Limits to Poisson's ratio in isotropic materials – general result for arbitrary deformation.

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Abstract

The lower bound customarily cited for Poisson's ratio ν, -1, is derived from the relationship between ν and the bulk and shear moduli in Lamé's theory of linear elasticity. However, experimental verification of the theory has been limited to materials having $\nu \geq 0.2$. From consideration of the longitudinal and biaxial moduli, we recently determined that the lower bound on ν for isotropic materials from this theory is actually 1/5. Since this value is consistent with experimental measurements on most real materials, the general presumption is that the theory has been validated. Herein we generalize our prior result, first by analyzing expressions for ν in terms of six common elastic constants, and then by considering arbitrary strains. The results corroborate that $\nu \geq 1/5$ for linear elasticity theory to be applicable. For materials that deviate from this bound ($\nu < 0.2$), Lamé's theory will yield erroneous results, and thus more sophisticated elasticity models must be used to analyze the mechanical behavior of such materials.

1. Introduction

The ratio of lateral strain ε_{22} to longitudinal strain ε_{11} defines the elastic constant

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} \tag{1}$$

for a material under uniaxial stress σ_{11}. This constant is named for Poisson, who defined it in 1829 in his single constant theory of linear elasticity, in which $\nu = ¼$ for all solids [1]. Recent interest in auxetic materials ($\nu < 0$) [2,3] and nano-composites, in which Poisson's ratio is used to characterize mechanical behavior [4,5,6,7,8], has renewed attention to this quantity.

Much of the experimental investigations of the mechanical behavior of isotropic solids in the early 19th century were devoted to measuring ν, in order to verify the single constant Poisson idea. Its refutation developed sporadically; the first evidence appeared in 1848, when ν was found to be *ca.* ⅓ for various oxide glasses and brasses [9], and in 1859, when experiments determined $\nu = 0.295$ for steel [10]. Unfortunately, other less accurate measurements supported the theory, and the controversy persisted into the 1860s. Lamé's two-constant linear elasticity theory for isotropic materials [11] was adopted by most researchers soon thereafter, in part because it accommodates variation in ν [12,13]; however, this did not prove the theory was valid.

According to the theory, for an isotropic material only two elastic constants are unique, so its validation requires measurement and comparison of three different constants. For example, the relation

$$\nu = \tfrac{1}{2} - \frac{E}{6B} \tag{2}$$

can be used to compare measured values of Poisson's ratio to that determined from Young's modulus E and the bulk modulus B. This approach presents two challenges: (i) highly precise data are required (see review [14]); and (ii) conventional solids are often non-linear even at strains as small as 10^{-5} [15,16]. Experimental verification appeared in the early 1900s [17,18], with data for iron, tin, aluminum, copper, silver, platinum, and lead [18] conforming to Lamé's two-constant theory (Fig. 1). In the past 100 years, the theory has been fully accepted and is universally applied in science and engineering. Thus, it is common practice to limit characterizations of isotropic solids to two elastic constants, obtained for example from shear and longitudinal wave speed measurements [19,20], with other parameters calculated from the Lamé relations.

The accepted theoretical limits on Poisson's ratio are much lower than the experimental range in Fig. 1, which means that the theory has actually been verified only for materials having $\nu \geq 0.2$. The conventional limits are found from [12]

$$G = B\frac{3(1-2\nu)}{2(1+\nu)} \tag{3}$$

where G is the shear modulus. To minimize the strain energy at equilibrium and avoid spontaneous deformation, G and B must be positive, leading to the oft-stated "thermodynamically admissible" range [12,21]

$$-1 < \nu < \tfrac{1}{2} \tag{4}$$

This derivation of the limits on ν is the obvious one, considering deformations involving changes in size and shape. The actual thermodynamic limits on ν have never been determined experimentally, and measurements for isotropic materials occupy a much narrower range than the conventional limits (Fig. 1). Reviews of the literature of more than 3,000 measurements on 596 different substances over a wide range of temperature and pressure, including pure elements, engineering alloys, polymers, ceramics, and glasses, show that with very few exceptions (e.g., porous quartz or very hard materials such as diamond and beryllium), $\nu \geq 0.2$ for isotropic, homogeneous materials [22,23]. Thus, the lower limit in eq. 4 does not represent the behavior of most real materials. This does not mean that real materials cannot have $\nu < 0.2$, but only that Lamé's theory has not been experimentally validated for $\nu < 0.2$

