambda;N_E}; N) \le g(\lambda N + (1-\lambda)N_E) - f(\lambda; N_E)$$

Proof:

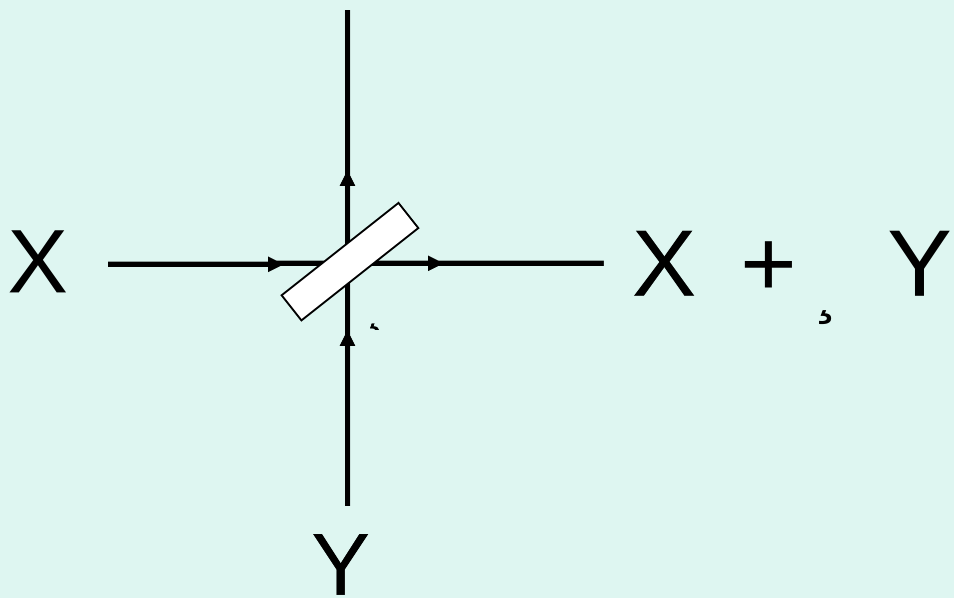
$$\chi(N^{\otimes n}; nN) = \max_{\{p_x;\rho_x\}: \langle n\rangle_\rho \le nN} H(N_{\lambda;N_E}^{\otimes n}(\rho)) - \sum_x p_x H(N_{\lambda;N_E}^{\otimes n}(\rho_x))$$

$$\le nH_{max}(N_{\lambda;N_E}) \qquad \ge H_{min}(N_{\lambda;N_E}^{\otimes n})$$

