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Abstract

We introduce a new class of dynamical systems that we call "linear complementarity systems". The evolution of these systems typically consists of a series of continuous phases separated by "events" which cause a change in dynamics and possibly a jump in the state vector. The occurrence of events is governed by certain inequalities similar to those appearing in the Linear Complementarity Problem of mathematical programming. The framework we describe is suitable for certain situations in which both differential equations and inequalities play a role, for instance in mechanics, electrical networks, and dynamic optimization. We present a precise definition of linear complementarity systems and give sufficient conditions for existence and uniqueness of solutions.

1 Introduction

In many technical and economic applications one encounters systems of differential equations and inequalities. For a quick roundup of examples, one may think of the following: motion of rigid bodies subject to unilateral constraints, electrical networks with ideal diodes, optimal control problems with inequality constraints in the states and/or controls, dynamic versions of linear and nonlinear programming problems, and dynamic Walrasian economies. It has to be noted that there is considerable inherent complexity in systems of differential equations and inequalities, since nonsmooth trajectories and possibly even jumps have to be taken into account;

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as a result of this, even basic issues such as existence and uniqueness of solutions are difficult to settle. Given the wealth of possible applications however, it is of interest to overcome these difficulties.

In the literature one can find several lines of research dealing with dynamics subject to inequality constraints, some mainly motivated by problems in mechanics, others more closely connected to operations research and economics. One way in which differential equations and inequalities can be combined is by means of differential inclusions, see [1] and the references given there. By their nature, differential inclusions usually have nonunique solutions; in this paper however we shall be interested in systems that are like ODEs in the sense that typically one will have uniqueness of solutions. A formulation of this type is provided by the "sweeping process" of Moreau [17] (see also [16] and [3]), which is mainly geared towards applications in mechanics. In a more economically oriented context, Dupuis and Nagurney [8] and Nagurney and Zhang [18] have recently discussed so-called "projected dynamical systems" for which they prove existence and uniqueness results. In [9] Filippov studies differential equations with discontinuous righthand sides. Van Bokhoven [2] has given a formulation for electrical networks with ideal diodes that fits within the description that will be used below; he mainly discusses equilibria and only briefly touches upon solving the equations in time.

The framework that we shall present in this paper is more general than the one discussed by Moreau and co-workers in that we do not constrain ourselves to mechanical systems; on the other hand, we shall discuss only (piecewise) linear systems whereas Moreau considers fully nonlinear systems. The limitation to linear dynamics is introduced here mainly because it simplifies the discussion of jump phenomena. It will be proven below that the formulation that we give agrees with the one given by Moreau within the class of systems to which both formulations apply. The solution concept proposed by Dupuis and Nagurney is such that no jumps can occur in state trajectories and so in general their formulation is different from ours. In the terminology of mechanics, we study inelastic rather than elastic collisions in this paper, since we want to allow transitions from constrained to unconstrained modes and vice versa. Elastic collisions bring their own problems in existence and uniqueness of solutions, see for instance [22] and [19].

The present paper continues a line of research begun in [20], where existence and uniqueness results were given for the case of systems with a single inequality constraint. The main result of this paper will be to give sufficient conditions for local existence and uniqueness of solutions for systems with several inequality constraints. We do this under a formulation of the mode transition rule that is different (for the multiconstrained case) than the one used in [20]. It seems to be difficult to obtain well-posedness results for the multiconstrained case using the rule of [20]; moreover, this rule is not consistent with Moreau's rule in the case of mechanical systems.

Obtaining well-posedness results for Moreau's sweeping process in the multiconstrained case is mentioned as an open problem by Monteiro Marques [16, p. 126]. Of course, this problem is posed in a general nonlinear setting and is certainly not completely solved here, since we consider only systems that have linear dynamics in each mode. Dupuis and Nagurney [8] prove global existence and uniqueness of solutions (under appropriate Lipschitz conditions) for systems with several inequality constraints. However, as noted above, the dynamics they consider is in general different from ours. To illustrate the difference, note that Dupuis and Nagurney also obtain continuous dependence of the solutions on initial conditions, which is a property that in our context need not hold (see Example 8.3 below). Filippov [9] presents existence and uniqueness results for differential equations with discontinuous righthand sides. In particular, the righthand side is assumed to be piecewise continuous with a countable number of domains of continuity with nonempty interior (separated by surfaces). The main difference with the work presented in this paper is that we deal with inequalities that are not allowed to be violated. Specification of the domains (called "mode selection" in our set-up) is a nontrivial task and has to be accomplished before formulating the dynamics of the linear complementarity systems as differential equations with discontinuous righthand sides. However, this results in domains of continuity ("modes") that have empty interior, which is different from the assumptions in Filippov's work. Furthermore, solutions in [9] are required to be absolutely continuous, while our solutions may contain jumps in order not to violate the inequalities. Continuous dependence of the solutions on initial conditions holds for systems considered in [9], but in general does not apply for linear complementarity systems.

This paper can be viewed as a continuation of the work of Lötstedt [14] who pioneered the application of the Linear Complementarity Problem (LCP) of mathematical programming to the simulation of the motion of systems of rigid bodies subject to unilateral constraints. There is some change of direction however, since we consider (piecewise) linear systems rather than (nonlinear) mechanical systems and aim for a complete specification of the system dynamics. Such a specification was not given by Lötstedt; in particular he does not precisely specify what trajectories should be chosen in case multiple constraints become active at the same time. One of the main objectives in this paper is to give a complete definition of the dynamics of linear complementarity systems, in a form that is suitable for simulation purposes.

The mode-switching behaviour that we study in this paper may also be looked at from a much more general viewpoint as an interaction of differential equations and switching rules. Systems in which continuous dynamics and discrete transition rules are connected to each other are sometimes called "hybrid systems"; these occur for instance when a discrete device, such as a computer program, interacts with a part of the outside world that has its own continuous dynamics, such as a chemical process. Hybrid systems have recently drawn considerable attention both from computer scientists and from control theorists, see for instance [15]. In this literature, existence and uniqueness of solutions is often simply assumed, and easily verifiable sufficient conditions for well-posedness in other than trivial cases are rarely given. The work presented in this paper may be seen as a contribution towards filling this gap.

The paper is organised as follows. We start with an example to motivate the definitions that will be given later. After having dealt with some mathematical preliminaries in Section 3, a formal definition of the class of linear complementarity systems with the corresponding solution concept is given in Section 4. Mode selection techniques are presented in Section 5. Sufficient conditions for local existence and uniqueness of solutions follow in Section 6. After that, we present a computational example to illustrate that our definition is suitable as a basis for the actual simulation of linear complementarity systems. In section 8, we establish the connection with the sweeping process formulation of Moreau. Finally, conclusions follow in Section 9.

In this paper, the following notational conventions will be in force. $\mathbb{R}$ denotes the real numbers,

$\mathbb{R}_+$ the nonnegative real numbers, $\mathbb{R}_* := \mathbb{R}_+ \cup \{\infty\}$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For a positive integer l, $\bar{l}$ denotes the set $\{1, 2, \ldots, l\}$. If a is a (column) vector with k real components, we write $a \in \mathbb{R}^k$ and denote the ith component by a_i. $M \in \mathbb{R}^{m \times n}$ means that M is a real matrix with size $m \times n$. $M^\top$ is the transpose of the matrix M. The kernel of M is denoted by Ker M and the image by Im M. Given $M \in \mathbb{R}^{k \times l}$ and two subsets $I \subseteq \bar{k}$ and $J \subseteq \bar{l}$, the (I, J)-submatrix of M is defined as $M_{IJ} := (m_{ij})_{i \in I, j \in J}$. In case $J = \bar{l}$, we also write $M_{I\bullet}$ and if $I = \bar{k}$, we write $M_{\bullet J}$. For a vector a, $a_I := a_{I\bullet} = (a_i)_{i \in I}$. $\operatorname{diag}(a_1, \ldots, a_k)$ denotes the diagonal matrix $A \in \mathbb{R}^{k \times k}$ with diagonal entries $a_1, \ldots, a_k$. Given two vectors $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^l$, then $\operatorname{col}(a, b)$ denotes the vector in $\mathbb{R}^{k+l}$ that arises from stacking a over b.

A rational matrix is a matrix with entries in the field $\mathbb{R}(s)$ of rational functions in one variable. A rational matrix is called proper, if for all entries the degree of the numerator is smaller than or equal to the degree of the denominator. A rational matrix is called biproper, if it is square, proper and has an inverse, that is proper too.

The set $C^\infty(\mathbb{R}, \mathbb{R})$ denotes the set of functions from $\mathbb{R}$ to $\mathbb{R}$ that are arbitrarily often differentiable.

A vector $u \in \mathbb{R}^k$ is called nonnegative, and we write $u \geqslant 0$, if $u_i \geqslant 0, i \in \bar{k}$ and positive ($u > 0$), if $u_i > 0, i \in \bar{k}$. If a vector u is not nonnegative, we write $u \not\geqslant 0$. A sequence of scalars is called lexicographically nonnegative, written as $(u^1, u^2, \ldots, u^k) \succeq 0$, if $(u^1, u^2, \ldots, u^k) = (0, 0, \ldots, 0)$ or $u^j > 0$ where $j := \min\{p \in \bar{k} \mid u^p \neq 0\}$. A sequence of scalars is called lexicographically positive, denoted by $(u^1, u^2, \ldots, u^k) \succ 0$, if $(u^1, u^2, \ldots, u^k) \succeq 0$ and $(u^1, u^2, \ldots, u^k) \neq (0, 0, \ldots, 0)$. For a sequence of vectors, we write $(u^1, u^2, \ldots, u^k) \succeq 0$ when $(u_i^1, u_i^2, \ldots, u_i^k) \succeq 0$ for all i. Likewise, we write $(u^1, u^2, \ldots, u^k) \succ 0$ when $(u_i^1, u_i^2, \ldots, u_i^k) \succ 0$ for all i.

For sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \setminus \mathcal{B} := \{x \in \mathcal{A} \mid x \notin \mathcal{B}\}$ and $\mathcal{P}(\mathcal{A})$ denotes the power set of $\mathcal{A}$, i.e. the collection of all subsets of $\mathcal{A}$. For two subspaces V, T of $\mathbb{R}^n$, the notation $V \oplus T = \mathbb{R}^n$ means that V and T form a direct sum decomposition of $\mathbb{R}^n$, i.e. $V + T := \{v + t \mid v \in V,\ t \in T\} = \mathbb{R}^n$ and $V \cap T = \{0\}$.

2 Example

Before we give a formal description of the class of systems under study, we will illustrate some of its hybrid dynamical aspects by considering the example of two carts connected by a spring (used also in [20]). The left cart is attached to a wall by a spring. The motion of the left cart is constrained by a completely inelastic stop. The system is depicted in figure 1.

For simplicity, the masses of the carts and the spring constants are set to 1. The stop is placed in the equilibrium position of the left cart. We now address the question how to model such a system. By x_1, x_2 we denote the deviation of the left and right cart, respectively, from their equilibrium positions and x_3, x_4 are the velocities of the left and right cart, respectively. By u, we denote the reaction force exerted by the stop. Furthermore, we set y equal to x_1. By using

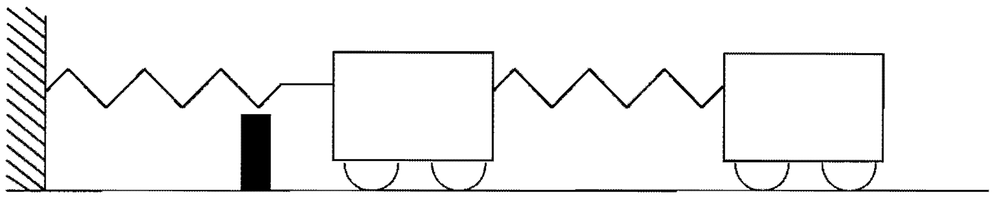

Figure 1: Two-carts system.

simple mechanical laws, we deduce the following dynamical relations for this system.

$$\begin{array}{rcl}\dot{x}_1(t) &=& x_3(t)\\ \dot{x}_2(t) &=& x_4(t)\\ \dot{x}_3(t) &=& -2x_1(t)+x_2(t)+u(t)\\ \dot{x}_4(t) &=& x_1(t)-x_2(t)\\ y(t) &:=& x_1(t)\end{array} \tag{1}$$

To model the stop in this setting, we reason as follows. The variable y should be nonnegative, because it is the position of the left cart with respect to the stop. The force exerted by the stop can only act in the positive direction, so that also u should be nonnegative. If the left cart is not at the stop at time t ($y(t) > 0$), the reaction force vanishes at those moments, i.e. $u(t) = 0$. Similarly, if $u(t) > 0$, the cart must necessarily be at the stop, i.e. $y(t) = 0$. This is expressed by

$$y(t) \geqslant 0,\ u(t) \geqslant 0,\ y(t)u(t) = 0. \tag{2}$$

The system has two modes, depending on whether the stop is active or not. We distinguish between the unconstrained mode ($u(t) = 0$) and the constrained mode ($y(t) = 0$). The dynamics of these modes are given by the Differential and Algebraic Equations (DAEs)

$$\begin{array}{rl}\underline{\text{unconstrained}} & \underline{\text{constrained}}\\ \dot{x}_1(t) = x_3(t) & \dot{x}_1(t) = x_3(t)\\ \dot{x}_2(t) = x_4(t) & \dot{x}_2(t) = x_4(t)\\ \dot{x}_3(t) = -2x_1(t)+x_2(t) & \dot{x}_3(t) = -2x_1(t)+x_2(t)+u(t)\\ \dot{x}_4(t) = x_1(t)+x_2(t) & \dot{x}_4(t) = x_1(t)+x_2(t)\\ u(t) = 0 & y(t) = x_1(t) = 0.\end{array}$$

When the system is in either of these modes, the triple (u, x, y) is given by the corresponding dynamics as long as the remaining inequalities in (2)

$$\begin{array}{rl}\underline{\text{unconstrained}} & \underline{\text{constrained}}\\ y(t) \geqslant 0 & u(t) \geqslant 0\end{array}$$

are satisfied. A mode change is triggered by violation of one of these inequalities. For this example, the following mode transitions are possible.

