LINEAR AUTOMATON TRANSFORMATIONS

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Let R be a nonempty set, let N consist of all non-negative rational integers, and denote by R^N the set of all functions on N to R. If R is a ring, a map M: $R^N \to R^N$ is *linear* if $M(r_1f_1+r_2f_2)=r_1(Mf_1)+r_2(Mf_2)$ for r_1, r_2 in R, f_1, f_2 in R^N. For a finite commutative ring with unit we determine which linear transformations M: $R^N \to R^N$ can be realized by finite automata.

More precisely, let A, B be finite nonempty sets. A map M: $A^N \to B^N$ is an *automaton transformation* if there exists a finite set Q, maps M_Q: $A \times Q \to Q$, M_B: $A \times Q \to B$, elements $\bar{b}$ in B, $\bar{q}$ in Q such that corresponding to each f in A^N there exists an h in Q^N satisfying

$$(1)\qquad \begin{gathered} h(0) = \bar{q}, \quad h(n+1) = M_Q(f(n), h(n)), \quad (Mf)(0) = \bar{b}, \\ (Mf)(n+1) = M_B(f(n), h(n)). \end{gathered}$$

(In automaton language, A is the input alphabet, B is the output alphabet, Q is the set of states, $\bar{q}$ is the initial state, $\bar{b}$ is the initial output, while $M_B(a, q)$ and $M_Q(a, q)$ are respectively the output and state resulting from input a and state q. For the case that A and B coincide with the set consisting of 0 and 1, the concept of automaton transformation is simply a variant of the concept of representable event of Kleene [1].)

Call a matrix u_{ij}: $N \times N \to R$ *eventually doubly-periodic* if for some positive integers P_1, P_2, p_1, p_2:

$$(2)\qquad u_{ij} = u_{(i+p_1)j} \quad \text{for all } i > P_1 \text{ and all } j,$$

$$(3)\qquad u_{ij} = u_{i(j+p_2)} \quad \text{for all } j > P_2 \text{ and all } i.$$

THEOREM 1. *Let R be a finite commutative ring with unit. Then M: $R^N \to R^N$ is a linear automaton transformation if and only if there exists a matrix u_{ij}: $N \times N \to R$ such that:*

(i) *for all j, $u_{0j}=0$;*

(ii) *for f in R^N and $n \geqq 0$, $(Mf)(n)=u_{n0}f(0)+u_{(n-1)1}f(1)+\cdots+u_{0n}f(n)$;*

(iii) *u_{ij} is eventually doubly-periodic.*

Define τ: $R^N \to R^N$ by $(\tau f)(0)=0$, $(\tau f)(n)=f(n-1)$, $n \geqq 1$. A map M: $R^N \to R^N$ is *translation invariant* if for f in R^N, $M\tau f=\tau Mf$. Call a sequence u_0, u_1, $\cdots$ *eventually periodic* if there exist positive integers

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P, p such that $u_{n+p}=u_n$ for $n \geqq P$, then p is a *period*.

COROLLARY. *Let R be a finite commutative ring with unit. Then $M: R^N \to R^N$ is a linear translation invariant automaton transformation if and only if there exists an eventually periodic sequence $u_0=0$, u_1, $u_2, \cdots$ of elements of R such that for f in R^N, $(Mf)(n)=u_0 f(n)+\cdots+u_n f(0)$.*

Consider a linear difference equation

$$\begin{aligned}(4)\quad & S_1(n-1)F(n-1)+\cdots+S_k(n-k)F(n-k)\\ &\qquad =G(n)+T_1(n-1)G(n-1)+\cdots+T_k(n-k)G(n-k),\end{aligned}$$

where $S_1, \cdots, S_k, T_1, \cdots, T_k, F, G$ are functions on the set of rational integers (positive and negative) to R which vanish for negative arguments. For fixed $S_1, \cdots, S_k, T_1, \cdots, T_k$, (4) induces a linear map $M: R^N \to R^N$ given by the requirement that whenever F, G jointly satisfy (4), and f is a member of R^N such that $f(n)=F(n)$ for $n \geqq 0$, then $(Mf)(n)=G(n)$ for $n \geqq 0$.