Notwithstanding its conceptual appeal, there is no mathematical or physical justification in Lamé's theory for preferring G and B over other pairs of constants in determining the limits on Poisson's ratio. For example, using Lamé's relation for ν in terms of E and the longitudinal modulus, M, we have shown from the roots of a quadratic expression that the range in eq. 4 is split into [23]

$$-1 < \nu \leq 1/5 \tag{5a}$$

$$1/5 \leq \nu < 1/2 \tag{5b}$$

Since elastic properties are unique, only one range can be valid; moreover, the lower limit of $1/5$ agrees with experimental data. Thus, this more restrictive upper range, $1/5 \leq \nu < 1/2$, appears to be the correct limit for Lamé elasticity, since values of ν still conform to eq. 4. The argument might be made that the range extending to $-1 < \nu$ in eq. 5 is mathematically valid, and hence represents an acceptable bound. However, rejection of spurious roots is common when an analysis produces two or more solutions; physical considerations are applied to eliminate roots that are false. Examples include the Landau-Lifshitz equation for the motion of a charge [24], analysis of projectile trajectories in air [25], Pythagoras' theorem for right triangles, and more generally in the solutions of ordinary differential equations [26]. We also note that a recent theoretical analysis [27], based on symmetry arguments from elastic constants that were restricted to linear combinations of the two Lamé constants, similarly found expressions for ν having multiple roots; the lower bound on Poisson's ratio was larger than -1, namely $\left(1-\sqrt{2}\right)/2$. Thus, two recent analyses [23,27] undermine the accepted range of ν for Lamé's theory to be valid.

This more restrictive lower bound on Poisson's ratio in eq. 5b is important because it means that whenever a material has $\nu < 0.2$, the equations of linear elasticity derived from the Lamé theory cannot apply; a more sophisticated model of elasticity must be invoked to provide relations between elastic constants for that material. In this work we first extend the analysis of ref. [23] to all commonly defined elastic constants, in order to obtain their associated limits for Poisson's ratio. We then generalize these results to arbitrary deformation mode. Our previous conclusion [23], that the minimum of ν for an isotropic material is 1/5, is shown to be general for materials for which the equations of the Lamé elasticity are valid.

2. Limits on ν from common elastic constants

For an isotropic solid with strain components ε_{ij}, the reversible work of deformation is [12]

$$2W = (\lambda + 2\mu)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 + \mu\left(\varepsilon_{12}{}^2 + \varepsilon_{13}{}^2 + \varepsilon_{23}{}^2 - 4\varepsilon_{22}\varepsilon_{33} - 4\varepsilon_{33}\varepsilon_{11} - 4\varepsilon_{11}\varepsilon_{22}\right) \qquad (6)$$

where λ and μ (=G) are the Lamé constants. (Note defining the shear strain as the , there is a factor Differentiation with respect to ε_{ij} defines the stress tensors σ_{ij}. When uniaxial loading is substituted (i.e., $\sigma = \sigma_{11}$ and all other $\sigma_{ij} = 0$),

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2\lambda + 2\mu} \qquad (7)$$

This procedure can be carried out for any deformation or loading geometry to define the corresponding stiffness [28]. These definitions are combined to obtain relations between the elastic constants. For example, for longitudinal loading ($\varepsilon = \varepsilon_{11}$ and all other $\varepsilon_{ij} = 0$) we obtain

$$M = \frac{1-\nu}{(1-2\nu)(1+\nu)} E \qquad (8)$$

where M is the longitudinal modulus.