- **Unconstrained → Constrained**: $y(t) \geqslant 0$ tends to get violated at a time instant $t = t'$. The left cart hits the stop and stays there. The velocity of the left cart is reduced to zero instantaneously at the time of impact: the kinetic energy of the left cart is totally absorbed by the stop due to a purely inelastic collision. A state for which this happens is for instance $x(t') = (0, -1, -1, 0)^\top$.
- **Constrained → Unconstrained**: $u(t) \geqslant 0$ tends to be violated at $t = t'$. The right cart is located at or moving to the right of its equilibrium position, so the spring between the carts is stretched and pulls the left cart away from the stop. This happens for example if $x(t') = (0, 0, 0, 1)^\top$.
- **Unconstrained → Unconstrained with re-initialisation according to Constrained mode.** $y(t) \geqslant 0$ tends to get violated at $t = t'$. As an example, consider $x(t') = (0, 1, -1, 0)^\top$. At the time of impact, the velocity of the left cart is put to zero just as in the first case. Hence, a state jump or re-initialisation to $(0, 1, 0, 0)^\top$ occurs. The right cart is to the right of its equilibrium position and pulls the left cart away from the stop. Stated differently, from $(0, 1, 0, 0)^\top$ smooth continuation in the unconstrained mode is possible.

This last transition is a special one in the sense that first the constrained mode is active causing the corresponding state jump. After the jump no smooth continuation is possible in the constrained mode resulting in a second mode change back to the unconstrained mode.

From state $x(t') = (0, -1, -1, 0)^\top$, we can enter the constrained mode by starting with an instantaneous jump to $x(t'+) = (0, -1, 0, 0)^\top$. This jump is caused by a (Dirac) pulse δ exerted by the stop. In fact, $u = \delta$ results in the state jump $x(t'+) - x(t') = (0, 0, 1, 0)^\top$. This motivates the usage of distributional theory as a feasible mathematical framework to describe physical phenomena like jumps and collisions.

To summarise, the motion of the carts is governed by a pair of Differential and Algebraic equations (DAEs), called the constrained and unconstrained mode. A change of mode is triggered by violation of certain inequalities corresponding to the current mode. The time instants at which this occurs, are called "event times," and one problem is to detect the instances that these events happen. At an event time, the system will switch to a new mode. A mode transition often calls for a state jump or re-initialisation. In the example, we observed velocity jumps, when the left cart arrived at the stop with negative velocity. In this paper, the above dynamics will be formalised for the complete class of linear complementarity systems and special attention is paid to the mode selection problem.

3 Mathematical Preliminaries

We consider a linear input-output system $\dot{x}(t) = Kx(t) + Lu(t)$, $y(t) = Mx(t) + Nu(t)$. The time arguments will often be suppressed. Throughout this section, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. The system parameters K, L, M and N are constant matrices of corresponding dimensions.

The set of distributions defined on $\mathbb{R}$ with support on $[0,\infty)$ is denoted by $\mathcal{D}'_+$. For more details on distributions, we refer to [23]. Particular examples of elements of $\mathcal{D}'_+$ are the δ-distribution and its derivatives. We denote convolution by juxtaposition like ordinary multiplication and denote the delta distribution by δ and its r-th derivative by $\delta^{(r)}$. Linear combinations of these particular distributions will be called *impulsive distributions*, that is, a function $u \in \mathcal{D}'_+$ is an impulsive distribution, if it can be written as $u = \sum_{i=0}^{l} u^{-l}\delta^{(l)}$. A special subclass of $\mathcal{D}'_+$ is the set of regular distributions in $\mathcal{D}'_+$. These are distributions that are smooth on $[0,\infty)$. Formally, a function $u \in \mathcal{D}'_+$ is *smooth* on $[0,\infty)$, if a function $v \in C^\infty(\mathbb{R},\mathbb{R})$ exists such that

$$u(t) = \begin{cases} 0 & (t < 0) \\ v(t) & (t \geqslant 0). \end{cases}$$

Definition 3.1 An *impulsive-smooth distribution* is a distribution $u \in \mathcal{D}'_+$ of the form $u = u_{imp} + u_{reg}$, where u_{imp} is impulsive and u_{reg} is smooth on $[0,\infty)$. The class of these distributions is denoted by C_{imp}.

Given an impulsive-smooth distribution $u = u_{imp} + u_{reg} \in C_{imp}$, we define the leading coefficient of its impulsive part by

$$\operatorname{lead}(u) := \begin{cases} 0, & \text{if } u_{imp} = 0 \\ u^{-l} & \text{if } u_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)} \text{ with } u^{-l} \neq 0. \end{cases} \tag{3}$$

To define the concept of a distributional solution to $\dot{x} = Kx + Lu, y = Mx + Nu$ given an initial condition x_0 and input $u \in C_{imp}^m$, we replace the differential equation by its distributional equivalent:

$$\dot{x} = Kx + Lu + x_0\delta \tag{4a}$$

$$y = Mx + Nu, \tag{4b}$$

where $\dot{x}$ denotes the distributional derivative of x.

Definition 3.2 [11] An element $(x_{x_0,u}, y_{x_0,u}) \in \mathcal{D}'^{(n+r)}_+$ is a (distributional) solution of $\dot{x} = Kx + Lu, y = Mx + Nu$ with initial condition x_0 and $u = \sum_{i=0}^{l} u^{-i}\delta^{(i)} + u_{reg} \in C_{imp}^m$, if $(x_{x_0,u}, y_{x_0,u})$ satisfies (4) as an equality of distributions.

In [11], it is shown that the solution $(x_{x_0,u}, y_{x_0,u})$ exists, is unique in $\mathcal{D}'^{(n+r)}_+$ and belongs to C_{imp}^{n+r}. The solution is given by

$$x_{x_0,u} = \underbrace{\sum_{i=1}^{l}\sum_{j=1}^{i} K^{i-j}Lu^{-i}\delta^{(j-1)}}_{x_{imp}} + x_{reg} \tag{5}$$

with

$$x_{reg}(t) = \begin{cases} e^{Kt}(x_0 + \sum_{i=0}^{l} K^i L u^{-i}) + \int_0^t e^{K(t-\tau)} L u_{reg}(\tau)d\tau & (t \geqslant 0) \\ 0 & (t < 0) \end{cases} \tag{6}$$

and

$$y_{x_0,u} = \underbrace{\sum_{i=1}^{l}\sum_{j=1}^{i} MK^{i-j}Lu^{-i}\delta^{(j-1)} + \sum_{i=0}^{l} Du^{-i}\delta^{(i)}}_{y_{imp}} + y_{reg} \tag{7}$$

with

$$y_{reg}(t) = Mx_{reg}(t) + Nu_{reg}(t),\ t \in \mathbb{R}. \tag{8}$$

If it is clear which x_0 and u are meant, we omit these subscripts in $x_{x_0,u}$ and $y_{x_0,u}$.

We introduce the notation

$$x_{x_0,u}(0+) := \lim_{t\downarrow 0} x_{reg}(t) = x_0 + \sum_{i=0}^{l} K^i Lu^{-i}. \tag{9}$$

Note that the jump $x_{x_0,u}(0+) - x_0$ of the state at time 0 only depends on the impulsive part of the input u. Furthermore, observe that

$$x_{reg}(x_0, u) = x_{x_{x_0,u}(0+),u_{reg}}\ ;\ y_{reg}(x_0, u) = y_{x_{x_0,u}(0+),u_{reg}}. \tag{10}$$

We now consider the system (4) under the additional condition that the output y is zero.

Definition 3.3 A state x_0 is said to be *consistent* for (K, L, M, N), if there exists a regular input u such that

$$\begin{array}{rcl} \dot{x} &=& Kx + Lu + x_0\delta \\ 0 &=& Mx + Nu \end{array} \tag{11}$$

is satisfied. The set of all consistent states for (K, L, M, N) is denoted by $V(K, L, M, N)$ and called the consistent subspace.

The following sequence of subspaces converges in at most n steps to $V(K, L, M, N)$ [11]:

$$\begin{array}{rcl} V_0 &=& \mathbb{R}^n \\ V_{i+1} &=& \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } Kx + Lu \in V_i,\ Mx + Nu = 0\}. \end{array} \tag{12}$$

Let $T_{x_0}(K, L, M, N)$ be the set of possible jumps from initial state x_0 caused by impulsive-smooth inputs u, that result in regular outputs $y_{x_0,u}$. Formally,

$$T_{x_0}(K, L, M, N) = \{x_{x_0,u}(0+) - x_0 \mid u \in C^m_{imp} \text{ such that } y_{x_0,u} \text{ regular}\}. \tag{13}$$

From (9), it is clear that $T_{x_0}(K, L, M, N)$ is actually independent of x_0. Therefore we omit the subscript x_0. The recursion

$$\begin{array}{rcl} T_0 &=& \{0\} \\ T_{i+1} &=& \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m,\ \exists \bar{x} \in T_i \text{ such that } x = K\bar{x} + Lu, M\bar{x} + Nu = 0\} \end{array} \tag{14}$$

converges in maximally n steps to $T(K,L,M,N)$ [11].

Note that $V(K,L,M,N)+T(K,L,M,N)$ is the set of states x_0 for which there exists a $u \in C^m_{imp}$ such that (11) holds (see [11, Prop. 3.23]).

Definition 3.4 The quadruple (K,L,M,N) is called *autonomous*, if for every consistent state x_0 there exists exactly one smooth solution $(u(\cdot), x(\cdot))$ to (11) with $x(0)=x_0$.

One can show that (K,L,M,N) is autonomous is iff for all $x_0 \in V(K,L,M,N)+T(K,L,M,N)$ there exists exactly one $u \in C^m_{imp}$ such that (11) is satisfied [11].

Lemma 3.5 *Consider the system (K,L,M,N) and suppose that the number of inputs (m) equals the number of outputs (r). Then the following statements are equivalent.*

1. *(K,L,M,N) is autonomous*
2. *$V(K,L,M,N) \oplus T(K,L,M,N) = \mathbb{R}^n$ and Ker $\begin{bmatrix} L \\ N \end{bmatrix} = \{0\}$*
3. *$G(s) := M(sI-K)^{-1}L + N$ is invertible as a rational matrix.*

Proof. The quadruple (K,L,M,N) is autonomous iff the system $\Sigma : \dot{x} = Kx+Lu, y = Mx+Nu$ is left invertible in the sense of [11]. In [11], it is proven that the statements

- the system Σ is left invertible
- $V(K,L,M,N) \cap T(K,L,M,N) = \{0\}$ and Ker $\begin{bmatrix} L \\ N \end{bmatrix} = \{0\}$
- $G(s)$ is left invertible

are equivalent. Since $G(s)$ is assumed to be square $(m=r)$, the last statement in fact implies that $G(s)$ is invertible. According to [11, Thm. 3.24], invertibility of $G(s)$ implies that $V(K,L,M,N)+T(K,L,M,N) = \mathbb{R}^n$. □

After these preliminaries from linear systems theory, we now recall some notions from mathematical programming that we shall use. The Linear Complementarity Problem (LCP) [4] is defined as follows.

Given a matrix $M \in \mathbb{R}^{k \times k}$ and $q \in \mathbb{R}^k$, find $w, z \in \mathbb{R}^k$ such that

$$w = q + Mz \tag{15}$$
$$w \geqslant 0, \; z \geqslant 0 \tag{16}$$
$$z^\top w = 0 \tag{17}$$

or show that no such z, w exist. We denote this problem by LCP(q, M).

Let a matrix M of size $k \times k$ and two subsets I and J of $\bar{k}$ of the same cardinality be given. The (I, J)-*minor* of M is the determinant of the square matrix $M_{IJ} := (m_{ij})_{i \in I, j \in J}$. The (I, I)-minors are also known as the principal minors. M is called a *P-matrix*, if all principal minors are (strictly) positive. A matrix M is said to be *positive definite*, if $x^\top M x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Note that a positive definite matrix is not necessarily symmetric according to this definition.

We state the following results.

Theorem 3.6 *For given M, the problem LCP(q, M) has a unique solution for all vectors q if and only if M is a P-matrix.*

Proof. See [4, Thm. 3.3.7]. □

Theorem 3.7 *A positive definite matrix is a P-matrix.*

Proof. [4, Thm. 3.1.6]. □

4 Linear Complementarity Systems

In this section, we will introduce linear complementarity systems and formulate the notion of solution for such systems.

