

LINEAR COMBINATIONS OF HARMONIC QUASICONFORMAL MAPPINGS CONVEX IN ONE DIRECTION

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Abstract

In this paper, we introduce a new class $\mathcal{S}_H(k, \gamma; \phi)$ of harmonic quasiconformal mappings, where $k \in [0, 1)$, $\gamma \in [0, \pi)$ and ϕ is an analytic function. Sufficient conditions for the linear combinations of mappings in such classes to be in a similar class, and convex in a given direction, are established. In particular, we prove that the images of linear combinations in this class, for special choices of γ and ϕ , are convex.

1. Introduction

A complex-valued function f defined in the open unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is called harmonic if f is twice continuously differentiable and satisfies $\Delta f = 4f_{z\bar{z}} = 0$. Let \mathcal{H} denote the class of all complex-valued harmonic functions f in \mathbf{D} normalized by the condition $f(0) = f_z(0) - 1 = 0$. Let \mathcal{S}_H be the subclass of \mathcal{H} consisting of univalent and sense-preserving functions. Such functions can be written in the form $f = h + \bar{g}$, where

$$(1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in \mathbf{D} and the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$, or equivalently, the analytic complex dilatation $\omega = g'/h'$ of f satisfies $|\omega| < 1$ in \mathbf{D} . The classical class \mathcal{S} of analytic univalent and normalized functions in \mathbf{D} is a subclass of \mathcal{S}_H with $g(z) \equiv 0$. The class of all functions $f \in \mathcal{S}_H$ with the additional property that $f_{\bar{z}}(0) = 0$ is denoted by \mathcal{S}_H^0 . We refer to [6, 9, 10] for the basic theory of harmonic mappings, and [2, 3, 5, 14, 16, 23] for some recent investigations on the topic.

If a univalent harmonic mapping $f = h + \bar{g}$ satisfies the condition

$$\left| \frac{g'(z)}{h'(z)} \right| \leq k < 1 \quad (z \in \mathbf{D}),$$

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then f is called a harmonic K -quasiconformal mapping in \mathbf{D} , where $K = \frac{1+k}{1-k}$. Let $\mathcal{S}_H(k)$ be the subclass of \mathcal{S}_H^0 consisting of harmonic K -quasiconformal mappings. Recently, several authors derived the conditions for univalent harmonic mappings to be K -quasiconformal, see (for example) the works [1, 11, 12, 18] and the references therein.

A domain $\Omega \subset \mathbf{C}$ is said to be convex in the direction $\gamma \in [0, \pi)$, if for all $a \in \mathbf{C}$, the set $\Omega \cap \{a + te^{i\gamma} : t \in \mathbf{R}\}$ is either connected or empty. In particular, a domain is convex in the direction of the real (imaginary) axis if every line parallel to the real (imaginary) axis has either an empty intersection or a connected intersection with the domain. A function is said to be convex in the direction γ if it maps \mathbf{D} univalently onto a domain convex in the direction γ .

Let $f_1 = h_1 + \bar{g}_1$ and $f_2 = h_2 + \bar{g}_2$ be two univalent harmonic mappings in \mathbf{D} with respective dilatations ω_1 and ω_2 . Then, the linear combination f of f_1 and f_2 is given by

$$(2) \quad \begin{aligned} f &= tf_1 + (1-t)f_2 = [th_1 + (1-t)h_2] + [t\bar{g}_1 + (1-t)\bar{g}_2] \\ &= h + \bar{g}, \quad (0 \leq t \leq 1). \end{aligned}$$

