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LINEAR BOUNDARY VALUE TYPE PROBLEMS  
FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS  
AND THEIR ADJOINTS

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0. INTRODUCTION

The paper deals with boundary value type problems for functional-differential equations

$$(0,1) \quad \dot{x}(t) = \int_{-r}^0 [d_{\vartheta}P(t, \vartheta)] x(t + \vartheta) + f(t) \quad \text{a.e. on } [a, b]$$

or

$$(0,2) \quad \dot{x}(t) = A(t)x(t) + B(t)x(t-r) + \int_{a-r}^b [d_sG(t, s)] x(s) + f(t) \quad \text{a.e. on } [a, b],$$

where  $-\infty < a < b < \infty$  and the functions  $P(t, \vartheta)$ ,  $G(t, s)$ ,  $A(t)$ ,  $B(t)$  and  $f(t)$  fulfil some natural assumptions. In particular, we derive their adjoints and in some special cases prove the Fredholm alternative. (The results of A. HALANAY [5] or E. A. LIFŠIC [9] on the existence of periodic solutions to the equation (0,1) and the results of [12] on integral boundary value problems for ordinary integro-differential equations are included.) Our approach is based on the ideas of D. WEXLER [14] and ŠT. SCHWABIK [11] and differs from that of A. Halanay [6] or D. HENRY [8] (cf. also J. K. HALE [7]). The adjoint problems obtained seem to be more natural than those of D. Henry [8] and follow directly from the principles of functional analysis. (It is shown that after some artificial steps our adjoint reduces to that of D. Henry.) Initial functions are continuous on  $[a-r, a]$  or of bounded variation on  $[a-r, a]$ . In § 4 boundary value type problems for hereditary differential equations considered in the sense of M. C. DELFOUR, S. K. MITTER [3] (with square integrable initial functions) are treated.

## 1. PRELIMINARIES

Let  $-\infty < \alpha < \beta < +\infty$ . The closed interval  $\alpha \leq t \leq \beta$  is denoted by  $[\alpha, \beta]$ , its interior  $\alpha < t < \beta$  by  $(\alpha, \beta)$  and the corresponding half-open intervals by  $[\alpha, \beta)$  and  $(\alpha, \beta]$ . Given a  $p \times q$ -matrix  $M = (m_{i,j})_{i=1,\dots,p,j=1,\dots,q}$ ,  $M'$  denotes its transpose and

$$\|M\| = \max_{i=1,\dots,p} \sum_{j=1}^q |m_{i,j}|.$$

$\mathcal{R}_n$  is the space of real column  $n$ -vectors with the norm  $\|x\| = \max_{i=1,\dots,p} |x_i|$ . The space of real row  $n$ -vectors is  $\mathcal{R}_n^*$ . (Elements of  $\mathcal{R}_n^*$  are denoted by  $x'$ , where  $x \in \mathcal{R}_n$ ;

$$\|x'\| = \sum_{i=1}^n |x_i|.)$$

$\mathcal{C}_n(\alpha, \beta)$  is the Banach space (B-space) of continuous functions  $u : [\alpha, \beta] \rightarrow \mathcal{R}_n$  with the norm  $\|u\|_{\mathcal{C}} = \sup_{t \in [\alpha, \beta]} \|u(t)\|$ ;  $\mathcal{BV}_n(\alpha, \beta)$  is the B-space of functions  $u : [\alpha, \beta] \rightarrow \mathcal{R}_n$  of bounded variation on  $[\alpha, \beta]$  with the norm  $\|u\|_{\mathcal{BV}} = \|u(\beta)\| + \text{var}_{\alpha}^{\beta} u$ ;  $\mathcal{V}_n^0(\alpha, \beta)$  is the set of functions  $u' : [\alpha, \beta] \rightarrow \mathcal{R}_n^*$  of bounded variation on  $[\alpha, \beta]$ , right continuous on  $(\alpha, \beta)$  and vanishing at  $\beta$  (being equipped with the norm  $\|u'\|_{\mathcal{BV}}$ ,  $\mathcal{V}_n^0(\alpha, \beta)$  becomes a B-space);  $\mathcal{AC}_n(\alpha, \beta)$  is the B-space of absolutely continuous functions  $u : [\alpha, \beta] \rightarrow \mathcal{R}_n$  with the norm  $\|u\|_{\mathcal{AC}} = \|u\|_{\mathcal{BV}}$ ;  $\mathcal{L}_n(\alpha, \beta)$  is the B-space of Lebesgue integrable (L-integrable) functions  $u : [\alpha, \beta] \rightarrow \mathcal{R}_n$  with the norm

$$\|u\|_{\mathcal{L}} = \int_{\alpha}^{\beta} \|u(t)\| dt;$$

$\mathcal{L}_n^{\infty}(\alpha, \beta)$  is the B-space of essentially bounded functions  $u' : [\alpha, \beta] \rightarrow \mathcal{R}_n^*$  with the norm  $\|u'\| = \sup_{t \in [\alpha, \beta]} \text{ess } \|u'(t)\|$ .

Given a B-space  $\mathcal{X}$ ,  $\mathcal{X}^*$  denotes its dual and the value of a functional  $y \in \mathcal{X}^*$  on  $x \in \mathcal{X}$  is denoted by  $\langle x, y \rangle_{\mathcal{X}}$ . The zero functional on  $\mathcal{X}$  is denoted by  $o_{\mathcal{X}}$ . Hereafter  $\mathcal{L}_n^*(\alpha, \beta)$  and  $\mathcal{C}_n^*(\alpha, \beta)$  are identified with  $\mathcal{L}_n^{\infty}(\alpha, \beta)$  and  $\mathcal{V}_n^0(\alpha, \beta)$ , respectively, while

$$\langle x, y' \rangle_{\mathcal{L}} = \int_{\alpha}^{\beta} y'(t) x(t) dt \quad \text{and} \quad \langle u, v' \rangle_{\mathcal{C}} = \int_{\alpha}^{\beta} [dv'(t)] u(t)$$

for  $x \in \mathcal{L}_n(\alpha, \beta)$ ,  $y' \in \mathcal{L}_n^{\infty}(\alpha, \beta)$ ,  $u \in \mathcal{C}_n(\alpha, \beta)$  and  $v' \in \mathcal{V}_n^0(\alpha, \beta)$ . (There are isometric isomorphisms between  $\mathcal{L}_n^*(\alpha, \beta)$  and  $\mathcal{L}_n^{\infty}(\alpha, \beta)$  and between  $\mathcal{C}_n^*(\alpha, \beta)$  and  $\mathcal{V}_n^0(\alpha, \beta)$ , cf. e.g. [4].)

Let  $\mathcal{X}, \mathcal{Y}$  be B-spaces. Given a linear bounded operator  $T : \mathcal{X} \rightarrow \mathcal{Y}$  (defined on the whole  $\mathcal{X}$ ),  $T^*$  denotes its adjoint ( $T^* : \mathcal{Y}^* \rightarrow \mathcal{X}^*$ ,  $\langle Tx, y \rangle_{\mathcal{Y}} = \langle x, T^*y \rangle_{\mathcal{X}}$  for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}^*$ ),  $\text{Ker}(T)$  is the set of all  $x \in \mathcal{X}$  such that  $Tx = 0 =$  zero element of  $\mathcal{Y}$  and  $\text{Im}(T)$  is the range of  $T$ . Given two operators  $T_1 : \mathcal{X}_1 \rightarrow \mathcal{Y}$ ,  $T_2 : \mathcal{X}_2 \rightarrow \mathcal{Y}$ ,

the homogeneous equations  $T_1x = 0$  and  $T_2z = 0$  are said to be equivalent if there is a one-to-one correspondence between  $\text{Ker}(T_1)$  and  $\text{Ker}(T_2)$ .

## 2. GENERAL BOUNDARY VALUE TYPE PROBLEM AND ITS ADJOINT

**2.1. Assumptions.** We assume  $-\infty < a < b < +\infty$ ,  $r > 0^*$ ).  $A(t)$  and  $B(t)$  are  $n \times n$ -matrix functions L-integrable on  $[a, b]$ ,  $G(t, s)$  is a Borel measurable in  $(t, s)$  on  $[a, b] \times [a - r, b]$   $n \times n$ -matrix function such that  $\text{var}_{a-r}^b G(t, \cdot) < \infty$  for any  $t \in [a, b]$  and

$$\int_a^b (\|G(t, b)\| + \text{var}_{a-r}^b G(t, \cdot)) dt < \infty,$$

$f(t) \in \mathcal{L}_n(a, b)$ .  $\Lambda$  is an arbitrary B-space,  $l \in \Lambda$  and the operators  $M : \mathcal{C}_n(a - r, a) \rightarrow \Lambda$ ,  $N : \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \Lambda$  are linear and bounded, while  $\text{Im}(N^*) \subset \mathcal{C}_n^*(a, b) = \mathcal{V}_n^0(a, b)$  (i.e., given  $\lambda \in \Lambda^*$ , there is a function  $(N^*\lambda)(t) \in \mathcal{V}_n^0(a, b)$  such that

$$\langle Nx, \lambda \rangle_\Lambda = \langle x, N^*\lambda \rangle_{\mathcal{A}\mathcal{C}} = \int_a^b [d(N^*\lambda)(t)] x(t) \text{ for all } x \in \mathcal{A}\mathcal{C}_n(a, b)).$$

Without any loss of generality we may also assume that, given  $t \in [a, b]$ , the function  $G(t, \cdot)$  is right continuous on  $(a - r, b)$ , while  $G(t, b) = 0$ . Given  $\lambda \in \Lambda^*$ , let us denote by  $(M^*\lambda)(t)$  the row  $n$ -vector function such that  $(M^*\lambda)(t) - (N^*\lambda)(a) \in \mathcal{V}_n^0(a - r, a)$  and

$$\langle Mu, \lambda \rangle_\Lambda = \langle u, M^*\lambda \rangle_{\mathcal{C}} = \int_{a-r}^a [d\{(M^*\lambda)(t) - (N^*\lambda)(a)\}] u(t)$$

for all  $u \in \mathcal{C}_n(a - r, a)$ .

We are interested in the following boundary value type problem:

**2.2. Problem (P).** Determine  $x \in \mathcal{A}\mathcal{C}_n(a, b)$  and  $u \in \mathcal{C}_n(a - r, a)$  such that

$$(2,1) \quad \dot{x}(t) = A(t)x(t) + \begin{cases} B(t)u(t-r), & t < a+r \\ B(t)x(t-r), & t \geq a+r \end{cases} + \int_{a-r}^a [d_s G(t, s)] u(s) + \\ + \int_a^b [d_s G(t, s)] x(s) + f(t) \text{ a.e. on } [a, b],$$

$$(2,2) \quad x(a) = u(a),$$

$$(2,3) \quad Mu + Nx = l,$$

where Assumptions 2,1 are fulfilled.

\*) If  $r = 0$ , the equation (2,1) reduces to an ordinary integro-differential equation with initial data in  $\mathcal{R}_n$ . The case of  $r = 0$  will be treated separately later on (cf. Sec. 5,5).