Quantum Entropy Power Inequality v1

$$\lambda H(X) + (1-\lambda)H(Y) \leq H(X +_\lambda Y)$$

for all $\rho_X \otimes \rho_Y$



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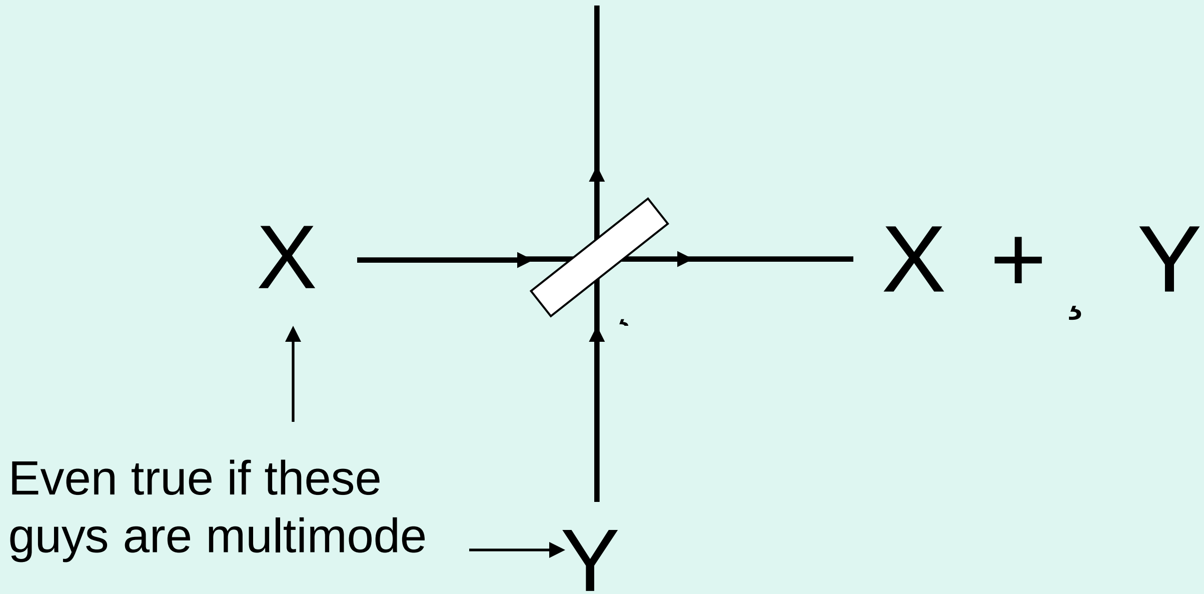
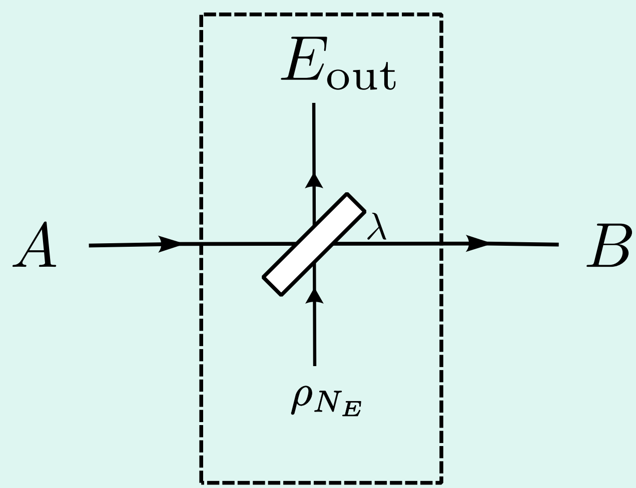


Quantum Entropy Power Inequality v1

$$\lambda H(A) + (1 - \lambda) H(E) \leq H(B)$$

for all $\rho_A \otimes \rho_E$

Single channel use:

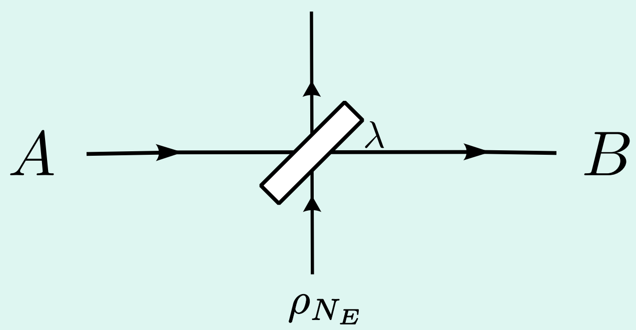


Quantum Entropy Power Inequality v1

¸ H(A) + (1 ¡ ¸)H(E) · H(B)

for all ½$_A$ - ½$_E$

Single channel use:



Quantum Entropy Power Inequality v1

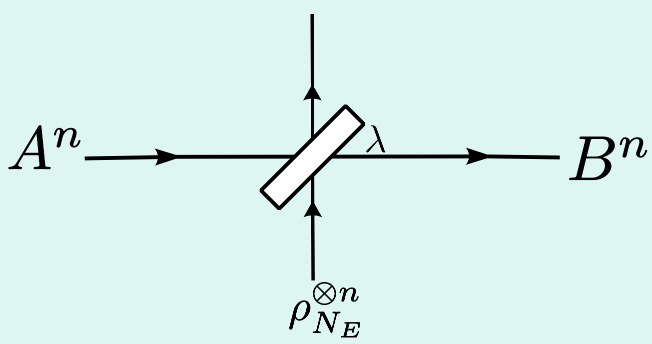
¸ H(A^n) + (1 ¡ ¸)H(E^n) · H(B^n)

zero

product

for all ½$_{A^n}$ - ½$_{E^n}$

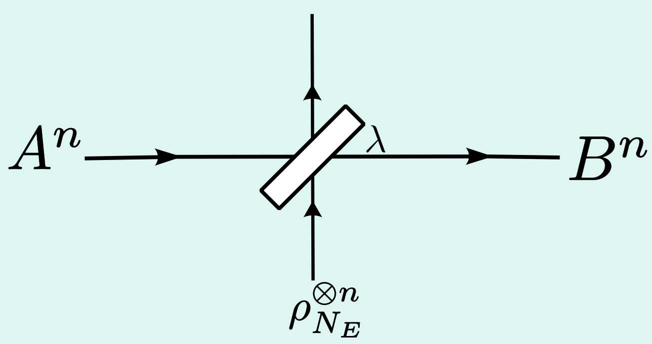
multiple channel uses:



