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Line Transversals to Disjoint Balls

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Abstract We prove that the set of directions of lines intersecting three disjoint balls in $\mathbb{R}^3$ in a given order is a strictly convex subset of $\mathbb{S}^2$. We then generalize this result to n disjoint balls in $\mathbb{R}^d$. As a consequence, we can improve upon several old and new results on line transversals to disjoint balls in arbitrary dimension, such as bounds on the number of connected components and Helly-type theorems.

Keywords Transversal · Geometric permutation · Convexity

1 Introduction

Helly's theorem [12] of 1923 opened a large field of inquiry designated now as *geometric transversal theory*. A typical concern is the study of all k-planes (also called k-flats) which intersect all sets of a given family of subsets (or *objects*) in $\mathbb{R}^d$. These are the k-*transversals* of the given family and they define a certain subspace of the corresponding Grassmannian. True to its origin, transversal theory usually implicates *convexity* in some form, either in its assumptions, its proofs or most likely, both.

In what follows, $k = 1$ and the objects will be pairwise disjoint closed balls with arbitrary radii in $\mathbb{R}^d$. Our main result is the following convexity theorem:

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Theorem 1 *The directions of all oriented lines intersecting a given finite family of disjoint balls in* $\mathbb{R}^d$ *in a specific order form a strictly convex subset of the sphere* $\mathbb{S}^{d-1}$.

As a first consequence, the connected components in the space of line transversals correspond to the possible *geometric permutations* of the given family, where a geometric permutation is understood as a pair of orderings defined by a single line transversal with its two orientations. This is not true in general, not even for $n \geq 4$ disjoint line segments in $\mathbb{R}^3$.

Before discussing other implications, we want to emphasize that the *key* to our theorem resides in the case of *three disjoint balls in* $\mathbb{R}^3$, and the approach we use to settle this case is geometrically quite revealing, in that it shows the nuanced dependency of the convexity property on the *curve of common tangents* to the three bounding spheres.

1.1 Relation to Previous Work

Helly's theorem [12] states that a finite family $\mathcal{S}$ of convex sets in $\mathbb{R}^d$ has non-empty intersection if and only if any subfamily of size at most $d+1$ has non-empty intersection. Passing from $k=0$ to $k=1$, one of the early results is due to Danzer [7] who proved that n disjoint *unit* disks in the plane have a line transversal if and only if every five of them have a line transversal. Hadwiger's theorem [11], which allows arbitrary disjoint convex sets in the plane as objects, showed the importance of the *order* in which oriented line transversals meet the objects: when every three objects have an oriented line transversal respecting some fixed order of the whole family, there must be a line transversal for the family.

This stimulated interest in comparing, for arbitrary dimension, two equivalence relations for line transversals: a coarse one, *geometric permutation*, determined by the order in which the given disjoint objects are met (up to reversal of orientation) and a finer one, *isotopy*, determined by the connected components of the space of transversals.

In general, for $d \geq 3$, the gap between the two notions may be wide [8], and families for which the two notions coincide are thereby "remarkable". The first examples of such families are "thinly distributed" balls[1] in arbitrary dimension, as observed by Hadwiger [9, 10]. Then, the work of Holmsen et al. [14] showed that disjoint *unit* balls in $\mathbb{R}^3$ provide remarkable cases as well. They verified the convexity property in the case of equal radii, and their method can be extended to the larger class of "pairwise inflatable" balls[2] in arbitrary dimension [6], inviting the obvious question regarding disjoint balls of arbitrary radii. The significance of this problem is also discussed in the recent notes [19, p. 191–195] where one can find ample references to related literature.

[1] A family of balls is *thinly distributed* if the distance between the centers of any two balls is at least twice the sum of their radii.

[2] A family of balls is *pairwise inflatable* if the squared distance between the centers of any two balls is at least twice the sum of their squared radii.

Our solution for the case of arbitrary radii is based on a new approach, suggested by the detailed study of the curve of common tangents to three spheres in $\mathbb{R}^3$ [2]. The main ideas are outlined in Sect. 3 as a preamble to the detailed proof in Sects. 4 to 6.

In dimension three particularly, there are connections with other problems in visibility and geometric computing. Changes of visibility (or "visual events") in a scene made of smooth obstacles typically occur for multiple tangencies between a line and some of the obstacles [20]. Tritangent and quadritangent lines play a prominent role in this picture as they determine the 1- and 0-dimensional faces of visibility structures. An attractive case is that of four balls in $\mathbb{R}^3$ which allow, generically, up to twelve common real tangents [17]. Degenerate configurations are identified in [3]. Variations on such problems, where reliance on algebraic geometry comes to the forefront, are surveyed in [22]. See also a brief account in [1].

1.2 Further Implications

Danzer's theorem [7] motivated several other attempts to generalize Helly's result for $k = 1$, that is, for line transversals. Whereas Helly's theorem only requires convexity, the case $k = 1$ appears to be more sensitive to the geometry of the objects. In particular, Holmsen and Matoušek [15] showed that no such theorem holds in general for families of disjoint translates of a convex set, not even with restriction on the ordering *à la* Hadwiger. Our Theorem 1 has consequences in this direction, presented below in Sect. 7.

Hadwiger's proof of his Transversal Theorem [11] relies on the observation that any *minimal pinning configuration*, that is, any family of objects with an isolated line transversal that would become non-isolated should any of the objects be removed, has size 3 if the objects are disjoint convex sets in the plane. Theorem 1 implies that any minimal pinning configuration of disjoint balls in $\mathbb{R}^d$ has size at most $2d - 1$ (Corollary 14). A generalization of Hadwiger's theorem for families of disjoint balls then follows (Corollary 15).