**2.3. Notation.** Let us put

$$\mathcal{X} = \mathcal{AC}_n(a, b) \times \mathcal{C}_n(a - r, a), \quad \mathcal{Y} = \mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$$

and

$$(2,4) \quad U : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X} \rightarrow \begin{pmatrix} Dx - Ax - B_1x - B_2u - G_1x - G_2u \\ Mu + Nx \\ u(a) - x(a) \end{pmatrix} \in \mathcal{Y},$$

where

$$D : x \in \mathcal{AC}_n(a, b) \rightarrow \dot{x}(t) \in \mathcal{L}_n(a, b),$$

$$A : x \in \mathcal{AC}_n(a, b) \rightarrow A(t)x(t) \in \mathcal{L}_n(a, b),$$

$$B_1 : x \in \mathcal{AC}_n(a, b) \rightarrow \begin{cases} 0, & t < a + r \\ B(t)x(t - r), & t \geq a + r \end{cases} \in \mathcal{L}_n(a, b),$$

$$B_2 : u \in \mathcal{C}_n(a - r, a) \rightarrow \begin{cases} B(t)u(t - r), & t < a + r \\ 0, & t \geq a + r \end{cases} \in \mathcal{L}_n(a, b),$$

$$G_1 : x \in \mathcal{AC}_n(a, b) \rightarrow \int_a^b [d_s G(t, s)] x(s) \in \mathcal{L}_n(a, b),$$

$$G_2 : u \in \mathcal{C}_n(a - r, a) \rightarrow \int_{a-r}^a [d_s G(t, s)] u(s) \in \mathcal{L}_n(a, b).$$

All these operators are linear and bounded. The given problem (P) can be reformulated as the operator equation

$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ 0 \end{bmatrix}.$$

Clearly,  $\mathcal{X}^* = \mathcal{AC}_n^*(a, b) \times \mathcal{V}_n^0(a - r, a)$ ,  $\mathcal{Y}^* = \mathcal{L}_n^\infty(a, b) \times \Lambda^* \times \mathcal{R}_n^*$  and

$$\left\langle \begin{pmatrix} x \\ u \end{pmatrix}, (g, h') \right\rangle_{\mathcal{X}} = \langle x, g \rangle_{\mathcal{AC}} + \int_{a-r}^a [dh'(t)] u(t),$$

$$\left\langle \begin{bmatrix} f \\ l \\ k \end{bmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathcal{Y}} = \int_a^b y'(t) f(t) ds + \langle l, \lambda \rangle_{\Lambda} + \gamma' k$$

for  $x \in \mathcal{AC}_n(a, b)$ ,  $u \in \mathcal{C}_n^*(a - r, a)$ ,  $g \in \mathcal{AC}_n^*(a, b)$ ,  $h' \in \mathcal{V}_n^0(a - r, a)$ ,  $f \in \mathcal{L}_n(a, b)$ ,  $l \in \Lambda$ ,  $k \in \mathcal{R}_n$ ,  $y' \in \mathcal{L}_n^\infty(a, b)$ ,  $\lambda \in \Lambda^*$  and  $\gamma' \in \mathcal{R}_n^*$ . Let  $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}$  and  $(y', \lambda, \gamma') \in \mathcal{Y}^*$ ,

then

$$\begin{aligned} \left\langle U \begin{pmatrix} x \\ u \end{pmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathcal{Y}} &= \langle Dx - Ax - B_1x - B_2u - G_1x - G_2u, y' \rangle_{\mathcal{L}} + \\ &+ \langle Mu + Nx, \lambda \rangle_{\Lambda} + \gamma'(u(a) - x(a)) = \end{aligned}$$

$$= \langle x, D^*y' - A^*y' - B_1^*y' - G_1^*y' + N^*\lambda + K_1^*\gamma' \rangle_{\mathcal{A}\mathcal{C}} + \\ + \langle u, -B_2^*y' - G_2^*y' + M^*\lambda + K_2^*\gamma' \rangle_{\mathcal{C}},$$

where

$$K_1 : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow -x(a) \in \mathcal{R}_n$$

and

$$K_2 : u \in \mathcal{C}_n(a - r, a) \rightarrow u(a) \in \mathcal{R}_n.$$

Consequently

$$U^* : (y', \lambda, \gamma') \in \mathcal{Y}^* \rightarrow \begin{bmatrix} D^*y' - A^*y' - B_1^*y' - G_1^*y' + N^*\lambda + K_1^*\gamma' \\ -B_2^*y' - G_2^*y' + M^*\lambda + K_2^*\gamma' \end{bmatrix} \in \mathcal{X}^*$$

and the adjoint to (P) is the system of equations for  $(y', \lambda, \gamma') \in \mathcal{Y}^*$

$$(2,5) \quad \begin{aligned} D^*y' - A^*y' - B_1^*y' - G_1^*y' + N^*\lambda + K_1^*\gamma' &= 0_{\mathcal{A}\mathcal{C}}, \\ -B_2^*y' - G_2^*y' + M^*\lambda + K_2^*\gamma' &= 0_{\mathcal{C}}. \end{aligned}$$

**2.4. An analytic form of the adjoint problem.** By the definition of an adjoint operator and by the unsymmetric Fubini theorem (2) it holds for all  $x \in \mathcal{A}\mathcal{C}_n(a, b)$ ,  $u \in \mathcal{C}_n(a - r, a)$ ,  $y' \in \mathcal{L}_n^\infty(a, b)$ ,  $\lambda \in A^*$  and  $\gamma' \in \mathcal{R}_n^*$

$$\begin{aligned} \left\langle \begin{pmatrix} x \\ u \end{pmatrix}, U^*(y', \lambda, \gamma') \right\rangle_x &= \left\langle U \begin{pmatrix} x \\ u \end{pmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathcal{Y}} = \\ &= \int_a^b y'(t) \dot{x}(t) dt - \int_a^b y'(t) A(t) x(t) dt - \int_{a+r}^b y'(t) B(t) x(t - r) dt - \\ &\quad - \int_a^{a+r} y'(t) B(t) u(t - r) dt - \int_a^b y'(t) \left( \int_a^b [d_s G(t, s)] x(s) \right) dt - \\ &\quad - \int_a^b y'(t) \left( \int_{a-r}^a [d_s G(t, s)] u(s) \right) dt + \langle Mu + Nx, \lambda \rangle_A + \gamma'(u(a) - x(a)) = \\ &= \int_a^b y'(t) \dot{x}(t) dt - \int_a^b [dg'(t)] x(t) - \int_{a-r}^a [dh'(t)] u(t), \end{aligned}$$

where

$$(2,6) \quad \begin{aligned} g'(t) &= - \int_t^b y'(s) A(s) ds + \int_a^b y'(s) G(s, t) ds - (N^*\lambda)(t) - \\ &\quad - \begin{cases} \int_{t+r}^b y'(s) B(s) ds, & t \leq b - r \\ 0, & t > b - r \end{cases} - \begin{cases} \gamma', & t = a \\ 0, & t > a \end{cases} \quad \text{for } t \in [a, b], \\ h'(t) &= - \int_{t+r}^{a+r} y'(s) B(s) ds + \int_a^b y'(s) (G(s, t) - G(s, a)) ds + \begin{cases} \gamma', & t < a \\ 0, & t = a \end{cases} - \\ &\quad - (M^*\lambda)(t) + (N^*\lambda)(a) \quad \text{for } t \in [a - r, a]. \end{aligned}$$

Now,  $(y', \lambda, \gamma') \in \text{Ker}(U^*)$  iff

$$(2,7) \quad 0 = \int_a^b y'(t) \dot{x}(t) dt - \int_a^b [dg'(t)] x(t) - \int_{a-r}^a [dh'(t)] u(t)$$

for all  $x \in \mathcal{AC}_n(a, b)$  and  $u \in \mathcal{C}_n(a-r, a)$ . In particular, if  $x(t) = 0$  on  $[a, b]$ , (2,7) means that

$$\int_{a-r}^a [dh'(t)] u(t) = 0 \quad \text{for all } u \in \mathcal{C}_n(a-r, a).$$

Since  $h' \in \mathcal{V}_n^0(a-r, a)$ , this is possible iff  $h'(t) = 0$  on  $[a-r, a]$ . Thus

$$(2,8) \quad \int_{t+r}^{a+r} y'(s) B(s) ds - \int_a^b y'(s) (G(s, t) - G(s, a)) ds + \\ + (M^* \lambda)(t) - (N^* \lambda)(a) - \gamma' = 0 \quad \text{on } [a-r, a].$$

The equality (2,7) now becomes (after integrating by parts)

$$(2,9) \quad \int_a^b y'(t) \dot{x}(t) dt = -g'(a) x(a) - \int_a^b g'(t) \dot{x}(t) dt \quad \text{for all } x \in \mathcal{AC}_n(a, b).$$

Since we may choose  $x(t) = x(a) \neq 0$  on  $[a, b]$ , (2,9) implies furthermore  $g'(a) = 0$  or

$$(2,10) \quad \gamma' = - \int_a^b y'(s) A(s) ds - \int_{a+r}^b y'(s) B(s) ds + \int_a^b y'(s) G(s, a) ds - (N^* \lambda)(a).$$

Consequently, (2,9) reduces to

$$\int_a^b y'(t) \dot{x}(t) dt = - \int_a^b g'(t) \dot{x}(t) dt \quad \text{for all } x \in \mathcal{AC}_n(a, b)$$

or

$$\int_a^b (y'(t) + g'(t)) z(t) dt = 0 \quad \text{for all } z \in \mathcal{L}_n(a, b).$$

Hence  $y'(t) = g'(t)$  a.e. on  $[a, b]$ , i.e.

$$(2,11) \quad y'(t) = \int_t^b y'(s) A(s) ds + \begin{cases} \int_{t+r}^b y'(s) B(s) ds, & t \leq b-r \\ 0, & t > b-r \end{cases} - \\ - \int_a^b y'(s) G(s, t) ds + (N^* \lambda)(t) \quad \text{a.e. on } [a, b].$$

Let  $z' \in \mathcal{L}_n^\infty(a, b)$ . Then  $(z', \lambda, \gamma') \in \text{Ker}(U^*)$  iff there exists  $y' \in \mathcal{L}_n^\infty(a, b)$  fulfilling (2,8) and (2,10) and such that  $y(t) = z(t)$  a.e. on  $[a, b]$  and (2,11) holds for all  $t \in$

$\in (a, b)$ . Finally, inserting (2,10) into (2,8) and taking into account that the right hand side of (2,11) is of bounded variation on  $[a, b]$  and right continuous on  $(a, b)$ , we complete the proof of the following

**2,5. Theorem.** Let  $z' \in \mathcal{L}_n^\infty(a, b)$ ,  $\lambda \in \Lambda^*$  and  $\gamma' \in \mathcal{R}_n^*$ . Then  $(z', \lambda, \gamma') \in \text{Ker}(U^*)$  iff there exists  $y \in \mathcal{BV}_n(a, b)$  right continuous on  $(a, b)$  (the values  $y(a), y(b)$  may be arbitrary) such that  $y(t) = z(t)$  a.e. on  $[a, b]$  and

$$(2,12) \quad \int_a^b y'(s) A(s) ds + \int_{t+r}^b y'(s) B(s) ds - \int_a^b y'(s) G(s, t) ds + (M^*\lambda)(t) = 0$$

for  $t \in [a - r, a)$ ,

$$(2,13) \quad y'(t) = \int_t^b y'(s) A(s) ds + \begin{cases} \int_{t+r}^b y'(s) B(s) ds, & t \leq b - r \\ 0, & t > b - r \end{cases} - \\ - \int_a^b y'(s) G(s, t) ds + (N^*\lambda)(t) \quad \text{for } t \in (a, b),$$

while  $\gamma'$  is given by (2,10).

**2,6. Definition.** The problem (P\*) of finding  $y \in \mathcal{BV}_n(a, b)$  right continuous on  $(a, b)$  and  $\lambda \in \Lambda^*$  such that (2,12) and (2,13) hold is called the *conjugate problem* to (P).

(In virtue of Theorem 2,5 the adjoint problem (2,5) to (P) and the problem (P\*) conjugate to (P) are equivalent.)

**2,7. Corollary.** The problem (P) has a solution only if

$$(2,14) \quad \int_a^b y'(s) f(s) ds + \langle l, \lambda \rangle_\Lambda = 0$$

for all solutions  $(y', \lambda)$  of the conjugate problem (P\*). If the operator  $U$  defined by (2,4) has a closed range  $\text{Im}(U)$  in  $\mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$ , then the condition (2,14) is also sufficient for the existence of a solution to the problem (P).

(The proof follows from Theorem 2,5 and from the fundamental "alternative" theorem concerning linear equations in B-spaces ([4], VI § 6).)