A linear complementarity system is governed by the simultaneous equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{18a}$$
$$y(t) = Cx(t) + Du(t) \tag{18b}$$
$$y(t) \geqslant 0, \quad u(t) \geqslant 0, \quad y^\top(t)u(t) = 0. \tag{18c}$$

The functions $u(\cdot)$, $x(\cdot)$, $y(\cdot)$ take values in $\mathbb{R}^k, \mathbb{R}^n$ and $\mathbb{R}^k$, respectively; A, B, C and D are constant matrices of appropriate dimensions. We shall use the above system description throughout the rest of this paper. The equations (18) will also be interpreted in the distributional sense with an initial condition as in (4). Equation (18c) implies that for every component $i = 1, \dots, k$ either $u_i = 0$ or $y_i = 0$. This results in a multimodal system with 2^k modes, where each mode is characterised by a subset I of $\bar{k}$, indicating that $y_i = 0$, $i \in I$ and $u_i = 0$, $i \in I^c$ with $I^c = \bar{k} \setminus I$. For each such mode the laws of motion are given by Differential and Algebraic Equations (DAEs). Specifically, in mode I they are given by

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \\ y_i(t) & = & 0, \ i \in I \\ u_i(t) & = & 0, \ i \in I^c, \end{array} \right. \tag{19}$$

or equivalently,

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_{\bullet I}u_I(t) \\ 0 &= C_{I\bullet}x(t) + D_{II}u_I(t) \\ y_{I^c}(t) &= C_{I^c\bullet}x(t) + D_{I^cI}u_I(t) \\ u_{I^c}(t) &= 0. \end{cases} \tag{20}$$

The set of consistent states for mode I, denoted by V_I, equals $V(A, B_{\bullet I}, C_{I^c\bullet}, D_{II})$. The jump space is given by $T_I := T(A, B_{\bullet I}, C_{I\bullet}, D_{II})$. The set of initial states for which an impulsive-smooth input exists such that (19) is satisfied in the distributional sense is $V_I + T_I$.

We call mode I *autonomous*, if the quadruple $(A, B_{\bullet I}, C_{I\bullet}, D_{II})$ is autonomous. A standing assumption in the remainder of this paper will be the following one.

Assumption 4.1 All modes are autonomous.

By Lemma 3.5 this is equivalent to saying that $G_{II}(s) := C_{I\bullet}(sI - A)^{-1}B_{\bullet I} + D_{II}$ is invertible for each subset $I \subseteq \bar{k}$. Note that $G_{II}(s)$ is indeed the (I, I)-submatrix of the rational matrix $G(s) := C(sI - A)^{-1}B + D$. By the same lemma, Assumption 4.1 implies $V_I \oplus T_I = \mathbb{R}^n$ for all $I \subseteq \bar{k}$.

Under Assumption 4.1, (19) has a unique impulsive-smooth solution for all individual modes given an arbitrary initial state.

4.1 DAE simulation

Definition 4.2 Given $x_0 \in \mathbb{R}^n$ and $I \in \mathcal{P}(\bar{k})$, we denote the unique solution to (19) for mode I and initial state x_0 by $(u(\cdot, x_0, I), x(\cdot, x_0, I), y(\cdot, x_0, I)) \in C_{imp}^{k+n+k}$. The *flow* $\phi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{P}(\bar{k}) \to \mathbb{R}^n$ is defined as

$$\phi(t, x_0, I) := x(t+, x_0, I),$$

where $x(0+, x_0, I)$ is given by (9).

The computation of this flow or solution in mode I for a *consistent* state of this mode is called *DAE simulation.* From [11, Thm. 3.10], it follows that the input satisfying a DAE of the form (11) can be represented by a linear state feedback. Substituting this feedback in (19) transforms the DAE into an ordinary differential equation (ODE). Hence, the regular part of a solution u satisfying (19) for some initial state is a *Bohl function,* i.e. a function of the form

$$u(t) = \begin{cases} 0 & (t < 0) \\ Fe^{Gt}v & (t \geqslant 0) \end{cases} \tag{21}$$

for real matrices F, G and a vector v depending on the initial state and the specific mode I.

4.2 Re-initialisation

If $x_0 \in V_I$, the corresponding solution $(u(\cdot, x_0, I), x(\cdot, x_0, I), y(\cdot, x_0, I))$ in mode I is regular. If $x_0 \notin V_I$, then a re-initialisation of the initial state will be necessary. Indeed, if $x_0 \notin V_I$, then the solution to (19) calls for a non-regular input $u(\cdot, x_0, I) \in C^k_{imp}$, i.e. an input u with nonzero impulsive part. This impulsive part results in an instantaneous jump or re-initialisation to $x(0+)$ as in (9). By definition, such impulsive inputs cause jumps along T_I. Using (10), we get $y_{x(0+),u_{reg}(\cdot,x_0,I)} = y_{reg}(\cdot, x_0, I)$, i.e. the output of the system (A, B, C, D) with initial state $x(0+)$ and input $u_{reg}(\cdot, x_0, I)$ equals $y_{reg}(\cdot, x_0, I)$. Hence, looking at the components $j \in I$, we get $y_{x(0+),u_{reg}(\cdot,x_0,I),j} = y_{reg,j}(\cdot, x_0, I) = 0, j \in I$. In words, this equation states that the output of the system $(A, B_{\cdot I}, C_{I\cdot}, D_{II})$ with initial state $x(0+)$ and input $u_{reg,I}(\cdot, x_0, I)$ is equal to zero. But this means that $x(0+)$ is a consistent state for mode I, i.e. $x(0+) \in V_I$. Summarizing: we have by definition of T_I that $x(0+) - x_0 \in T_I$ and $x(0+) \in V_I$. Since $V_I \oplus T_I = \mathbb{R}^n$, the jump along T_I from x_0 to $x(0+) \in V_I$ can be done in only one way. The re-initialised vector $x(0+)$ is the projection of x_0 onto V_I along T_I. The projection operator is denoted by $P^{T_I}_{V_I}$.

4.3 Event detection

If the current time, state and mode are τ, x_0 and I, respectively, then we can stay in mode I as long as the inequalities in (18c)

$$u_I(t, x_0, I) \geqslant 0 \text{ and } y_{I^c}(t, x_0, I) \geqslant 0 \tag{22}$$

remain satisfied for $t \geqslant \tau$. The function $\theta : \mathbb{R}^n \times \mathcal{P}(\bar{k}) \to \mathbb{R}_+$ gives the length of the time interval during which the system stays in mode I from initial state x_0. Formally, θ is defined as follows.

Definition 4.3 The time-to-next-event function $\theta : \mathbb{R}^n \times \mathcal{P}(\bar{k}) \to \mathbb{R}_+$ is defined as

$$\theta(x_0, I) := \inf\{t > 0 \mid u_I(t, x_0, I) \not\geqslant 0 \text{ or } y_{I^c}(t, x_0, I) \not\geqslant 0\}.$$

We call $\tau + \theta(x_0, I)$ an *event time*. Since continuation is not possible in mode I after the event time $\tau + \theta(x_0, I)$, a transition to another mode must occur.

To illustrate the definition of θ, consider Example 4.4 and 4.5 of the two-carts system in the next subsection. It is clear that $\theta((0, -1, 0, 0)^\top, \{1\}) = \frac{\pi}{2}$ and $\theta((0, 1, -1, 0)^\top, \{1\}) = 0$.

4.4 Mode selection

The mode selection procedure that we propose is built on the concept of *initial solution*. Loosely speaking, an initial solution with initial state x_0 is a triple $(u, x, y) \in C^{k+n+k}_{imp}$ satisfying (19) on $[0, \infty)$ for some mode I and satisfying (22) either on a time interval of positive length or on a time instant at which delta distributions are active. The idea will be that an initial solution is a starting trajectory for the "global" solution to (18). Indeed, we will build up the global solution to (18) by concatenation of initial solutions from different initial states.

Example 4.4 Consider the two-carts system with initial state $(0,-1,0,0)^\top$. The solution to the unconstrained mode is $u(t) = \cos t$ and $y(t) = 0$. Hence, it satisfies (19) for $I = \{1\}$ on $[0,\infty)$ and (22) on $[0,\frac{\pi}{2})$. So, this solution satisfies (18) on $[0,\frac{\pi}{2})$.

Example 4.5 From initial state $x_0 = (0,1,-1,0)^\top$ first a state jump occurs governed by the laws of the constrained mode, but no regular continuation is possible in the constrained mode. Solving the dynamics corresponding to the constrained mode, i.e. (19) with $I = \{1\}$, gives $u(t) = \delta - \cos t$. Although (22) is not satisfied on a positive time interval, incorporation of this solution in the definition of initial solutions as well seems well-motivated on physical grounds.

We will make our proposal for mode selection more specific.

Definition 4.6 We call a scalar-valued impulsive-smooth distribution $v \in C_{imp}$ *initially nonnegative*, if

$$\begin{cases} \mathrm{lead}(v) > 0, & \text{in case } \mathrm{lead}(v) \neq 0 \\ v_{reg}(t) \geqslant 0, \text{for all } t \in [0,\varepsilon) \text{ for certain } \varepsilon > 0, & \text{otherwise.} \end{cases}$$

An impulsive-smooth distribution in C_{imp}^k is called initially nonnegative, if each of its components is initially nonnegative.

Definition 4.7 We call $(u,x,y) \in C_{imp}^{k+n+k}$ an *initial solution* to (18) with initial state x_0, if

1. there exists an $I \subseteq \bar{k}$ such that (u,x,y) satisfies (19) on $[0,\infty)$ with initial state x_0 in the distributional sense; and
2. u, y are initially nonnegative.

By the fact that the parameters in (18) are constant, it is sufficient to consider only initial time zero.

Given a state x_0, we define $\mathcal{S}(x_0)$ by

$$\mathcal{S}(x_0) := \{I \subseteq \mathcal{P}(\bar{k}) \mid \text{there exists an initial solution } (u,x,y) \text{ to (18) that satisfies (19)for mode } I \text{ on } [0,\infty)\}. \tag{23}$$

The set $\mathcal{S}(x_0)$ denotes the set of possible modes that can be selected from x_0.

4.5 Solution concept

Definition 4.8 A *solution* to (18) on $[0, T_e)$, $T_e > 0$ with initial state x_0 consists of a 6-tuple $(\mathcal{D}, \tau, x_e, x_c, u_c, y_c)$ where $\mathcal{D}$ is either $\{0, \dots, N\}$ for some $N \geqslant 0$ or $\mathbb{N}$,

$$\begin{array}{rcl} \tau: & \mathcal{D} & \to \mathbb{R}_* \\ x_e: & \mathcal{D} & \to \mathbb{R}^n \\ x_c: & (0, T_e) \setminus \tau(\mathcal{D}) & \to \mathbb{R}^n \\ u_c: & (0, T_e) \setminus \tau(\mathcal{D}) & \to \mathbb{R}^k \\ y_c: & (0, T_e) \setminus \tau(\mathcal{D}) & \to \mathbb{R}^k, \end{array}$$

that satisfies the following.

1. There exists a function $I: \mathcal{D} \to P(\bar{k})$ with $I(i) \in \mathcal{S}(x_e(i))$.
2. (a) $\tau(0) = 0$
   (b) $\tau(i+1) = \tau(i) + \theta(x_e(i), I(i))$, $i \in \mathcal{D}$, $i+1 \in \mathcal{D}$
   (c) $\sup_{i \in \mathcal{D}} \tau(i) + \theta(x_e(i), I(i)) \geqslant T_e$.
3. $x_e(0) = x_0$ and $x_e(i+1) = \phi(\tau(i+1) - \tau(i), x_e(i), I(i))$, $i \in \mathcal{D}$, $i+1 \in \mathcal{D}$.
4. On an interval $(a, b) \subseteq [0, T_e)$ with $a = \tau(i)$ for certain $i \in \mathcal{D}$ and $(a, b) \cap \tau(\mathcal{D}) = \varnothing$, $(u_c(t), x_c(t), y_c(t))$ satisfies (19) for $t \in (a, b)$ in mode $I(i)$ with $x_c(a+) = P_{V_{I(i)}}^{T_{I(i)}} x_e(i)$.

Note that $(u_c(t), x_c(t), y_c(t))$ is smooth on an interval $(a, b) \subseteq (0, T_e)$ with $(a, b) \cap \tau(\mathcal{D}) = \varnothing$.

The above definition describes how the solution is built up by concatenation of initial solutions. A solution can be constructed by using flow chart 2 and the following description. Taking x_0 as the initial state starts the procedure. The state x_0 is presented to the mode selection block resulting in a selected mode. If mode I is selected, there are two possibilities indicated by the question in the decision block:

1. From the state x_0 smooth continuation is possible in the selected mode $I \in \mathcal{S}(x_0)$, i.e. $x_0 \in V_I$ (answer is "Yes"). Go to the DAE simulation with this initial state and mode I.
2. No smooth continuation is possible in the selected mode I from x_0 (answer is "No"), i.e. $x_0 \notin V_I$. The right arrow leads to the re-initialisation block performing the projection along T_I onto V_I. The re-initialised state is returned to the mode-selection block. After solving the mode selection problem, the same two possibilities have to be considered again.

If we arrive in a state where the answer to the question in the decision block is "Yes," the DAE simulation block leads to a smooth part of the solution until an event time τ_{event} is reached. The state x_0 is set to $x_c(\tau_{event}-)$ and again given to the mode selection block. Next, the whole cycle starts again.

The construction does not lead to a solution on $[0, T_e)$, if we end up in a state $\tilde{x}$ with $\mathcal{S}(\tilde{x}) = \varnothing$ at time $\tau < T_e$ (deadlock), or we end up in an infinite loop, where only re-initialisations and mode selections occur without smooth continuation.