Quantum Entropy Power Inequality v1

n(1 ¡ ¸)g(N_E) · H(B^n)

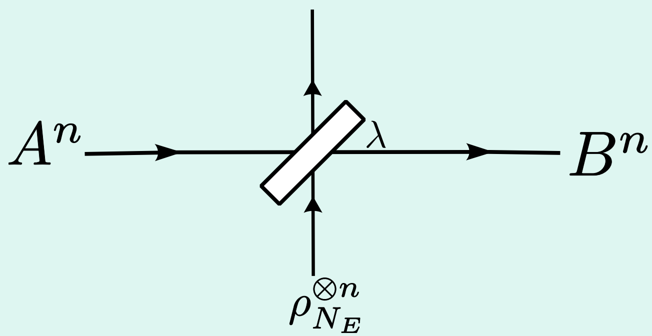
multiple channel uses:



Quantum Entropy Power Inequality v1

$$(1-\lambda)g(N_E) \le \frac{1}{n}H(B^n)$$

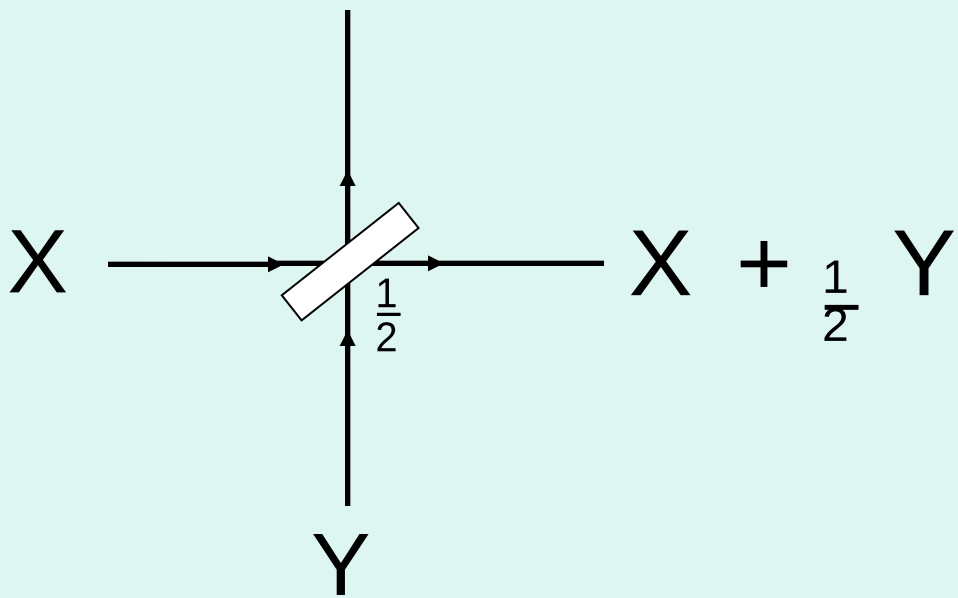
multiple channel uses:



Quantum Entropy Power Inequality v2

$$\frac{1}{2}\exp\left(\frac{1}{n}H(X)\right) + \frac{1}{2}\exp\left(\frac{1}{n}H(Y)\right) \cdot \exp\left(\frac{1}{n}H(X +_{\frac{1}{2}} Y)\right)$$