**2,8. Remark.** Let  $\mathcal{X}, \mathcal{Y}$  be B-spaces and let  $L: \mathcal{X} \rightarrow \mathcal{Y}$  be linear and bounded. A set  $\mathcal{Y}^+ \subset \mathcal{Y}^*$  of linear continuous functionals on  $\mathcal{Y}$  is said to be total in  $\mathcal{Y}^*$  if  $\langle y, g \rangle_{\mathcal{Y}} = 0$  for all  $g \in \mathcal{Y}^+$  implies  $y = 0$ . Furthermore, if  $L^+ : \mathcal{Y}^+ \rightarrow \mathcal{X}^*$  is a linear operator such that  $\langle Lx, g \rangle_{\mathcal{Y}} = \langle x, L^+g \rangle_{\mathcal{X}}$  for all  $x \in \mathcal{X}$  and  $g \in \mathcal{Y}^+$ , we shall say that  $L^+$  is a conjugate operator to  $L$  with respect to  $\mathcal{Y}^+$ . Clearly,  $L^+$  is a restriction of the adjoint operator  $L^*$  to  $L$  on  $\mathcal{Y}^+$ . Hence  $\text{Ker}(L^+) \subset \text{Ker}(L^*)$ . (For some more details concerning conjugate operators see [11].)



Now, let  $\mathcal{V}_n(a, b)$  be the space of row  $n$ -vector functions of bounded variation on  $[a, b]$  and right continuous on  $(a, b)$ . Then  $\mathcal{V}_n(a, b)$  is a total subset in  $\mathcal{L}_n^\infty(a, b)$ . (In fact, let  $f \in \mathcal{L}_n^\infty(a, b)$  and

$$0 = \int_a^b y'(t) f(t) dt \quad \text{for all } y' \in \mathcal{V}_n(a, b).$$

Then

$$(2,15) \quad 0 = \int_a^b y'(t) dg(t) \quad \text{for all } y' \in \mathcal{V}_n(a, b),$$

where  $g \in \mathcal{AC}_n(a, b)$  is an indefinite integral of  $f$  on  $[a, b]$ . Let  $g_i(t_1) \neq g_i(t_2)$  for a component  $g_i$  of the vector  $g = (g_1, g_2, \dots, g_n)'$  and for some  $t_1, t_2 \in [a, b]$ ,  $t_1 < t_2$ . Analogously to the second part of the proof of Lemma 5,1 in [10] we put  $y'(t) = (y_1(t), y_2(t), \dots, y_n(t))$ , where  $y_j(t) = 0$  on  $[a, b]$  for  $j \neq i$ ,  $y_i(t) = 0$  for  $t \in [a, t_1)$ ,  $y_i(t) = 1$  for  $t \in [t_1, t_2)$  and  $y_i(t) = 0$  for  $t \in [t_2, b]$ . Then  $y' \in \mathcal{V}_n(a, b)$  and

$$\int_a^b y'(t) dg(t) = \sum_{j=1}^n \int_a^b y_j(t) dg_j(t) = \int_a^b y_i(t) dg_i(t) = \int_{t_1}^{t_2} dg_i(t) = g_i(t_2) - g_i(t_1) \neq 0$$

which contradicts (2,15). Hence  $g(t) = \text{const.}$  on  $[a, b]$  and  $f(t) = 0$  a.e. on  $[a, b]$ .

The operator  $D : x \in \mathcal{AC}_n(a, b) \rightarrow \dot{x} \in \mathcal{L}_n^\infty(a, b)$  is linear and bounded. It is easy to verify that its conjugate operator  $D^+$  with respect to  $\mathcal{V}_n(a, b)$  is given by

$$D^+ : y' \in \mathcal{V}_n(a, b) \rightarrow \begin{cases} 0, & t = a \\ -y'(t), & t \in (a, b) \\ 0, & t = b \end{cases} \in \mathcal{V}_n^0(a, b).$$

Let us put  $\mathcal{Y}^+ = \mathcal{V}_n(a, b) \times \Lambda^* \times \mathcal{R}_n^*$ . Then  $\mathcal{Y}^+$  is a total subset in  $\mathcal{Y}^* = \mathcal{L}_n^\infty(a, b) \times \Lambda^* \times \mathcal{R}_n^*$  and the conjugate operator  $U^+$  to  $U$  with respect to  $\mathcal{Y}^+$  is given by

$$U^+ : (y', \lambda, \gamma') \in \mathcal{Y}^+ \rightarrow (\xi'(t), \eta'(t)) \in \mathcal{V}_n^0(a, b) \times \mathcal{V}_n^0(a - r, a),$$

where

$$\xi'(t) = \begin{cases} 0, & t = a \\ -y'(t), & t \in (a, b) \\ 0, & t = b \end{cases} + \int_t^b y'(s) A(s) ds + \begin{cases} \int_{t+r}^b y'(s) B(s) ds, & t \leq b - r \\ 0, & t > b - r \end{cases} - \\ - \int_a^b y'(s) G(s, t) ds + (N^* \lambda)(t) + \begin{cases} \gamma', & t = a \\ 0, & t > a \end{cases} \quad \text{for } t \in [a, b],$$

$$\eta'(t) = \int_{t+r}^{a+r} y'(s) B(s) ds - \int_a^b y'(s) (G(s, t) - G(s, a)) ds + (M^* \lambda)(t) - (N^* \lambda)(a) - \\ - \begin{cases} \gamma', & t < a \\ 0, & t = a \end{cases} \quad \text{for } t \in [a - r, a].$$

The equation  $U^+(y', \lambda, \gamma') = 0$  is identical with the system of equations (2,8), (2,10), (2,13) and hence it is equivalent also with the problem (P\*) introduced in Definition 2,6. In Section 2,4 we proved actually that  $\text{Ker}(U^*) \subset \mathscr{W}^+$  and hence  $\text{Ker}(U^+) = \text{Ker}(U^*)$ .

**2,9. Remark.** The above procedure can be also applied to the case of initial functions of bounded variation on  $[a - r, a]$ . This means that instead of  $u \in \mathscr{C}_n(a - r, a)$  we are looking for  $u \in \mathscr{BV}_n(a - r, a)$ . The adjoint problem is again equivalent to the system of the form (2,12), (2,13). Only we have to suppose in addition that  $\text{Im}(M^*) \subset \mathscr{V}_n^0(a - r, a)$ .

**2,10. Remark.** Some examples of spaces  $\Lambda$  and operators  $M, N$  fulfilling Assumptions 2,1 are given in the following § 3. Some conditions on the closedness of  $\text{Im}(U)$  are given in § 5.

**2,11. Remark.** The couple  $(y', \lambda)$  being a solution to (P\*), the values  $y'(a), y'(b)$  may be arbitrary. We can require e.g.  $y'(a) = y'(b) = 0$  or  $y'(a+) = y'(a), y'(b-) = y'(b)$ . In the latter case we add to the system (2,12), (2,13) the conditions

$$(2,16) \quad y'(a) = - \int_a^b y'(s) (G(s, a+) - G(s, a-)) ds + (N^*\lambda)(a+) - (M^*\lambda)(a-),$$

$$y'(b) = \int_a^b y'(s) G(s, b-) ds - (N^*\lambda)(b-).$$

(Indeed, by (2,12)

$$\int_a^b y'(s) A(s) ds + \int_{a+r}^b y'(s) B(s) ds = \int_a^b y'(s) G(s, a-) ds - (M^*\lambda)(a-).$$

**2,12. Remark.**  $\mathscr{AC}_n^*(a, b)$  is isometrically isomorphic with  $\mathscr{L}_n^\infty(a, b) \times \mathscr{R}_n^*$ . Given  $g \in \mathscr{AC}_n^*(a, b)$ , there exist uniquely determined  $\beta' \in \mathscr{R}_n^*$  and  $y'(t) \in \mathscr{L}_n^\infty(a, b)$  such that

$$\langle x, g \rangle_{\mathscr{AC}} = \beta' x(a) + \int_a^b y'(t) \dot{x}(t) dt$$

for all  $x \in \mathscr{AC}_n(a, b)$ . (See [4] IV, 13, 29.) By a similar argument as in 2,4 we could derive the analytic form of the adjoint problem also in the case that  $N$  is a general linear bounded operator  $\mathscr{AC}_n(a, b) \rightarrow \Lambda$  without supposing  $\text{Im}(N^*) \subset \mathscr{V}_n^0(a, b)$ .

If  $N^* : \lambda \in \Lambda^* \rightarrow (N^*\lambda, \tilde{N}^*\lambda) \in \mathscr{L}_n^\infty(a, b) \times \mathscr{R}_n^*$  and  $(M^*\lambda)(t) - (\tilde{N}^*\lambda) \in \mathscr{V}_n^0(a - r, a)$  for any  $\lambda \in \Lambda^*$ , then the problem of finding  $(y'(t), \lambda) \in \mathscr{L}_n^\infty(a, b) \times \Lambda^*$  such that (2,12) holds on  $[a - r, a)$  and (2,13) holds a.e. on  $[a, b]$  is equivalent to the adjoint of the given problem (P).

### 3. SOME SPECIAL CASES

Let us mention some special cases of the given problem (P) which arise by a special choice of the boundary operators  $M, N$  and of the terminal space  $A$ .

**3.1. The case  $A = \mathcal{L}_m(c, d)$ .** Let  $A = \mathcal{L}_m(c, d)$  ( $-\infty < c < d < +\infty$ ) and

$$(3,1) \quad M : u \in \mathcal{C}_n(a-r, a) \rightarrow \int_{a-r}^a [d_s M(\alpha, s)] u(s) \in A,$$

$$(3,2) \quad N : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \int_a^b [d_s N(\alpha, s)] x(s) \in A,$$

where  $M(\alpha, s)$  is a Borel measurable in  $(\alpha, s) \in [c, d] \times [a-r, a]$   $m \times n$ -matrix function such that  $\text{var}_{a-r}^a M(\alpha, \cdot) < \infty$  for any  $\alpha \in [c, d]$  and

$$\int_c^d (\|M(\alpha, a)\| + \text{var}_{a-r}^a M(\alpha, \cdot)) d\alpha < \infty$$

and  $N(\alpha, s)$  is a Borel measurable in  $(\alpha, s) \in [c, d] \times [a, b]$   $m \times n$ -matrix function such that  $\text{var}_a^b N(\alpha, \cdot) < \infty$  for any  $\alpha \in [c, d]$  and

$$\int_c^d (\|N(\alpha, b)\| + \text{var}_a^b N(\alpha, \cdot)) d\alpha < \infty.$$

Without any loss of generality we may assume that for any  $\alpha \in [c, d]$ ,  $M(\alpha, \cdot)$  is right continuous on  $(a-r, a)$ ,  $N(\alpha, \cdot)$  is right continuous on  $(a, b)$ ,  $M(\alpha, a) = N(\alpha, a)$  and  $N(\alpha, b) = 0$ .

Let  $x \in AC_n(a, b)$ ,  $u \in C_n(a-r, a)$ ,  $\lambda' \in \mathcal{L}_m^\infty(c, d)$ . Then by the unsymmetric Fubini theorem ([2])

$$\begin{aligned} \langle Mu, \lambda' \rangle_{\mathcal{L}} &= \int_c^d \lambda'(\alpha) \left( \int_{a-r}^a [d_s M(\alpha, s)] u(s) \right) d\alpha = \\ &= \int_{a-r}^a \left[ d_s \int_c^d \lambda'(\alpha) (M(\alpha, s) - M(\alpha, a)) d\alpha \right] u(s) \end{aligned}$$

and

$$\langle Nx, \lambda' \rangle_{\mathcal{L}} = \int_c^d \lambda'(\alpha) \left( \int_a^b [d_s N(\alpha, s)] x(s) \right) d\alpha = \int_a^b \left[ d_s \int_c^d \lambda'(\alpha) N(\alpha, s) d\alpha \right] x(s),$$

where

$$(3,3) \quad (N^* \lambda') (t) = \int_c^d \lambda'(\alpha) N(\alpha, t) d\alpha \in \mathcal{V}_n^0(a, b)$$

and

$$(3,4) \quad (M^*\lambda')(t) - (M^*\lambda')(a) = \int_c^d \lambda'(\alpha) (M(\alpha, t) - M(\alpha, a)) d\alpha \in \mathcal{V}_n^0(a-r, a).$$

Hence in this case the adjoint problem is equivalent to the system (2,12), (2,13), where  $M^*$  and  $N^*$  have the special form (3,4) and (3,3), respectively.

