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LINE BUNDLES FOR WHICH A PROJECTIVIZED JET BUNDLE IS A PRODUCT

SANDRA DI ROCCO AND ANDREW J. SOMMESE

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ABSTRACT. We characterize the triples (X, L, H), consisting of line bundles L and H on a complex projective manifold X, such that for some positive integer k, the k-th holomorphic jet bundle of L, $J_k(X, L)$, is isomorphic to a direct sum $H \oplus \cdots \oplus H$.

INTRODUCTION

Let X be a complex projective manifold. A large amount of information on the geometry of an embedding $i : X \hookrightarrow \mathbb{P}^N$ is contained in the "bundles of jets" of the line bundles $k\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^N}(k)$ for $k \geq 1$. The k-th jet bundle of a line bundle L (sometimes called the k-th principal part of L) is usually denoted by $J_k(X, L)$, or by $J_k(L)$ when the space X is clear from the context. The bundle $J_k(k\mathcal{L})$ is spanned by the k-jets of global sections of $k\mathcal{L}$. If $\alpha : \mathbb{P}(J_1(\mathcal{L})) \to \mathbb{P}^N$ is the map given by the 1-jets of elements of $H^0(X, \mathcal{L})$, then $\alpha(\mathbb{P}(J_1(\mathcal{L}))_x) = \mathbb{T}_x$, where $\mathbb{T}_x$ is the embedded tangent space at x. Similarly $\mathbb{P}(J_k(k\mathcal{L}))_x$ is mapped to the "k-th embedded tangent space" at x (see [GH] for more details). Given this interpretation of the projectivized jet bundle, it is natural to expect that a projectivized jet bundle is a product of the base and a projective space only under very rare circumstances.

In this paper we analyze the more general setting where L is a line bundle on X (with no hypothesis on its positivity) and $J_k(L) = \bigoplus H$, with H a line bundle on X. Some years ago the second author [So] analyzed the pairs (X, L) under the stronger hypothesis that $J_k(L)$ is a trivial bundle. Though the results in this paper do not follow from [So], the only possible projective manifolds with a projectivized jet bundle of some line bundle being a product of the base manifold and a projective space, turn out to be the same, i.e., Abelian varieties and projective space. We completely characterize the possible triples (X, L, H). In fact, on Abelian varieties we have the necessary and sufficient condition that $L \in \mathrm{Pic}_0(X)$, and if $L = \mathcal{O}_{\mathbb{P}^n}(a)$, then only the range $a \geq k$ or $a \leq -1$ can occur.

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1. Some preliminaries

We follow the usual notation of algebraic geometry. Often we denote the direct sum of m copies of a vector bundle $\mathcal{E}$, for some integer $m > 0$, by $\bigoplus_m \mathcal{E}$. We freely use the additive notation for line bundles. We often use the same symbols for a vector bundle and its associated locally free sheaf of germs of holomorphic sections. We say a vector bundle is spanned if the global sections generate each fiber of the bundle. For general references we refer to [BSo] and [KSp].

By X we will always denote a projective manifold over the complex numbers $\mathbb{C}$ and by L a holomorphic line bundle on X.

Let k be a nonnegative integer. The *k-th jet bundle* $J_k(L)$ associated to L is defined as the vector bundle of rank $\binom{k+n}{n}$ associated to the sheaf

$$p^*L/(p^*L \otimes \mathcal{J}_\Delta^{k+1}),$$

where $p : X \times X \to X$ is the projection on the first factor and $\mathcal{J}_\Delta$ is the sheaf of ideals of the diagonal, Δ, of $X \times X$. Let $j_k : H^0(X, L) \times X \to J_k(L)$ be the map which associates to each section $s \in H^0(X, L)$ its k-th jet. That means that locally, choosing coordinates $(x_1, ..., x_n)$ and a trivialization of L in a neighborhood of x, $j_k(s, x) = (a_1, ..., a_{\binom{n+k}{n}})$, where the a_i's are the coefficients of the terms of degree up to k in the Taylor expansion of s around x. Notice that the map j_k is surjective if and only if $H^0(X, L)$ generates all the k-jets at all points $x \in X$. For example j_1 being surjective is equivalent to $|L|$ defining an immersion of X in $\mathbb{P}^{h^0(L)-1}$.