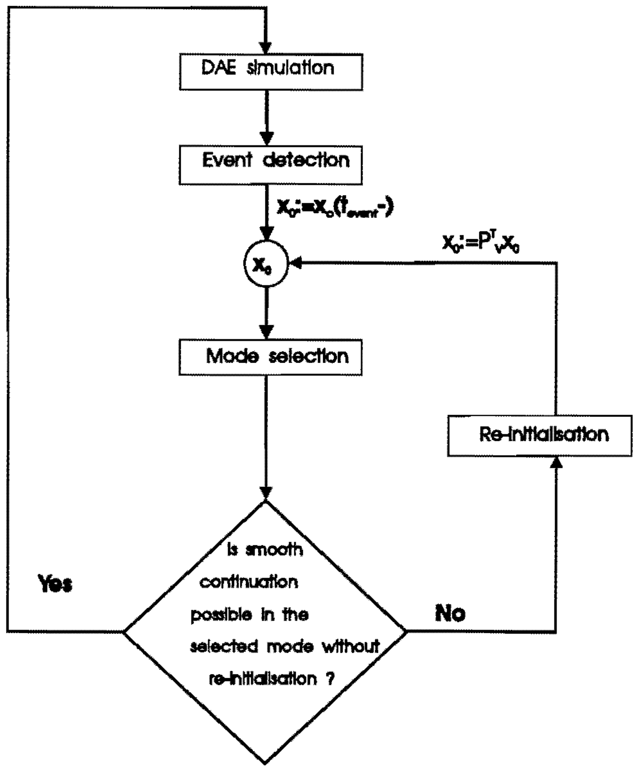

Figure 2: Schematic description of complete dynamics

Before presenting conditions on the complementarity system to guarantee the existence and uniqueness of solutions, we have to introduce two algebraic mode selection procedures.

5 Mode selection methods

In this section, we present two mode selection methods. It will turn out that these two methods are equivalent and that the selected mode set for both selection methods equals $\mathcal{S}(x_0)$ based on the concept of initial solutions.

As noticed in section 4, the solutions to (19) are impulsive-smooth distributions whose regular part is a Bohl function. Such "Bohl distributions" have rational Laplace transforms. Specifically, the Laplace transform of $u = \sum_{i=0}^{l} u^{-i}\delta^{(i)} + u_{reg}$ with u_{reg} as in (21) equals [10]

$$\hat{u}(s) = \sum_{i=0}^{l} u^{-i}s^{i} + F(sI - G)^{-1}v \ .$$

Observe that the polynomial part of the Laplace transform corresponds to the impulsive part and the strictly proper part to the smooth part of the Bohl distribution.

Lemma 5.1 *A Bohl distribution $u \in C^k_{imp}$ is initially nonnegative if and only if there exists* *an $s_0 \in \mathbb{R}$ such that $\hat{u}(s) \geqslant 0$ for all $s \in \mathbb{R}, s \geqslant s_0$. A Bohl distribution $v \in C_{imp}$ is the zero distribution if and only if $\hat{v}$ is the zero function.*

Proof. Evident. □

Let (u, x, y) be an initial solution to (18) with initial state x_0. The Laplace transforms of u, y, denoted by $\hat{u}, \hat{y}$ are rational and satisfy

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + (C(sI - A)^{-1}B + D)\hat{u}(s) \text{ and } \hat{y}^\top(s)\hat{u}(s) = 0 \tag{24}$$

for all $s \in \mathbb{R}$ and

$$\hat{y}(s) \geqslant 0, \ \hat{u}(s) \geqslant 0 \tag{25}$$

for all $s \in \mathbb{R}$ larger than some $s_0 \in \mathbb{R}_+$. This is even an if-and-only-if statement: the Laplace transforms are rational and satisfy (24)-(25) iff the corresponding time functions define an initial solution of (18). Indeed, since $\hat{y}_i$ and $\hat{u}_i$ are rational, they have only a finite number of zeros or are identically zero. Hence, since their product $\hat{y}_i(s)\hat{u}_i(s)$ vanishes for $s \geqslant s_0$ at least one of the two factors has to be identically zero, implying that Item 1. in Definition 4.7 holds.

We formulate the *Rational Complementarity Problem* (nomenclature introduced in [21]):

Rational Complementarity Problem. (RCP(x_0)) Let a system description (A, B, C, D) and initial state x_0 be given. Find rational functions $\hat{y}(s)$ and $\hat{u}(s)$ such that the equalities

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + (C(sI - A)^{-1}B + D)\hat{u}(s) \text{ and } \hat{y}^\top(s)\hat{u}(s) = 0 \tag{26}$$

hold for all $s \in \mathbb{R}$, and there exists an $s_0 \in \mathbb{R}_+$ such that for all $s \geqslant s_0$ we have

$$\hat{y}(s) \geqslant 0, \ \hat{u}(s) \geqslant 0. \tag{27}$$

If $(\hat{u}, \hat{y})$ is a solution to RCP(x_0), any mode J satisfying $\hat{u}_{J^c}(s) = 0$ and $\hat{y}_J(s) = 0$, for all $s \in \mathbb{R}$ is a a mode for which an initial solution exists satisfying (19) for $I = J$. Such modes may hence be selected as continuation modes.

Remark 5.2 The new mode corresponding to a given solution of RCP(x_0) is not unique. Indeed, define

$$I := \{i \in \bar{k} \mid \hat{u}_i(s) > 0 \text{ for almost all } s > s_0\}. \tag{28}$$

Then I is the set of indices i for which u_i as a time function (inverse Laplace transform of $\hat{u}_i$) is initially positive[1] and consequently, y_i is zero. Hence, an initial solution to (18) satisfying (19)

[1] We call an impulsive-smooth distribution u *initially positive*, if u is initially nonnegative and additionally if u_i is regular, then for some $\epsilon > 0$ $u_i(t) > 0, t \in (0, \epsilon)$.

for mode I exists. Consider now the "undetermined index set"

$$K := \{i \in \bar{k} \mid \hat{u}_i(s) = 0 \text{ and } \hat{y}_i(s) = 0 \text{ for all } s\}.$$

Any mode $I \subseteq J \subseteq I \cup K$ may also legally be selected. However, the solution to (19) in mode J with initial state x_0 is the same solution as for mode I. This follows from the fact that the inverse Laplace transform u, y of $\hat{u}, \hat{y}$ satisfies (19) for mode I and mode J. Since all modes are assumed to be autonomous, u, y is the only solution for both modes. Hence, solving (19) for I or J leads to the same triple (u, x, y). Summarizing, given a solution of the RCP(x_0), the freedom in the choice of the modes corresponding to this solution is exactly described by K. Moreover, all choices lead to the same (u, x, y).

The set of modes I that can be selected, or equivalently, the set of modes for which an initial solution exists satisfying (20) for mode I is defined as $\mathcal{S}_{\mathrm{RCP}}(x_0)$. More specifically,

$$\mathcal{S}_{\mathrm{RCP}}(x_0) = \{I \subseteq \mathcal{P}(\bar{k}) \mid \exists (\hat{u}, \hat{y}) \text{ solution to RCP}(x_0) \text{ such that} \hat{u}_{I^c}(s) = 0,\ \hat{y}_I(s) = 0, \text{ for all } s\}. \tag{29}$$

This completes the formulation of the RCP as a mode selection method. By using the power series expansion of the solution to RCP(x_0), we will now derive an alternative mode selection method.

If $(\hat{u}, \hat{y})$ is a solution to RCP(x_0), then it necessarily has to satisfy $\hat{u}_{I^c} = 0$, $\hat{y}_I = 0$ for some $I \subseteq \bar{k}$. Consequently,

$$\begin{aligned} 0 &= R_{I\bullet}(s)x_0 + G_{II}(s)\hat{u}_I(s) \\ \hat{y}_{I^c}(s) &= R_{I^c\bullet}(s)x_0 + G_{I^cI}(s)\hat{u}_I(s), \end{aligned}$$

where $G(s)$ is the proper transfer function $C(sI - A)^{-1}B + D$ and $R(s)$ is the strictly proper rational matrix $C(sI - A)^{-1}$. Note that $G_{II}(s)$ is invertible by Assumption 4.1. This implies that $\hat{u}_I(s) = -G_{II}^{-1}(s)R_{I\bullet}(s)x_0$ and

$$\hat{y}_{I^c}(s) = R_{I^c\bullet}(s)x_0 - G_{I^cI}(s)G_{II}^{-1}(s)R_{I\bullet}(s)x_0.$$

The maximal degree of the polynomial part of $G_{II}^{-1}(s)$ is n, because the underlying state space dimension is n. Hence, the maximal degree of the polynomial part of the rational functions $\hat{u}_I(s)$ and $\hat{y}_{I^c}(s)$ is $n-1$. So, for initial solutions we only have to consider polynomial parts with at most degree $n-1$, or equivalently, derivatives of the Dirac function up to order $n-1$.

In terms of the power series expansion of $\hat{y}(s)$ around infinity,

$$\hat{y}(s) = \sum_{i=-n+1}^{\infty} y^i s^{-i}, \tag{30}$$

$\hat{y}(s)$ is nonnegative for all sufficiently large real s if and only if

$$(y^{-n+1}, y^{-n+2}, \ldots) \succeq 0. \tag{31}$$

Given the system description (A, B, C, D), the *Markov parameters* of the system are defined by

$$\begin{cases} H^i = D, & \text{if } i = 0 \\ H^i = CA^{i-1}B, & \text{if} i = 1, 2, \ldots \end{cases} \tag{32}$$

Note that

$$G(s) = \sum_{i=0}^{\infty} H^i s^{-i}. \tag{33}$$

Using the power series expansions of $\hat{y}$ and $\hat{u}$ and (33), we can reformulate RCP(x_0) as the *Linear Dynamic Complementarity Problem* (nomenclature introduced in [21]) by considering the coefficients corresponding to equal powers of s.

Linear Dynamic Complementarity Problem (LDCP$_\kappa(x_0)$) Let a system description (A, B, C, D), an integer $\kappa \geqslant -n+1$ and an initial state x_0 be given. Let H^i, $i \geqslant 0$ be given by (32). Find sequences $(y^{-n+1}, y^{-n+2}, \ldots, y^{\kappa})$ and $(u^{-n+1}, u^{-n+2}, \ldots, u^{\kappa})$ such that the equations

$$y^i = \sum_{j=-n+1}^{i} H^{i-j} u^j, \qquad \text{if } -n+1 \leqslant i \leqslant 0 \tag{34a}$$

$$y^i = CA^{i-1}x_0 + \sum_{j=-n+1}^{i} H^{i-j} u^j, \qquad \text{if } 1 \leqslant i \leqslant \kappa \tag{34b}$$

are satisfied, and for all indices $i \in \bar{k}$ at least one of the following is true:

$$(y_i^{-n+1}, y_i^{-n+2}, \ldots, y_i^{\kappa}) = 0 \quad \text{and} \quad (u_i^{-n+1}, u_i^{-n+2}, \ldots, u_i^{\kappa}) \succeq 0 \tag{35}$$

$$(y_i^{-n+1}, y_i^{-n+2}, \ldots, y_i^{\kappa}) \succeq 0 \quad \text{and} \quad (u_i^{-n+1}, u_i^{-n+2}, \ldots, u_i^{\kappa}) = 0 \tag{36}$$

LDCP$_\infty(x_0)$ denotes the problem of finding a solution $(u^{-n+1}, u^{-n+2}, \ldots)$ and $(y^{-n+1}, y^{-n+2}, \ldots)$ that satisfies LDCP$_\kappa(x_0)$ for all $\kappa \geqslant -n+1$ (or showing that no such solution exists).

If $(u^{-n+1}, u^{-n+2}, \ldots)$ and $(y^{-n+1}, y^{-n+2}, \ldots)$ is a solution to LDCP$_\infty(x_0)$, then modes J satisfying $(u_i^{-n+1}, u_i^{-n+2}, \ldots) = 0$, $i \in J^c$, $(y_i^{-n+1}, y_i^{-n+2}, \ldots) = 0$, $i \in J$ are candidates for selection.

The complete set of candidates for selection, denoted by $\mathcal{S}^{\kappa}_{\mathrm{LDCP}}(x_0)$, is defined by

$$\mathcal{S}^{\kappa}_{\mathrm{LDCP}}(x_0) := \{I \in \mathcal{P}(\bar{k}) \mid \exists (u^j)_{j=-n+1}^{\kappa}, (y^j)_{j=-n+1}^{\kappa} \text{ solution to LDCP}_\kappa(x_0) \text{ such that } (35) \text{ holds for } i \in I \text{ and } (36) \text{ holds for } i \in I^c\}.$$

In some cases, it suffices to consider $\mathrm{LDCP}_n(x_0)$ instead of $\mathrm{LDCP}_\infty(x_0)$ (see Theorem 6.10 below). In [7], it has been shown that $\mathrm{LDCP}_\kappa(x_0)$ is a special case of the Generalized Linear Complementarity Problem [5] and the Extended Linear Complementarity Problem [6]. In [5], an algorithm is proposed to find all solutions to GLCP.

