LINEAR COMPLEMENTARITY PROBLEMS
SOLVABLE BY A SINGLE LINEAR PROGRAM

by

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Abstract

It is shown that the linear complementarity problem
of finding a z in R^n such that $Mz + q \geq 0$, $z \geq 0$ and
$z^T(Mz+q) = 0$ can be solved by a single linear program in
some important special cases such as when M or its
inverse is a Z-matrix, that is a real square matrix
with nonpositive off-diagonal elements. As a conse-
quence certain problems in mechanics, certain problems
of finding the least element of a polyhedral set and
certain quadratic programming problems, can each be
solved by a single linear program.

We consider the linear complementarity problem of finding a z in R^n such that

(1) $Mz + q \geqq 0$, $z \geqq 0$, $z^T(Mz+q) = 0$

where M is a given real $n \times n$ matrix and q is a given vector in R^n. A number of authors [3,19,20,14,4,5,18,9] have recently considered an important special case of this problem under the restriction that M is a Z-matrix, that is a real square matrix with nonpositive off-diagonal elements, and have proposed a variety of methods for its solution. Originally Chandrasekaran [3] proposed solving a sequence of linear inequalities, Saigal [20] proposed Lemke's method, and Cottle, Golub and Sacher [4,5,18] proposed a modification of the principal pivoting method [6], a specialization of Chandrasekaran's method and a modification of the point successive overrelaxation technique [9]. Part of the significance of this problem arises from the fact that a number of free boundary problems of fluid mechanics can be solved by solving a linear complementarity problem (1) where M is a Z-matrix [8,5]. It is anticipated that many more physically significant free boundary value problems governed by elliptic partial differential equations will lead to complementarity problems (1) for which M is a Z-matrix [10].

The principal and somewhat surprising result of this paper is that each solution z of the linear program

(2) minimize p^Tz subject to $Mz + q \geqq 0$, $z \geqq 0$

for an easily determined p in R^n, solves the linear complementarity problem (1) for a number of special cases, including those when M or its inverse (if it exists) are Z-matrices (Theorems 1 and 2). In addition if M is a Z-matrix with a nonnegative inverse (or equivalently a Z-matrix with positive principal minors), we show that the

least element of the polyhedral set $\{z \mid Mz+q \geq 0, z \geq 0\}$ in the sense of Cottle-Veinott [7] can be obtained by a single linear program (Theorem 3). Finally, because the quadratic programming problem of minimizing $\frac{1}{2}z^T Mz + q^T z$ subject to $z \geq 0$ is equivalent [9, p. 386, 15, p. 111] to the linear complementarity problem (1) when M is symmetric and positive semidefinite, we will show (Theorem 4) that this quadratic programming problem can be solved by a single linear program whenever M or its inverse is a Z-matrix. We state now our principal result.

THEOREM 1: Let the set $\{z \mid Mz+q \geq 0, z \geq 0\}$ be nonempty, and let M satisfy

(3) $\quad MZ_1 = Z_2$

(4) $\quad r^T Z_1 + s^T Z_2 > 0 \quad (r,s) \geq 0$

where Z_1 and Z_2 are $n \times n$ Z-matrices, $r \in R^n$ and $s \in R^n$. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with

(5) $\quad p = r + M^T s$

To prove this theorem it is convenient to first write the dual linear program to (2)

(6) $\quad$ maximize $-q^T y$ subject to $-M^T y + p \geq 0$, $y \geq 0$

and to establish the following simple but key lemma.

LEMMA 1: If z solves the linear program (2) and if the corresponding optimal dual variable y satisfies

$\quad (I-M^T)y + p > 0$

where I is the identity matrix, then z solves the linear complementarity problem (1).

Proof: $\quad y^T(Mz+q) + z^T(-M^Ty+p) = y^Tq + z^Tp = 0$

Since $\quad y \geq 0$, $Mz + q \geq 0$, $z \geq 0$ and $-M^Ty + p \geq 0$ it follows that

$$y_i(Mz+q)_i = 0, \quad z_i(-M^Ty+p)_i = 0 \quad i = 1,\ldots,n$$

where subscripted quantities denote the ith element of a vector. But $y_i + (-M^Ty+p)_i > 0$, $i = 1,\ldots,n$, hence either $y_i > 0$ or $(-M^Ty+p)_i > 0$, $i = 1,\ldots,n$, and consequently $(Mz+q)_i = 0$ or $z_i = 0$, $i = 1,\ldots,n$. $\quad\square$