**3.2. The case  $A = \mathcal{C}_m(c, d)$ .** Similarly we can treat the case of  $A = \mathcal{C}_m(c, d)$  ( $-\infty < c < d < +\infty$ ) with the operators  $M, N$  given by (3,1), (3,2), where  $M(\cdot, s)$  and  $N(\cdot, \sigma)$  are continuous on  $[c, d]$  for any  $s \in [a-r, a]$  and  $\sigma \in [a, b]$ . (Let us note that in this case any linear bounded operator  $M : \mathcal{C}_n(a-r, a) \rightarrow A$  can be expressed in the form (3,1), where  $M(\alpha, s)$  fulfils all our assumptions.) Analogously as in 3,1 we obtain

$$M^* : \lambda' \in \mathcal{V}_m^0(c, d) \rightarrow \int_c^d [d\lambda'(\alpha)] M(\alpha, t) \in \mathcal{V}_n(a-r, a),$$

$$N^* : \lambda' \in \mathcal{V}_m^0(c, d) \rightarrow \int_c^d [d\lambda'(\alpha)] N(\alpha, t) \in \mathcal{V}_n^0(a, b).$$

**3.3. Finite dimensional terminal space.** Let  $A = \mathcal{R}_m$  and

$$M : u \in \mathcal{C}_n(a-r, a) \rightarrow \int_{a-r}^a [dM(s)] u(s) \in \mathcal{R}_m,$$

$$N : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \int_a^b [dN(s)] x(s) \in \mathcal{R}_m,$$

where  $M(t)$  and  $N(t)$  are  $m \times n$ -matrix functions of bounded variation on  $[a-r, a]$  and  $[a, b]$ , respectively. We may assume also  $M$  right continuous on  $(a-r, a)$ ,  $N$  right continuous on  $(a, b)$ ,  $M(a) = N(a)$  and  $N(b) = 0$ .

Let  $x \in \mathcal{A}\mathcal{C}_n(a, b)$ ,  $u \in \mathcal{C}_n(a-r, a)$  and  $\lambda' \in \mathcal{R}_m^*$ , then

$$\langle Mu, \lambda' \rangle_{\mathcal{R}} = \lambda'(Mu) = \int_{a-r}^a [d\{\lambda'(M(s) - M(a))\}] u(s)$$

and

$$\langle Nx, \lambda' \rangle_{\mathcal{R}} = \lambda'(Nx) = \int_a^b [d(\lambda' N(s))] x(s),$$

where  $(M^*\lambda')(t) - (M^*\lambda')(a) = \lambda'(M(t) - M(a)) \in \mathcal{V}_n^0(a-r, a)$  and  $(N^*\lambda')(t) = \lambda' N(t) \in \mathcal{V}_n^0(a, b)$ .

The adjoint problem is equivalent to the conjugate problem (P\*) given by (2,12), (2,13) with  $M^*$  and  $N^*$  defined above. Moreover, we may write it in the form more

similar to the adjoint of the boundary value problem for ordinary integro-differential equation ([12]). Let us put for  $t \in [a, b]$

$$\begin{aligned} \tilde{M} &= N(a+) - M(a-), \quad \tilde{N} = -N(b-), \\ C(t) &= G(t, a+) - G(t, a-), \quad D(t) = -G(t, b-), \\ L(s) &= \begin{cases} N(a+) & \text{for } s = a, \\ N(s) & \text{for } a < s < b, \\ N(b-) & \text{for } s = b, \end{cases} \quad G_0(t, s) = \begin{cases} G(t, a+) & \text{for } s = a, \\ G(t, s) & \text{for } a < s < b, \\ G(t, b-) & \text{for } s = b. \end{cases} \end{aligned}$$

Then, requiring  $y'(a+) = y'(a)$ ,  $y'(b-) = y'(b)$  (cf. Remark 2,11) we obtain the conjugate problem (P\*) to (P) in the following form:

$$\begin{aligned} \int_a^b y'(s) A(s) ds + \int_{t+r}^b y'(s) B(s) ds - \int_a^b y'(s) \dot{G}(s, t) ds + \lambda' M(t) &= 0, \\ &\text{on } [a-r, a), \\ y'(t) &= y'(b) + \int_t^b y'(s) A(s) ds + \left\{ \begin{array}{l} \int_{t+r}^b y'(s) B(s) ds, \quad t \leq b-r \\ 0, \quad t > b-r \end{array} \right\} - \\ &- \int_a^b y'(s) (G_0(s, t) - G_0(s, b)) ds + \lambda'(L(t) - L(b)) \quad \text{on } [a, b], \\ y'(a) &= \lambda' \tilde{M} - \int_a^b y'(s) C(s) ds, \quad y'(b) = -\lambda' \tilde{N} + \int_a^b y'(s) D(s) ds. \end{aligned}$$

**3.4. Boundary value type problems for functional-differential equations of retarded type.** In this section we shall deal with boundary value problems for standard functional-differential equation

$$(3,5) \quad \dot{x}(t) = \int_{-r}^0 [d_\vartheta P(t, \vartheta)] x(t + \vartheta) + f(t) \quad \text{a.e. on } [a, b],$$

$$(3,6) \quad x(t) = u(t) \quad \text{on } [a-r, a],$$

$$(3,7) \quad Mu + Nx = l \in A,$$

where the initial functions  $u(t)$  are continuous on  $[a-r, a]$  and the following assumptions are fulfilled:

$P(t, \vartheta)$  is a Borel measurable in  $(t, \vartheta) \in [a, b] \times (-\infty, +\infty)$   $n \times n$ -matrix function such that  $P(t, \vartheta) = P(t, -r)$  for  $\vartheta \leq -r$ ,  $P(t, \vartheta) = P(t, 0)$  for  $\vartheta \geq 0$ ,  $\text{var}_{-r}^0 P(t, \cdot) < \infty$  for all  $t \in [a, b]$  and

$$\int_a^b (\|P(t, 0)\| + \text{var}_{-r}^0 P(t, \cdot)) dt < \infty.$$

$A$  is a B-space and the operators  $M: \mathcal{C}_n(a-r, a) \rightarrow A$  and  $N: \mathcal{A}\mathcal{C}_n(a, b) \rightarrow A$  are linear and bounded, while  $\text{Im}(N^*) \subset \mathcal{V}_n^0(a, b)$ . Furthermore,  $l \in A$  and  $f(t) \in \mathcal{L}_n(a, b)$ . We may also assume that  $P(t, \cdot)$  is right continuous on  $(-r, 0)$  and  $P(t, 0) = 0$  for any  $t \in [a, b]$ .

Let us put for  $t \in [a, b]$

$$B(t) = P(t, -r+) - P(t, -r), \quad G(t, s) = \begin{cases} P(t, -r+) & \text{if } s \leq t-r, \\ P(t, s-t) & \text{if } t-r \leq s \leq t, \\ P(t, 0) = 0 & \text{if } s \geq t. \end{cases}$$

Then  $B(t)$  and  $G(t, s)$  fulfil Assumptions 2.1. Moreover, given  $t \in [a, b]$ ,  $G(t, \cdot)$  is right continuous on  $(a-r, b)$ ,  $G(t, b) = 0$  and

$$\begin{aligned} \int_{-r}^0 [d_{\vartheta} P(t, \vartheta)] x(t + \vartheta) &= \int_{t-r}^t [d_s P(t, s-t)] x(s) = \\ &= B(t) x(t-r) + \int_{a-r}^b [d_s G(t, s)] x(s). \end{aligned}$$

The problem (3.5)–(3.7) is reduced to the problem of the type (P). Furthermore, for  $t \in [a-r, a]$

$$\begin{aligned} \int_{t+r}^b y'(s) B(s) ds - \int_a^b y'(s) G(s, t) ds &= \int_{t+r}^b y'(s) (P(s, -r+) - P(s, -r)) ds - \\ - \int_a^{t+r} y'(s) P(s, t-s) ds - \int_{t+r}^b y'(s) P(s, -r+) ds &= - \int_a^b y'(s) P(s, t-s) ds. \end{aligned}$$

Analogously for  $t \in (a, b-r)$

$$\int_{t+r}^b y'(s) B(s) ds - \int_a^b y'(s) G(s, t) ds = - \int_t^b y'(s) P(s, t-s) ds$$

and

$$- \int_a^b y'(s) G(s, t) ds = - \int_t^b y'(s) P(s, t-s) ds \quad \text{for } t \in [b-r, b].$$

The following theorem is now a direct consequence of Theorem 2.5.

**3.5. Theorem.** *The problem of finding  $y \in \mathcal{B}\mathcal{V}_n(a, b)$  right continuous on  $(a, b)$  (the values  $y(a), y(b)$  may be arbitrary) and  $\lambda \in A^*$  such that*

$$(3.8) \quad - \int_a^b y'(s) P(s, t-s) ds + (M^* \lambda)(t) = 0 \quad \text{on } [a-r, a),$$

$$(3.9) \quad y'(t) = - \int_t^b y'(s) P(s, t-s) ds + (N^* \lambda)(t) \quad \text{on } (a, b)$$

*is equivalent to the adjoint problem to the problem (3.5)–(3.7).*

(The functions  $(M^*\lambda)(t)$  and  $(N^*\lambda)(t)$  are again such that for any  $\lambda \in A^*$   $(M^*\lambda)(t) - (N^*\lambda)(a) \in \mathcal{V}_n^0(a-r, a)$ ,  $(N^*\lambda)(t) \in \mathcal{V}_n^0(a, b)$  and

$$\langle Mu, \lambda \rangle_A = \int_{a-r}^a [d\{(M^*\lambda)(t) - (M^*\lambda)(a)\}] u(t),$$

$$\langle Nx, \lambda \rangle_A = \int_a^b [d(N^*\lambda)(t)] x(t)$$

for all  $u \in \mathcal{C}_n(a-r, a)$ ,  $x \in \mathcal{A}\mathcal{C}_n(a, b)$  and  $\lambda \in A^*$ .)

**3.6. Two-point boundary value type problem.** Let us consider the "two-point" boundary value type problem given by the system (3,5), (3,6) and

$$(3,10) \quad Mu + N_b x = l \in A,$$

where the functions  $P(t, \vartheta)$ ,  $f(t)$  and the operator  $M$  satisfy the corresponding assumptions of Section 3.4. Given  $\lambda \in A^*$ , let  $(M^*\lambda)(t)$  denote now a function from  $\mathcal{V}_n^0(a-r, a)$  such that

$$\langle Mu, \lambda \rangle_A = \int_{a-r}^a [d(M^*\lambda)(t)] u(t)$$

for all  $u \in \mathcal{C}_n(a-r, a)$  and  $\lambda \in A^*$ . The operator  $N_b = NS_b : \mathcal{A}\mathcal{C}_n(a, b) \rightarrow A$  is the composition of a linear bounded operator  $N : \mathcal{C}_n(b-r, b) \rightarrow A$  and of a shift operator  $S_b : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow x/[b-r, b] \in \mathcal{C}_n(b-r, b)$  (which is also linear and bounded). Let  $0 < r \leq b-a$ .

Let  $x \in \mathcal{A}\mathcal{C}_n(a, b)$  and  $\lambda \in A^*$ . Then

$$\langle N_b x, \lambda \rangle_A = \langle S_b x, N\lambda \rangle_{\mathcal{C}} = \int_{b-r}^b [d(N^*\lambda)(t)] x(t)$$

where  $(N^*\lambda)(t) \in \mathcal{V}_n^0(b-r, b)$ , and putting

$$(\tilde{N}^*\lambda)(t) = \begin{cases} (N^*\lambda)(b-r+) & \text{for } t = b-r, \\ (N^*\lambda)(t) & \text{for } b-r < t \leq b \end{cases}$$

and

$$(N_b^*\lambda)(t) = \begin{cases} (N^*\lambda)(b-r) & \text{for } a \leq t < b-r \\ (\tilde{N}^*\lambda)(t) & \text{for } b-r \leq t \leq b \end{cases} \in \mathcal{V}_n^0(a, b),$$

we get finally

$$\langle N_b x, \lambda \rangle_A = \int_a^b [d(N_b^*\lambda)(t)] x(t).$$

Since all the assumptions of Section 3.4 are satisfied, the following assertion is an immediate consequence of Theorem 3.5.