2 Preliminaries

2.1 Notations and Prerequisites

For any two vectors $\mathbf{a}, \mathbf{b}$ of $\mathbb{R}^3$, we denote by $\langle \mathbf{a}, \mathbf{b} \rangle$ their dot product and by $\mathbf{a} \times \mathbf{b}$ their cross product. These expressions will retain their algebraic meaning when $\mathbf{a}$ and $\mathbf{b}$ are complex vectors.

The space of directions in $\mathbb{R}^3$ is the real projective space $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{R})$ envisaged either as the space of lines through the origin (and then the direction of a line is given by its parallel through the origin) or as the "plane at infinity" in the completion $\mathbb{P}^3 = \mathbb{R}^3 \cup \mathbb{P}^2$ (and then the direction of a line is simply its point of intersection with the plane at infinity). A non-zero vector $\mathbf{u} \in \mathbb{R}^3$ may also stand for the direction $(u_1 : u_2 : u_3)$ it defines in $\mathbb{P}^2$.

Convexity in $\mathbb{P}^2$ is relative to the metric induced by the standard metric of the sphere through the identification $\mathbb{S}^2/\mathbb{Z}_2 = \mathbb{P}^2$. All considerations can be pulled-back to $\mathbb{S}^2$ by orienting the lines.

In following our convexity arguments related to three disjoint balls in $\mathbb{R}^3$, it may be helpful to bear in mind that the regions of $\mathbb{P}^2$ determined by directions of line transversals are always contained in the simply-connected side of some smooth conic[3]. When testing convexity, one may use affine charts $\mathbb{R}^2$, and verify locally, then globally, that the boundary curve "stays on the same side of its tangent". If this property were to fail at some point, one must have an *inflection point* there or, in one word, a *flex*.

We denote by B_0, B_1, B_2 three balls in $\mathbb{R}^3$ with respective centers $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ and squared radii s_0, s_1, s_2, $s_k = r_k^2$. Since degenerate cases are eventually shown to follow from the generic case (Lemma 10), we assume here that we have a non-degenerate *triangle of centers*.

2.2 Direction-sextic

The directions of common tangent lines to B_0, B_1, B_2 make up an algebraic curve of degree six in $\mathbb{P}^2$, which we call the *direction-sextic* and denote by σ. To take advantage of symmetries in expressing σ, we introduce the edge vectors $\mathbf{e}_{ij} = \mathbf{c}_j - \mathbf{c}_i$ and denote by $\delta_{ij} = \langle \mathbf{e}_{ij}, \mathbf{e}_{ij} \rangle$ their squared norms. For a direction $\mathbf{u} \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, we put:

$$q = q(\mathbf{u}) = \langle \mathbf{u}, \mathbf{u} \rangle,$$

$$t_{ij} = t_{ji} = \langle \mathbf{e}_{ij} \times \mathbf{u}, \mathbf{e}_{ij} \times \mathbf{u} \rangle = \delta_{ij} q - \langle \mathbf{e}_{ij}, \mathbf{u} \rangle^2.$$

Thus in $\mathbb{P}^2(\mathbb{C})$, the equation $t_{ij} = 0$ gives the two tangents from e_{ij} to the imaginary conic $q = 0$.

Proposition 2 *The direction-sextic for B_0, B_1, B_2 can be given by means of the Cayley determinant*:

$$\sigma = \sigma(\mathbf{u}) = \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & qs_0 & qs_1 & qs_2 \\ 1 & qs_0 & 0 & t_{01} & t_{02} \\ 1 & qs_1 & t_{01} & 0 & t_{12} \\ 1 & qs_2 & t_{02} & t_{12} & 0 \end{pmatrix}.$$

Proof One way to find the equation of the direction curve is to begin with a description of lines in $\mathbb{R}^3$ by parameters $(\mathbf{p}, \mathbf{u}) \in \mathbb{R}^3 \times \mathbb{P}^2$, where $\mathbf{p}$ is the orthogonal projection of the origin on the given line, and $\mathbf{u}$ is the direction of the line. With $\mathbf{c}_0 = 0$ and abbreviations:

$$a_i = a_i(\mathbf{u}) = \langle \mathbf{c}_i \times \mathbf{u}, \mathbf{c}_i \times \mathbf{u} \rangle + (s_0 - s_i)\langle \mathbf{u}, \mathbf{u} \rangle = t_{0i} + (s_0 - s_i)q, \quad i = 1, 2,$$

[3]The complement of any proper non-empty conic in the real projective plane consists of two connected components, one homeomorphic to a Möbius strip and the other to a disc.

affine common tangents obey the system (see e.g. [3] or [17]):

$$\langle \mathbf{p}, \mathbf{c}_i \rangle = \frac{a_i(\mathbf{u})}{2\langle \mathbf{u}, \mathbf{u} \rangle}, \quad i = 1, 2, \qquad \langle \mathbf{p}, \mathbf{u} \rangle = 0, \qquad \langle \mathbf{p}, \mathbf{p} \rangle = s_0.$$

The direction-sextic is obtained by eliminating $\mathbf{p}$ from this system. The fact that the resulting equation allows the stated Cayley determinant expression is given a natural explanation in [2], but can be directly verified by computation. □

The direction of an *oriented line* can be represented either by a point on the unit sphere or, by the whole *ray* emanating from the origin and passing through that point. Our expression "cone of directions" stems from the latter representation, which converts questions of convexity in $\mathbb{S}^2$ into equivalent questions of convexity in $\mathbb{R}^3$. In the projective context, it will be understood that we mean the image via $\mathbb{S}^2/\mathbb{Z}_2 = \mathbb{P}^2$.