Even if f and g are convex analytic functions, Macgregor [15] has shown that $tf + (1-t)g$ ($0 \leq t \leq 1$) need not be univalent. For results on the analytic linear combination, see (for example) [4, 21]. For linear combinations of harmonic functions, Dorff and Rolf [8] provided sufficient conditions for the linear combination $f = tf_1 + (1-t)f_2$ to be univalent and convex in the direction of the imaginary axis under the assumption that $\omega_1 = \omega_2$. Furthermore, Wang *et al.* [22] proved that the linear combination $f = tf_1 + (1-t)f_2$ with $h_j + g_j = \frac{z}{1-z}$ ($j = 1, 2$) is univalent and convex in the direction of the real axis. Recently, Kumar *et al.* [13] established that the linear combination $f = tf_1 + (1-t)f_2$ with $h_j + g_j = \frac{z(1-\alpha_j z)}{1-z^2}$ ($-1 \leq \alpha_j \leq 1; j = 1, 2$) is univalent and convex in the direction of the imaginary axis.

Let \mathcal{A} be the subclass of \mathcal{S}_H^0 consisting of analytic functions. For $k \in [0, 1)$, $\gamma \in [0, \pi)$ and $\phi \in \mathcal{A}$, consider the following subclass $\mathcal{S}_H(k, \gamma; \phi)$ of \mathcal{S}_H defined by

$$\mathcal{S}_H(k, \gamma; \phi) := \{f = h + \bar{g} \in \mathcal{S}_H(k) : h - e^{2i\gamma}g = \phi\}.$$

For simplicity, we write $\mathcal{S}_H(k, 0; \phi) =: \mathcal{S}_H^-(k; \phi)$ and $\mathcal{S}_H\left(k, \frac{\pi}{2}; \phi\right) =: \mathcal{S}_H^+(k; \phi)$.

These subclasses of harmonic mappings were introduced in [17, 24] for specific choices of γ and ϕ .

In this paper, we derive sufficient conditions for the linear combinations of harmonic quasiconformal mappings to be univalent and convex in a given direction. In particular, we prove that the images of linear combinations in this subclass, for special choices of γ and ϕ , are convex.

2. Preliminary results

The proofs of our main results are based on the following lemmas.

LEMMA 1 (see [6]). *A sense-preserving harmonic function $f = h + \bar{g}$ in \mathbf{D} is a univalent mapping of \mathbf{D} onto a domain convex in the direction of the real (resp. imaginary) axis if and only if $h - g$ (resp. $h + g$) is an analytic univalent mapping of \mathbf{D} onto a domain convex in the direction of the real (resp. imaginary) axis.*

It is clear that Lemma 1 of Clunie and Sheil-Small can easily be generalized to a domain convex in the direction γ .

LEMMA 2. *A sense-preserving harmonic function $f = h + \bar{g}$ in \mathbf{D} is a univalent mapping of \mathbf{D} onto a domain convex in the direction γ if and only if $h - e^{2i\gamma}g$ is an analytic univalent mapping of \mathbf{D} onto a domain convex in the direction γ .*

LEMMA 3 (see [19]). *Let f be an analytic function in \mathbf{D} with $f(0) = 0$ and $f'(0) \neq 0$ and let*

$$(3) \quad \kappa(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})} \quad (\theta \in \mathbf{R}).$$

If

$$\Re\left(\frac{zf'(z)}{\kappa(z)}\right) > 0 \quad (z \in \mathbf{D}),$$

then f is convex in the direction of the real axis.

LEMMA 4 (see [20]). *Let $\varphi(z)$ be a non-constant function regular in \mathbf{D} . The function $\varphi(z)$ maps \mathbf{D} univalently onto a domain convex in the direction of imaginary axis, if and only if there are numbers μ and ν , $0 \leq \mu < 2\pi$ and $0 \leq \nu \leq \pi$ such that*

$$(4) \quad \Re(-ie^{i\mu}(1 - 2ze^{-i\mu} \cos \nu + z^2e^{-2i\mu})\varphi'(z)) \geq 0 \quad (z \in \mathbf{D}).$$