Table 1 lists all of the equations for Poisson's ratio from commonly defined moduli. Included are expressions that involve the biaxial stress modulus H, defined when $\sigma = \sigma_{11} = \sigma_{22}$ and all other $\sigma_{ij} = 0$, and the biaxial strain modulus I, defined when $\varepsilon = \varepsilon_{11} = \varepsilon_{22}$ and all other $\varepsilon_{ij} = 0$. I is unusual, but is included here as the counterpart to H. The second column in the Table shows the restrictions on ν arising from the requirement that all elastic moduli are greater than zero. It is seen that the conventional limits, $-1 < \nu < ½$, follow from eqs. T1 and T2. The other linear expressions lead to wider ranges for v. Of course, the more restrictive limits for Poisson's ratio is the governing range, since all broader ranges are also satisfied.

Of special interest are the four quadratic relations, eqs. T12 – T15. These arise from stress-strain counterparts, such as E (defined from a stress) and M (defined from a strain). Note that if E/M is substituted for H/I, eq. T13 becomes eq. T12, and therefore the two equations are identical; thus,

$$\frac{E}{M} = \frac{H}{I} \tag{9}$$

Each quadratic relation in Table 1 has two roots that limit the span of Poisson's ratio. These relations are plotted in Fig. 2, with the positive roots denoted by a solid line and the negative with a dashed line. The roots converge at smoothly continuous maxima; only the bounds of −1 and ½ encompass the allowable range for all relations. Restricting v to real numbers means that:

1. Eqs. T12 and T13: $0 < E/M \leq 1$ with the same range for H/I. The two roots of this expression have ranges $-1 < v \leq 0$ and $0 \leq v < ½$. This equation also produces real values if $E/M \geq 9$, which has two roots with ranges $1 < v \leq 2$ and $2 \leq v < \infty$; however, this solution is discarded because it falls beyond the bounds of eq. 4.
2. Eq. T14: $0 < E/I \leq 9/8$; the two roots have the ranges $-1 < v \leq -¼$ and $-¼ \leq v < ½$.
3. Eq. T15: $0 < H/M \leq 9/8$; the two roots have the ranges $-1 < v \leq 1/5$ or $1/5 \leq v < ½$.

There are companion relations for G and B, and these quadratic equations have interconnected roots. For example, the counterpart to eq. T15 for the bulk modulus is

$$B = \frac{M}{6}\left[3 \pm \left(9 - 8\frac{H}{M}\right)^{½}\right] \tag{10}$$

and, having the same argument for the square root as in eq. T15, restricts $0 < H/M \leq 9/8$ for this expression to be real. The negative root has the range $0 < B/M \leq ½$, $½ \leq B/M < 1$ for the positive root. It can be shown that the positive root is linked to the positive root of eq. T15 and vice-versa; that is, if $½ \leq B/M < 1$, then $1/5 \leq v < ½$.

Quadratic expressions with two possible solutions for G, B, and v are at odds with the behavior of real materials, which have unique elastic constants for any thermodynamic state. Therefore, only one set of solutions can be valid.

3. Limits on ν for arbitrary deformations

The considered elastic constants – shear G, hydrostatic pressure or dilatation B, uniaxial stress E, uniaxial strain M, biaxial stress H, and biaxial strain I – permute a single stress or strain through the available tensor combinations for an isotropic material. However, the possibility exists that more restrictive limits on ν can be found from other elastic constants derived from more complex combinations of stress or strain. To examine this, we introduce two, continuously variable elastic constants. The first is a biaxial stress with $\sigma_{11} = \sigma$ and $\sigma_{22} = y\sigma$, where y is a constant describing the fraction of biaxial stress, $0 \le y \le 1$; all other $\sigma_{ij} = 0$. The elastic constant for this variable stress geometry is

$$H_y = \frac{E}{1 - y\nu} \tag{11}$$

When $y = 0$ (uniaxial loading), $H_0 = E$; when $y = 1$ (biaxial stress), eq. 11 becomes eq. T8.