2.3 Cone of Directions

The *cone of directions* $K(B_0B_1B_2)$ of B_0, B_1, B_2 is the set of directions of all oriented line transversals to these balls which meet them in the stated order: $B_0 \prec B_1 \prec B_2$. The boundary of $K(B_0B_1B_2)$ consists of [6, Lemma 9] certain arcs of the direction-sextic σ and certain arcs of directions of *inner special bitangents* i.e. tangents to two of the balls passing through their inner similitude center [13]. Figure 1 offers an illustration of a cone of directions. The plane of the picture must be conceived as an affine piece $\mathbb{R}^2 \subset \mathbb{P}^2$.

We recall the fact that a common tangent (here called bitangent) for two disjoint spheres (more precisely, the boundary of two disjoint balls) passes through their inner similitude center if and only if it is contained in a common tangent plane which has the two spheres on opposite sides. If a transversal for the two balls has the direction of an inner special bitangent, it must actually be that bitangent. The cone of directions

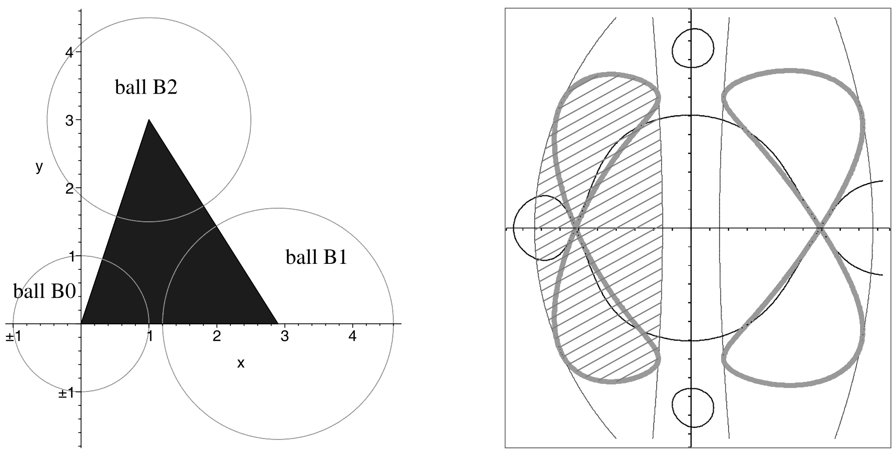

Fig. 1 *Left*: The trace of three balls B_0, B_1, B_2 on their plane of centers. *Right*: A planar depiction (hatched area) of $\mathcal{K}(B_1B_0B_2)$. The direction-sextic is drawn in *thick grey*, the Hessian in *black*, and the conics of inner special bitangents in *thin grey*

for a pair of disjoint balls is bounded precisely by their inner special bitangents. In $\mathbb{P}^2$ they trace a (circular) conic.

The points of σ that appear on the boundary $\partial K(B_0B_1B_2)$ can be characterized as follows:

Proposition 3 *The direction of a tritangent ℓ meeting the three balls B_0, B_1, B_2 in the prescribed order belongs to $\partial K(B_0B_1B_2)$ if and only if ℓ intersects the triangle of centers $\mathbf{c}_0\mathbf{c}_1\mathbf{c}_2$.*

Proof The set of directions of common transversals to disjoint balls is a proper subset of $\mathbb{P}^2$.

Assume that ℓ is neither parallel to the plane of centers, nor contained in it.

If ℓ does not intersect the triangle of centers, then, in the projected configuration on $\ell^\perp$, there is a line λ through two of the projected centers, separating the foot of ℓ from the third projected center. When moving ℓ parallel to itself and closer to λ, along a perpendicular to the latter, all distances to centers decrease. This shows that there are lines parallel to ℓ intersecting the open balls, and therefore the direction of ℓ is not on the boundary.

On the other hand, when the tritangent ℓ intersects the triangle of centers in a point P, there is no motion of ℓ parallel to itself which can decrease all distances to the centers. Indeed, reasoning in $\ell^\perp$ with respect to the triangle of projected centers, this would decrease all areas over edges, while these areas have a constant sum. This shows that no other transversal but ℓ can have its direction.[4] Looking now in the plane spanned by ℓ and the normal ν to the plane of centers at P, the rotation of ℓ, with center P, brings its direction inside $K(B_0B_1B_2)$ when approaching the plane of centers, and takes it outside $K(B_0B_1B_2)$ when approaching ν. Indeed, when rotating towards the plane of centers all distances to centers decrease, while increasing in the opposite sense. Some other transversal with direction between ℓ and ν (and parallel to the ℓ,ν-plane) cannot exist since by the same argument of rotating towards the plane of centers, one would obtain a realization of the direction of ℓ not passing through P. Thus, the direction of ℓ is in $\partial K(B_0B_1B_2)$.

If ℓ is parallel to the plane of centers (but not contained in it), we may consider any parallel plane which is closer to $\mathbf{c}_0\mathbf{c}_1\mathbf{c}_2$ than ℓ is, and find in this plane transversals to the open balls parallel to ℓ. Thus, ℓ cannot be on the boundary.