We will often use the associated exact sequence

$$0 \to T_X^{*(k)} \otimes L \to J_k(L) \to J_{k-1}(L) \to 0 \qquad (j_k)$$

where $T_X^{*(k)}$ denotes the k-th symmetric power of the cotangent bundle of X. There is an injective bundle map [KSp, p. 52]

$$\gamma_{i,j} : J_{i+j}(L) \to J_i(J_j(L)).$$

Using the sequence (j_k) it is easy to see that

$$\det J_k(L) = \frac{1}{n+1}\binom{n+k}{n}(kK_X + (n+1)L).$$

Lemma 1.1. *Let X be a compact Kähler variety and L a holomorphic line bundle on X. Then $c_1(L) = 0$ in $H^1(T_X^*)$ if and only if the bundle sequence (j_1) splits. If $c_1(L) = 0$ in $H^1(T_X^*)$, then $J_k(L) \cong J_k(\mathcal{O}_X) \otimes L$.*

Proof. A local computation, using Cech coverings, shows that the Atiyah class defined by the sequence (j_1) is the cocycle $c_1(L) \in H^1(X, T_X^*)$. Thus $c_1(L) = 0$ in $H^1(T_X^*)$ if and only if the bundle sequence (j_1) splits.

If $c_1(L) = 0$ in $H^1(T_X^*)$, then L has constant transition functions. From this, the isomorphism $J_k(L) \cong J_k(\mathcal{O}_X) \otimes L$ follows. $\square$

Let A and B be vector bundles on X. We recall that if $\alpha \in H^1(A \otimes B^*)$ represents the vector bundle extension E, then any nonzero multiple $\lambda\alpha$ gives an isomorphic extension. If $\lambda = 0$ this is of course false since we would get the trivial extension. It follows that:

Lemma 1.2. *Let λ be a nonzero integer. Then $J_1(\lambda L) \cong J_1(L) \otimes (\lambda - 1)L$.*

Proof. Consider the extension

$$0 \to T_X^* \otimes \lambda L \to J_1(L) \otimes (\lambda - 1)L \to \lambda L \to 0 \qquad (j_1) \otimes (\lambda - 1)L$$

represented by $c_1(L) + c_1((\lambda - 1)L) = \lambda c_1(L) = c_1(\lambda L)$. Then, from what was observed above, $J_1(\lambda L)$, which is the vector bundle extension given by $c_1(\lambda L)$, is isomorphic to $J_1(L) \otimes (\lambda - 1)L$. □

2. The basic examples

In this section we will characterize line bundles with splitting k-th jet bundles on Abelian varieties and on $\mathbb{P}^n$. These turn out to be the only possible examples.

Proposition 2.1. *Let L and H be line bundles on an Abelian variety X. Then the following assertions are equivalent:*

- *$J_k(L) \cong \bigoplus H$ for some k;*
- *$H \cong L$ and $L \in \mathrm{Pic}_0(X)$.*

In particular $J_k(L)$ splits in the sum of spanned line bundles only when $L = \mathcal{O}_X$ and $J_k(L) = \bigoplus \mathcal{O}_X$.

Proof. Let X be an Abelian variety and assume that $J_k(L) = \bigoplus H$ for some line bundle H. Then the sequences (j_m) for $m \leq k$ imply that there is a surjection of the trivial bundle $J_k(L) \otimes (-H)$ onto $L - H$. Thus the line bundle $L - H$ is a spanned line bundle with

$$\binom{n+k}{n} c_1(L - H) = 0,$$

which implies $L = H$. Using this we can find a direct summand L of $J_k(L)$ whose image in $J_1(L)$ maps onto L under the map $J_1(L) \to L \to 0$ of the sequence (j_1). Thus the sequence (j_1) splits and therefore $c_1(L) = 0$ in $H^1(T_X^*)$ by Lemma 1.1.