Proof of Theorem 1: Since $y = s$ is a dual feasible point, the dual linear programs (2) and (6) must have solutions, which we denote by z and y respectively. Let $Z_1 = D - V$ and $Z_2 = D - U$, where V and U are nonnegative matrices and D is a positive diagonal matrix. Then

$$0 < r^TZ_1 + s^TZ_2 = (r^T+s^TM)Z_1 = p^T(D-V)$$

$$= p^T(D-V) + y^T(-MD+MV+D-U) \quad (\text{Since } M(D-V)=D-U)$$

$$= (-y^TM+p^T)(D-V) + y^T(D-U)$$

$$\leq (y^T(I-M)+p^T)D \qquad\qquad (\text{Since } -y^TM+p^T \geq 0, \; V \geq 0,$$
$$y \geq 0, \; U \geq 0)$$

Since D is a positive diagonal matrix, it follows that $y^T(I-M) + p^T > 0$ and by Lemma 1, z solves the linear complementarity problem (1). $\quad\square$

Remark 1: The proof of Theorem 1 shows that conditions (3),(4) and (5) imply that there exists a dual feasible point for (6) and for each dual feasible point the condition $(I-M^T)y + p > 0$ of Lemma 1 is satisfied. The converse (which is not needed in the sequel of this paper) is also true and can be shown by using Motzkin's theorem of the alternative [15, p. 28]. Conditions (3), (4) and (5) are also equivalent to $MZ_3 \leq I$, $p^TZ_3 > 0$, $p = r + M^Ts$, $Z_3 \in Z$ and $(r,s) \geq 0$. The condition $p = r + M^Ts$, $(r,s) \geq 0$ is equivalent to dual feasibility.

Remark 2: The set Z of Z-matrices contains an important subset K which has been extensively characterized by Fiedler and Pták [13, p. 387]. This set K will play an important role in obtaining useful special cases of Theorem 1. In particular we shall employ the following equivalent characterizations of a K-matrix A: (a) A is a Z-matrix with a nonnegative inverse, (b) A is a Z-matrix with positive principal minors, (c) A is a Z-matrix and there exists an $r \in R^n$, $r \geq 0$, such that $r^T A > 0$, and (d) A is a Z-matrix and $z_i(Az)_i \leq 0$, $i = 1,\ldots,n$, implies that $z = 0$. (A K-matrix is also called an M-matrix by some authors.)

The following immediate consequence of Theorem 1, is an existence result for the linear complementarity problem (1) which also provides a linear program (2) for solving (1) for important special cases such as when M or its inverse are Z-matrices.

THEOREM 2: Let $\{z \mid Mz+q \geq 0, z \geq 0\}$ be nonempty, and let e be any positive vector in R^n (in particular it may be taken as a vector of ones.) Then for each of the cases when

(a) $M = Z_2 Z_1^{-1}$, $Z_1 \in K$, $Z_2 \in Z$ ($p=r\geq 0, r^T Z_1 > 0$)

(b) $M = Z_2 Z_1^{-1}$, $Z_1 \in Z$, $Z_2 \in K$ ($p=M^T s, s\geq 0, s^T Z_2 > 0$)

(c) $M \in Z$ ($p=e$)

(d) $M^{-1} \in Z$ ($p=M^T e$)

(e) $-M \in K$ ($p=-e$ or $p=M^T e$)

(f) $-M^{-1} \in K$ ($p=-M^T e$ or $p=e$)

the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the p indicated above.

Proof:

(a) Follows from Theorem 1 by setting $s = 0$, and from Remark 2(c).

(b) Follows from Theorem 1 by setting $r = 0$, and from Remark 2(c).

(c) Follows from part (a) of this Theorem by setting $Z_1 = I$, $M = Z_2$ and $r = e$.

(d) Follows from part (b) of this Theorem by setting $Z_2 = I$, $M = Z_1^{-1}$ and $s = e$.

(e) The case $p = -e$ follows from part (b) of this Theorem by setting $Z_1 = -I$, $M = -Z_2$ and $s = -(M^T)^{-1}e$. The case $p = M^T e$ follows from part (d) of this Theorem by observing that, by Remark 2(a), $M^{-1} \leq 0$ and hence $M^{-1} \in Z$.

(f) The case $p = -M^T e$ follows from part (a) of this Theorem by setting $Z_2 = -I$, $M = -Z_1^{-1}$ and $r = -M^T e$. The case $p = e$ follows from part (c) of this Theorem by observing that, by Remark 2(a), $M \leq 0$, and hence $M \in Z$. $\square$

Remark 3: Some special nonnegative matrices can be handled as special cases of Theorem 2 above. For example the case $M^{-1} \in K$ (and hence $M \geq 0$) is a special case of part (d) of Theorem 2. Also if in part (a) of Theorem 2 we impose the additional restriction that $Z_2 \geq Z_1$ then $M = (Z_2 - Z_1)Z_1^{-1} + I \geq I$. Similarly some special matrices with nonnegative inverses can be handled as special cases of Theorem 2. Thus the case $M \in K$ (and hence $M^{-1} \geq 0$) is a special case of part (c) of Theorem 2. Also if in part (b) of Theorem 2 we impose the additional restriction that $Z_1 \geq Z_2$ then $M^{-1} = (Z_1 - Z_2)Z_2^{-1} + I \geq I$.