THEOREM 2. *Let R be a finite commutative ring with unit. Then $M: R^N \to R^N$ is a linear automaton transformation if and only if induced by a linear difference equation* (4) *with $S_1, \cdots, S_k, T_1, \cdots, T_k$ eventually periodic for $n \geqq 0$.*

COROLLARY. *Let R be a finite commutative ring with unit. Then $M: R^N \to R^N$ is a translation invariant linear automaton transformation if and only if induced by a linear difference equation* (4) *with $S_1, \cdots, S_k, T_1, \cdots, T_k$ constant for $n \geqq 0$.*

We will need three lemmas to prove Theorems 1 and 2.

LEMMA 1. *Let R be a finite commutative ring with unit. Endow R with the discrete, R^N with the product topology. Then $L: R^N \to R$ is linear and continuous if and only if there exists a finite sequence $W_0, \cdots, W_m$ of elements of R such that for f in R^N, $Lf=W_0 f(0)+\cdots+W_m f(m)$.*

PROOF. It is an easy consequence of the compactness of R^N and the continuity of L that there exists an m such that $Lf_1=Lf_2$ whenever f_1, f_2 are in R^N and agree for $n \leqq m$. If we put $\delta_k(n)=1$ or 0 as $n=k$ or not, then we may take $W_k=L\delta_k$ for $k \leqq m$.

Call $M: A^N \to B^N$ *causal* if: for f_1, f_2 in A^N, $(Mf_1)(0)=(Mf_2)(0)$; for f_1, f_2 in A^N and $k>0$, if $f_1(n)=f_2(n)$ for $n<k$, then $(Mf_1)(k)=(Mf_2)(k)$. Denote by $\sigma(A)$ the set of finite sequences $(x_0, \cdots, x_j)$ consisting of elements from a finite set A. Call two such sequences

$(x_0, \cdots, x_j)$, $(y_0, \cdots, y_k)$ *state-equivalent* (relative to M) if for any f in A^N, $(Mf_1)(n+j+1) = (Mf_2)(n+k+1)$ for all $n \geqq 0$, where f_1, f_2 are chosen satisfying: $f_1(n) = x_n$ for $0 \leqq n < j$, $f_1(n) = f(n-j)$ for $n \geqq j$, $f_2(n) = y_n$ for $0 \leqq n < k$, $f_2(n) = f(n-k)$ for $n \geqq k$. (Note that the state-equivalence of two sequences does not depend on the last member of either.) Define an *intrinsic state* for M to be an equivalence class under state-equivalence.

LEMMA 2. *Let A, B be finite nonempty sets. Then M: $A^N \to B^N$ is an automaton transformation if and only if M is causal and M possesses only a finite number of intrinsic states. Further, the least number of states required in order to induce M as in* (1) *is the number of intrinsic states.*

PROOF. Suppose that M is an automaton transformation. Then M is certainly causal due to (1). We show that M possesses no more intrinsic states than the number of elements of Q. If $X = (x_0, \cdots, x_j)$ is in $\sigma(A)$, define q_X to be the $h(j)$ determined from (1) by letting $f(n) = x_n$ for all $n < j$. Then X, Y in $\sigma(A)$ are state-equivalent whenever $q_X = q_Y$.

Conversely, if M is causal and possesses only a finite set Q of intrinsic states, define $\bar{b}$, $\bar{q}$, M_B, M_Q as follows.

(i) Let $\bar{b} = (Mf)(0)$ for any f in A^N.

(ii) Let $\bar{q}$ be the intrinsic state of any finite sequence of length 1.

(iii) Let $M_Q(a, q_1) = q_2$ if for some X in q_1, Y in q_2, we have $X = (x_0, \cdots, x_j)$, $Y = (y_0, \cdots, y_{j+1})$, $x_n = y_n$ for all $n < j$, $y_j = a$. Let $M_B(a, q_1) = (Mf)(j+1)$ if f is a member of A^N such that $f(n) = y_n$ for $n \leqq j$.