Finally, if ℓ is in the plane of centers, we look at the "section configuration" traced in that plane. *Either* all three discs are on one side of ℓ and then ℓ does not cross the triangle of centers and is not on the boundary, *or* ℓ has two discs on one side with the third on the other side and must cross the triangle of centers. Then, it is actually an inner special bitangent for two pairs of balls (and an outer special bitangent for the third pair) and belongs to the boundary. □

[4] One could conclude from here using [6, Lemma 9], which shows that a direction of $K(B_0B_1B_2)$ is in the interior if and only if there is a line transversal to the open balls with that direction.

Proposition 4 *For three disjoint balls, we have:*

(i) *The cone of directions $K(B_0B_1B_2)$ consists of a single point if and only if there is a tritangent contained in the plane of centers and tracing in it a pinned planar configuration, that is, the disc traced by B_1 is on the opposite side of the tritangent from the discs traced by B_0 and B_2;*
(ii) *In all other cases, the cone of directions $K(B_0B_1B_2)$ is the closure of its interior.*

Proof (i) Sufficiency: the plane intersecting the plane of centers along the tritangent and perpendicular to it, will have B_1 on one side, and B_0 and B_2 on the other. An oriented transversal meeting B_0 first, then B_1, and then B_2 must be contained in this separating perpendicular plane, and thus coincide with the given tritangent. Necessity is covered by our arguments in (ii).

(ii) Suppose we are not in case (i), and the centers are not aligned. If we have a transversal ℓ with direction belonging to the boundary of $K(B_0B_1B_2)$, we may assume the transversal is not in the plane of centers, since a non-pinned planar case is clear. But then ℓ and its reflection in the plane of centers define a plane perpendicular to the latter and all lines between them (passing through their intersection) have directions belonging to the interior, because all distances from centers decrease.

The case of collinear centers is trivial; there is only one geometric permutation (given by the line of centers) and the cone of directions is a disc-like region bounded by a conic. □

Corollary of the proof Cones of directions and connected components of transversals for three disjoint balls in $\mathbb{R}^3$ are *contractible*.

Indeed, the argument above shows that we may contract first to the segment in $K(B_0B_1B_2)$ consisting of directions in the plane of centers, and then contract this segment.

Obviously, the same holds true at the level of the connected components in the space of transversals.

2.4 Hessian and Flexes

The *Hessian* of the direction-sextic σ is defined as the determinant of the matrix of second derivatives:

$$H(\sigma) = H(\sigma)(\mathbf{u}) = \det\left(\frac{\partial^2\sigma}{\partial u_i \partial u_j}\right).$$

The Hessian curve, or simply "the Hessian", is the projective curve defined by the zero-set of this determinant.

The Hessian of a direction-sextic for three balls in $\mathbb{R}^3$ is thus an algebraic curve of degree twelve. The intersection between σ and its Hessian $H(\sigma)$ consists of all singular points of σ and all flexes of σ [4].

3 Outline of the Proof

For $d = 2$ the convexity theorem is elementary, and for $d \geq 3$ it is easily reduced to the case of three disjoint balls in $\mathbb{R}^3$. The *key property* used to settle this case is the following:

Proposition 5 *For disjoint balls B_0, B_1, B_2, any arc of their direction-sextic σ which belongs to the boundary $\partial K(B_0B_1B_2)$ contains no flex or singularity of σ between its endpoints.*

The convexity of the cone of directions $K(B_0B_1B_2)$ can then be inferred from the known fact that a simple C^1-loop in $\mathbb{R}^2 \subset \mathbb{P}^2$ with no inflection (in Euclidean terms: with positive curvature on its algebraic arcs) bounds a convex interior [23].

Thus, what is essential for this approach, is to obtain sufficient control over the flexes of σ. At first sight, the fact that the intersection of σ and the Hessian $H(\sigma)$ in $\mathbb{P}^2(\mathbb{C})$ has, counting multiplicities, $6 \times 12 = 72$ points, leaves little hope for the possibility of "tracking" all flexes. However, there is another way to exploit the Hessian: fix a direction and consider the ball configurations which have a tritangent with that direction and give the same planar configuration of four points when projecting, tangent and centers, on some orthogonal plane; evaluate the Hessians of the corresponding direction-sextics and determine which can vanish for the given direction.

The important point is that one can anticipate, from the form of the equations, that the computations must result in polynomials of low degree, which will be subject, in their turn, to geometric control.

The unfolding of this scenario is presented below and involves a certain amount of explicit computations. Although no part is too complicated to be done by hand, we have relied on Maple [18] in a few instances.

4 Absence of Flexes and Singularities

4.1 The Hessian Test

Following Proposition 3, we need only consider directions of tangents to the three balls that cross the triangle of centers and are not directions of inner special bitangents. When projecting along such a tangent on a perpendicular plane, the projected centers form a triangle containing the point image of the tangent as an interior point. One may start with the latter planar configuration, a triangle and an interior point, and ask which ball configurations yield this picture (by projection along a common tangent intersecting at the interior point)? Since the radii of the balls are given, one has only to "lift" the vertices of the triangle in the normal direction and obtain all the desired configurations.