Remark 4: Note that whenever $p \geq 0$, as is the case in parts (a), (c) and (f) of Theorem 2 above, $y = 0$ is a dual feasible point, and hence the dual simplex algorithm [11] should be used.

Remark 5: Note that when $M^{-1} \geq 0$, as is the case for example when $M \in K$, the set $\{z \mid Mz+q \geq 0, z \geq 0\}$ is nonempty for any q. For, choose $a \in R^n$ such that $a \geq 0$ and $a \geq q$, and define $z = M^{-1}(a-q) \geq 0$. Then $Mz + q = a \geq 0$.

Our next result shows how to find by a single linear program the least element, in the sense of Cottle and Veinott [7], of a polyhedral set defined by a K-matrix.

THEOREM 3: If $M \in K$, then for each q the polyhedral set $\{z \mid Mz+q \geq 0, z \geq 0\}$ contains a unique least element $\bar{z}$, that is $\bar{z} \leq z$ for all $z \in \{z \mid Mz+q \geq 0, z \geq 0\}$, which is also the unique solution of the linear complementarity problem (1), and which can be obtained by solving the linear program (2) with any $p > 0$.

Proof: Because the condition $\bar{z} \leq z$ is equivalent to $p^T(z-\bar{z}) \geq 0$ for all $p > 0$, it follows that $\bar{z}$ is the desired least element of $\{z \mid Mz+q \geq 0, z \geq 0\}$ if and only if it solves the linear program (2) for all $p > 0$. By Remark 5 and Theorem 2(c), for any q the linear complementarity problem has a solution which can be obtained by solving the linear program (2) with any $p > 0$. Suppose $\bar{z}$ and $\hat{z}$ are solutions to the linear program (2) with $p = \bar{p} > 0$ and $p = \hat{p} > 0$ respectively. (We do not exclude the possibility that $\bar{p} = \hat{p}$.) Then by Theorem 2(c) both $\bar{z}$ and $\hat{z}$ solve the linear complementarity problem (1). Hence, similarly to Gale-Murty [17, p. 76], we have that for $i = 1, \ldots, n$

$$(\bar{z}-\hat{z})_i (M(\bar{z}-\hat{z}))_i = (\bar{z}-\hat{z})_i (M\bar{z}+q-(M\hat{z}+q))_i$$
$$= -\bar{z}_i (M\hat{z}+q)_i - \hat{z}_i (M\bar{z}+q)_i \leq 0$$

Thus by Remark 2(d), $\bar{z} - \hat{z} = 0$ and $\bar{z}$ is a unique solution of the linear program (2) no matter what $p > 0$ is used. Hence $\bar{z}$ is the desired least element. □

Remark 6: For very large linear complementarity problems such as those arising from discretization of numerical analysis problems [9,10,5] and in which M is a Z-matrix, we propose the use of large scale linear programming codes for solving (2) or alternatively the use of relaxation methods or pro- jection methods [16,1,2,12] to solve the system of linear inequalities and equalities $Mz + q \geq 0$, $z \geq 0$, $-M^T y + p \geq 0$, $y \geq 0$, $p^T z + q^T y = 0$, which constitutes the Kuhn-Tucker conditions of the linear program (2). Since these methods do not disturb the sparsity, if any, of the matrix M, it would be interesting to compare them with the methods proposed in [5] for large sparse matrices.

We conclude by showing that under suitable conditions the quadratic program

(7) $\quad$ minimize $\frac{1}{2}z^T Mz + q^T z$ $\quad$ subject to $\quad z \geq 0$

can be solved by solving the linear program (2).

THEOREM 4: (a) Let M be a symmetric positive semidefinite matrix, let $\{z \mid Mz+q \geq 0, z \geq 0\}$ be nonempty and let either $M \in Z$ or $M^{-1} \in Z$. Then the quadratic program (7) has a solution which can be obtained by solving the linear program (2) with $p = e$, any positive vector in R^n, when $M \in Z$; and $p = M^T e$ when $M^{-1} \in Z$. (b) Let M be symmetric. If $M \in K$, or in particular if $M \in Z$ and M has a positive strictly dominant diagonal, then the quadratic program (7) has a unique solution which can be obtained by solving the linear program (2) with $p = e$, any positive vector in R^n.

Proof: (a) The necessary and sufficient Kuhn-Tucker optimality conditions for (7) are precisely conditions (1) [9, p. 386, 15, p. 111]. This part of the theorem follows then from Theorem 2, parts (c) and (d).

(b)   By Remark 2(c),   M   is a K-matrix , and by
Theorem 3, the conditions (1) have a unique solution for
each   q   which is also the unique solution of the quadratic
program (7) and the linear program (2) for any   p > 0.   □

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