LEMMA 3. *If $S_1, \cdots, S_k, T_1, \cdots, T_k$ are eventually periodic for $n \geqq 0$, then* (4) *induces a linear automaton transformation.*

PROOF. We wish to apply Lemma 2; it suffices to show that M has only a finite number of intrinsic states, since any M induced by Equation (4) is causal. Let p_i, p_i' be periods for S_i, T_i, $i = 1, \cdots, k$. Then for n_1 sufficiently large, the intrinsic state of a finite sequence $(x_0, \cdots, x_{n+1})$ is determined for $n \geqq n_1$ by $F(n-1), \cdots, F(n-k)$, $G(n-1), \cdots, G(n-k)$, n mod $p_1, \cdots, n$ mod p_k, n mod $p_1', \cdots$, n mod p_k'. Thus for $n \geqq n_1$, finite sequences fall into at most $z^{2k} p_1 \cdots p_s p_1' \cdots p_s'$ distinct intrinsic states, where z is the number of elements of R. Thus M has altogether only a finite number of intrinsic states.

We now prove Theorems 1 and 2. If M is a linear automaton transformation, then for each $n \geqq 0$, the map L_n: $R^N \to R$ given by

$L_n f = (Mf)(n)$ is linear and continuous. Thus Lemma 1 applies and there exists a matrix W_{nk}: $N \times N \rightarrow R$ such that for each $n \geqq 0$ we can find an $m \geqq 0$ satisfying $(Mf)(n) = W_{n0}f(0) + \cdots + W_{nm}f(m)$, for all f in R^N. Causality implies $W_{nk} = 0$ for $k \geqq n$. Setting $u_{ij} = W_{(i+j)j}$ we need only verify (2) and (3) to satisfy Theorem 1.

(5) Suppose that $M: A^N \rightarrow B^N$ is an automaton transformation, and that f is a member of A^N such that $f(0), f(1), f(2), \cdots$ is eventually periodic. Then $(Mf)(0), (Mf)(1), (Mf)(2), \cdots$ is eventually periodic. Moreover, if q_n is the intrinsic state of $(f(0), \cdots, f(n))$, then $q_0, q_1, q_2, \cdots$ is eventually periodic.

We employ (5) to prove (2) and (3). Since the kth column of u_{ij} consists of the entries $0, (M\delta_k)(k+1), (M\delta_k)(k+2), (M\delta_k)(k+3), \cdots$ it follows that this column is completely determined by the intrinsic state of a k-term sequence consisting of $k-1$ zero entries followed by a one. Since this sequence has the same intrinsic state as a k-term sequence consisting of zeros, (5) applies to show that this intrinsic state is an eventually periodic function of k, and hence proves (3).

With this done, (2) is easy since it now suffices to show that the kth column is itself eventually periodic. But (5) applied to $M\delta_k$ yields this.

Conversely, suppose that M is defined by a matrix u_{ij} satisfying (i), (ii), (iii) of Theorem 1. Define functions U_i by $U_i(j) = u_{ij}$ for $j \geqq 0$, $U_i(j) = 0$ for $j < 0$. Then the following linear difference equation induces M when recast in form (4). (In the notation of (2), put $k = p_1 + P_1$.)

$$\begin{aligned} &U_1(n-1)F(n-1) + \cdots + U_k(n-k)F(n-k) \\ &\quad - U_1(n-p_1-1)F(n-p_1-1) - \cdots - U_{P_1}(n-k)F(n-k) \\ &\quad = G(n) - G(n-p_1). \end{aligned}$$

By (3), $U_1, \cdots, U_k$ are eventually periodic for $n \geqq 0$; hence by Lemma 3, M is an automaton transformation. This proves both Theorem 1 and Theorem 2.

Reference

1. S. C. Kleene, *Representation of events in nerve nets and finite automata*, Automata Studies, Princeton University Press, 1956, pp. 3–41.

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