LEMMA 5. *If $f_j \in \mathcal{S}_H(k, \gamma; \phi)$ ($j = 1, 2$), then the dilatation ω of the linear combination $f = tf_1 + (1 - t)f_2$ ($0 \leq t \leq 1$) satisfies*

$$|\omega| = \left| \frac{tg'_1 + (1 - t)g'_2}{th'_1 + (1 - t)h'_2} \right| \leq k < 1.$$

Proof. Since $h_j - e^{2i\gamma}g_j = \phi$ and $g'_j = \omega_j h'_j$ ($j = 1, 2$), we get

$$h'_j = \frac{\phi'}{1 - e^{2i\gamma}\omega_j} \quad (j = 1, 2).$$

We obtain a new harmonic mapping as follows

$$f = tf_1 + (1 - t)f_2 = [th_1 + (1 - t)h_2] + \overline{[tg_1 + (1 - t)g_2]} = h + \bar{g},$$

and the dilatation $\omega = g'/h'$ satisfies the condition

$$(5) \quad |\omega| = \left| \frac{tg'_1 + (1-t)g'_2}{th'_1 + (1-t)h'_2} \right| = \left| \frac{\frac{t\omega_1\phi'}{1-e^{2i\gamma}\omega_1} + \frac{(1-t)\omega_2\phi'}{1-e^{2i\gamma}\omega_2}}{\frac{t\phi'}{1-e^{2i\gamma}\omega_1} + \frac{(1-t)\phi'}{1-e^{2i\gamma}\omega_2}} \right|$$

$$= \left| \frac{\frac{t\omega_1}{1-e^{2i\gamma}\omega_1} + \frac{(1-t)\omega_2}{1-e^{2i\gamma}\omega_2}}{\frac{t}{1-e^{2i\gamma}\omega_1} + \frac{1-t}{1-e^{2i\gamma}\omega_2}} \right|.$$

From (5) it follows that $|\omega| \leq k$ if and only if

$$k^2 \left| \frac{t}{1-e^{2i\gamma}\omega_1} + \frac{1-t}{1-e^{2i\gamma}\omega_2} \right|^2 - \left| \frac{t\omega_1}{1-e^{2i\gamma}\omega_1} + \frac{(1-t)\omega_2}{1-e^{2i\gamma}\omega_2} \right|^2 \geq 0.$$

Let

$$\omega_j = \rho_j e^{i\theta_j} \quad (0 \leq \rho_j \leq k < 1, \theta_j \in \mathbf{R}; j = 1, 2)$$

and

$$\Phi := \frac{2t(1-t)}{|1-e^{2i\gamma}\omega_1|^2 |1-e^{2i\gamma}\omega_2|^2} \geq 0.$$

Then we have

$$k^2 \left| \frac{t}{1-e^{2i\gamma}\omega_1} + \frac{1-t}{1-e^{2i\gamma}\omega_2} \right|^2 - \left| \frac{t\omega_1}{1-e^{2i\gamma}\omega_1} + \frac{(1-t)\omega_2}{1-e^{2i\gamma}\omega_2} \right|^2$$

$$= \frac{t^2(k^2 - |\omega_1|^2)}{|1-e^{2i\gamma}\omega_1|^2} + \frac{(1-t)^2(k^2 - |\omega_2|^2)}{|1-e^{2i\gamma}\omega_2|^2}$$

$$+ 2t(1-t) \Re \left(\frac{k^2 - \omega_1 \overline{\omega_2}}{(1-e^{2i\gamma}\omega_1)(1-e^{-2i\gamma}\overline{\omega_2})} \right)$$

$$\geq \frac{2t(1-t)}{|1-e^{2i\gamma}\omega_1|^2 |1-e^{2i\gamma}\omega_2|^2} \Re((k^2 - \omega_1 \overline{\omega_2})(1 - e^{-2i\gamma}\overline{\omega_1})(1 - e^{2i\gamma}\omega_2))$$

$$= \Phi((k^2 - \rho_1^2 \rho_2^2) + \rho_1(\rho_2^2 - k^2) \cos(2\gamma + \theta_1)$$