**3.7. Corollary.** *The problem of finding  $y \in \mathcal{BV}_n(a, b)$  right continuous on  $(a, b)$  (the values  $y(a), y(b)$  may be arbitrary) and  $\lambda \in A^*$  such that*

$$(3,11) \quad \int_a^b y'(s) P(s, t-s) ds - (M^*\lambda)(t) = (N^*\lambda)(b-r) \quad \text{on } [a-r, a],$$

$$(3,12) \quad y'(t) + \int_t^b y'(s) P(s, t-s) ds = (N^*\lambda)(b-r) \quad \text{on } (a, b-r),$$

$$(3,13) \quad y'(t) + \int_t^b y'(s) P(s, t-s) ds - (\tilde{N}^*\lambda)(t) = 0 \quad \text{on } [b-r, b]$$

is equivalent to the adjoint problem to the two-point problem (3,5), (3,6), (3,10).

**3.8. Relationship with the adjoint of D. Henry.** Let us continue the investigation of the two-point boundary value type problem (3,5), (3,6), (3,10). We shall show that the adjoint problem (3,11) derived in 3,6 can be reduced to the form of D. Henry [8]. Let us put for  $\vartheta \in [-r, 0]$   $P(t, \vartheta) = P(t+b-a, \vartheta)$  if  $t \in [a-r, a]$ . Given a function  $z(t)$  defined on  $[a-r, b]$  and  $t \in [a, b]$ , we put

$$z_t^0(\alpha) = \begin{cases} z(t+\alpha) & \text{if } \alpha \in [-r, 0], \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Let  $\mathcal{V}_n(-r, 0)$  be the space of all row  $n$ -vector functions of bounded variation on  $[-r, 0]$  and right continuous on  $(-r, 0)$ . Let  $R(\beta, \alpha)$  be the resolvent kernel for the Volterra integral equation

$$z'(\alpha) + \int_\alpha^0 z'(\beta) P(b+\beta, \alpha-\beta) d\beta = 0, \quad \alpha \in [-r, 0].$$

Gronwall's inequality applied to the „resolvent equation”

$$R(\beta, \alpha) + \int_\beta^\alpha R(\beta, \gamma) P(b+\gamma, \alpha-\gamma) d\gamma = P(b+\beta, \alpha-\beta); \quad \alpha, \beta \in [-r, 0]$$

yields analogously as in the proof of Lemma 1 in [14] that  $\text{var}_-^0 R(\beta, \cdot) < \infty$  for any  $\beta \in [-r, 0]$ , while the function  $r(\beta) = \text{var}_-^0 R(\beta, \cdot)$  is bounded on  $[-r, 0]$ . Hence the resolvent operator

$$R : w'(\alpha) \in V_n(-r, 0) \rightarrow \int_\alpha^0 w'(\beta) R(\beta, \alpha) d\beta \in \mathcal{V}_n(-r, 0)$$

is linear and bounded and for any  $w' \in \mathcal{V}_n(-r, 0)$ , the unique solution  $z'(\alpha)$  on  $[-r, 0]$  to

$$z'(\alpha) + \int_\alpha^0 z'(\beta) P(b+\beta, \alpha-\beta) d\beta = w'(\alpha)$$



is given by

$$z' = w' - R w' = (I - R) w',$$

where  $I$  denotes the identity operator.

Now, let  $(y', \lambda)$  be a solution to (3,11)–(3,13). Let us extend the function  $y'(t)$  on the interval  $[a - r, a]$  in such a way that

$$(3,15) \quad y'(t) + \int_t^a y'(s) P(s, t - s) ds = -(M^* \lambda)(t) \quad \text{for } t \in [a - r, a]$$

and

$$(3,16) \quad y'(a) + \int_a^b y'(s) P(s, a - s) ds = (N^* \lambda)(b - r).$$

Since

$$\begin{aligned} \int_t^a y'(s) P(s, t - s) ds &= \int_{t-a}^0 y'(a + \beta) P(a + \beta, t - a - \beta) d\beta = \\ &= \int_x^0 y'(a + \beta) P(b + \beta, \alpha - \beta) d\beta, \quad \text{where } \alpha = t - a, \end{aligned}$$

(3,15) yields

$$(3,17) \quad y_a^{\circ} = -(I - R)(M^* \lambda).$$

The last equation (3,13) in our conjugate system is obviously equivalent to the condition

$$(3,18) \quad y_b^{\circ} = (I - R)(\tilde{N}^* \lambda).$$

Finally, owing to (3,15) and (3,16) the equations (3,11) and (3,12) can be replaced by the single equation

$$(3,19) \quad y'(t) + \int_t^b y'(s) P(s, t - s) ds = (N^* \lambda)(b - r) \quad \text{on } [a - r, b - r].$$

The system (3,17)–(3,19) is just the adjoint problem of D. Henry from [8]. (Only we have the expression depending on  $\lambda$  instead of an arbitrary constant on the right hand side of the Volterra integral equation on  $[a - r, b - r]$ .)

Obviously the couple  $(y', \lambda)$  being a solution to the system (3,17)–(3,19), it is a solution to (3,11)–(3,13).

**3,9. Periodic solutions.** Let  $a = 0$ ,  $b = T < \infty$  ( $r \leq T$ ). Let  $P(\cdot, \vartheta)$  be for any  $\vartheta \in [-r, 0]$  a  $T$ -periodic function on  $(-\infty, +\infty)$ . Let us consider the periodic problem consisting of the equations (3,5), (3,6) and

$$(3,20) \quad u(t) - x(T + t) = 0 \quad \text{for } t \in [-r, 0]$$

(i.e., in (3,10) we have  $A = \mathcal{C}_n(-r, 0)$ ,  $l = 0$ ,  $M = I$ ,  $N_T = NS_T$ ,  $N : z(t) \in \mathcal{C}_n(T-r, T) \rightarrow -z(T+s) \in \mathcal{C}_n(-r, 0)$ .)

By Corollary 3,7 the adjoint problem is equivalent to the system of equations for  $y'(t)$  of bounded variation on  $[-r, T]$  and right continuous on  $(-r, T-r) \cup (T-r, T)$  and for  $\lambda'(t) \in \mathcal{V}_n^0(-r, 0)$ ,

$$(3,21) \quad y'(t) + \int_t^T y'(s) P(s, t-s) ds = -\lambda'(-r) \quad \text{on } [-r, T-r],$$

$$(3,22) \quad y'(t) + \int_t^0 y'(s) P(s, t-s) ds = -\lambda'(t) \quad \text{on } [-r, 0],$$

$$(3,23) \quad y'(t) + \int_t^T y'(s) P(s, t-s) ds = -\lambda'(t-T) \quad \text{on } [T-r, T].$$

Indeed, since actually we are looking for  $y'(t)$  in the space  $\mathcal{L}_n^\infty(-r, T)$ , we may change the values of  $y'$  on a set of measure zero in  $[-r, T]$ . Hence we may put

$$y'(0) + \int_0^T y'(s) P(s, -s) ds = -\lambda'(-r)$$

and

$$y'(T-r) + \int_{T-r}^T y'(s) P(s, T-r-s) ds = -\lambda'(-r).$$

$$(P(s, -s+) = P(s, -s) \text{ for any } s \neq r \text{ and thus } y'(0+) = y'(0).)$$

Furthermore, since by the periodicity assumption on  $P(\cdot, \vartheta)$

$$\int_t^T y'(s) P(s, t-s) ds = \int_{t-T}^0 y'(T+\beta) P(\beta, t-T-\beta) d\beta \quad \text{for } t \in [T-r, T],$$

the system (3,22), (3,23) is equivalent to the condition

$$(3,24) \quad y'(t) = y'(T+t) \quad \text{for } t \in [-r, 0].$$

**3,10. Corollary.** *The adjoint to the periodic problem (3,5), (3,6), (3,20) is equivalent to the problem of finding  $y(t) \in \mathcal{BV}_n(-r, T)$  right continuous on  $(-r, T-r) \cup (T-r, T)$  which satisfies (3,21) and (3,24), where  $\lambda'(-r)$  stands for an arbitrary constant  $n$ -vector.*

(In other words, the problem of finding  $T$ -periodic solutions to the equation

$$y'(t) + \int_t^T y'(s) P(s, t-s) ds = \text{const}$$

is a well posed adjoint problem to the problem of finding  $T$ -periodic solutions to the equation (3,5).)

4. BOUNDARY VALUE TYPE PROBLEMS FOR HEREDITARY  
DIFFERENTIAL EQUATIONS OF THE DELFOUR-MITTER TYPE

**4.1. Notation.** Let  $-\infty < \alpha < \beta < +\infty$ .  $\mathcal{L}_n^2(\alpha, \beta)$  is the Hilbert space of square integrable (column)  $n$ -vector functions on  $[\alpha, \beta]$  with the inner product

$$u, v \in \mathcal{L}_n^2(\alpha, \beta) \rightarrow (u, v)_{\mathcal{L}} = \int_{\alpha}^{\beta} u^{\vee}(s) v(s) ds = \int_{\alpha}^{\beta} v^{\vee}(s) u(s) ds.$$

(The corresponding norm on  $\mathcal{L}_n^2(\alpha, \beta)$  is given by

$$u \in \mathcal{L}_n^2(\alpha, \beta) \rightarrow \|u\|_{\mathcal{L}^2} = \left( \int_{\alpha}^{\beta} \|u(s)\|^2 ds \right)^{1/2}.$$

$\mathcal{W}_n^{1,2}(\alpha, \beta)$  is the Hilbert space of functions  $x : [\alpha, \beta] \rightarrow \mathcal{R}_n$  which are absolutely continuous on  $[\alpha, \beta]$  and whose derivatives  $Dx$  are square integrable on  $[\alpha, \beta]$ . The inner product and the corresponding norm are on  $\mathcal{W}_n^{1,2}(\alpha, \beta)$  given by

$$x, y \in \mathcal{W}_n^{1,2}(\alpha, \beta) \rightarrow (x, y)_{\mathcal{W}} = (Dx, Dy)_{\mathcal{L}} + (x, y)_{\mathcal{L}}$$

and

$$x \in \mathcal{W}_n^{1,2}(\alpha, \beta) \rightarrow \|x\|_{\mathcal{W}} = (\|Dx\|_{\mathcal{L}^2}^2 + \|x\|_{\mathcal{L}^2}^2)^{1/2}.$$

The corresponding spaces of row vector functions will be denoted also by  $\mathcal{L}_n^2(\alpha, \beta)$  and  $\mathcal{W}_n^{1,2}(\alpha, \beta)$ . No misunderstanding may arise.

**4.2. Assumptions.** Let  $-\infty < a < b < +\infty$  and  $r > 0$ . Let  $A(t)$  and  $B(t)$  be  $n \times n$ -matrix functions essentially bounded on  $[a, b]$  and  $f(t) \in \mathcal{L}_n^2(a, b)$ , let  $M$  and  $N$  be constant  $m \times n$ -matrices and  $l \in \mathcal{R}_m$ . Let  $\mathcal{A}$  be an arbitrary  $\mathcal{B}$ -space,  $w \in \mathcal{A}$  and let  $P : \mathcal{L}_n^2(a - r, a) \rightarrow \mathcal{A}$  and  $Q : \mathcal{W}_n^{1,2}(a, b) \rightarrow \mathcal{A}$  be linear and bounded operators.