For the second constant, consider a variable biaxial strain $\varepsilon_{11} = \varepsilon$; $\varepsilon_{22} = \beta\varepsilon$, where β is the fraction of biaxial strain, $0 \le \beta \le 1$; and all other $\varepsilon_{ij} = 0$. The elastic constant for this variable strain geometry is

$$I_\beta = \frac{1 - \nu\,(1-\beta)}{1-\nu} M \tag{12}$$

Similarly, when $\beta = 0$, $I_0 = M$ (longitudinal deformation), and when $\beta = 1$, eq. 12 becomes eq. T9, corresponding to biaxial strain. These expressions define the elastic stiffness for any mixture of one or two dimensional stress or strain.

From the equations in Table 1, many other relations that involve H_y and I_β can be derived. Of particular interest is

$$\nu = \frac{I_\beta}{4I_\beta + 2y(1-\beta)H_y}\left\{(1-\beta+y)\frac{H_y}{I_\beta} - 1 \pm \left[9 - (10 - 2\beta - 2y + 4\beta y)\frac{H_y}{I_\beta} + (1-\beta-y)^2\frac{H_y{}^2}{I_\beta{}^2}\right]^{1/2}\right\} \tag{13}$$

This equation combines the four quadratic expressions for Poisson's ratio into a single, continuous function. Each of the four quadratic expressions for ν, T12 – T15 in Table 1, can be recovered by substituting the respective values for y and β. Intermediate values y and β produce curves that lie between these extremes. Shown in Fig. 2 is the curve for $y = ½$ and $\beta = 0$, which falls between the H/M and E/M curves. Likewise, the two roots of eq. 13 meet without discontinuity. This common point is defined as $\nu^*(y,\beta)$ at H_y^*/I_β^*; it divides Poisson's ratio into the ranges $-1 < \nu \le \nu^*$ and $\nu^* \le \nu < ½$. Since the upper span corresponds to experimental data

[29,30], it is of interest to determine the lower limit ν^*. This point is found when the two roots are equal, which occurs when

$$9-(10-2\beta-2y+4\beta y)\frac{H_y^*}{I_\beta^*}+(1-\beta-y)^2\Big(\frac{H_y^*}{I_\beta^*}\Big)^2=0 \tag{14}$$

This expression has the solutions

$$\frac{H_y^*}{I_\beta^*}=\frac{5-y-\beta+2\beta y\pm 2[(\beta^2-\beta-2)(y^2-y-2)]^{1/2}}{(1-\beta-y)^2} \tag{15}$$

The positive root is rejected because it returns $H_y^*/I_\beta^* \geq 9$, producing $\nu > 1$, which is beyond the bounds from eq. 4. Note this corresponds to $E/M \geq 9$, which was also discarded in Section 2 above.

The values of ν^* satisfying eq. 15 for given y and β have the range $-1/4 < \nu^* \leq 1/5$, with $\nu^* = 0$ for $y = \beta$. In terms of the common elastic constants, (i) $\nu^* = 1$ at $\beta = 0$, $y = 0$, corresponding to $\nu^*(E,M)$; (ii) $\nu^* = 1/5$ at $\beta = 0$, $y = 1$, corresponding to $\nu^*(H,M)$; (iii) $\nu^* = -1/4$ at $\beta = 1$, $y = 0$, corresponding to $\nu^*(E,I)$; and (iv) $\nu^* = 0$ at $\beta = 1$, $y = 1$, corresponding to $\nu^*(H,I)$. Thus, eq. 15 merges the ranges of ν for specific conditions of stress and strain (Fig. 1) into a single continuous function describing arbitrary stress and strain. Fractional values of y and β in eq. 13 determine ν^* for any combination of two-dimensional stress or strain. Again, the most restrictive range is the correct range, because it accommodates the other ranges, and the lower bound for Lamé's theory to be applicable is 1/5 for any stress and strain.