Theorem 5.3 *The following statements are equivalent.*

1. *The equations (18) have an initial solution for initial state x_0.*
2. *RCP(x_0) has a solution.*
3. *$LDCP_\infty(x_0)$ has a solution.*

Proof. From the derivation of RCP, it follows that *1.* and *2.* are equivalent. If $(\hat{u}, \hat{y})$ is a solution to $\mathrm{RCP}(x_0)$, then the coefficients of the power series expansion of this solution around infinity form a solution to $\mathrm{LDCP}_\infty(x_0)$. Hence, *2.* implies *3.* To see that *3.* implies 1., suppose that $(y^{-n+1}, y^{-n+2}, \ldots)$, $(u^{-n+1}, u^{-n+2}, \ldots)$ is a solution of $\mathrm{LDCP}_\infty(x_0)$. Take $I \in \mathcal{S}^\infty_{\mathrm{LDCP}}(x_0)$ corresponding to this solution of $\mathrm{LDCP}_\infty(x_0)$. Define $x(0+) := x_0 + \sum_{i=0}^{n-1} A^i B u^{-i}$. We first show that $x(0+) \in V_I$. To this end, note that $y^i_I = 0$ and $u^i_{I^c} = 0$ for all i. From (34b), it follows that $x(0+)$ satisfies

$$\begin{array}{rcl}
0 = y^1_I & = & C_{I\bullet} x(0+) + D_{II} v^0 \\
0 = y^2_I & = & C_{I\bullet} A x(0+) + D_{II} v^1 + C_{I\bullet} B_{\bullet I} v^0 \\
\vdots & \vdots & \ddots \\
0 = y^\kappa_I & = & C_{I\bullet} A^{\kappa-1} x(0+) + D_{II} v^{\kappa-1} + C_{I\bullet} B_{\bullet I} v^{\kappa-2} + \ldots + C_{I\bullet} A^{\kappa-2} B_{\bullet I} v^0 \\
\vdots & \vdots & \vdots
\end{array} \tag{37}$$

with $v^i = u^{i+1}_I$, $i \geqslant 0$. Combining algorithm (12) and the equations above, it follows that for $l \geqslant 0$ the states $A^l x(0+) + \sum_{i=0}^{l-1} A^i B_{\bullet I} v^{l-1-i}$ belongs to V_j, $j \geqslant 0$ for $(A, B_{\bullet I}, C_{I\bullet}, D_{II})$ and so in particular for $l = 0$, $x(0+) \in \lim V^j = V_I$. Hence, there exists a regular solution $(u_{reg}(\cdot), x_{reg}(\cdot), y_{reg}(\cdot))$ to (19) for mode I with initial state $x(0+)$.

By differentiating (19) in time and evaluating the resulting equalities at time instant 0, we see that $\tilde{v}^i := u^{(i)}_{reg,I}(0)$, $i = 0, 1, \ldots$ satisfies (37) as well. To show that this implies that $\tilde{v}^i = v^i$ for all i, interpret both sequences as inputs of the discrete time system

$$x(l+1) = Ax(l) + B_{\bullet I} v^l;\ y_I(l) = C_{I\bullet} x(l) + D_{II} v^l,\ l = 0, 1, 2, \ldots$$

with initial state $x(0+)$ and output $y_I(l)$. Then by (37), $y_I(l) = 0$ for all $l \geqslant 0$ and the difference $w^i := v^i - \tilde{v}^i$ is an input that keeps the output of the discrete time system with initial state 0 identically zero. We introduce the z-transform

$$w(z) := \sum_{i=0}^{\infty} w^i z^{-i}.$$

Using the z-transform $G_{II}(z)$ of the transfer function of the discrete time system (see e.g. [13]), we get $0 = G_{II}(z) w(z)$. The invertibility of $G_{II}(z)$ implies that $w(z) = 0$ and hence, $v^i = \tilde{v}^i$, $i \geqslant 0$, or equivalently, $u^{i+1}_I = u^{(i)}_{reg,I}(0)$, $i \geqslant 0$. This also implies that $y^{i+1} = y^{(i)}_{reg}(0), i \geqslant 0$.

We define $u := \sum_{i=0}^{n-1} u^{-i}\delta^{(i)} + u_{reg}$, $x := \sum_{i=1}^{n-1}\sum_{j=1}^{i} A^{i-j}Bu_i\delta^{(j-1)} + x_{reg}$, $y := \sum_{i=1}^{n-1} y^i\delta^{(i-1)} + y_{reg}$. Obviously, (u, x, y) satisfies *1.* in Definition 4.7. We only have to show that *2.* in Definition 4.7 is satisfied. Since $(y^{-n+1}, y^{-n+2}, \ldots) = (y^{-n+1}, \ldots, y^0, y_{reg}^{(0)}(0), y_{reg}^{(1)}(0), \ldots)$, $(u^{-n+1}, u^{-n+2}, \ldots) = (u^{-n+1}, \ldots, u^0, u_{reg}^{(0)}, u_{reg}^{(1)} \ldots)$ is a solution of LDCP$_\infty(x_0)$, (35) or (36) is satisfied for all $i \in \bar{k}$. Hence, u and y are initially nonnegative and so (u, x, y) is an initial solution. □

Corollary 5.4 *There is a one-to-one correspondence between initial solutions to (18), solutions to RCP(x_0), and solutions to LDCP$_\infty(x_0)$. Furthermore, $\mathcal{S}(x_0) = \mathcal{S}_{RCP}(x_0) = \mathcal{S}_{LDCP}^{\infty}(x_0)$.*

Proof. This follows from the proof above. The second statement is a result of the one-to-one correspondence. □

Remark 5.5 Notice that in the proof of Theorem 5.3, a direct link between initial solutions and solutions to LDCP$_\infty(x_0)$ is given. If $u = \sum_{i=0}^{n-1} u^{-i}\delta^{(i)} + u_{reg}$ and $y = \sum_{i=0}^{n-1} y^{-i}\delta^{(i)} + y_{reg}$, define $\tilde{u}^i := u^i$, $i = -n+1, \ldots, 0$ and $\tilde{u}^{i+1} = u_{reg}^{(i)}(0)$, $i \geqslant 0$ and let $\tilde{y}^i$, $i \geqslant -n+1$ be defined analogously. Then $(\tilde{u}^{-n+1}, \tilde{u}^{-n+2}, \ldots)$, $(\tilde{y}^{-n+1}, \tilde{y}^{-n+2}, \ldots)$ is a solution to LDCP$_\infty(x_0)$. We shall use the transformations between LDCP$_\infty(x_0)$, RCP(x_0) and initial solutions frequently. Note that the above proof also yields an alternative way of deriving the LDCP: differentiate the initial solution with incorporation of the impulsive part and evaluate the results at time instant zero. For smooth continuations this method can be directly generalized to the nonlinear case as in [14,21].

6 Well-posedness results

There exist linear complementarity systems, for which no solution exists from certain initial conditions (due to deadlock or infinitely many jumps without smooth continuation) or for which the solution is not unique (see [20]).

Definition 6.1 The complementarity system (18) is *(locally) well-posed* if from each initial state there exists an $\varepsilon > 0$ such that a unique solution on $[0, \varepsilon)$ in the sense of Definition 4.8 exists.

This definition can be reformulated as follows. The system is well-posed if and only if from each state there exists a unique solution on an interval of positive length starting with at most a finite number of jumps followed by smooth continuation on that interval.

Definition 6.2 Let (A, B, C, D) be a system with Markov parameters H^i, $i = 0, 1, 2, \ldots$. The leading column indices $\eta_1, \ldots, \eta_k$ of (A, B, C, D) are defined for $j \in \bar{k}$ as

$$\eta_j := \inf\{i \in \mathbb{N} \mid H^i_{\bullet j} \neq 0\}$$

with the convention $\inf \emptyset = \infty$. The leading row indices $\rho_1, \ldots, \rho_k$ of the linear system (A, B, C, D) are defined for $j \in \bar{k}$ as

$$\rho_j := \inf\{i \in \mathbb{N} \mid H^i_{j\bullet} \neq 0\}.$$

Since we consider only invertible transfer functions (see Assumption 4.1 and Lemma 3.5), the leading row and column indices are all finite. Due to the Cayley-Hamilton theorem, we even have $\rho_i \leqslant n$ and $\eta_i \leqslant n$. The *leading row coefficient matrix* $M(A, B, C, D)$ and *leading column coefficient matrix* $N(A, B, C, D)$ for the system (A, B, C, D) are defined as

$$M(A,B,C,D) := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \text{ and } N(A,B,C,D) := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k}) \tag{38}$$

respectively. We omit the arguments (A, B, C, D), if this does not give rise to confusion.

Recall the definition of a P-matrix at the end of section 3. We now present the main result of this section.

Theorem 6.3 *If the leading column coefficient matrix N and the leading row coefficient matrix M are both P-matrices, then the linear complementarity system (18) is well-posed. From each initial condition, at most one state jump occurs before smooth continuation is possible.*

Remark 6.4 Although we prove existence and uniqueness of solutions under the above conditions, the system may display discontinuous dependence on initial conditions. An illustration of this effect is presented in Section 8.

Remark 6.5 We allow that the event times $\tau(i)$ may have a finite accumulation point, i.e. $\lim_{i\to\infty} \tau(i) = \tau^* < \infty$. The largest interval on which a solution of (18) exists is $[0, \tau^*)$ in this case. Hence, a solution on $[0, \infty)$ does not exists in the sense of Definition 4.8. Well-posedness in the sense of Definition 6.1 implies that for each initial state $\tau^* > 0$, so that not infinitely many events occur at one time instant. At present, there are no nontrivial conditions known to the authors that will prevent such accumulations of event times.

To prove the main result, we first need some auxiliary results.

Lemma 6.6 *If the leading row coefficient matrix M or the leading column coefficient matrix N has only nonzero principal minors, then assumption 4.1 is satisfied, i.e. all modes are autonomous.*

Proof. From Lemma 3.5, it is sufficient to show that $G_{II}(s)$ is invertible for all $I \subseteq \bar{k}$. For notational convenience, we assume $I = \bar{l}$, $l \in \bar{k}$. If M has only nonzero principal minors, then M_{II} is invertible. Hence, $G_{II}(s) = \mathrm{diag}(s^{-\rho_1}, \ldots, s^{-\rho_l})V(s)$ where $V(s)$ is a biproper matrix, because $V(\infty) = M_{II}$ is invertible [10, Thm. 4.5]. The reasoning proceeds analogously for N. $\square$

Definition 6.7 A state x_0 is called *regular for mode I* if $x_0 \in V_I$ and the corresponding smooth solution in this mode satisfies (22) for a time interval of positive length. A state is called *regular* if it is regular for at least one mode of the system.

A state x_0 is regular if and only if $\mathrm{RCP}(x_0)$ has a strictly proper solution. Or equivalently, x_0 is regular if and only if $\mathrm{LDCP}_\infty(x_0)$ has a solution with $u^{-n+1} = \ldots = u^0 = 0$.

Theorem 6.8 *Suppose that the leading row coefficient matrix M is a P-matrix. Then $x_0 \in \mathbb{R}^n$ is a regular state if and only if for all $i \in \bar{k}$*

$$(C_i x_0, C_i A x_0, \ldots, C_i A^{\rho_i - 1} x_0) \succeq 0. \tag{39}$$

Moreover, the smooth continuation is unique.

Proof. Note that $y_i^{(j)}(0) = C_i A^j x_0, j = 0, \ldots, \rho_i - 1$, $i = 1, \ldots, k$, which cannot be influenced by a regular input u. Hence, the above condition is necessary to guarantee $y(t) \geqslant 0$, $t \in [0, \varepsilon)$ for some positive ε.

To prove the converse statement, we will show that if for all $i \in \bar{k}$ (39) holds, the corresponding $\mathrm{LDCP}_\infty(x_0)$ has a solution with $u^{-n+1} = \ldots = u^0 = 0$. The idea of the proof is to reduce the $\mathrm{LDCP}_\infty(x_0)$ to a series of LCP's, that all can be solved uniquely.

We will show that $\mathrm{LDCP}_\infty(x_0)$ has a unique solution with $y^{-n+1} = \ldots = y^0 = 0$, $u^{-n+1} = \ldots = u^0 = 0$. Observe that (34a) is satisfied automatically. The remaining equalities can be written as

$$y_i^j = C_i A^{j-1} x_0, \; j = 1, 2, \ldots, \rho_i, \; i = 1, \ldots, k \tag{40}$$

and

$$\begin{pmatrix} y_1^{\rho_1 + p} \\ \vdots \\ y_k^{\rho_k + p} \end{pmatrix} = \xi_p(x_0, u^1, \ldots, u^{p-1}) + M u^p, \; p = 1, 2, \ldots \tag{41}$$

for linear functions ξ_p, $p \geqslant 1$. We denote by $L(l)$, $l \in \mathbb{N}$ the problem of finding a solution $(u^1, \ldots, u^l)$, $\{y_j^i \mid j = 1, \ldots, k, \; i = 1, \ldots, \rho_j + l\}$ to (40) and (41), $p = 1, 2, \ldots, l$ together with the requirement that for all indices $i \in \bar{k}$ at least one of the following statements is true:

$$(y_i^1, y_i^2, \ldots, y_i^{\rho_i + l}) = 0 \quad \text{and} \quad (u_i^1, u_i^2, \ldots, u_i^l) \succeq 0 \tag{42}$$

$$(y_i^1, y_i^2, \ldots, y_i^{\rho_i + l}) \succeq 0 \quad \text{and} \quad (u_i^1, u_i^2, \ldots, u_i^l) = 0. \tag{43}$$

Note that $L(l)$ is a subproblem of $\mathrm{LDCP}_\infty(x_0)$ and that if we find a solution $(y^1, y^2, \ldots)$, $(u^1, u^2, \ldots)$ satisfying $L(l)$ for all $l \geqslant 0$, then this solution is a solution to the corresponding $\mathrm{LDCP}_\infty(x_0)$.