We equip $\mathbb{R}^3$ with a coordinate frame such that the triangle lies in the plane $\mathbf{e}_3^{\perp} \subset \mathbb{R}^3$ and has its vertices at $\tilde{\mathbf{c}}_0 = 0, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2$, with the understanding that there is a point inside, with squared distances s_i to these vertices. Then, we use three real parameters, x_0, x_1 and x_2, to describe the possible positions of the three centers:

$$\mathbf{c}_0 = \tilde{\mathbf{c}}_0 + x_0\mathbf{e}_3, \qquad \mathbf{c}_1 = \tilde{\mathbf{c}}_1 + x_1\mathbf{e}_3, \qquad \mathbf{c}_2 = \tilde{\mathbf{c}}_2 + x_2\mathbf{e}_3.$$

We use Proposition 2 to express the corresponding direction-sextic σ and its Hessian $H(\sigma)$ as functions of $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ depending on $\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2, s_0, s_1, s_2$. Proposition 5 is now equivalent to proving that

$$H(\sigma)(0, 0, 1) \neq 0$$

holds for all initial data (triangle and interior point) and all (x_0, x_1, x_2) corresponding to disjoint balls.

4.2 A Quadric and a Quartic

We have reduced the probe for flexes to the study of a polynomial function of $\mathbf{x}$ (and parameters) which can be explicitly computed.

The parameters involved are the following:

$$\tilde{\mathbf{c}}_0 = (0,0,0), \qquad \tilde{\mathbf{c}}_1 = (a,0,0), \qquad \tilde{\mathbf{c}}_2 = (b,c,0),$$

the triangle of centers $(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2)$ having interior point:

$$p = \frac{\sum p_i \tilde{\mathbf{c}}_i}{\sum p_i} = \frac{p_1\tilde{\mathbf{c}}_1 + p_2\tilde{\mathbf{c}}_2}{\sum p_i}, \quad p_0, p_1, p_2 > 0.$$

Let $\mathbf{v}_k = \mathbf{p} - \tilde{\mathbf{c}}_k$. Then $s_k = r_k^2 = \langle \mathbf{v}_k, \mathbf{v}_k \rangle$.

The computation gives the result:

$$H(\sigma)(0,0,1) = \frac{2^{12}5^2a^6c^6}{(\sum p_i)^5}[H_2(\mathbf{x}) + H_4(\mathbf{x})],$$

where H_2 and H_4 have degree respectively 2 and 4 in $\mathbf{x} = (x_0, x_1, x_2)$:

$$H_2 = H_2(\mathbf{x}) = -a^2c^2 \left(\prod p_k\right) \sum p_i p_j (x_i - x_j)^2,$$
$$H_4 = H_4(\mathbf{x}) = \sum p_k^3 s_k (x_i - x_k)^2 (x_j - x_k)^2,$$

with cyclic products and sums for $\{i, j, k\} = \{0, 1, 2\}$. Thus, away from $(0,0,0)$, H_2 is negative and H_4 is positive. The aim is now to show that ball disjointness is enough to ensure the positivity of $H_2 + H_4$.

4.3 Hyperboloid and Octant

We can further transform these expressions by retaining as parameters the (positive numbers) p_i and $q_j = p_j r_j$, and renaming the squares $z_k = (x_i - x_j)^2$. This gives:

$$H_2 = H_2(\mathbf{z}) = -a^2c^2 \left(\prod p_k\right) \sum p_i p_j z_k,$$
$$H_4 = H_4(\mathbf{z}) = \sum p_k q_k^2 z_i z_j.$$

From now on, assume that $\sum p_i = 1$. We have to replace $\Delta = a^2c^2$, which is four times the squared area of the triangle $\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2$, by its expression in terms of p_i and q_j.

Lemma 6 *We have*:

$$\Delta = a^2c^2 = \frac{Q}{4\prod p_k^2}, \quad \textit{with } Q = \sum (2q_i^2 q_j^2 - q_k^4).$$

Proof This is an elementary computation, which may be conducted as follows. By the definition of $\mathbf{v}_i$, we have

$$\sum p_i \mathbf{v}_i = 0.$$

From $\langle \sum p_i \mathbf{v}_i, \mathbf{v}_j \rangle = 0$, we obtain a linear system for $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$, $i \neq j$:

$$p_i \langle \mathbf{v}_i, \mathbf{v}_k \rangle + p_j \langle \mathbf{v}_j, \mathbf{v}_k \rangle = -p_k \langle \mathbf{v}_k, \mathbf{v}_k \rangle = -p_k s_k,$$

with solutions:

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \frac{p_k^2 s_k - p_i^2 s_i - p_j^2 s_j}{2 p_i p_j} = \frac{q_k^2 - q_i^2 - q_j^2}{2 p_i p_j}.$$

Four times the squared area of a triangle $\mathbf{p}, \tilde{\mathbf{c}}_i, \tilde{\mathbf{c}}_j$ is a Gram determinant:

$$\begin{vmatrix} \langle \mathbf{v}_i, \mathbf{v}_i \rangle & \langle \mathbf{v}_i, \mathbf{v}_j \rangle \\ \langle \mathbf{v}_i, \mathbf{v}_j \rangle & \langle \mathbf{v}_j, \mathbf{v}_j \rangle \end{vmatrix} = s_i s_j - \langle \mathbf{v}_i, \mathbf{v}_j \rangle^2 = \frac{Q}{4 p_i^2 p_j^2},$$

where $Q = \sum (2 q_i^2 q_j^2 - q_k^4)$. Hence the area of the triangle $\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2$ is:

$$\frac{1}{4} Q^{1/2} \sum \frac{1}{p_i p_j} = \frac{Q^{1/2}}{4 \prod p_k},$$

resulting in:

$$\Delta = a^2 c^2 = \frac{Q}{4 \prod p_k^2}. \qquad \square$$

Several new substitutions will be in order for the study of $H_2 + H_4$. Since a positive factor won't affect sign considerations, we will use the symbol $*H$ for any positive multiple of $H_2 + H_4$. We have found above:

$$*H = *H(\mathbf{z}) = -\frac{1}{4} Q \sum \frac{z_k}{p_k} + \sum p_k q_k^2 z_i z_j,$$

with the shorthand $Q = \sum (2 q_i^2 q_j^2 - q_k^4)$. We put $p_i p_j z_k = q_k^2 w_k$ and obtain, up to a positive factor:

$$*H = *H(\mathbf{w}) = -\frac{1}{4} Q \sum q_k^2 w_k + \prod q_k^2 \sum w_i w_j.$$