for all ½x - ½y



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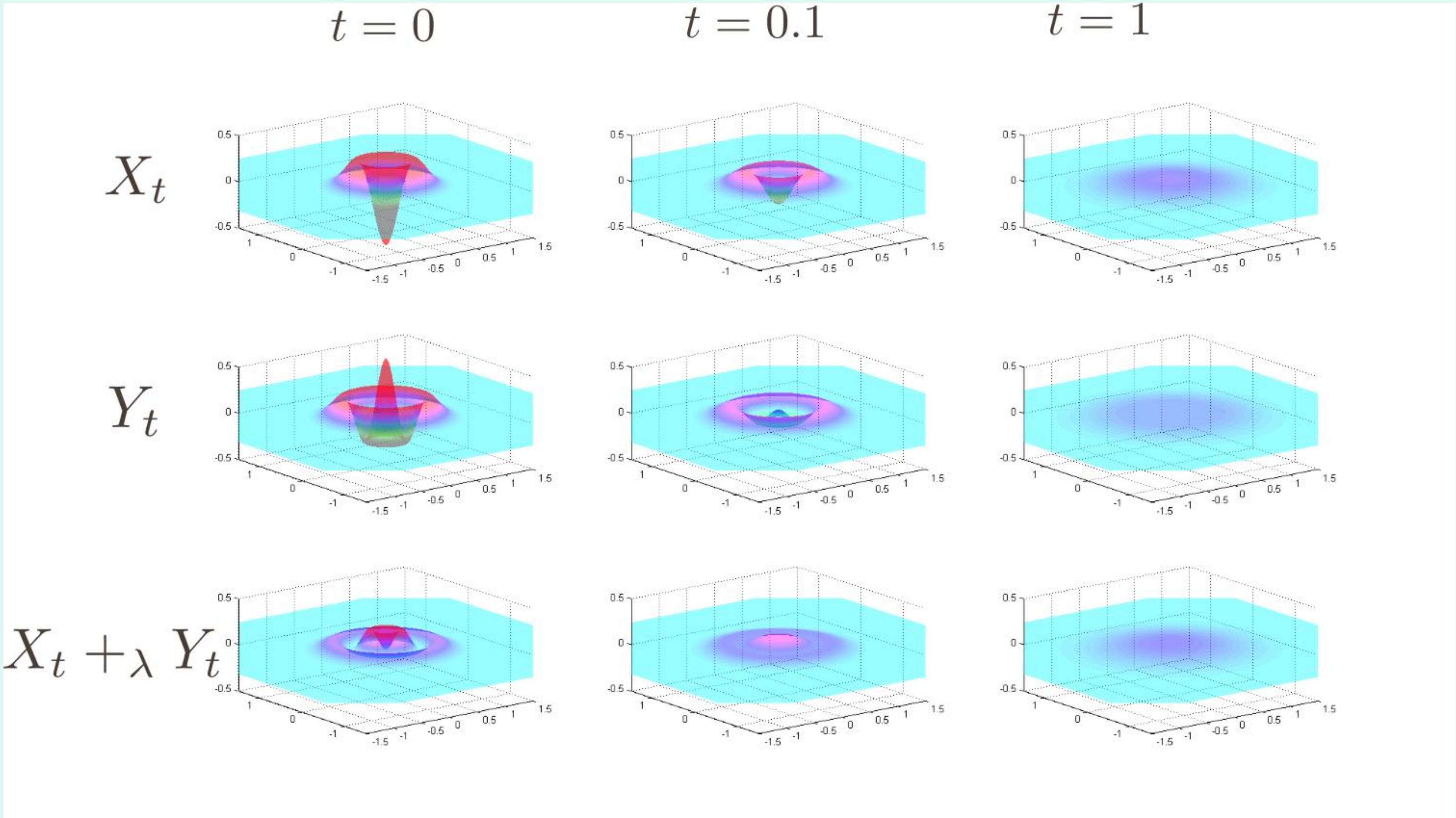
Idea: Smooth out the differences