We claim that $L(l)$ has a unique solution u^j, $j = 1, \dots, l-1$ and y_i^j, $i \in \bar{k}$, $j = 1, \dots, \rho_i + l - 1$, for all $l \geqslant 0$. Note that this holds for $l = 0$. We will prove this by induction, similarly as in [14,21].

We write I_l, J_l, K_l for the active (input) index set, the inactive index set and the undecided index set, respectively, determined by $L(l)$. Formally, for $l \geqslant 1$, $I_l = \{i \in \bar{k} \mid (u_i^1, \dots, u_i^l) \succ 0\}$, $J_l = \{i \in \bar{k} \mid (y_i^1, \dots, y_i^{\rho_i + l}) \succ 0\}$ and $K_l = \bar{k} \setminus (I_l \cup J_l)$ with y_i^j, $i = 1, \dots, k$, $j = 1, \dots, \rho_i + l$ and u^i, $i = 1, \dots, l$ determined by $L(l)$. For convenience we also define $I_0 := \varnothing, J_0 = \{i \in \bar{k} \mid (y_i^1, \dots, y_i^{\rho_i}) \succ 0\}$ and $K_0 = \bar{k} \setminus J_0$.

Note that $L(l-1)$ is a subproblem of $L(l)$, so variables uniquely determined by $L(l-1)$ are automatically uniquely specified for $L(l)$. As a consequence, I_{l-1}, J_{l-1}, K_{l-1} are determined as well. Comparing $L(l)$ with $L(l-1)$, we observe that $L(l)$ has one additional equation: (41) for $p = l$. We divide this equation into the three parts given by I_{l-1}, J_{l-1} and K_{l-1}. For notational convenience, we omit all indices depending on l and all superscripts:

$$\begin{pmatrix} y_I \\ y_J \\ y_K \end{pmatrix} = \begin{pmatrix} z_I \\ z_J \\ z_K \end{pmatrix} + \begin{pmatrix} M_{II} & M_{IJ} & M_{IK} \\ M_{JI} & M_{JJ} & M_{JK} \\ M_{KI} & M_{KJ} & M_{KK} \end{pmatrix} \begin{pmatrix} u_I \\ u_J \\ u_K \end{pmatrix} \tag{44}$$

with $z = \xi_l(x_0, u^1, \dots, u^{l-1})$. From the definition of I_{l-1}, J_{l-1} and K_{l-1}, we get $y_I = 0$ and $u_J = 0$. By substituting this result in (44), we obtain

$$\begin{aligned} 0 &= z_I + M_{II}u_I + M_{IK}u_K && (45) \\ y_J &= z_J + M_{JI}u_I + M_{JK}u_K && (46) \\ y_K &= z_K + M_{KI}u_I + M_{KK}u_K. && (47) \end{aligned}$$

Since M_{II} is a principal submatrix of a P-matrix, it is invertible and hence we get from (45) that $u_I = -M_{II}^{-1}(z_I + M_{IK}u_k)$. Substituting this expression in (47) leads to

$$y_K = z_K - M_{KI}M_{II}^{-1}z_I + (M_{KK} - M_{KI}M_{II}^{-1}M_{IK})u_K \tag{48}$$

Due to (42) and (43) and the definition of K_{l-1}, the complementarity conditions

$$u_K \geqslant 0,\ y_K \geqslant 0,\ y_K^\top u_K = 0 \tag{49}$$

hold. So, (48) and (49) constitute an LCP. Since $M_{KK} - M_{KI}M_{II}^{-1}M_{IK}$ is a Schur complement of a P-matrix, it follows from Proposition 2.3.5 in [4] that it is a P-matrix as well. Theorem 3.6 states that the corresponding LCP has a unique solution. From u_K we can compute u_I and y_J. Hence, the induction hypothesis has been proven for l. So we find a solution of LDCP$_\infty(x_0)$ with $u^{-n+1} = \dots = u^0 = 0$, $y^{-n+1} = \dots = y^0 = 0$. Since this solution is unique, the one-to-one correspondence between initial solutions and solutions of LDCP$_\infty(x_0)$ implies that the corresponding smooth initial solution is unique. □

Theorem 6.9 *If the leading column coefficient matrix N is a P-matrix, then for every state x_0 LDCP$_\kappa(x_0)$, $\kappa \geqslant 0$ has a solution that is unique except for u_i^j, $i \in \bar{k}$, $j = \kappa - \eta_i + 1, \dots, \kappa$, which are left undetermined. Furthermore, $u_i^{-n+1} = u_i^{-n+2} = \dots = u_i^{-\eta_i} = 0$, $i \in \bar{k}$ and $y^{-n+1} = \dots = y^0 = 0$.*

Proof. The proof is based on separation of the equalities (34) in two parts, (34a) and (34b), providing the equations for y^i, $i = -n+1, \dots, 0$ and y^i, $i = 1, \dots, \kappa$, respectively. For both parts we start an induction that is analogous to the previous proof: we reduce the LDCP to a series of LCP's which can be solved uniquely. To do so, we constitute successive LCP's by selecting certain equations from (34). This is done in such a way that only principal submatrices of the leading column coefficient matrix N appear in these LCP's.

We introduce the index sets $O_j := \{i \in \bar{k} \mid \eta_i = j\}$, $j = 0, 1, \dots, n$ and $S_j := \bigcup_{i=0}^{j} O_i$, $j = 0, 1, \dots, n$. In words, the η_j-th Markov parameter is the first Markov parameter in which the j-th column is nonzero. O_j is the set of indices i for which the i-th column in the sequence of Markov parameters $(H^0, H^1, \dots)$ is nonzero for the first time in H^j. S_j is the set of indices i for which at least one of the sequence of columns $(H_i^0, H_i^1, \dots, H_i^j)$ is nonzero. As remarked before, $\eta_i \leqslant n$. Hence, $S_n = \bar{k}$. By definition, $H^i_{\bullet S_j^c} = 0$, $i \leqslant j$.

By suitable permutation of rows and columns, we get the existence of $0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \dots k_n \leqslant k_{n+1} = k$ such that $O_j = \{k_j + 1, \dots, k_{j+1}\}$, $j = 0, 1, \dots, n$. Then

$$N = [H^0_{\bullet O_0}\ H^1_{\bullet O_1} \cdots\ H^{n-1}_{\bullet O_n}].$$

We claim that for $1 \leqslant r \leqslant n$ $\mathrm{LDCP}_{-n+r}(x_0)$ has a solution with

$$u^{-n+1}_{S_{r-1}} = u^{-n+2}_{S_{r-2}} = \dots = u^{-n+r}_{S_0} = 0 \tag{50}$$

$$y^{-n+1} = y^{-n+2} = \dots = y^{-n+r} = 0. \tag{51}$$

The remaining variables $u^{-n+1}_{S^c_{r-1}}, u^{-n+2}_{S^c_{r-2}}, \dots, u^{-n+r}_{S^c_0}$ are left undetermined. This is our induction hypothesis.

For $r = 1$, we only have the equation

$$y^{-n+1} = H^0 u^{-n+1} \tag{52}$$

with the complementarity conditions $y^{-n+1} \geqslant 0$, $u^{-n+1} \geqslant 0$, $y^{-n+1\ \top} u^{-n+1} = 0$. The complementarity conditions follow from the fact that for each index either (35) or (36) should hold. Since $H^0_{S_0^c} = 0$, (52) reduces to

$$y^{-n+1} = H^0_{\bullet S_0} u^{-n+1}_{S_0}. \tag{53}$$

Since $u^{-n+1}_{S_0^c}$ does not appear in this equation, it is left completely undetermined (except for the condition $u^{-n+1}_{S_0^c} \geqslant 0$). Considering (53) and the complementarity conditions only for y_i^{-n+1}, $i \in S_0$ results in the LCP

$$\begin{aligned} y^{-n+1}_{S_0} &= H^0_{S_0 S_0} u^{-n+1}_{S_0} = N_{S_0 S_0} u^{-n+1}_{S_0} \\ y^{-n+1}_{S_0} &\geqslant 0, \quad u^{-n+1}_{S_0} \geqslant 0, \quad (y^{-n+1}_{S_0})^\top u^{-n+1}_{S_0} = 0. \end{aligned}$$

Since $N_{S_0 S_0}$ is a principal submatrix of N, it is a P-matrix. Theorem 3.6 then states that the above LCP has a unique solution.Obviously, $y^{-n+1}_{S_0} = 0$, $u^{-n+1}_{S_0} = 0$ is this unique solution. From (53), $y^{-n+1} = 0$ follows immediately. This proves the first step of our induction.

Suppose that the induction hypothesis above holds for $r-1$, where $2 \leqslant r \leqslant n$. Since $\mathrm{LDCP}_{-n+r-1}(x_0)$ is a subproblem of $\mathrm{LDCP}_{-n+r}(x_0)$, we consider only the additional equality in (34):

$$\begin{aligned} y^{-n+r} &= H^0 u^{-n+r} + H^1 u^{-n+r-1} + \ldots + H^{r-1} u^{-n+1} \\ &= H^0_{\bullet S_0} u^{-n+r}_{S_0} + H^1_{\bullet S_1} u^{-n+r-1}_{S_1} + \ldots + H^{r-1}_{\bullet S_{r-1}} u^{-n+1}_{S_{r-1}} \\ &= H^0_{\bullet S_0} u^{-n+r}_{S_0} + H^1_{\bullet S_1 \setminus S_0} u^{-n+r-1}_{S_1 \setminus S_0} + \ldots + H^{r-1}_{\bullet S_{r-1} \setminus S_{r-2}} u^{-n+1}_{S_{r-1} \setminus S_{r-2}} \\ &= H^0_{\bullet O_0} u^{-n+r}_{O_0} + H^1_{\bullet O_1} u^{-n+r-1}_{O_1} + \ldots + H^{r-1}_{\bullet O_{r-1}} u^{-n+1}_{O_{r-1}}. \end{aligned} \tag{54}$$

The second equality follows from $H^i_{S_i^c} = 0$, the third one follows from the induction hypothesis (50). The last equality is a consequence of $S_j \setminus S_{j-1} = O_j$. Since $u^{-n+1}_{S^c_{r-1}}, u^{-n+2}_{S^c_{r-2}}, \ldots, u^{-n+r}_{S^c_0}$ do not appear in this additional equation, these variables remain undetermined.

Equation (54) consists of k scalar equations. Considering only the equalities for y_i^{-n+r}, $i \in S_{r-1}$, we find

$$\begin{aligned} y^{-n+r}_{S_{r-1}} &= \begin{pmatrix} H^0_{S_{r-1}O_0} & H^1_{S_{r-1}O_1} \cdots H^{r-1}_{S_{r-1}O_{r-1}} \end{pmatrix} \begin{pmatrix} u^{-n+r}_{O_0} \\ u^{-n+r-1}_{O_1} \\ \vdots \\ u^{-n+1}_{O_{r-1}} \end{pmatrix} \\ &= N_{S_{r-1}S_{r-1}} \underbrace{\begin{pmatrix} u^{-n+r}_{O_0} \\ u^{-n+r-1}_{O_1} \\ \vdots \\ u^{-n+1}_{O_{r-1}} \end{pmatrix}}_{=:v_{-r}}. \end{aligned}$$

Since (35) or (36) should hold for all i, it follows that

$$y^{-n+r}_{S_{r-1}} \geqslant 0, \; v_{-r} \geqslant 0, \; v_{-r}^\top \, y^{-n+r}_{S_{r-1}} = 0.$$

This is the LCP we are looking for. Since $N_{S_{r-1}S_{r-1}}$ (as a submatrix of N) is also a P-matrix, the above LCP has a unique solution (Theorem 3.6). Hence, this solution must be $v_{-r} = y^{-n+r}_{S_{r-1}} = 0$. Using this in (54) shows that $y^{-n+r} = 0$. In combination with the induction hypothesis for $r-1$, this yields the hypothesis for r. This completes our induction step and hence the proof of our first claim.

To complete the proof, we start a second induction with hypothesis as stated in the formulation of the theorem. Note that this is equivalent to saying: $\mathrm{LDCP}_\kappa(x_0)$ has a unique solution for every state x_0, only $u^\kappa_{S^c_0}, u^{\kappa-1}_{S^c_1}, \ldots, u^{\kappa-n+1}_{S^c_{n-1}}$ are left undetermined. For $\kappa = 0$ this hypothesis is true, for it follows from the previous induction by taking $r = n$. Suppose the hypothesis is true for $\kappa - 1, \kappa \geqslant 1$. Since $\mathrm{LDCP}_{\kappa-1}(x_0)$ is a subproblem of $\mathrm{LDCP}_\kappa(x_0)$, the variables $u^{\kappa-1}_{S_0}$, …,

$u_{S_{n-1}}^{\kappa-n}$, $u^{\kappa-n-1}$, ..., u^{-n+1} are already uniquely determined. We set

$$\begin{array}{rcl} I & := & \{i \in \bar{k} \mid (u_i^{-n+1}, u_i^{-n+2}, \ldots, u_i^{\kappa-\eta_i-1}) \succ 0\}, \\ J & := & \{i \in \bar{k} \mid (y_i^{-n+1}, y_i^{-n+2}, \ldots, y_i^{\kappa-1}) \succ 0\} \text{ and} \\ K & := & \bar{k} \setminus (I \cup J). \end{array}$$

In comparison with $\mathrm{LDCP}_{\kappa-1}(x_0)$, $\mathrm{LDCP}_{\kappa}(x_0)$ has the additional equality

$$y^{\kappa} = \sigma(x_0, u_{S_0}^{\kappa-1}, u_{S_1}^{\kappa-2}, \ldots, u_{S_{n-1}}^{\kappa-n}, u^{\kappa-n-1}, \ldots, u^{-n+1}) + N \begin{pmatrix} u_{O_0}^{\kappa} \\ u_{O_1}^{\kappa-1} \\ \vdots \\ u_{O_{n-1}}^{\kappa-n+1} \end{pmatrix}$$

for some function σ. Splitting this equation into three parts according to the index sets I, J, K, we can follow the same reasoning as in the proof of Theorem 6.8 to conclude that $y^{\kappa}, u_{O_0}^{\kappa}, u_{O_1}^{\kappa-1}, \ldots, u_{O_{n-1}}^{\kappa-n+1}$ are uniquely determined and thus prove the induction hypothesis for κ. □

We are now in a position to prove Theorem 6.3.