With one more positive rescaling, and $a_k = \frac{Q}{4 q_i^2 q_j^2}$, we have:

$$*H = *H(\mathbf{w}) = \sum w_i w_j - \sum a_k w_k.$$

We can turn now to the conditions expressing the fact that the spheres with centers $\mathbf{c}_i = \tilde{\mathbf{c}}_i + x_i \mathbf{e}_3$ and radii r_i are disjoint. They are:

$$z_k = (x_i - x_j)^2 > (r_i + r_j)^2 - \delta_{ij} = (r_i + r_j)^2 - \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{v}_i - \mathbf{v}_j \rangle,$$

that is,

$$z_k > \frac{q_k^2 - (q_i - q_j)^2}{p_i p_j}.$$

In $\mathbf{w}$-coordinates, the "*disjointness conditions*" become

$$w_k > 1 - \left(\frac{q_i - q_j}{q_k}\right)^2.$$

Note that from $\sum p_i \mathbf{v}_i = 0$ it follows that $q_k = \|p_i \mathbf{v}_i\| > 0$ are the lengths of the three edges in a triangle, and therefore the latter expressions are positive by the triangle inequality.

The purpose now is to study the position of the octant defined by the disjointness conditions relative to the affine quadric in $\mathbb{R}^3$ defined by $*H(\mathbf{w}) = 0$. We use first a translation by β, in order to absorb the linear part in $*H$:

$$*H = *H(\mathbf{w}) = \sum (w_i - \beta_i)(w_j - \beta_j) - \sum \beta_i \beta_j,$$

with β respecting:

$$\beta_i + \beta_j = a_k, \quad \text{that is } \beta_k = \frac{1}{2}(a_i + a_j - a_k).$$

This makes

$$\sum \beta_i \beta_j = \frac{1}{4} \sum (a_k + a_i - a_j)(a_k - a_i + a_j) = \frac{1}{4} \sum (2a_i a_j - a_k^2),$$

and results in

$$\sum \beta_i \beta_j = \frac{1}{4} \left(\frac{Q}{4 \prod q_k^2}\right)^2 \sum (2q_i^2 q_j^2 - q_k^4) = \frac{Q^3}{4^3 \prod q_k^4} > 0.$$

Thus, with translated coordinates $t_k = w_k - \beta_k$ we have a *hyperboloid of two sheets*:

$$*H = *H(\mathbf{t}) = \sum t_i t_j - \frac{Q^3}{4^3 \prod q_k^4} = 0,$$

which lies on the positive side of its asymptotic cone $\sum t_i t_j = 0$.

Lemma 7 $\sum t_i t_j = 0$ *is a circular cone with axis* $t_0 = t_1 = t_2$. *The two components of its smooth points circumscribe the positive and negative open octants*, *which are both contained in the positive part* $\sum t_i t_j > 0$.

The open octant defined by our disjointness conditions $w_k > 1 - (\frac{q_i - q_j}{q_k})^2$ is a translate of the open positive octant, and its position relative to the hyperboloid $*H(\mathbf{w}) = 0$ is determined by the position of its vertex $\mathbf{V}$. Continuing to refer here to $\mathbf{w}$-coordinates, we have:

Lemma 8 *The point* $\mathbf{V} = (1 - (\frac{q_i - q_j}{q_k})^2)_{0 \le k \le 2}$ *is on the "positive side" of the hyperboloid* $*H(\mathbf{w}) = 0$ *and on the "positive side" of the plane* $\sum t_k = \sum (w_k - \beta_k) = 0$, *that is*:

$$*H(\mathbf{V}) > 0 \quad \textit{and} \quad \sum\left(1 - \left(\frac{q_i - q_j}{q_k}\right)^2\right) > \frac{Q}{8\prod q_k^2} \sum q_k^2.$$

Proof A Maple assisted computation shows that $*H(\mathbf{V})$ factors as

$$*H(\mathbf{V}) = \frac{3\prod(q_i + q_j - q_k)^2}{4\prod q_k^2},$$

from which the first inequality follows.

The second inequality, which determines on which of the two components of the positive side of the hyperboloid $\mathbf{V}$ lies, is satisfied for $q_0 = q_1 = q_2$, and by continuity, must be satisfied for any other triangle edges, since vertex $\mathbf{V}$ cannot "jump" from one component to the other. □

It is now clear, geometrically, that the octant where the disjointness conditions are satisfied and the hyperboloid indicating a flex or a singularity for the corresponding configuration *have no point in common*. This completes the proof of Proposition 5.

5 Convexity of the Cone for 3 Balls in $\mathbb{R}^3$

We consider now three *disjoint* closed balls B_0, B_1, B_2 described by parameters: centers $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ and radii r_0, r_1, r_2. We shall prove first the convexity of any cone of directions in the *generic* case i.e. when the centers and radii are in the complement of a proper algebraic subset. Then, we will show that the generic case implies the general case.