$$+ \rho_2(\rho_1^2 - k^2) \cos(2\gamma + \theta_2) + \rho_1 \rho_2 (k^2 - 1) \cos(\theta_2 - \theta_1))$$

$$\geq \Phi((k^2 - \rho_1^2 \rho_2^2) - \rho_1(k^2 - \rho_2^2) - \rho_2(k^2 - \rho_1^2) - \rho_1 \rho_2 (1 - k^2))$$

$$= \Phi(k^2 - \rho_1 \rho_2)(1 - \rho_1)(1 - \rho_2) \geq 0.$$

The proof of Lemma 5 is thus completed. ■

Obviously, we may generalize Lemma 5 as follows.

LEMMA 6. *If $f_j \in \mathcal{S}_H(k, \gamma; \phi)$ ($j = 1, 2, \dots, n$), then the dilatation ω of the linear combination $f = t_1 f_1 + t_2 f_2 + \dots + t_n f_n$ satisfies*

$$|\omega| = \left| \frac{t_1 g'_1 + t_2 g'_2 + \dots + t_n g'_n}{t_1 h'_1 + t_2 h'_2 + \dots + t_n h'_n} \right| \leq k < 1,$$

where $0 \leq t_j \leq 1$ and $t_1 + t_2 + \dots + t_n = 1$.

3. Main results

We begin by presenting sufficient conditions for the linear combinations for the class $\mathcal{S}_H(k, \gamma; \phi)$ to preserve certain properties of mappings.

THEOREM 1. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H(k, \gamma; \phi)$ ($j = 1, 2$). If ϕ is convex in the direction γ , then $f = t f_1 + (1 - t) f_2 \in \mathcal{S}_H(k, \gamma; \phi)$ ($0 \leq t \leq 1$), and it is convex in the direction γ .*

Proof. In view of Lemma 5, we know that the dilatation ω of $f = t f_1 + (1 - t) f_2$ satisfies $|\omega| \leq k$. Since $h_j - e^{2i\gamma} g_j = \phi$ ($j = 1, 2$), we have

$$\begin{aligned} h - e^{2i\gamma} g &= [t h_1 + (1 - t) h_2] - e^{2i\gamma} [t g_1 + (1 - t) g_2] \\ &= t(h_1 - e^{2i\gamma} g_1) + (1 - t)(h_2 - e^{2i\gamma} g_2) = \phi, \end{aligned}$$

which is convex in the direction γ by the assumption. Thus, from Lemma 2, we see that $f \in \mathcal{S}_H(k, \gamma; \phi)$ and convex in the direction γ . ■

In view of Theorem 1 and Lemma 6, we have the following result.

COROLLARY 1. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H(k, \gamma; \phi)$ ($j = 1, 2, \dots, n$). If ϕ is convex in the direction γ , then $f = \sum_{j=1}^n t_j f_j \in \mathcal{S}_H(k, \gamma; \phi)$ ($0 \leq t_j \leq 1, \sum_{j=1}^n t_j = 1$), and it is convex in the direction γ .*

Remark 1. If we set $n = 2$, $\gamma = 0$ and $\phi = \frac{z}{1 - z}$ in Corollary 1, then it reduces to the result of Wang *et al.* [22, Theorem 3].

By making use of Theorem 1, we can obtain some interesting results for specific choices of γ and ϕ .

COROLLARY 2. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H(k, \gamma; \phi)$ ($j = 1, 2$), where*

$$(6) \quad \phi(z) = \int_0^z \frac{e^{i\gamma} d\zeta}{(1 + \zeta e^{i\theta})(1 + \zeta e^{-i\theta})} \quad (\theta \in \mathbf{R}).$$

Then $f = t f_1 + (1 - t) f_2 \in \mathcal{S}_H(k, \gamma; \phi)$ ($0 \leq t \leq 1$), and it is convex in the direction γ .