Conversely assume that we have a line bundle $L \in \mathrm{Pic}_0(X)$. Then by Lemma 1.1, we have that $J_k(L) \cong J_k(\mathcal{O}_X) \otimes L$. Using the triviality of $T_X^{(j)}$ for all $j \geq 0$, it follows that $J_k(\mathcal{O}_X)$ is trivial. □

Proposition 2.2. *$J_k(\mathcal{O}_{\mathbb{P}^n}(a))$ is isomorphic to a direct sum $\bigoplus_{\binom{k+n}{n}} \mathcal{O}_{\mathbb{P}^n}(q)$ for some $k, a, q \in \mathbb{Z}$ with $k > 0$ if and only if $q = a - k$ and either $a \geq k$ or $a \leq -1$.*

Proof. Let $L = \mathcal{O}_{\mathbb{P}^n}(a)$. First notice that if the k-th jet bundle splits as $J_k(L) = \bigoplus_{\binom{k+n}{n}} \mathcal{O}_{\mathbb{P}^n}(q)$ for some nonnegative k and some integer q, then

$$\det J_k(L) = \mathcal{O}_{\mathbb{P}^n}\left(\binom{n+k}{k}(a-k)\right),$$

which implies that $q = a - k$. Then the fact that the k-th jet bundle of $\mathcal{O}_{\mathbb{P}^n}(a)$ always has sections for $a \geq 0$ rules out the cases $a = 0, ..., k-1$.

Lemma 1.2 gives $J_1(\mathcal{O}_{\mathbb{P}^n}(a)) \cong J_1(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(a-1)$. Then $J_1(\mathcal{O}_{\mathbb{P}^n}(1)) = \bigoplus \mathcal{O}_{\mathbb{P}}$ (see, e.g., [So]) implies

$$J_1(\mathcal{O}_{\mathbb{P}^n}(a)) = \bigoplus_{n+1} \mathcal{O}_{\mathbb{P}^n}(a-1).$$

Dualizing $\gamma_{1,1}$ and using

$$J_1\left(J_1(\mathcal{O}_{\mathbb{P}^n}(a))\right) = J_1\left(\bigoplus_{n+1}\mathcal{O}_{\mathbb{P}^n}(a-1)\right) = \bigoplus_{n+1}\left(\bigoplus_{n+1}\mathcal{O}_{\mathbb{P}^n}(a-2)\right)$$

we obtain the quotient

$$\bigoplus_{n+1}\left(\bigoplus_{n+1}\mathcal{O}_{\mathbb{P}^n}(a-2)\right)^* \to J_2(\mathcal{O}_{\mathbb{P}^n}(a))^* \to 0$$

and thus, tensoring by $\mathcal{O}_{\mathbb{P}^n}(a-2)$

$$\bigoplus_{(n+1)^2}\mathcal{O}_{\mathbb{P}^n} \to J_2(\mathcal{O}_{\mathbb{P}^n}(a))^* \otimes \mathcal{O}_{\mathbb{P}^n}(a-2) \to 0.$$

Then the vector bundle $J_2(\mathcal{O}_{\mathbb{P}^n}(a))^* \otimes \mathcal{O}_{\mathbb{P}^n}(a-2)$ is spanned with

$$\det(J_2(\mathcal{O}_{\mathbb{P}^n}(a))^* \otimes \mathcal{O}_{\mathbb{P}^n}(a-2)) = \mathcal{O}_{\mathbb{P}^n}$$

which implies that $J_2(\mathcal{O}_{\mathbb{P}^n}(a))^* \otimes \mathcal{O}_{\mathbb{P}^n}(a-2) = \bigoplus \mathcal{O}_{\mathbb{P}^n}$. Iterating this argument one gets $J_k(\mathcal{O}_{\mathbb{P}^n}(a))^* \otimes \mathcal{O}_{\mathbb{P}^n}(a-k) = \bigoplus_{\binom{k+n}{n}}\mathcal{O}_{\mathbb{P}^n}$. □

3. The main result

In this section we characterize complex projective manifolds having a line bundle whose projectivized k-jet bundle is a product of the base manifold and a projective space.