Note that eq. 15 is undefined when $\beta + y = 1$. For this condition, the solution for H_y^*/I_β^* is found by substituting $a - y = \beta$ and taking the limit $a \to 1$ by twice applying L'Hôpital's rule. The result is

$$H_y^*/I_\beta^* = 1-\frac{(1-2y)^2}{4(y^2-y-2)} \tag{16}$$

This demonstrates that there is no discontinuity when $\beta + y = 1$.

The companion quadratic relations for G and B are

$$G=\frac{I_\beta}{4-8\beta}\Big\{3+(1-\beta+y)\frac{H_y}{I_\beta}\mp\Big[9-(10-2\beta-2y+4\beta y)\frac{H_y}{I_\beta}+(1-\beta-y)^2\frac{H_y{}^2}{I_\beta{}^2}\Big]^{1/2}\Big\} \tag{17}$$

$$B=\frac{I_\beta}{6+6\beta}\Big\{3-(1-\beta+y)\frac{H_y}{I_\beta}\pm\Big[9-(10-2\beta-2y+4\beta y)\frac{H_y}{I_\beta}+(1-\beta-y)^2\frac{H_y{}^2}{I_\beta{}^2}\Big]^{1/2}\Big\} \tag{18}$$

The inverted ± sign in eq. 17 denotes that its negative root is linked to the positive roots of eqs. 13 and 18.

4. **Exceptions**

As stated in the introduction, isotropic materials exist for which $\nu < 1/5$, although they are rare. Homogenous materials which show this behavior include pyrite [31], α-cristobalite [32], diamond [33,34,35], a $TiNb_{24}Zr_4Sn_{7.9}$ (β-type titanium) alloy [36], boron nitride [37], α-beryllium [38], and certain silicate glasses [39]. In the former cases (pyrite, cristobalite, diamond), elastic properties have been determined from vibrational measurements of single crystals, and aggregate isotropic behavior is inferred. For the titanium alloy, boron nitride, beryllium, and SiO_2 glasses, elastic properties of the aggregate were determined by vibrational methods, in which two elastic constants are measured, with Poisson's ratio in turn found from the expressions in table 1. It can be seen that while homogeneous solids having $\nu < 1/5$ have been identified, for none have the Lamé relations been tested.

There are recent reports of auxetic behavior in crystalline materials that exhibit negative ν in certain directions [40,41]. However, when the aggregate isotropic behavior is examined, these substances show the conventional behavior, $\nu \geq 1/5$. There is also a class of open-cell foams that have negative Poisson's ratio, due to debuckling of the cell walls [3]. These auxetic foams exhibit non-linear mechanical properties [42], so that the application of linear elasticity is problematic. Fitting their behavior to more complicated elasticity models has had limited success [43], although recently we showed that the equations of Lamé elasticity theory fail for such materials [44]. Recent investigations of larger scale, two-dimensional skeletal structures, both experimental [45,46] and theoretical [47], also discovered auxetic behavior, but linear elasticity does not apply to deformations larger than mathematically infinitesimal, so that the theory cannot be tested.

5. **Summary**

The equations of Lamé elasticity impose restrictions on the values of Poisson's ratio. Any pair of elastic constants leads to various expressions for the bounds on ν, but for mutual consistency, the most restrictive limits are the correct ones. The result, $1/5 \leq \nu < ½$, is shown to be the valid range for any isotropic material subjected to arbitrary loading or deformation. This range comports with the values of v for the vast majority of isotropic materials, although materials having $v < 1/5$ do exist. However, the equations of Lamé elasticity cannot be applied for the latter.