Proof of Theorem 6.3 Lemma 6.6 implies that all modes are autonomous. Take an arbitrary initial state x_0. Theorem 6.9 states that $\mathrm{LDCP}_{\infty}(x_0)$ has a unique solution which satisfies $u_i^{-n+1} = u_i^{-n+2} = \ldots = u_i^{-\eta_i} = 0$, $i \in \bar{k}$ and $y^{-n+1} = \ldots = y^0 = 0$. Due to the one-to-one correspondence between initial solutions and solutions to $\mathrm{LDCP}_{\infty}(x_0)$, the initial solution must be unique as well. In case this initial solution is regular, we proved smooth continuation without jumps. To prove that at most one state jump is needed before smooth continuation is possible, we have to show that the re-initialised state $x(0+)$ is regular. The re-initialisation is given by the impulsive part $u_{imp} = \sum_{i=0}^{n-1} u^{-i}\delta^{(i)}$, where the coefficients u^{-i} follow from $\mathrm{LDCP}_{\infty}(x_0)$. Since the impulsive part is unique, the re-initialisation is unique and results in $x(0+) := x_0 + \sum_{i=0}^{n-1} A^i B u^{-i}$. The complementarity conditions (35) and (36) imply that $(y^1, y^2, \ldots, y^n) \succeq 0$. The right hand side of (34) contains for $y_i^1, \ldots, y_i^{\rho_i}$, $i \in \bar{k}$ only coefficients corresponding to the impulsive part, i.e. only $u^0, \ldots, u^{-n+1}$. Hence, observe that $(C_i x(0+), \ldots, C_i A^{\rho_i - 1} x(0+)) = (y_i^1, \ldots, y_i^{\rho_i}) \succeq 0, i \in \bar{k}$. According to lemma 6.8, $x(0+)$ is a regular state. So after at most one re-initialisation, (unique) smooth continuation is guaranteed. □

The next theorem states that in case N is a P-matrix, it is sufficient to consider $\mathrm{LDCP}_n(x_0)$ (instead of $\mathrm{LDCP}_{\infty}(x_0)$) to select a mode. Hence, only an algebraic problem with a finite number of constraints has to be solved to fulfil the mode selection criterion that has been proposed above. Obviously, $\mathrm{LDCP}_n(x_0)$ is to be preferred over $\mathrm{LDCP}_{\infty}(x_0)$ from a computational point of view.

Theorem 6.10 *If the leading column coefficient matrix N is a P-matrix, then from every initial state there exists a unique initial solution to (18). This solution evolves in mode I where $I := \{i \in \bar{k} \mid (u_i^{-n+1}, u_i^{-n+2}, \ldots, u_i^{n-\eta_i}) \succ 0\}$ with the vectors $\{u^j\}$ constituting a solution to $LDCP_n(x_0)$.*

Proof. Let $(y^{-n+1}, y^{-n+2}, \dots, y^n)$ and $(u^{-n+1}, u^{-n+2}, \dots, u^n)$ be a solution to $\mathrm{LDCP}_n(x_0)$ and let I be defined as in the formulation of the theorem. The state after re-initialisation $x(0+)$ is defined by $x_0 + \sum_{i=0}^{n-1} A^i B u^{-i}$. The jump is induced by the impulsive input $u_{imp} = \sum_{i=0}^{n-1} u^{-i}\delta^{(i)}$. It follows from the definition of I that $(u_i^{-n+1}, \dots, u_i^{n-\eta_i}) = 0$, $i \in I^c$, and in combination with (35), (36) the same definition yields $(y_i^{-n+1}, \dots, y_i^n) = 0$, $i \in I$. Using (34b), we conclude that $x(0+)$ satisfies

$$\begin{array}{rcl} 0 = y_I^1 & = & C_{I\bullet}x(0+) + D_{II}v^1 \\ 0 = y_I^2 & = & C_{I\bullet}Ax(0+) + D_{II}v^2 + C_{I\bullet}B_{\bullet I}v^1 \\ \vdots & & \vdots \qquad\qquad \ddots \\ 0 = y_I^n & = & C_{I\bullet}A^{n-1}x(0+) + D_{II}v^n + C_{I\bullet}B_{\bullet I}v^{n-1} + \dots + C_{I\bullet}A^{n-2}B_{\bullet I}v^1. \end{array} \tag{55}$$

with $v^i = u_I^i$. By using (12) and the equations above, we can show that $x(0+) \in V_j$, $j = 0, 1, 2, \dots, n$ for $(A, B_{\bullet I}, C_{I\bullet}, D_{II})$ and so $x(0+) \in \lim V_j = V_I$. Hence, there exists a regular solution $(u_{reg}(\cdot), x_{reg}(\cdot), y_{reg}(\cdot))$ to (19) in mode I with initial state $x(0+)$. We define

$$\begin{array}{rcl} \tilde{u} & := & \displaystyle\sum_{i=0}^{n-1} u^{-i}\delta^{(i)} + u_{reg}, \\ \tilde{x} & := & \displaystyle\sum_{i=1}^{n-1}\sum_{j=1}^{i} A^{i-j}Bu_{-i}\delta^{(j-1)} + x_{reg}, \\ \tilde{y} & := & \displaystyle\sum_{i=0}^{n-1} y^{-i}\delta^{(i)} + y_{reg}. \end{array}$$

Obviously, this is a solution to (18a) and (18b); so it only remains to show that (18c) is satisfied. We shall do this by proving that we must have $u_{reg,j}^{(i)}(0) = u_j^{i+1}$ for all j and for $i = 0, 1, \dots, n - \eta_i - 1$.

Notice that both $v^i = u_{reg,I}^{(i-1)}(0)$, $i = 1, \dots, n$ and $v^i = u_I^i$, $i = 1, \dots, n$ satisfy (55). We extend the solution of $\mathrm{LDCP}_n(x_0)$ with zeros to get an infinite series $(u^{-n+1}, u^{-n+2}, \dots, u^n, 0, 0, \dots)$. The difference $w^i = u_{reg,I}^{(i)}(0) - u_I^{i+1}$, $i \geqslant 0$ can be taken as an input to the discrete time system

$$\begin{array}{rcl} x(i+1) & = & Ax(i) + B_{\bullet I}w^i, \quad x(0) = 0 \\ \bar{y}(i) & = & C_{I\bullet}x(i) + D_{II}w^i \end{array} \tag{56}$$

satisfying $\bar{y}(0) = \dots = \bar{y}(n-1) = 0$. Taking the z-transform of the discrete time system (56) (see e.g. [13]) with input w^i gives

$$G_{II}(z)w(z) = \sum_{i=0}^{\infty} \bar{y}(i)z^{-i} = z^{-n}p(z) \tag{57}$$

for some proper function $p(z)$, where $w(z)$ denotes the Laplace transform of w. For notational simplicity, we set $I = \bar{l}$, $l \in \bar{k}$. Since N_{II} is a P-matrix (and hence invertible), G_{II} can be written as

$$G_{II}(z) = V_2(z)\mathrm{diag}(z^{-\eta_1}, \dots, z^{-\eta_l}) \tag{58}$$

with V_2 biproper, because $V_2(\infty) = N_{II}$ is invertible (Theorem 4.5 in [10]). Hence, (57) yields

$$w(z) = G_{II}^{-1}(z)p(z) = \mathrm{diag}(z^{-\eta_1 - n}, \dots, z^{-\eta_l - n})\tilde{p}(z),$$

where $\tilde{p}(z) = V_2^{-1}(z)p(z)$ is proper. The definition of w^i now implies that $u_{reg,j}^{(i)}(0) = u_j^{i+1}$, $j \in I$, $i = 0, 1, \dots, n - \eta_i - 1$.

Since for $j \in I$,

$$(u_j^{-n+1}, \dots, u_j^0, u_{reg,j}^{(0)}(0), \dots, u_{reg,j}^{(n-\eta_i-1)}(0)) = (u_j^{-n+1}, \dots, u_j^{n-\eta_i}) \succ 0$$

the distribution $\tilde{u}_j \in C_{imp}$ is initially positive. Note that $\tilde{y}_I = 0$ by construction of $\tilde{y}$. For $j \in I^c$, $\tilde{u}_j = 0$ by definition. Note that

$$(y^{-n+1}, \dots, y^0, y_{reg}^{(0)}, \dots, y_{reg}^{(n-1)}) = (y^{-n+1}, \dots, y^n) \succeq 0$$

due to the equality between $u_{reg}^{(i-1)}$ and u^i. Hence, if $(y_i^{-n+1}, \dots, y_i^n) \succ 0$, then $\tilde{y}_i \in C_{imp}$ is initially positive. For $j \in I^c$, it can also happen that $(y_j^{-n+1}, \dots, y_j^n) = 0$; however, this implies that $\tilde{y}_j$ is identically zero. To see this, note that y_{reg,I^c} can be written as the output of the system

$$\begin{aligned} \dot{x} &= (A + B_I F)x \\ y_{reg,I^c} &= (C_{I^c} + D_{I^c I} F)x, \end{aligned}$$

because the input u satisfying (19) can be given in feedback form $u = Fx$ (see section 4). By the Cayley-Hamilton theorem and because the underlying state space dimension of the system is equal to n, $(y_i^{-n+1}, \dots, y_i^n) = 0$ implies $(y_i^{-n+1}, y_i^{-n+2}, \dots) = 0$. Since $y_{reg,i}$ is real-analytic (it is even a Bohl function) $\tilde{y}_i = y_{reg,i} \in C_{imp}$ is identically zero. Hence, $(\tilde{u}, \tilde{x}, \tilde{y})$ is an initial solution to (18).

Uniqueness follows from the fact that that $\mathrm{LDCP}_\infty(x_0)$ has a unique solution (Theorem 6.9). Indeed, the one-to-one correspondence between initial solutions and solutions to $\mathrm{LDCP}_\infty(x_0)$ implies that there is only one initial solution, which must be given by the above mode. □

Remark 6.11 Since $\mathrm{LDCP}_\infty(x_0)$ has a unique solution, it necessarily has the same solution as selected by $\mathrm{LDCP}_n(x_0)$ as indicated in the above theorem. The equivalence between $\mathrm{LDCP}_\infty(x_0)$ and $\mathrm{RCP}(x_0)$ shows that $\mathrm{RCP}(x_0)$ also selects the correct mode.

Remark 6.12 Solving the $\mathrm{LDCP}_n(x_0)$ can be simplified by using Theorem 6.9. This theorem states that the variables $y^{-n+1}, y^{-n+2}, \dots, y^0$ and $u_i^{-n+1}, u_i^{-n+2}, \dots, u_i^{-\eta_i}$, $i \in \bar{k}$ can immediately be set to zero.

Remark 6.13 Note that it is not claimed in the proof that the *regular* part of $(\tilde{u}, \tilde{y})$ is initially nonnegative; actually this may not be true as shown in the example below. In such cases the initial solution constructed in the proof just serves as a re-initialisation.

Next we shall illustrate the above procedure by the two-carts example.

7 Computational example

In this section, we illustrate the computation of trajectories of the example of section 2 by means of the flow chart 2. Suppose that the initial state equals

$$x_e(0) = x_0 = (0.3202, -0.4335, 0.3716, -1.0915)^\top.$$

Presenting this state to the mode selection block will result in selection of the unconstrained mode ($I(0) = \varnothing$), because the distance to the stop is strictly positive. We will show how to go through the flow chart 2.

DAE simulation Since the unconstrained dynamics is specified by an ordinary differential equation (ODE), a solution can be computed by an ODE solver.

Event detection At time $t = 1$, we arrive at state $(0, -1, -1, 0)^\top$, which is not regular for the unconstrained mode. Note that $y(1) = 0$, $\dot{y}(1) < 0$, so continuing in the unconstrained mode would violate the inequality constraint $y(t) \geqslant 0$. So $\tau(1) = 1$ is an event time and $x_e(1) = (0, -1, -1, 0)^\top$. We have to select a new mode.