**4.3. Problem ( $\pi$ ).** The subject of this paragraph is the following boundary value type problem ( $\pi$ )

Determine  $x \in \mathcal{W}_n^{1,2}(a, b)$ ,  $\xi \in \mathcal{R}_n$  and  $u \in \mathcal{L}_n^2(a - r, a)$  in such a way that

$$(4.1) \quad \dot{x}(t) - A(t)x(t) - \begin{cases} B(t)u(t-r), & t < a+r \\ B(t)x(t-r), & t \geq a+r \end{cases} = f(t) \quad \text{a.e. on } [a, b],$$

$$(4.2) \quad Pu + Qx = w,$$

$$(4.3) \quad M\xi + Nx(b) = l,$$

$$(4.4) \quad x(a) - \xi = 0.$$

Let  $\mathcal{W} = \mathcal{W}_n^{1,2}(a, b) \times \mathcal{R}_n \times \mathcal{L}_n^2(a - r, a)$ ,  $\mathcal{L} = \mathcal{L}_n^2(a, b) \times \mathcal{A} \times \mathcal{R}_m \times \mathcal{R}_n$  and

let the operators  $D, A, B_1 : \mathcal{W}_n^{1,2}(a, b) \rightarrow \mathcal{L}_n^2(a, b)$  and  $B_2 : \mathcal{L}_n^2(a - r, a) \rightarrow \mathcal{L}_n^2(a, b)$  be defined analogously as in 2,3 and

$$U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} \in \mathcal{W} \rightarrow \begin{bmatrix} Dx - Ax - B_1x - B_2u \\ Pu + Qx \\ M\xi + Nx(b) \\ x(a) - \xi \end{bmatrix} \in \mathcal{Z}.$$

The operator  $U$  is clearly linear and bounded and the given problem  $(\pi)$  is equivalent to the operator equation

$$U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} = \begin{bmatrix} f \\ w \\ l \\ 0 \end{bmatrix}.$$

**4.4. Remark.** The corresponding initial value problem (4,1) and (4,4) (with  $u \in \mathcal{L}_n^2(a - r, a)$  and  $\xi \in \mathcal{R}_n$  fixed) was studied in [3].

**4.5. Theorem.** Let  $\eta' \in \mathcal{L}_n^2(a, b)$ ,  $\lambda \in A^*$ ,  $\gamma' \in \mathcal{R}_m^*$  and  $\delta' \in \mathcal{R}_n^*$ . Then  $(\eta', \lambda, \gamma', \delta') \in \text{Ker}(U^*)$  iff there exists  $y' \in \mathcal{L}_n^2(a, b)$  such that  $y' + (d/dt)(Q^*\lambda) \in \mathcal{AC}_n(a, b)$ ,  $y(t) = \eta(t)$  a.e. on  $[a, b]$  and

$$\frac{d}{dt} \left[ y' + \frac{d}{dt}(Q^*\lambda) \right] (t) = -y'(t) A(t) - \begin{cases} y'(t+r) B(t+r), & t < b-r \\ 0, & t > b-r \end{cases} + \\ + (Q^*\lambda)(t) \quad \text{a.e. on } [a, b],$$

$$\left[ y' + \frac{d}{dt}(Q^*\lambda) \right] (a) = \gamma' M, \quad \left[ y' + \frac{d}{dt}(Q^*\lambda) \right] (b) = -\gamma' N,$$

$$y'(t+r) B(t+r) - (P^*\lambda)(t) = 0 \quad \text{a.e. on } [a-r, a],$$

while  $\delta' = \gamma' M (P^* : A^* \rightarrow \mathcal{L}_n^2(a - r, a))$  and  $Q^* : A^* \rightarrow \mathcal{W}_n^{1,2}(a, b)$  are the adjoints to  $P$  and  $Q$ .

*Proof.* Let  $\eta' \in \mathcal{L}_n^2(a, b)$ ,  $\lambda \in A^*$ ,  $\gamma' \in \mathcal{R}_m^*$  and  $\delta' \in \mathcal{R}_n^*$ . Then  $(\eta', \lambda, \gamma', \delta') \in \text{Ker}(U^*)$  iff for any  $(x, \xi, u) \in \mathcal{W}$

$$0 = \left( \begin{bmatrix} x \\ \xi \\ u \end{bmatrix}, U^*(\eta', \lambda, \gamma', \delta') \right)_{\mathcal{W}} = \left( U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix}, (\eta', \lambda, \gamma', \delta') \right)_{\mathcal{Z}} = \\ = \int_a^b \eta'(t) \dot{x}(t) dt - \int_a^b \eta'(t) A(t) x(t) dt - \int_a^{b-r} \eta'(t+r) B(t+r) x(t) dt - \\ - \int_{a-r}^a \eta'(t+r) B(t+r) u(t) dt + \gamma'(M\xi + Nx(b)) + \delta'(x(a) - \xi) +$$

$$\begin{aligned}
& + (u, P^*\lambda)_{\mathcal{L}} + (x, Q^*\lambda)_{\mathcal{H}} = \\
= & \int_a^b \eta'(t) \dot{x}(t) dt + \int_a^b \left[ \frac{d}{dt} (Q^*\lambda)(t) \right] \dot{x}(t) dt - \int_a^b p'(t) x(t) dt + \\
& + \gamma' N x(b) + \delta' x(a) - \int_{a-r}^a q'(t) u(t) dt + (\gamma' M - \delta') \xi,
\end{aligned}$$

where

$$p'(t) = \eta'(t) A(t) + \begin{cases} \eta'(t+r) B(t+r), & t < b-r \\ 0, & t \geq b-r \end{cases} - (Q^*\lambda)(t) \quad \text{on } [a, b]$$

and

$$q'(t) = \eta'(t+r) B(t+r) - (P^*\lambda)(t) \quad \text{on } [a-r, a].$$

In particular, putting  $\xi = 0$  and  $x(t) = 0$  on  $[a, b]$ , we get

$$(4,5) \quad \eta'(t+r) B(t+r) - (P^*\lambda)(t) = 0 \quad \text{a.e. on } [a-r, a].$$

Furthermore, putting  $x(t) = 0$  on  $[a, b]$  and  $u(t) = 0$  on  $[a-r, a]$ , we get

$$(4,6) \quad \gamma' M - \delta' = 0.$$

Let us put

$$g'(t) = \begin{cases} \int_a^b p'(s) ds - \gamma' N - \delta', & t = a \\ \int_t^b p'(s) ds - \gamma' N & , a < t < b \\ 0 & , t = b \end{cases} \in \mathcal{V}_n^0(a, b).$$

Then, in virtue of the integration-by-parts formula,

$$\begin{aligned}
0 & = \int_a^b \left\{ \eta'(t) + \left[ \frac{d}{dt} (Q^*\lambda)(t) \right] \right\} \dot{x}(t) dt + \int_a^b [dg'(t)] x(t) = \\
& = \int_a^b \left\{ \eta'(t) + \left[ \frac{d}{dt} (Q^*\lambda)(t) \right] - g'(t) \right\} \dot{x}(t) dt - g'(a) x(a)
\end{aligned}$$

for all  $x \in \mathcal{A}\mathcal{C}_n(a, b)$ . Again, we deduce that

$$(4,7) \quad g'(a) = \int_a^b y'(s) A(s) ds + \int_{a+r}^b y'(s) B(s) ds - \int_a^b (Q^*\lambda)(s) ds - \gamma' N - \delta' = 0$$

and

$$(4,8) \quad y'(t) + \left[ \frac{d}{dt} (Q^*\lambda)(t) \right] = \int_t^b y'(s) A(s) ds - \int_t^b (Q^*\lambda)(s) ds - \gamma' N +$$

$$+ \left\{ \begin{array}{ll} \int_{t+r}^b y'(s) B(s) ds, & t < b-r \\ 0 & , t \geq b-r \end{array} \right\} \text{ on } (a, b)$$

for some  $y' \in \mathcal{L}_n^2(a, b)$ ,  $y'(t) = \eta'(t)$  a.e. on  $[a, b]$ .

By (4,8),  $[y' + (d/dt)(Q^*\lambda)](a+)$  and  $[y' + (d/dt)(Q^*\lambda)](b-)$  exist,

$$\left[ y' + \frac{d}{dt}(Q^*\lambda) \right](b-) = -\gamma'N$$

and according to (4,6) and (4,7)

$$\left[ y' + \frac{d}{dt}(Q^*\lambda) \right](a+) = \delta' = \gamma'M.$$

The theorem easily follows.

**4,6. Corollary.** Let the operator  $Q$  in (4,2) be a linear and bounded mapping of  $\mathcal{L}_n^2(a, b)$  into  $A$ . Then  $(\eta', \lambda, \gamma', \delta') \in \text{Ker}(U^*)$  iff there is  $y' \in \mathcal{AC}_n(a, b)$  such that  $y'(t) = \eta'(t)$  a.e. on  $[a, b]$  and

$$\dot{y}'(t) = -y'(t)A(t) - \begin{cases} y'(t+r)B(t+r), & t < b-r \\ 0 & , t > b-r \end{cases} + (Q^*\lambda)(t) \text{ a.e. on } [a, b],$$

$$y'(a) = \gamma'M, \quad y'(b) = -\gamma'N,$$

$$-y'(t+r)B(t+r) + (P^*\lambda)(t) = 0 \text{ a.e. on } [a-r, a]$$

( $P^* : A^* \rightarrow \mathcal{L}_n^2(a-r, a)$  and  $Q^* : A^* \rightarrow \mathcal{L}_n^2(a, b)$  are adjoints of  $P$  and  $Q$ ).

Proof. Since for all  $x \in \mathcal{L}_n^2(a, b)$  and  $\lambda \in A^*$

$$\langle Qx, \lambda \rangle_A = (x, Q^*\lambda)_{\mathcal{L}} = \int_a^b (Q^*\lambda)(t) x(t) dt,$$

the term  $[(d/dt)(Q^*\lambda)]$  does not appear in the formula (4,8).

**4,7. Remark.** Let  $Q : \mathcal{L}_n^2(a, b) \rightarrow A$  be linear and bounded. Then  $Q$  is also bounded as an operator  $\mathcal{W}_n^{1,2}(a, b) \rightarrow A$  and apparently we have two possible adjoint problems, defined in Theorem 4,5 and Corollary 4,6, respectively. We must take into account that in this case we should write  $\tilde{Q}^*$  instead of  $Q^*$  in the former adjoint, where  $\tilde{Q} = QE$  and  $E : x \in \mathcal{W}_n^{1,2}(a, b) \rightarrow x \in \mathcal{L}_n^2(a, b)$  is a continuous imbedding of  $\mathcal{W}_n^{1,2}(a, b)$  into  $\mathcal{L}_n^2(a, b)$ . (Given  $\lambda \in A^*$  and  $x \in \mathcal{W}_n^{1,2}(a, b)$ ),

$$\int_a^b (Q^*\lambda)(t) x(t) dt = \int_a^b \left\{ \left[ \frac{d}{dt}(\tilde{Q}^*\lambda)(t) \right] \dot{x}(t) + (\tilde{Q}^*\lambda)(t) x(t) \right\} dt.$$

5. REMARKS ON THE CLOSEDNESS OF  $\text{Im}(U)$

All the boundary value type problems which occur in paragraphs 2 and 3 of this paper may be formulated as operator equations of the type

$$U\xi = \eta,$$

where  $U$  is a linear bounded mapping of either  $\mathcal{X}_c = \mathcal{A}\mathcal{C}_n(a, b) \times \mathcal{C}_n(a - r, a)$  or  $\mathcal{X}_v = \mathcal{A}\mathcal{C}_n(a, b) \times \mathcal{B}\mathcal{V}_n(a - r, a)$  into  $\mathcal{Y} = \mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$  and  $\Lambda$  is a B-space. The aim of this paragraph is to characterize in some special cases the range  $\text{Im}(U)$  of the operator  $U$  and, in particular, to find some conditions guaranteeing the closedness of  $\text{Im}(U)$ .

Let  $(\mathcal{C} + \mathcal{B}\mathcal{V})_n(a - r, a)$  denote the set of all functions  $w : [a - r, a] \rightarrow \mathcal{R}_n$  for which there exist functions  $u \in \mathcal{C}_n(a - r, a)$  and  $v \in \mathcal{B}\mathcal{V}_n(a - r, a)$  such that  $w(t) = u(t) + v(t)$  on  $[a - r, a]$ .

In what follows we make use of the following lemma which is a slight modification of the variation-of-constants formula due to H. T. Banks [1].