Proof. By setting $\kappa(z)$ by (3), we find that

$$\begin{aligned} \Re\left(\frac{ze^{-i\gamma}(h' - e^{2i\gamma}g')}{\kappa(z)}\right) &= \Re\left(\frac{ze^{-i\gamma}}{\kappa(z)}[t(h'_1 - e^{2i\gamma}g'_1) + (1-t)(h'_2 - e^{2i\gamma}g'_2)]\right) \\ &= t \cdot \Re\left(\frac{ze^{-i\gamma}\phi'(z)}{\kappa(z)}\right) + (1-t) \cdot \Re\left(\frac{ze^{-i\gamma}\phi'(z)}{\kappa(z)}\right) \\ &= t + (1-t) = 1 > 0. \end{aligned}$$

Therefore, by Lemma 3, we see that $e^{-i\gamma}(h - e^{2i\gamma}g)$ is convex in the direction of the real axis, and hence the function $h - e^{2i\gamma}g$ is convex in the direction γ . Furthermore, by Lemma 2 and Lemma 5, we deduce that $f \in \mathcal{S}_H(k, \gamma; \phi)$ and convex in the direction γ . ■

COROLLARY 3. *Suppose that $\alpha \in [-1, 1]$, $\theta \in (0, \pi)$ and $A, B \geq 0$, $A + B \neq 0$. Let $f_j = h_j + \bar{g}_j \in \mathcal{S}_H^+(k; \phi)$ ($j = 1, 2$), where*

$$(7) \quad \phi = A \cdot \frac{z(1 - \alpha z)}{1 - z^2} + B \cdot \frac{1}{2i \sin \theta} \log\left(\frac{1 + ze^{i\theta}}{1 + ze^{-i\theta}}\right),$$

then $f = tf_1 + (1-t)f_2 \in \mathcal{S}_H^+(k; \phi)$ ($0 \leq t \leq 1$), and it is convex in the direction of the imaginary axis.

Proof. By taking $\mu = \nu = \frac{\pi}{2}$ in (4), we find that

$$\begin{aligned} \Re((1 - z^2)\phi'(z)) &= A \cdot \Re\left(\frac{1 - 2\alpha z + z^2}{1 - z^2}\right) + B \cdot \Re\left(\frac{1 - z^2}{(1 + ze^{i\theta})(1 + ze^{-i\theta})}\right) \\ &= A \cdot \frac{(1 - |z|^2)(1 - 2\alpha\Re(z) + |z|^2)}{|1 - z^2|^2} \\ &\quad + B \cdot \frac{(1 - |z|^2)(1 + 2 \cos \theta \Re(z) + |z|^2)}{|1 + ze^{i\theta}|^2 \cdot |1 + ze^{-i\theta}|^2} > 0. \end{aligned}$$

Therefore, by Lemma 4, ϕ is convex in the direction of the imaginary axis, and hence by Theorem 1 with $\gamma = \frac{\pi}{2}$, we see that $f \in \mathcal{S}_H^+(k; \phi)$ and f is convex in the direction of the imaginary axis. ■

Remark 2. The main results of Kumar *et al.* [13] reduce to special cases of Corollary 3.

Since the function defined by (8) is convex in the direction of the real axis (see [7]), we can obtain the following result.

COROLLARY 4. *Suppose that $A, B \geq 0$, $A + B \neq 0$ and $c \in [-2, 2]$. Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H^-(k; \phi)$ ($j = 1, 2$), where*

$$(8) \quad \phi = A \cdot \log\left(\frac{1+z}{1-z}\right) + B \cdot \frac{z}{1+cz+z^2},$$

then $f = tf_1 + (1-t)f_2 \in \mathcal{S}_H^-(k; \phi)$ ($0 \leq t \leq 1$), and it is convex in the direction of the real axis.