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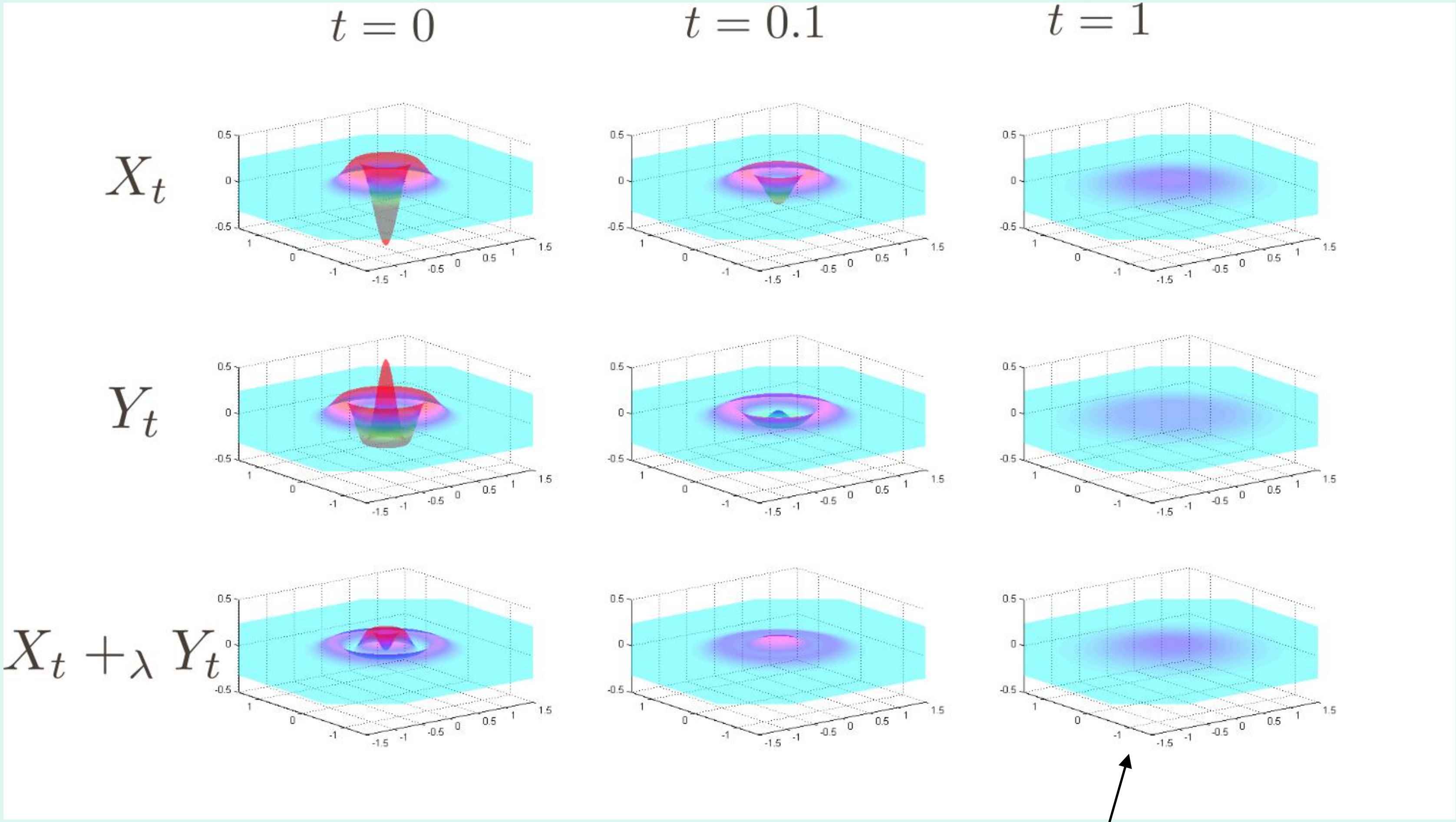
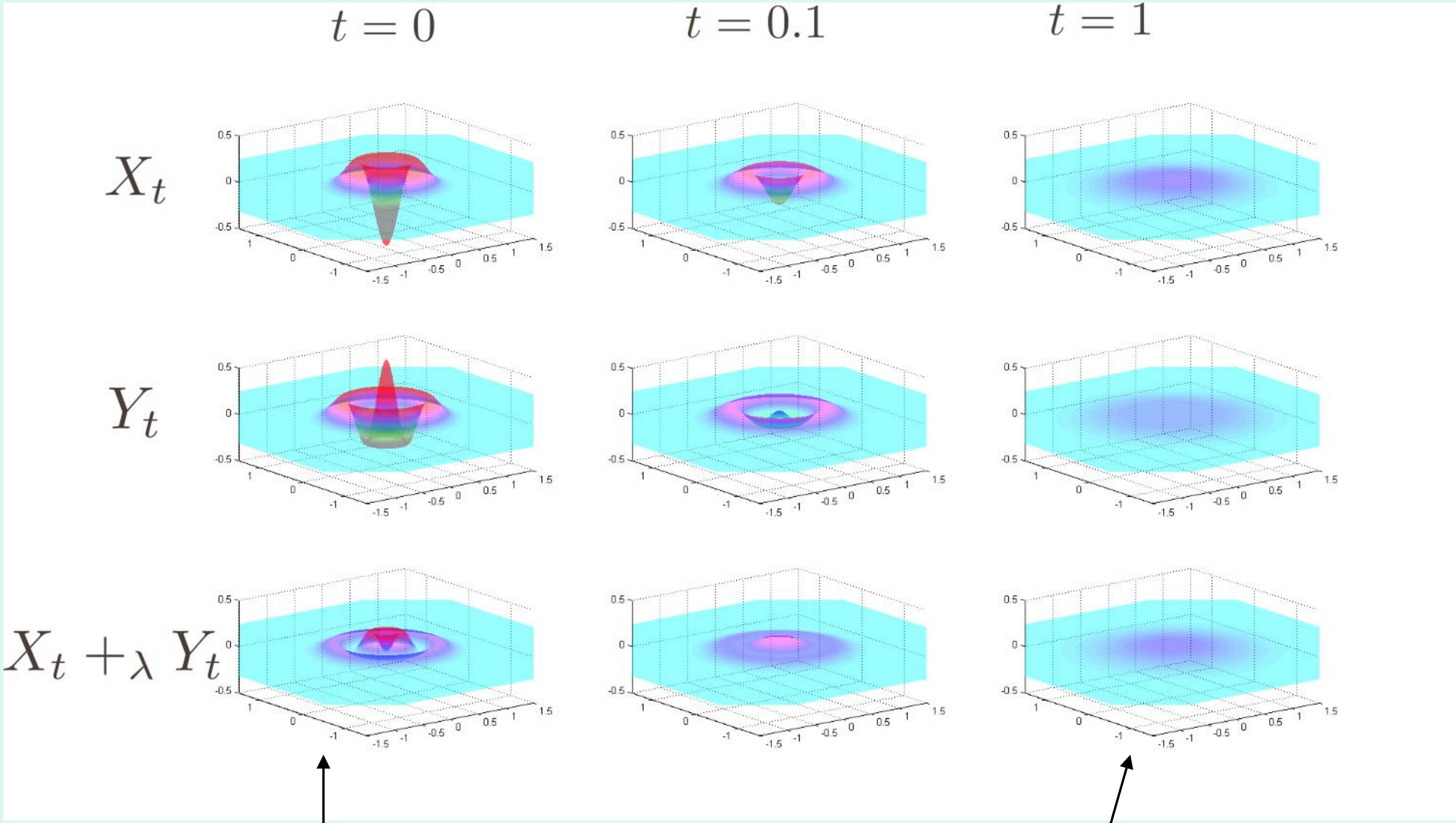