Lemma 9 *The direction cone* $K(B_0B_1B_2)$ *of a generic triple of disjoint balls in* $\mathbb{R}^3$ *is strictly convex.*

Proof If $\partial K(B_0B_1B_2)$ is made only of directions of inner special bitangents, strict convexity is immediate, since $K(B_0B_1B_2)$ is then an intersection of convex regions bounded by conics. Otherwise, genericity allows us to assume that the direction-sextic σ is non-singular at all its contacts with any of the three conics determined by inner special tangents. Since the direction-sextic necessarily lies on the simply-connected side of each of the three conics, these contacts are tangency points at which $\partial K(B_0B_1B_2)$ is locally convex. Thus, if we start at some point of $\partial K(B_0B_1B_2)$ and follow the boundary curve, we obtain, by Proposition 5, a differentiable simple loop of class C^1, which is, locally, always on the same side of its tangent. For any affine plane $\mathbb{R}^2 \subset \mathbb{P}^2$ covering the loop, and any Euclidean metric in it, this means positive curvature on all its algebraic arcs and this implies [23] that our simple loop bounds a compact convex set. In fact *strictly convex*, because of non-vanishing curvature. By Proposition 4 and its Corollary, this strictly convex set is $K(B_0B_1B_2)$. □

The passage from the generic case to the general case is based on:

Lemma 10 *Let $\mathcal{B} = (B_0, B_1, B_2)$ be a configuration of three disjoint closed balls, and suppose $K(B_0B_1B_2)$ has non-empty interior. If $\mathcal{B}$ is the limit of a sequence of configurations $\mathcal{B}^{(\nu)}$ with a convex cone of directions for the given ordering, then $K(B_0B_1B_2)$ is convex as well.*

Proof By Proposition 4, it is enough to prove that, for any two points in the interior, the (geodesic) segment joining them is contained in $K(B_0B_1B_2)$.

Take two interior points. By assumption, for sufficiently large ν, the segment joining them is contained in all corresponding cones for $\mathcal{B}^{(\nu)}$. Consider one point of the segment, and project the sphere configuration along the direction defined by the point, on a perpendicular plane. We have to prove that the disks representing the projected balls have at least one point in common.

Suppose they don't. Then so would discs with the same centers and radii increased by a small $\epsilon > 0$. But then we can find, for sufficiently large ν, configurations $\mathcal{B}^{(\nu)}$ with centers projecting less than $\epsilon/2$ away from those of $\mathcal{B}$ and corresponding radii with less than $\epsilon/2$ augmentation. Then the point of the segment cannot be in the respective cones of directions, a contradiction.

Note that strict convexity still follows from non-zero curvature on smooth arcs for non-collinear centers, while for collinear centers it is obvious because of rotational symmetry. □

Lemmas 9 and 10 immediately imply Theorem 1 for the case of three balls in $\mathbb{R}^3$:

Proposition 11 *The directions of all oriented lines intersecting three disjoint balls in $\mathbb{R}^3$ in a specific order form a strictly convex subset of the sphere $\mathbb{S}^2$.*

6 Convexity of the Cone for *n* Balls in $\mathbb{R}^d$

The convexity result of Proposition 11 generalizes to arbitrary n and d as follows:

Proof of Theorem 1 Recall that, for any collection of balls in $\mathbb{R}^3$, a direction will be realized by some transversal if and only if the orthogonal projection of the balls on a perpendicular plane has non-empty intersection. By Helly's Theorem in the plane, the direction cone for a sequence of $n \geq 3$ balls is the intersection of the direction cones of all its triples. Thus, the direction cone of n ordered 3-dimensional disjoint balls is strictly convex for any n.

Given a sequence $\mathcal{S}$ of n disjoint balls in $\mathbb{R}^d$, let K be its direction cone for a prescribed order of intersection. Let $\mathbf{u}$ and $\mathbf{v}$ be two directions in K, $\ell_\mathbf{u}$ and $\ell_\mathbf{v}$ be two corresponding line transversals and let E denote the 3-dimensional affine space these two lines span (or a 3-space containing their planar span, should the lines be coplanar).

$E \cap \mathcal{S}$ is a collection of 3-dimensional disjoint balls whose corresponding direction cone is convex on $\mathbb{S}^2$. Thus, for any direction on the small arc of great circle joining $\mathbf{u}$ and $\mathbf{v}$ there exists an order-respecting transversal to $\mathcal{S}$, because it already exists