THEOREM 2. *Let $f_1 = h_1 + \overline{g_1} \in \mathcal{S}_H(k, \gamma; \phi)$ and $f_2 = h_2 + \overline{g_2} \in \mathcal{S}_H(k, \gamma; \psi)$. Suppose that*

$$\Re(k^2 h'_1 \overline{h'_2} - g'_1 \overline{g'_2}) \geq 0$$

and $t\phi + (1-t)\psi$ is convex in the direction γ , then $f = tf_1 + (1-t)f_2 \in \mathcal{S}_H(k)$ ($0 \leq t \leq 1$), and it is convex in the direction γ .

Proof. For $g'_j = \omega_j h'_j$ satisfy the conditions $|\omega_j| \leq k < 1$ ($j = 1, 2$), we have

$$(9) \quad |\omega| = \left| \frac{tg'_1 + (1-t)g'_2}{th'_1 + (1-t)h'_2} \right| = \left| \frac{t\omega_1 h'_1 + (1-t)\omega_2 h'_2}{th'_1 + (1-t)h'_2} \right|.$$

By assumption, it follows that

$$(10) \quad \begin{aligned} & k^2 |th'_1 + (1-t)h'_2|^2 - |t\omega_1 h'_1 + (1-t)\omega_2 h'_2|^2 \\ &= t^2 |h'_1|^2 (k^2 - |\omega_1|^2) + (1-t)^2 |h'_2|^2 (k^2 - |\omega_2|^2) \\ &\quad + 2t(1-t) \cdot \Re((k^2 - \omega_1 \overline{\omega_2}) h'_1 \overline{h'_2}) \\ &\geq 2t(1-t) \cdot \Re(k^2 h'_1 \overline{h'_2} - g'_1 \overline{g'_2}) \geq 0. \end{aligned}$$

Hence $|\omega| \leq k < 1$. Since $h_1 - e^{2i\gamma} g_1 = \phi$ and $h_2 - e^{2i\gamma} g_2 = \psi$, we have

$$\begin{aligned} h - e^{2i\gamma} g &= [th_1 + (1-t)h_2] - e^{2i\gamma} [tg_1 + (1-t)g_2] \\ &= t(h_1 - e^{2i\gamma} g_1) + (1-t)(h_2 - e^{2i\gamma} g_2) = t\phi + (1-t)\psi, \end{aligned}$$

which is convex in the direction γ by the assumption. Thus, from Lemma 2, we know that $f \in \mathcal{S}_H(k)$ and is convex in the direction γ . \blacksquare

THEOREM 3. *Let*

$$f_1 = h_1 + \overline{g_1} \in \mathcal{S}_H(k, \gamma; \phi) \quad \text{and} \quad f_2 = h_2 + \overline{g_2} \in \mathcal{S}_H\left(k, \gamma + \frac{\pi}{2}; \phi\right),$$

where

$$(11) \quad \phi(z) = \int_0^z \frac{e^{i\gamma} d\zeta}{(1+\zeta e^{i\theta})(1+\zeta e^{-i\theta})} \quad (\theta \in \mathbf{R}).$$

Suppose that

$$\Re(k^2 h'_1 \overline{h'_2} - g'_1 \overline{g'_2}) \geq 0,$$

then $f = tf_1 + (1 - t)f_2 \in \mathcal{S}_H(k)$ ($0 \leq t \leq 1$) and convex in the direction γ .

Proof. Making use of the similar arguments as in the proof of Theorem 2, in view of (9) and (10), we obtain that the dilatation ω of $f = tf_1 + (1 - t)f_2$ satisfies $|\omega| \leq k < 1$.