**5.1. Lemma.** *Let the  $n \times n$ -matrix function  $P(t, \vartheta)$  fulfil the corresponding assumptions from Sec. 3,4. Given  $f \in \mathcal{L}_n(a, b)$  and  $u \in (\mathcal{C} + \mathcal{B}\mathcal{V})_n(a - r, a)$ , there is just one solution to the initial value problem ((3,5), (3,6))*

$$\begin{aligned} \dot{x}(t) &= \int_{-r}^0 [d_\vartheta P(t, \vartheta)] x(t + \vartheta) + f(t) \quad \text{a.e. on } [a, b], \\ x(t) &= u(t) \quad \text{on } [a - r, a]. \end{aligned}$$

There exist a linear operator  $\Phi : (\mathcal{C} + \mathcal{B}\mathcal{V})_n(a - r, a) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  and a linear bounded operator  $\Psi : \mathcal{L}_n(a, b) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  such that this solution is given by

$$(5,1) \quad x = \Phi u + \Psi f.$$

The operator  $\Phi$  as a mapping  $\mathcal{B}\mathcal{V}_n(a - r, a) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  is completely continuous and as a mapping  $\mathcal{C}_n(a - r, a) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  bounded. Moreover, if  $b - r \geq a$  and if  $S_b : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow x|_{[b - r, b]} \in \mathcal{C}_n(b - r, b)$ , then the operator  $T = S_b \Phi : \mathcal{C}_n(a - r, a) \rightarrow \mathcal{C}_n(b - r, b)$  is completely continuous.

(The compactness of  $\Phi : \mathcal{B}\mathcal{V}_n(a - r, a) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  was shown in [13] and the proof of the compactness of  $T$  can be find in [7], Remark 8,9.)

**5.2. Remark.** It follows from the special form of the operator  $\Phi$  (cf. [1]) that for any  $u \in (\mathcal{C} + \mathcal{B}\mathcal{V})_n(a - r, a)$

$$(5,2) \quad \Phi u = \Phi^0 u(a) + \Phi^1 u,$$

where  $\Phi^0 : \mathcal{R}_n \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  is linear and bounded and  $\Phi^1 : (\mathcal{C} + \mathcal{B}\mathcal{V})_n(a - r, a) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  is linear and completely continuous as an operator  $\mathcal{B}\mathcal{V}_n(a - r, a) \rightarrow$

$\rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  and bounded as an operator  $\mathcal{C}_n(a - r, a) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$ . Moreover, if  $v$  is a simple jump function  $v(t) = 0$  on  $[a - r, a)$  and  $v(a) = d$ , then  $\Phi^1 v = 0$ .

**5.3. Problem (3,5)–(3,7).** Let us turn back to the problem (3,5)–(3,7) whose adjoint was derived in Sec. 3,4. Let  $\Lambda$  be an arbitrary B-space and let the operators  $M : \mathcal{C}_n(a - r, a) \rightarrow \Lambda$  and  $N : \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \Lambda$  and the  $n \times n$ -matrix function  $P(t, \vartheta)$  fulfil the assumptions of Sec. 3,4. Let  $f \in \mathcal{L}_n(a, b)$  and  $l \in \Lambda$ . Let us put  $\mathcal{X}_c = \mathcal{A}\mathcal{C}_n(a, b) \times \mathcal{C}_n(a - r, a)$ ,  $\mathcal{Y} = \mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$ ,

$$P_1 : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \int_{\max(-r, a-t)}^0 [d_\vartheta P(t, \vartheta)] x(t + \vartheta) \in \mathcal{L}_n(a, b),$$

$$P_2 : u \in \mathcal{C}_n(a - r, a) \rightarrow \int_{-r}^{\max(-r, a-t)} [d_\vartheta P(t, \vartheta)] u(t + \vartheta) \in \mathcal{L}_n(a, b),$$

and

$$(5,3) \quad U : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_c \rightarrow \begin{bmatrix} Dx - P_1 x - P_2 u \\ Mu + Nx \\ u(a) - x(a) \end{bmatrix} \in \mathcal{Y}$$

(where again  $D : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \dot{x} \in \mathcal{L}_n(a, b)$ ). The system (3,5)–(3,7) is equivalent to the operator equation

$$(5,4) \quad U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ 0 \end{bmatrix}.$$

**5,3,1. Theorem.** Let  $\text{Im}(M + N\Phi)$  be closed in  $\Lambda$ , then the operator  $U$  defined by (5,3) has closed range  $\text{Im}(U)$  in  $\mathcal{Y}$ .

*Proof.* Let  $(f, l, d) \in \mathcal{Y}$ . According to the variation-of-constants formula (5,1) a couple  $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_c$  is a solution to the equation

$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ d \end{bmatrix}$$

iff

$$x = \Phi \tilde{u} + \Psi f = \Phi^0(u(a) + d) + \Phi^1 u + \Psi f = \Phi^0 d + \Phi u + \Psi f,$$

where  $\tilde{u} = u + u_d$ ,  $u_d(t) = 0$  on  $[a - r, a)$ ,  $u_d(a) = d$  ( $\Phi^1 u_d = 0$ , cf. Remark 5,2) and  $u \in \mathcal{C}_n(a - r, a)$  is a solution to the operator equation

$$[M + N\Phi] u = -N\Psi f + l - N\Phi^0 d.$$



Let us denote

$$S : \begin{bmatrix} f \\ l \\ d \end{bmatrix} \in \mathcal{Y} \rightarrow -N\Psi f + l - N\Phi^0 d \in \Lambda.$$

Then  $S(\text{Im}(U)) = \text{Im}(M + N\Phi)$  and since the operator  $S$  is linear and bounded, our assertion readily follows.

**5,3,2. Corollary.** *If  $\Lambda = \mathcal{R}_m$ , then  $\text{Im}(U)$  is closed in  $\mathcal{Y}$ .*

(In this case  $\text{Im}(M + N\Phi)$  is a  $k$ -dimensional ( $0 \leq k \leq m$ ) linear subspace of  $\mathcal{R}_m$ .)

**5,3,3. Corollary.** *Let  $0 \leq r \leq b - a$ ,  $S_b : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow x | [b - r, b] \in \mathcal{C}_n(b - r, b)$  and let  $\tilde{N} : \mathcal{C}_n(b - r, b) \rightarrow \Lambda$  be linear and bounded. Let the operator  $U$  be given by (5,3), where  $N$  is replaced by  $N_b = \tilde{N}S_b$ . Then, if the operator  $M$  possesses a bounded inverse  $M^{-1} : \Lambda \rightarrow \mathcal{C}_n(a - r, a)$ , the range  $\text{Im}(U)$  of  $U$  is closed in  $\mathcal{Y}$ .*

*Proof.* By Theorem 5,3,1  $\text{Im}(U)$  is closed in  $\mathcal{Y}$  if the range of the operator

$$M + \tilde{N}S_b\Phi = M + \tilde{N}T : \mathcal{C}_n(a - r, a) \rightarrow \Lambda$$

is closed. Since by Lemma 5,1 the operator  $T = S_b\Phi : \mathcal{C}_n(a - r, a) \rightarrow \mathcal{C}_n(b - r, b)$  is completely continuous, the existence of a bounded  $M^{-1}$  implies the closedness of  $\text{Im}(M + \tilde{N}T)$  and hence also of  $\text{Im}(U)$ .

**5,3,4. Remark.** Our restriction to two-point boundary value type problems in Corollary 5,3,3 does not mean an essential loss of generality (cf. [8]).

**5,3,5. Corollary.** *The  $T$ -periodic problem (3,5), (3,6), (3,20) (cf. Sec. 3,9) has a solution iff*

$$\int_0^T y'(s)f(s) ds = 0$$

for all  $T$ -periodic solutions  $y'(t)$  (i.e.,  $y'(t) = y'(T + t)$  on  $[-r, 0)$ ) of the equation

$$y'(t) + \int_t^b y'(s)P(s, t - s) ds = \text{const. on } [-r, T - r].$$

(Proof follows from Corollaries 3,10 and 5,3,3.)

**5,3,6. Remark.** Let  $\Lambda_1$  be a B-space and let the operators  $M_1 : \mathcal{C}_n(a - r, a) \rightarrow \Lambda_1$  and  $N_1 : \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \Lambda_1$  be linear and bounded. If  $\Lambda = \mathcal{C}_n(a - r, a) \times \Lambda_1$  and

$$M : u \in \mathcal{C}_n(a - r, a) \rightarrow \begin{bmatrix} u \\ M_1 u \end{bmatrix} \in \Lambda, \quad N : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \begin{bmatrix} 0 \\ N_1 x \end{bmatrix} \in \Lambda,$$

then the operator  $U$  given by (5,3) has closed range  $\text{Im}(U)$  in  $\mathcal{Y} = \mathcal{L}_n(a, b) \times \mathcal{C}_n(a-r, a) \times A_1 \times \mathcal{R}_n$ . (Indeed, according to Lemma 5,1 an element  $(f, h, l, d)$  of  $\mathcal{Y}$  belongs to  $\text{Im}(U)$  iff

$$F(f, h, l, d) = N_1 \Psi f + (M_1 + N_1 \Phi) h - l + N_1 \Phi^0 d = 0.$$

It is easy to see that the operator  $F : \mathcal{Y} \rightarrow A_1$  is linear and bounded. Consequently, the set  $\text{Im}(U) = \text{Ker}(F)$  is closed in  $\mathcal{Y}$ .)

**5,3,7. Remark.** All the assertions of this section will remain true if we replace the initial space  $\mathcal{C}_n(a-r, a)$  by  $\mathcal{BV}_n(a-r, a)$ . Moreover, Corollary 5,3,3 could be now formulated directly for a general linear bounded operator  $N : \mathcal{AC}_n(a, b) \rightarrow A$ . ( $N$  need not be of the two-point character  $N = \tilde{N}S_b$ .) This is possible in virtue of the compactness of the operator  $\Phi : \mathcal{BV}_n(a-r, a) \rightarrow \mathcal{AC}_n(a, b)$  in the variation-of-constants formula (5,1) (cf. Lemma 5,1).

**5,4. Problem (2,1)–(2,3).** The subject of this section is the general problem of finding  $x \in \mathcal{AC}_n(a, b)$  and  $u \in \mathcal{BV}_n(a-r, a)$  which satisfy (2,1)–(2,3). Let Assumptions 2,1 be fulfilled. We make use of the notation introduced in Sec. 2,3. (Only  $\mathcal{C}_n(a-r, a)$  should be replaced everywhere by  $\mathcal{BV}_n(a-r, a)$ .)

**5,4,1. Lemma.** Let  $-\infty < c < d < +\infty$  and let  $K(t, s)$  be an  $n \times n$ -matrix function defined and Borel measurable in  $(t, s)$  on  $[a, b] \times [c, d]$  and such that  $\text{var}_c^d K(t, \cdot) < \infty$  for any  $t \in [a, b]$ , while

$$\int_a^b (\text{var}_c^d K(t, \cdot) + \|K(t, d)\|) dt < \infty.$$

Then the operator

$$K : u \in \mathcal{BV}_n(c, d) \rightarrow \int_c^d [d_s K(t, s)] u(s) \in \mathcal{L}_n(a, b)$$

is completely continuous.