Theorem 3.1. *Let L and H be holomorphic line bundles on X. Then $J_k(L) = H \oplus \cdots \oplus H$ if and only if the triple (X, L, H) is one of the two below:*

(1) *(X, L, L) where X is Abelian and $L \in \mathrm{Pic}_0(X)$,*
(2) *$(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(a-k))$ and $a \geq k$ or $a \leq -1$.*

Proof. Assume $J_k(L) = H \oplus \cdots \oplus H$. Tensoring the sequence (j_k) by H^* gives the quotient $\bigoplus_{\binom{n+k}{n}}\mathcal{O}_X \to L \otimes H^* \to 0$. Tensoring the sequence (j_k) by H^* and then dualizing it gives the quotient $\bigoplus_{\binom{n+k}{n}}\mathcal{O}_X \to T_X^{(k)} \otimes (H \otimes L^*) \to 0$. It follows that $L \otimes H^*, T_X^{(k)} \otimes (H \otimes L^*)$ and thus $T_X^{(k)}$ are spanned bundles over X.

First assume that the canonical bundle K_X is nef. Since $T_X^{(k)}$ is spanned, we conclude that $\det T_X^{(k)}$, which is a spanned negative multiple of K_X is trivial. Thus $T_X^{(k)}$ is a trivial bundle. From this we conclude that $L - H$ is trivial. It follows that the trivial bundle $J_k(L)^* \otimes L$ has a filtration with quotient bundles $T_X^{(j)}$ for $0 \leq j \leq k$. This shows that $T_X^{(j)}$ is trivial for all $j > 0$. Thus under the assumption that K_X is nef, we conclude that X would be an Abelian variety. Proposition 2.1 implies also that $L \in \mathrm{Pic}_0(X)$. This gives the first case of the theorem.

We are thus left with the case when K_X is not nef. Since $T_X^{(k)}$ is spanned $-K_X$ is nef. The cone theorem then yields the existence of an extremal ray $\mathbb{R}_+[\gamma]$, with $1 \leq -K_X \cdot \gamma \leq n+1$. Let l be the normalization of γ and let

$$T_{X|l} = \bigoplus \mathcal{O}_l(a_i), \quad (L \otimes H^*)_l = \mathcal{O}_l(b),$$

where by abuse of notation we denote the pullback of a bundle $\mathcal{E}$ on γ to l by $\mathcal{E}_l$. Writing for simplicity $T^{(k)}_{X|l} = (\bigoplus_i \mathcal{O}_l(ka_i)) \oplus P$ we get

$$T^{(k)}_{X|l} \otimes (L \otimes H^*)^*_l = (\bigoplus_i \mathcal{O}(ka_i - b)) \oplus (P \otimes \mathcal{O}_l(-b)).$$

Since $T^{(k)}_{X|l} \otimes (L \otimes H^*)^*_l$ is spanned we conclude that $ka_i \geq b \geq 0$. Note that $b > 0$. Indeed if $b = 0$, then

$$0 = \deg(\det J_k(L)_l \otimes (-H_l)) = \frac{1}{n+1}\binom{n+k}{n} kK_X \cdot l \neq 0.$$

Thus $a_i > 0$ for all i's. Moreover from sheaf injection $0 \to T_l \to T_{X|l}$ we see that $T_{X|l}$ must contain a factor $\mathcal{O}_l(a_i)$ with $a_i \geq 2$. Then $-K_X \cdot \gamma = \sum_{i=1}^n a_i \leq n+1$ implies that $a_j = 2$ for one j and $a_i = 1$ for $i \neq j$, i.e., $-K_X \cdot \gamma = n+1$. Now from Mori's proof of Hartshorne conjecture (see [L, §4]) we deduce that X must be $\mathbb{P}^n$. At this point we have recovered case (2) by applying Proposition 2.2.

Propositions 2.1 and 2.2 show that if we are in cases (1) and (2) respectively, then $J_k(L) = H \oplus \cdots \oplus H$ is satisfied. □

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Department of Mathematics, KTH, Royal Institute of Technology, S-100 44 Stockholm, Sweden

E-mail address: sandra@math.kth.se

URL: http://www.math.kth.se/∼sandra

Current address: Department of Mathematics, Yale University, P.O. Box 208229, New Haven, Connecticut 06520

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556

E-mail address: sommese@nd.edu

URL: http://www.nd.edu/∼sommese