Now we show that f is convex in the direction γ . Note that

$$h'_2 - e^{2i\gamma} g'_2 = (h'_2 + e^{2i\gamma} g'_2) \left(\frac{h'_2 - e^{2i\gamma} g'_2}{h'_2 + e^{2i\gamma} g'_2} \right) = \phi'(z) \left(\frac{1 - e^{2i\gamma} \omega_2}{1 + e^{2i\gamma} \omega_2} \right) = \phi'(z) p(z),$$

where

$$p(z) = \frac{1 - e^{2i\gamma} \omega_2}{1 + e^{2i\gamma} \omega_2}$$

satisfies $\Re(p(z)) > 0$. By setting $\kappa(z)$ by (3), we find that

$$\begin{aligned} \Re\left(\frac{ze^{-i\gamma}(h' - e^{2i\gamma}g')}{\kappa(z)}\right) &= \Re\left(\frac{ze^{-i\gamma}}{\kappa(z)}[t(h'_1 - e^{2i\gamma}g'_1) + (1 - t)(h'_2 - e^{2i\gamma}g'_2)]\right) \\ &= t \cdot \Re\left(\frac{ze^{-i\gamma}\phi'(z)}{\kappa(z)}\right) + (1 - t) \cdot \Re\left(\frac{ze^{-i\gamma}\phi'(z)p(z)}{\kappa(z)}\right) \\ &= t + (1 - t)\Re(p(z)) > 0. \end{aligned}$$

Therefore, by Lemma 3, we see that $e^{-i\gamma}(h - e^{2i\gamma}g)$ is convex in the direction of the real axis, and hence the function $h - e^{2i\gamma}g$ is convex in the direction γ . Furthermore, by Lemma 2 and Lemma 5, we conclude that $f \in \mathcal{S}_H(k)$, and it is convex in the direction γ . ■

Next, we prove the convexity of the linear combinations $f = tf_1 + (1 - t)f_2$ for the classes $\mathcal{S}_H^-(k; \phi)$ and $\mathcal{S}_H^+(k; \phi)$ for special ϕ .

THEOREM 4. Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H^-(k; \phi)$ ($j = 1, 2$), where

$$\phi(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) \quad (z \in \mathbf{D}).$$

Then $f = tf_1 + (1 - t)f_2 \in \mathcal{S}_H^-(k; \phi)$ ($0 \leq t \leq 1$), and $f(\mathbf{D})$ is convex.

Proof. In view of Corollary 4, we have $f = h + \overline{g} \in \mathcal{S}_H^-(k; \phi)$, then by Lemma 2 the set $f(\mathbf{D})$ will be convex if and only if the analytic functions $h - e^{2i\theta}g$ are univalent and convex in the direction θ for all θ , $0 \leq \theta < \pi$. To show the latter, it is sufficient to show that the functions $F_\theta = ie^{-i\theta}(h - e^{2i\theta}g)$ are univalent and convex in the direction of the imaginary axis.

Note that

$$\begin{aligned} h'(z) - g'(z) &= [th'_1(z) + (1-t)h'_2(z)] - [tg'_1(z) + (1-t)g'_2(z)] \\ &= t(h'_1(z) - g'_1(z)) + (1-t)(h'_2(z) - g'_2(z)) \\ &= \frac{1}{1-z^2}. \end{aligned}$$

Taking $\mu = \nu = \pi/2$ in (4), we have

$$\begin{aligned} \Re((1-z^2)F'_\theta(z)) &= -\Im(e^{-i\theta}[h'(z) - e^{2i\theta}g'(z)](1-z^2)) \\ &= -\Im([e^{-i\theta}h'(z) - e^{i\theta}g'(z)](1-z^2)) \\ &= -\Im([(h'(z) - g'(z)) \cos \theta - i(h'(z) + g'(z)) \sin \theta](1-z^2)) \\ &= -\Im\left(\cos \theta - i \sin \theta \frac{h'(z) + g'(z)}{h'(z) - g'(z)}\right) \\ &= \Re(p(z)) \sin \theta \geq 0, \end{aligned}$$

where

$$p(z) = \frac{h'(z) + g'(z)}{h'(z) - g'(z)}$$

satisfies $\Re(p(z)) > 0$. Thus by Lemma 4, we see that the function F_θ is univalent and convex in the direction of the imaginary axis. \blacksquare

In view of Theorem 4 and Lemma 6, we have the following result.