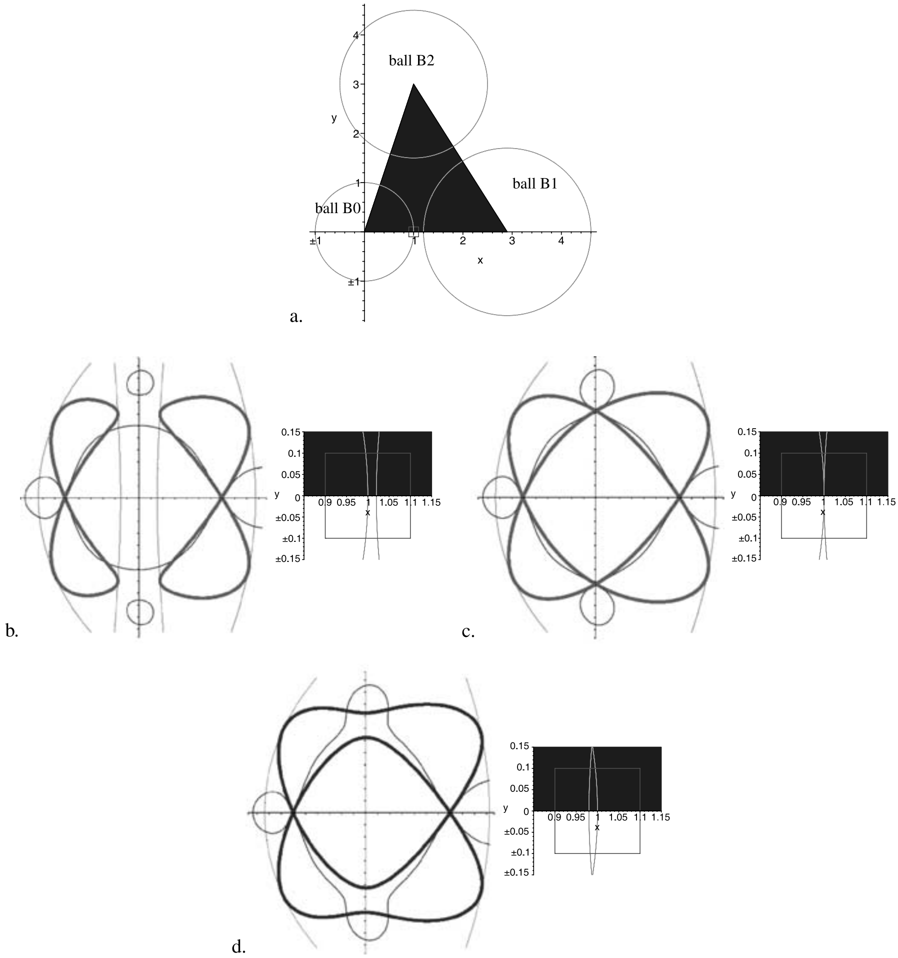

Fig. 2 **a** The trace of three disjoint balls *on the plane of centers*, with ball B_1 moving *on the horizontal axis* towards ball B_0. *The small square* is used for close-ups below. **b**, **c**, **d** The direction-sextic (*in thick gray*), its Hessian (*in black*) and arcs of inner special bitangent conics, when balls B_0 and B_1 are disjoint (**b**), tangent (**c**) and intersecting (**d**)

in E. It follows that K is convex, and again, from the three dimensional case, strictly convex. □

Let us emphasize the importance of the assumption that the balls are disjoint. Figure 2 illustrates a transition from convex to non-convex direction cones as three disjoint balls move and allow an overlap.

7 Implications

This section explores some consequences of Theorem 1. Similar results were proven for the case of unit balls in [6] and, with Theorem 1, the proofs carry through. We thus omit all arguments here and point to the relevant lemmata in [6].

7.1 Isotopy and Geometric Permutations

An immediate corollary of Theorem 1 is the correspondence of isotopy and geometric permutations for line transversals to disjoint balls:

Corollary 12 *The set of line transversals to n disjoint balls in $\mathbb{R}^d$ realizing the same geometric permutation is contractible.*

The proof given by Cheong et al. [6, Lemma 14] for disjoint unit balls immediately extends, with Theorem 1, to the case of disjoint balls.

Smorodinsky et al. [21] showed that in the worst case n disjoint balls in $\mathbb{R}^d$ admit $\Theta(n^{d-1})$ geometric permutations. The same bound thus applies for the number of connected components of line transversals, improving on the previous bounds of $O(n^{3+\epsilon})$ for $d=3$ and of $O(n^{2d-2})$ for $d \geq 4$ due to Koltun and Sharir [16]. If the radii of the balls are in some interval $[1,\gamma]$ where γ is independent of n and d, then the number of components of transversals is $O(\gamma^{\log\gamma})$, following the bound on the number of geometric permutations obtained by Zhou and Suri [24]. These results are summarized as follows:

Corollary 13 *In the worst case, n disjoint balls in $\mathbb{R}^d$ have $\Theta(n^{d-1})$ connected components of line transversals. If the radii of the balls are in the interval $[1,\gamma]$, where γ is independent of n and d, this number becomes $O(\gamma^{\log\gamma})$.*

7.2 Minimal Pinning Configurations

A *minimal pinning configuration* is a collection of objects having an isolated line transversal that ceases to be isolated if any of the objects is discarded. An important step in the proof of Hadwiger's transversal theorem [11] is the observation that, in the plane, any minimal pinning configuration consisting of disjoint convex objects has cardinality 3. Cheong et al. [6, Proposition 13] proved that any minimal pinning configuration consisting of disjoint unit balls in $\mathbb{R}^d$ has cardinality at most $2d-1$. With Theorem 1, the same holds for disjoint balls of arbitrary radii:

Corollary 14 *Any minimal pinning configuration consisting of disjoint balls in $\mathbb{R}^d$ has cardinality at most $2d-1$.*

7.3 A Hadwiger-Type Result

A result in the flavor of Hadwiger's Transversal Theorem [6, Theorem 1] generalizes to disjoint balls of arbitrary radii:

Corollary 15 *A sequence of n disjoint balls in $\mathbb{R}^d$ has a line transversal if any subsequence of size at most $2d$ has an order-respecting line transversal.*

The "pure" generalizations [6, 14] of Helly's theorem, i.e. without additional constraints on the ordering à la Hadwiger, use the fact that $n \geq 9$ disjoint unit balls have at most 2 geometric permutations [5]. Since the latter is not true for balls of arbitrary radii [21], obtaining a Helly-type theorem for line transversals in this case requires different arguments.

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