At late times, satisfied with equality

Idea: Smooth out the differences



Show violations only get worse as process runs

At late times, satisfied with equality

Quantum diffusion process

$$\frac{d\rho_t}{dt} = \mathcal{L}(\rho_t) = -[P,[P,\rho_t]] - [Q,[Q,\rho_t]]$$

$$\rho_t = e^{\mathcal{L}t}(\rho_0)$$

Quantum diffusion process

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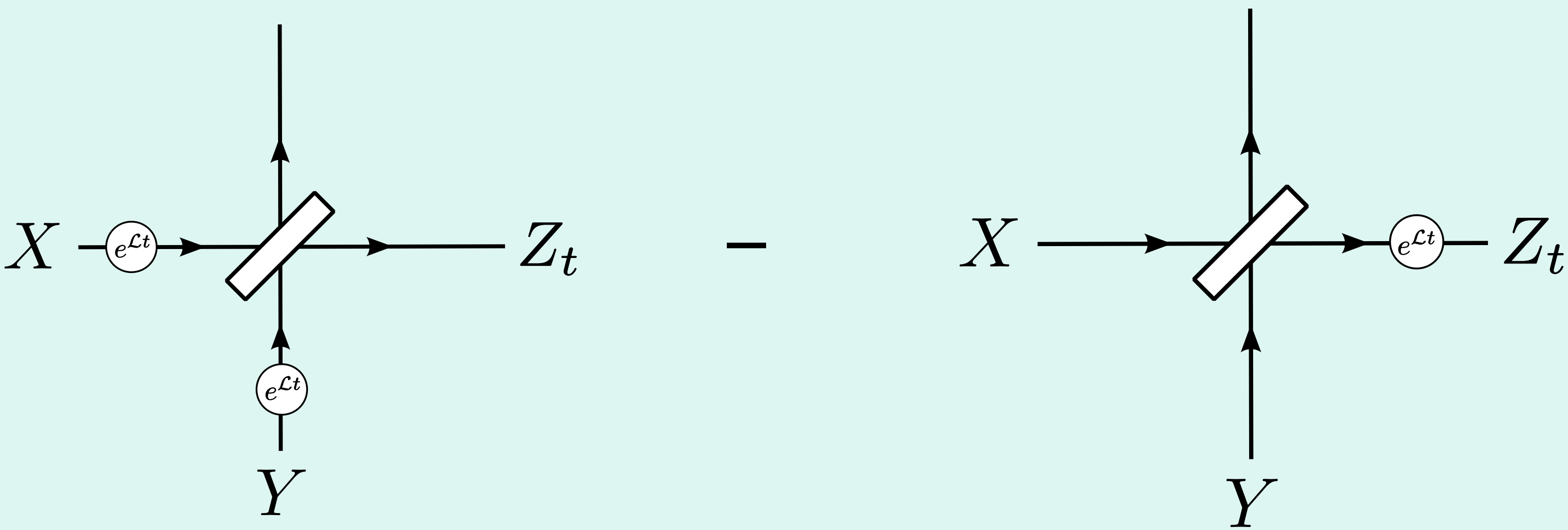