COROLLARY 5. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H^-(k; \phi)$ ($j = 1, 2, \dots, n$), where*

$$\phi(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right) \quad (z \in \mathbf{D}).$$

Then $f = \sum_{j=1}^n t_j f_j \in \mathcal{S}_H^-(k; \phi)$ ($0 \leq t_j \leq 1, \sum_{j=1}^n t_j = 1$), and $f(\mathbf{D})$ is convex.

By similarly applying the method as in the proof of Theorem 4, we can easily get the following result for the class $\mathcal{S}_H^+(k; \phi)$ for special ϕ .

THEOREM 5. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_H^+(k; \phi)$ ($j = 1, 2$), where*

$$\phi(z) = \frac{z}{1-z} \quad (z \in \mathbf{D}).$$

Then $f = tf_1 + (1-t)f_2 \in \mathcal{S}_H^+(k; \phi)$ ($0 \leq t \leq 1$), and $f(\mathbf{D})$ is convex.

Proof. By Corollary 3 with $A = 1, B = 0$ and $\alpha = -1$, we have $f = h + \overline{g} \in \mathcal{S}_H^+(k; \phi)$. In order to prove that $f(\mathbf{D})$ is convex, by Lemma 2, it suffices to

show that the analytic function $h - e^{2i\theta}g$ is convex in the direction θ for every $\theta \in [0, \pi)$. The function $h - e^{2i\theta}g$ is convex in the direction θ if and only if $F_\theta = ie^{-i\theta}(h - e^{2i\theta}g)$ is convex in the direction of the imaginary axis.

Note that

$$\begin{aligned} h'(z) + g'(z) &= [th'_1(z) + (1-t)h'_2(z)] + [tg'_1(z) + (1-t)g'_2(z)] \\ &= t(h'_1(z) + g'_1(z)) + (1-t)(h'_2(z) + g'_2(z)) \\ &= \frac{1}{(1-z)^2}. \end{aligned}$$

For $\theta \in [0, \pi/2)$, taking $\mu = \nu = 0$ in (4), we have

$$\begin{aligned} \Re(-iF'_\theta(z)(1-z)^2) &= \Re(e^{-i\theta}[h'(z) - e^{2i\theta}g'(z)](1-z)^2) \\ &= \Re([e^{-i\theta}h'(z) - e^{i\theta}g'(z)](1-z)^2) \\ &= \Re([(h'(z) - g'(z)) \cos \theta - i(h'(z) + g'(z)) \sin \theta](1-z)^2) \\ &= \Re\left(\frac{h'(z) - g'(z)}{h'(z) + g'(z)} \cos \theta - i \sin \theta\right) \\ &= \Re(p(z)) \cos \theta \geq 0, \end{aligned}$$

where

$$p(z) = \frac{h'(z) - g'(z)}{h'(z) + g'(z)}$$

satisfies $\Re(p(z)) > 0$. Therefore, by Lemma 4, the function F_θ is convex in the direction of the imaginary axis for $\theta \in [0, \pi/2)$. The same conclusion can be drawn for the function F_θ with $\theta \in [\pi/2, \pi)$ if we apply Lemma 4 with $\mu = \nu = \pi$. ■

In view of Theorem 5 and Lemma 6, we have the following result.

COROLLARY 6. *Let $f_j = h_j + \bar{g}_j \in \mathcal{S}_H^+(k; \phi)$ ($j = 1, 2, \dots, n$) with*

$$\phi(z) = \frac{z}{1-z} \quad (z \in \mathbf{D}).$$

Then $f = \sum_{j=1}^n t_j f_j \in \mathcal{S}_H^+(k; \phi)$ ($0 \leq t_j \leq 1, \sum_{j=1}^n t_j = 1$), and $f(\mathbf{D})$ is convex.

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