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Table 1: Relations between elastic constants that include Poisson's ratio

RELATION	(eq.)	RESTRICTIONS ON ν
$\nu = \frac{3B-2G}{6B+2G}$	(T1)	$-1<\nu<\frac{1}{2}$
$\nu = \frac{H-3B}{H-6B}$	(T2)	$-1<\nu<\frac{1}{2}$
$\nu = \frac{3B-M}{3B+M}$	(T3)	$-1<\nu<1$
$\nu = \frac{H-2G}{H+2G}$	(T4)	$-1<\nu<1$
$\nu = \frac{M-2G}{2M-2G}$	(T5)	$-\infty<\nu<\frac{1}{2}$
$\nu = \frac{1}{2}-\frac{E}{6B}$	(T6)	$-\infty<\nu<\frac{1}{2}$
$\nu = \frac{1}{2}-\frac{G}{I}$	(T7)	$-\infty<\nu<\frac{1}{2}$
$\nu = 1-\frac{E}{H}$	(T8)	$-\infty<\nu<1$
$\nu = 1-\frac{M}{I}$	(T9)	$-\infty<\nu<1$
$\nu = \frac{E}{2G}-1$	(T10)	$-1<\nu<\infty$
$\nu = \frac{3B}{I}-1$	(T11)	$-1<\nu<\infty$
$\nu = \frac{1}{4}\left[\frac{E}{M}-1\pm\left(\frac{E^2}{M^2}-10\frac{E}{M}+9\right)^{½}\right]$	(T12)	$0<\frac{E}{M}\le 1$: $-1<\nu<0$ *or* $0<\nu<\frac{1}{2}$
$\nu = \frac{1}{4}\left[\frac{H}{I}-1\pm\left(\frac{H^2}{I^2}-10\frac{H}{I}+9\right)^{½}\right]$	(T13)	$0<\frac{H}{I}\le 1$: $-1<\nu<0$ *or* $0<\nu<\frac{1}{2}$
$\nu = -\frac{1}{4}\left[1\mp\left(9-8\frac{E}{I}\right)^{½}\right]$	(T14)	$0<\frac{E}{I}\le\frac{9}{8}$: $-1<\nu\le-\frac{1}{4}$ *or* $-\frac{1}{4}\le\nu<\frac{1}{2}$
$\nu = \frac{M}{2H+4M}\left[2\frac{H}{M}-1\pm\left(9-8\frac{H}{M}\right)^{½}\right]$	(T15)	$0<\frac{H}{M}\le\frac{9}{8}$: $-1<\nu\le\frac{1}{5}$ *or* $\frac{1}{5}\le\nu<\frac{1}{2}$

Figure Captions

Figure 1. Experimental data from Grüneisen [18], demonstrating the validity of Lamé's quadratic theory of linear elasticity for $\nu \geq 1/5$.

Figure 2. Poisson's ratio as a function of the ratio of the indicated elastic constants, with positive roots shown indicated by the solid lines and negative roots with dashed lines. Also included are the two roots of eq. 13 with $y = ½$ and $\beta = 0$. The limits encompassing all moduli is $1/5 \leq \nu < ½$.

Figure 1

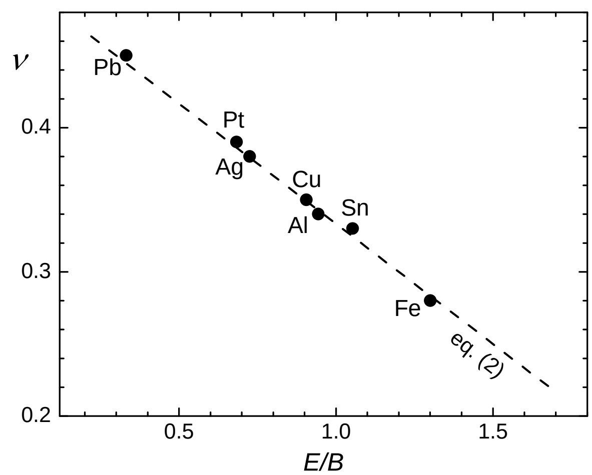

Figure 2