**Proof.** The operator  $K$  is surely linear and bounded.

Let  $\{u^j\}_{j=1}^\infty \subset \mathcal{BV}_n(c, d)$  and  $\|u^j\|_{\mathcal{BV}} < 1$  ( $j = 1, 2, \dots$ ). Then by Helly's Choice Theorem there exists a subsequence  $\{u^{j_l}\} \subset \{u^j\}$  and  $u^0 \in \mathcal{BV}_n(c, d)$  such that

$$\lim_{l \rightarrow \infty} u^{j_l}(s) = u^0(s) \quad \text{for all } s \in [c, d].$$

Let us put for  $s \in [c, d]$  and  $l = 1, 2, \dots$

$$v^l(s) = \|u^{j_l}(s) - u^0(s)\|$$

and for  $t, s \in [a, b] \times [c, d]$

$$k(t, s) = \text{var}_c^d K(t, \cdot).$$

Then  $\|v^l(s)\| \leq \|u^0\|_{\mathcal{B}\mathcal{V}} + 1$  on  $[c, d]$  for any  $l = 1, 2, \dots$ ,  $\text{var}_c^d k(t, \cdot) = \text{var}_c^d K(t, \cdot)$  for any  $t \in [a, b]$  and by the unsymmetric Fubini theorem

$$\begin{aligned} \int_a^b \left\| \int_c^d [d_s K(t, s)] (u^{j_l}(s) - u^0(s)) \right\| dt &\leq \int_a^b \left( \int_c^d [d_s k(t, s)] v^l(s) \right) dt = \\ &= \int_c^d \left[ d_s \int_a^b k(t, s) dt \right] v^l(s). \end{aligned}$$

Given a subdivision  $\{c = s_0 < s_1 \dots < s_m = d\}$  of  $[c, d]$ ,

$$\begin{aligned} \sum_{i=1}^m \left\| \int_a^b (k(t, s_i) - k(t, s_{i-1})) dt \right\| &\leq \int_a^b \left( \sum_{i=1}^m \|k(t, s_i) - k(t, s_{i-1})\| \right) dt \leq \\ &\leq \int_a^b (\text{var}_c^d k(t, \cdot)) dt < \infty. \end{aligned}$$

Thus

$$\text{var}_c^d \left( \int_a^b k(t, \cdot) dt \right) < \infty$$

and according to the dominated convergence theorem for Perron-Stieltjes integrals

$$\lim_{l \rightarrow \infty} \int_c^d \left[ d_s \int_a^b k(t, s) dt \right] v^l(s) = 0$$

or

$$\lim_{l \rightarrow \infty} \|Ku^{j_l} - Ku^0\|_{\mathcal{L}} = \lim_{l \rightarrow \infty} \int_a^b \left\| \int_c^d [d_s K(t, s)] (u^{j_l}(s) - u^0(s)) \right\| dt = 0$$

which completes the proof.

**5,4,2. Remark.** The operator

$$u \in \mathcal{C}_n(c, d) \rightarrow \int_c^d [d_s K(t, s)] u(s) \in \mathcal{L}_n(a, b)$$

(with  $K(t, s)$  fulfilling the assumptions of Lemma 5,4,1) need not be generally completely continuous.

**5,4,3. Theorem.** *If the operator  $M : \mathcal{B}\mathcal{V}_n(a - r, a) \rightarrow \Lambda$  has a bounded inverse  $M^{-1}$ , then the operator  $U$  given by (2,4) (with  $\mathcal{C}_n(a - r, a)$  replaced by  $\mathcal{B}\mathcal{V}_n(a - r, a)$ ) has closed range in  $\mathcal{Y}$ .*

*Proof.* By Lemma 5,1 applied to initial value problems of the type

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)x(t - r) + g(t) \quad \text{a.e. on } [a, b], \\ x(t) &= u(t) \quad \text{on } [a - r, a], \end{aligned}$$

the triple  $(f, l, d) \in \mathcal{Y}$  belongs to  $\text{Im}(U)$  iff there is a solution  $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_v = \mathcal{AC}_n(a, b) \times \mathcal{BV}_n(a-r, a)$  to the system of operator equations

$$(5,5) \quad \begin{aligned} x - \Psi G_1 x - \Phi u - \Psi G_2 u &= \Psi f + \Phi^0 d, \\ Mu + Nx &= l, \end{aligned}$$

where the operator  $\Phi : \mathcal{BV}_n(a-r, a) \rightarrow \mathcal{AC}_n(a, b)$  is linear and completely continuous and the operators  $\Phi^0 : \mathcal{R}_n \rightarrow \mathcal{AC}_n(a, b)$  and  $\Psi : \mathcal{L}_n(a, b) \rightarrow \mathcal{AC}_n(a, b)$  are linear and bounded. Since there exists a bounded inverse  $M^{-1}$  of  $M$ , the latter equation in (5,5) yields  $u = M^{-1}l - M^{-1}Nx$ , while the former becomes

$$x - \{\Phi M^{-1}N + \Psi G_1 + \Psi G_2 M^{-1}N\} u = \Psi f + (\Phi + \Psi G_2) M^{-1}l + \Phi^0 d.$$

Let us put  $K = \Phi M^{-1}N + \Psi G_1 + \Psi G_2 M^{-1}N$ ,  $S(f, l, d) = \Psi f + (\Phi + \Psi G_2) M^{-1}l + \Phi^0 d$  and let  $I$  denote the identity operator on  $\mathcal{AC}_n(a, b)$ . Then  $S(\text{Im}(U)) = \text{Im}(I - K)$  and since  $S : \mathcal{Y} \rightarrow \mathcal{AC}_n(a, b)$  is linear and bounded,  $\text{Im}(U)$  is closed if  $\text{Im}(I - K)$  is closed. The operators  $G_1, G_2$  are completely continuous by Lemma 5,4,1 and since the operators  $M^{-1}, N$  and  $\Psi$  are bounded the operator  $K$  is also completely continuous and  $\text{Im}(I - K)$  is closed.

**5,4,4. Remark.** As an easy consequence of Theorem 5,4,3 we obtain that in the case of the  $T$ -periodic problem (i.e.  $a = 0, b = T, r < T, A = \mathcal{AC}_n(-r, 0), M = I, N : x \in \mathcal{AC}_n(0, T) \rightarrow x_T(s) = x(T + s) \in \mathcal{AC}_n(-r, 0)$  and  $l = 0$ ) the range  $\text{Im}(U)$  of  $U$  is closed in  $\mathcal{Y}$ .

**5,5. Boundary value problems for ordinary integrodifferential equations.** If  $r = 0$  and  $A = \mathcal{R}_m$ , then the given problem (2,1)–(2,3) reduces to the boundary value problem for an ordinary integrodifferential equation of the form

$$(5,6) \quad \dot{x}(t) = A(t)x(t) + \int_a^b [d_s G(t, s)] x(s) + f(t) \quad \text{a.e. on } [a, b],$$

$$(5,7) \quad Nx = l,$$

where the  $n \times n$ -matrix function  $A(t)$  is  $\mathcal{L}$ -integrable on  $[a, b]$ ,  $\text{var}_a^b G(t, \cdot) < \infty$  for any  $t \in [a, b]$ ,

$$\int_a^b (\text{var}_a^b G(t, \cdot) + \|G(t, b)\|) dt < \infty,$$

$f \in \mathcal{L}_n(a, b)$ ,  $l \in \mathcal{R}_m$  and the operator  $N : \mathcal{AC}_n(a, b) \rightarrow \mathcal{R}_m$  is linear and bounded. (The initial space reduces to  $\mathcal{R}_n$ .)

Let us reformulate the problem (5,6), (5,7) as the operator equation

$$Ux = \begin{pmatrix} f \\ l \end{pmatrix},$$

where

$$(5,8) \quad U : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \begin{pmatrix} Dx - Ax - Gx \\ Nx \end{pmatrix} \in \mathcal{L}_n(a, b) \times \mathcal{R}_m$$

and the symbols  $D, A, G$  have the obvious meaning.

**5,5,1. Theorem.** *The operator  $U$  defined by (5,8) has closed range in  $\mathcal{L}_n(a, b) \times \mathcal{R}_m$ .*

*Proof.* There exist linear and bounded operators  $\Phi^0 : \mathcal{R}_n \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  and  $\Psi : \mathcal{L}_n(a, b) \rightarrow \mathcal{A}\mathcal{C}_n(a, b)$  such that an  $n$ -vector function  $x(t)$  is a solution to the given problem iff

$$x = \Phi^0 c + \Psi h + \Psi f,$$

where the couple  $(h, c) \in \mathcal{L}_n(a, b) \times \mathcal{R}_n$  ( $h = Gx$ ) is a solution to the system

$$(5,9) \quad \begin{aligned} h - (G\Phi^0) c - (G\Psi) h &= (G\Psi) f, \\ (N\Phi^0) c + (N\Psi) h &= l - (N\Psi) f. \end{aligned}$$

$(N\Phi^0)$  is a constant  $m \times n$ -matrix. Let e.g.  $m < n$ . Putting

$$\begin{aligned} Q &= I_n - \begin{bmatrix} N\Phi^0 \\ 0_{n-m,n} \end{bmatrix}, \quad l = \begin{bmatrix} l \\ 0_{n-m,1} \end{bmatrix} \in \mathcal{R}_n, \\ R &: h \in \mathcal{L}_n(a, b) \rightarrow \begin{bmatrix} (N\Psi) h \\ 0_{n-m,1} \end{bmatrix} \in \mathcal{R}_n \end{aligned}$$

( $0_{p,q}$  denotes the zero  $p \times q$ -matrix and  $I_n$  is the identity  $n \times n$ -matrix),

$$K : \begin{pmatrix} h \\ c \end{pmatrix} \in \mathcal{L}_n(a, b) \times \mathcal{R}_n \rightarrow \begin{bmatrix} (G\Phi^0) c + (G\Psi) h \\ Qc - Rh \end{bmatrix} \in \mathcal{L}_n(a, b) \times \mathcal{R}_n$$

and

$$S : \begin{pmatrix} f \\ l \end{pmatrix} \in \mathcal{L}_n(a, b) \times \mathcal{R}_m \rightarrow \begin{bmatrix} (G\Psi) f \\ l - Rf \end{bmatrix} \in \mathcal{L}_n(a, b) \times \mathcal{R}_n,$$

the system (5,9) becomes

$$(I - K) \begin{pmatrix} h \\ c \end{pmatrix} = S \begin{pmatrix} f \\ l \end{pmatrix}$$

and  $S(\text{Im}(U)) = \text{Im}(I - K)$ . Since by Lemma 5,4,1 the operator  $G$  is completely continuous, it is easy to verify that the operator  $K$  is completely continuous. It means that  $\text{Im}(I - K)$  is closed and taking into account that the operator  $S$  is linear and bounded we complete the proof. The case  $m > n$  can be treated analogously.

Let  $N(t)$  be an  $m \times n$ -matrix function of bounded variation on  $[a, b]$  and let the operator  $N$  be given by

$$(5,10) \quad N : x \in \mathcal{A}\mathcal{C}_n(a, b) \rightarrow \int_a^b [dN(s)] x(s) \in \mathcal{R}_m.$$

Without any loss of generality we may assume that for any  $t \in [a, b]$  the functions  $G(t, \cdot)$  and  $N$  are right-continuous on  $(a, b)$ . Let us put for  $t \in [a, b]$

$$C(t) = G(t, a+) - G(t, a), \quad D(t) = G(t, b) - G(t, b-),$$

$$G_0(t, s) = \begin{cases} G(t, a+) & \text{for } s = a, \\ G(t, s) & \text{for } a < s < b, \\ G(t, b-) & \text{for } s = b, \end{cases} \quad L(s) = \begin{cases} N(a+) & \text{for } s = a, \\ N(s) & \text{for } a < s < b, \\ N(b-) & \text{for } s = b, \end{cases}$$

$$M = N(a+) - N(a), \quad N = N(b) - N(b-).$$

Then similarly as in Sec. 3,3 we obtain that the adjoint problem to (5,6), (5,7) is equivalent to the problem of finding  $y \in \mathcal{B}\mathcal{V}_n(a, b)$ , right-continuous on  $[a, b]$  and left-continuous at  $b$  and  $\lambda \in \mathcal{R}_n$  such that

$$(5,11) \quad y'(t) = y'(b) + \int_t^b y'(s) A(s) ds - \int_a^b y'(s) (G_0(s, t) - G_0(s, b)) ds + \lambda'(L(t) - L(b)) \quad \text{on } [a, b],$$

$$(5,12) \quad y'(a) = \lambda'M - \int_a^b y'(s) C(s) ds, \quad y'(b) = -\lambda'N + \int_a^b y'(s) D(s) ds.$$

The following theorem is then a direct corollary of Theorem 5,5,1.

**5,5,2. Theorem.** *The problem (5,6), (5,7) possesses a solution iff*

$$\int_a^b y'(s) f(s) ds + \lambda'l = 0$$

for any solution  $(y'(t), \lambda')$  of the adjoint problem (5,11), (5,12).

Theorem 5,5,2 generalizes Theorem 3,1 from [12].

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- Remark 2,12 was added in the proofs. Its assertion was proved in [15] (Theorem 4,4). The results of sec. 5,5 were shown in another way also in [16].
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