SIR RONALD A. FISHER, 1961

Quantum de Bruijin identity

$$\rho_t = e^{\mathcal{L} t}(\rho_0)$$

$$\frac{dH(\rho_t)}{dt} = J(\rho_t)$$

Quantum Fisher Information: $$J(\rho) = \sum_i \partial^2_{\theta_i} S(\rho \| \rho^{\theta_i}_{R_i})$$

$$\rho^{\theta}_{R_i} = e^{i\theta R_i/2} \rho e^{-i\theta R_i/2} \qquad S(\rho \| \sigma) = \mathrm{Tr}\,\rho(\log \rho - \log \sigma)$$

Quantum de Bruijin identity

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Quantum Fisher Information: $J(\rho) = \sum_i -\partial^2_{\theta_i} S(\rho \| \rho^{\theta_i}_{R_i})$

$\rho^{\theta}_{R_i} = e^{-i\theta R_i/2}\rho e^{i\theta R_i/2}$ $\quad S(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$

crucial property:

$$J(X + \lambda Y) \le \lambda J(X) + (1-\lambda)J(Y)$$

Proof ingredients

$$\pm(t) = H(X_t + ¸\, Y_t)\ ¡\ ¸\, H(X_t)\ ¡\ (1\ ¡\ ¸)H(Y_t)$$

Proof ingredients

$$\delta(t) = H(X_t + \lambda Y_t) - \lambda H(X_t) - (1-\lambda) H(Y_t)$$

$H(\rho_t) \to g(t)$ as $t \to \infty$, so $\delta(\infty) = 0$

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Proof ingredients

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$$\delta'(t) = J(X_t + \lambda Y_t) - \lambda J(X_t) - (1-\lambda)J(Y_t) \leq 0$$

$$\delta(0) \geq 0 \text{ so that}$$

$$H(X + \lambda Y) \geq \lambda H(X) + (1-\lambda)H(Y)$$

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Summary

- Bosonic Gaussian Channels model real systems (thermal noise, amplification)
- Lower bound to classical capacity from displaced coherent states
- Gave upper bounds that are close to this lower bound: 1) bottlenecking 2) EPIs
- Entropy power inequality controls entropy production as two states combine at a beamsplitter
- Proof of EPI uses diffusion process that smooths arbitrary state towards gaussians, de Bruijin identity and Fisher information

Questions

- Entropy photon-number inequality: we showed λ E(X) +(1-λ)E(Y) $\leq$ E(X+$_\lambda$ Y) for E(X) = H(X) and E(X) = $e^{H(X)/n}$ for E(X) = g^{-1}(H(X)) we would get capacity exactly
- Quantum Fisher information is not unique: is there a semigroup/EPI pair for each FI?
- Application: supports rough estimates in discrete quadrature model
- Further applications. For classical: gaussian broadcast channel, quadratic gaussian distributed source coding/CEO problem, multiple-description coding, gaussian wiretap channel, …
- Semi-groups as proof tool: more information-theoretic problems solved by physical smoothing process?

THANK YOU