Mode selection Transforming the dynamical system to the Laplace domain, leads to

$$(s^4 + 3s^2 + 1)\hat{y}(s) = \begin{pmatrix} s(s^2+1) & s & s^2+1 & 1 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{pmatrix} + (s^2+1)\hat{u}(s). \tag{59}$$

Substituting $x_e(1)$ for $(x_{10}, x_{20}, x_{30}, x_{40})^\top$ results in

$$(s^4 + 3s^2 + 1)\hat{y}(s) = -s - s^2 - 1 + (s^2+1)\hat{u}(s).$$

Since $\hat{y}(s)$ or $\hat{u}(s)$ should be zero, there are only two possibilities:

$$\begin{aligned} \text{unconstrained mode: } \hat{u}(s) = 0; \quad & \hat{y}(s) = \frac{-s^2 - s - 1}{s^4 + 3s^2 + 1} \\ \text{constrained mode: } \hat{y}(s) = 0; \quad & \hat{u}(s) = 1 + \frac{s}{s^2+1}. \end{aligned}$$

Since the RCP requires nonnegativeness for sufficiently large s, $\hat{y}(s) = 0$, $\hat{u}(s) = 1 + \frac{s}{s^2+1}$ is the only solution to RCP($x_e(1)$), so $\mathcal{S}_{\mathrm{RCP}}(x_e(1)) = \{\{1\}\}$. Hence, the constrained mode must be selected ($I(1) = \{1\}$). Since the solution to RCP($x_e(1)$) is not strictly proper, the answer to the question in the decision block is negative, so we have to re-initialise.

Re-initialisation Using (12) and (14, we can compute the consistent states and the jump space:

$$T_{\{1\}} = \mathrm{Im} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}; \; V_{\{1\}} = \mathrm{Ker} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathrm{Im} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

To re-initialise we have to project $x_e(1)$ onto $V_{\{1\}}$ along $T_{\{1\}}$, which results in

$$x(1+) = P_{V_{\{1\}}}^{T_{\{1\}}} x_e(1) = (0, -1, 0, 0)^\top.$$

In the flow chart, the next step is a new mode selection.

Mode selection We have to solve RCP($x(1+)$):

$$(s^4 + 3s^2 + 1)\hat{y}(s) = -s + (s^2 + 1)\hat{u}(s)$$

together with the complementarity conditions. The only solution is $\hat{y}(s) = 0$, $\hat{u}(s) = \frac{s}{s^2+1}$, which is strictly proper. Hence, the question in the decision block is answered positively and we can go to the DAE simulation. The physical interpretation is clear: the left cart hits the stop. Instantaneously, the velocity is put to zero and the right cart keeps the left cart pushed against the stop.

DAE simulation The dynamics of the constrained mode is given by a set of DAEs. However, these can easily be translated into an ODE. The input u must be chosen in such a way, that it keeps y identically zero. Since $y = x_1$, $\dot{y} = x_3$, $\ddot{y} = 2x_1 + x_2 + u$, u should equal $-2x_1 - x_2$. Hence, the dynamics is given by $x_1 = x_3 = 0$, $\ddot{x_2} = -x_2$, $u = -x_2$. Incorporating $x(1+)$ as new initial condition, we get $x_2(t) = -\cos(t-1)$, $u(t) = \cos(t-1)$ for t in an interval starting at 1. Note that we could also have concluded this by taking the inverse Laplace transform of $\hat{u}$ in the previous mode selection. We can continue in this mode as long as $u(t) \geqslant 0$.

Event detection An event is detected at $\tau(2) = \inf\{t \geqslant 1 \mid \cos(t-1) < 0\} = 1 + \frac{\pi}{2} = 1 + \theta(x(1+), \{1\})$. The corresponding event state is $x_e(2) = (0, 0, 0, 1)^\top$. Again we have to select a new mode.

Mode selection This time, LDCP will be demonstrated as a mode selection method. Since the conditions of Theorem 6.10 are satisfied, we can use $\text{LDCP}_4(x_e(2))$ for mode selection:

$$\begin{aligned}
y^{-3} &= 0\\
y^{-2} &= 0\\
y^{-1} &= u^{-3}\\
y^{0} &= u^{-2}\\
y^{1} &= u^{-1} - 2u^{-3}\\
y^{2} &= u^{0} - 2u^{-2} + u^{-3}\\
y^{3} &= u^{1} - 2u^{-1} + u^{-2} + 3u^{-3}\\
y^{4} &= 1 + u^{2} - 2u^{0} + u^{-1} + 3u^{-2} - 3u^{-3},
\end{aligned}$$

with complementarity conditions (35) and (36). Setting $y^i = 0$, $i \in \{-3, \dots, 4\}$ leads to $(u^{-3}, \dots, u^1, u^2) = (0, \dots, 0, -1) \prec 0$. Hence, (35) does not hold. It is obvious that setting $u^i = 0$, $i \in \{-3, \dots, 4\}$ leads to $(y^{-3}, \dots, y^3, y^4) = (0, \dots, 0, 1) \succeq 0$ implying that (36) holds. Hence, $\mathcal{S}^4_{\text{LDCP}}(x_e(2)) = \{\varnothing\}$ and the unconstrained mode must be selected ($I(2) = \varnothing$).

Since the impulsive part of u is zero, i.e. $u^{-3} = u^{-2} = u^{-1} = u^0 = 0$, the arrow marked "yes" in the flow chart 2 must be followed and leads to the DAE simulation. This could also be observed from the fact that $(0,0,0,1)^\top$ is a consistent state for the unconstrained mode. In terms of the physical system: the right cart was on the right of its equilibrium and pulled the left cart away from the stop. The simulated trajectory is plotted in figure 3. Note the complementarity between u and x_1 and the discontinuity in the derivative of x_1 at time $t = 1$.

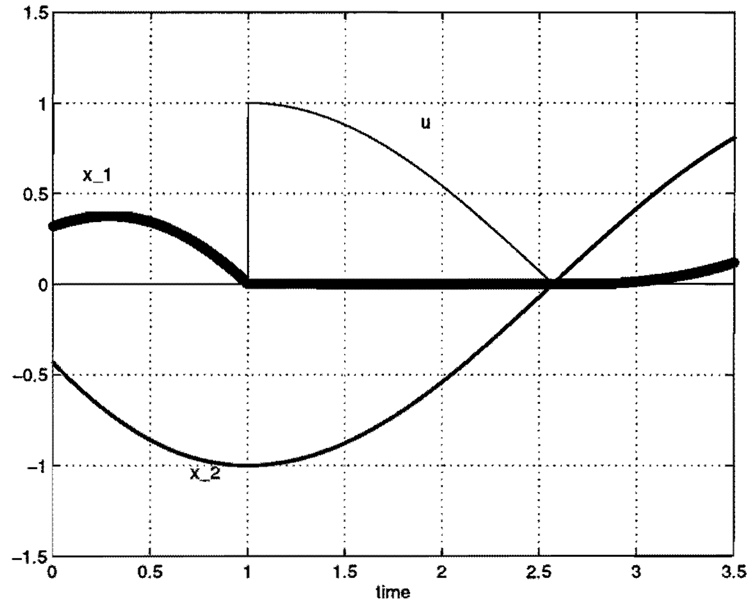

Figure 3: Simulation of two-carts system.

To consider the special case of section 2, we take the initial state $x_e(0) = x_0 = (0,1,-1,0)^\top$. Substituting this initial condition in (59) results in

$$(s^4 + 3s^2 + 1)\hat{y}(s) = s - s^2 - 1 + (s^2 + 1)\hat{u}(s).$$

Solving RCP(x_0) leads to $\hat{y}(s) = 0$ and $\hat{u}(s) = 1 - \frac{s}{s^2+1}$ and so $\mathcal{S}_{\mathrm{RCP}}(x_0) = \{\{1\}\}$. We select the constrained mode ($I(1) = \{1\}$).

The decision block question is answered negatively, because the solution to RCP is not strictly proper. Re-initialisation leads to $x(0+) = (0,1,0,0)^\top$. RCP$(x(0+))$ has to be considered:

$$(s^4 + 3s^2 + 1)\hat{y}(s) = s + (s^2 + 1)\hat{u}(s).$$

Notice that setting $\hat{y}(s)$ equal to zero results in $\hat{u}(s) = -\frac{s}{s^2+1}$, the strictly proper part of the solution of RCP(x_0). This is not a valid choice. The solution is $\hat{u}(s) = 0$ and $\hat{y}(s) = \frac{s}{(s^4+3s^2+1)}$, which corresponds to the unconstrained mode. Since the constrained mode cannot be chosen, we get $x_e(1) = x(0+)$. Since the solution of RCP$(x(0+))$ is strictly proper, the decision block question is answered positively.

8 Mechanical Systems

In this section, we show that the mode selection rule that we propose coincides with the one proposed by Moreau [16, 17] when both rules are applied to the class of systems that are covered by both our and Moreau's framework, to wit, linear mechanical systems.

We will focus on linear mechanical systems whose dynamics in free motion is given by the differential equations

$$M\ddot{q} + D\dot{q} + Kq = 0, \tag{60}$$

where q denotes the vector of generalized coordinates. M denotes the generalized mass matrix, which is assumed to be positive definite, D denotes the damping matrix and K the elasticity matrix. The system is subject to frictionless unilateral constraints given by

$$Fq \geqslant 0 \tag{61}$$

with F of full row rank. Furthermore, we assume that impacts are purely inelastic.

To obtain a complementarity formulation, we introduce the constraint forces u needed to satisfy the unilateral constraints, and the state vector $x = \mathrm{col}(q, \dot{q})$. According to the rules of classical mechanics, the system can then be written as follows

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ M^{-1}F^{\top} \end{pmatrix}}_{B} u \tag{62a}$$

$$y = \underbrace{(F \ \ 0)}_{C} x \tag{62b}$$

$$y \geqslant 0, \quad u \geqslant 0, \quad y^{\top} u = 0. \tag{62c}$$

This systems satisfies $\rho_i = \eta_i = 2$, $i \in \bar{k}$; note that $M(A,B,C,D) = N(A,B,C,D) = FM^{-1}F^{\top}$ is positive definite and hence a P-matrix (Theorem 3.7).

We consider only initial states $x_0 = \mathrm{col}(q_0, \dot{q}_0)$ with $Fq_0 \geqslant 0$. We call these points feasible. In the two-carts system, this means that we do not consider initial states for which the left cart starts on the left of the stop. In Moreau's sweeping process (see [16, 17]) no jumps occur in q itself, but jumps can occur in the velocities $\dot{q}$. These jumps are governed by the following minimization problem, where $J := \{i \in \bar{k} \mid F_i q_0 = 0\}$.

Minimization Problem 8.1 Let an initial state $x_0 = \mathrm{col}(q_0, \dot{q}_0)$ be given. The new state after re-initialization, denoted by $x(0+) = \mathrm{col}(q(0+), \dot{q}(0+))$, is determined by

$$\begin{aligned} q(0+) &= q_0 \\ \dot{q}(0+) &= \arg \min_{\{w \mid F_i w \geqslant 0,\ i \in J\}} \frac{1}{2}(w - \dot{q}_0)^{\top} M (w - \dot{q}_0). \end{aligned}$$

Note that the minimization problem has a unique solution. The problem reflects a kind of "principle of economy": among the kinematically admissible right velocities, the nearest one is chosen in the kinetic metric [16, p. 75]. Observe that if we have proven that jumps in our formulation correspond to the above minimization problem, then it follows that the feasible set $\{x \in \mathbb{R}^n \mid Cx \geqslant 0\}$ is invariant under the dynamics as introduced in section 4.

The Kuhn-Tucker conditions [12] for the minimization problem give necessary conditions for optimality. The vector $\dot{q}(0+)$ is the minimizing argument only if there exists a Lagrange multiplier λ such that

$$M(\dot{q}(0+) - \dot{q}_0) - F_J^\top \lambda = 0 \tag{63}$$

$$\lambda \geqslant 0,\ F_J\dot{q}(0+) \geqslant 0,\ \lambda^\top F_J\dot{q}(0+) = 0. \tag{64}$$

The equality (63) is equivalent to

$$\dot{q}(0+) = \dot{q}_0 + M^{-1}F_J^\top \lambda \tag{65}$$

and therefore $\dot{y}(0+) = F\dot{q}(0+)$ and λ satisfy the following LCP with $\dot{y}_0 := F\dot{q}_0$:

$$\dot{y}_J(0+) = \dot{y}_0 + F_J M^{-1} F_J^\top \lambda \tag{66}$$

$$\dot{y}_J(0+) \geqslant 0,\ \lambda \geqslant 0,\ \dot{y}_J^\top(0+)\lambda = 0. \tag{67}$$

According to Theorem 3.6, this LCP has a unique solution, because $F_J M^{-1} F_J^\top$ is a P-matrix. Since the minimization problem (8.1) is convex, the Kuhn-Tucker conditions are even sufficient for optimality. Hence, the formulated LCP is equivalent to the minimization problem for determining the jumps. Notice that once this LCP is solved, the required jumps are known, because $\dot{q}(0+)$ follows from (65).

We will prove now that $\mathrm{LDCP}_n(x_0)$ (and hence also $\mathrm{LDCP}_\infty(x_0)$ and $\mathrm{RCP}(x_0)$) are equivalent to the optimization problem in the sense that both methods result in the same jumps of state and select the same mode.

Theorem 8.2 *For linear mechanical systems of the form (62) with M positive definite and F of full row rank, the re-initialisation by means of $LDCP_n(x_0)$ (or $LDCP_\infty(x_0)$ or $RCP(x_0)$) agrees with Moreau's sweeping process [16],[17] for feasible initial states. Linear mechanical complementarity systems are well-posed.*

Proof. Since the row coefficient matrix and the column coefficient matrix are P-matrices, well-posedness follows from Theorem 6.3. Furthermore, Theorem 6.9 states that $u^{-2} = u^{-3} = \ldots = u^{-n} = 0$. Because we start from a feasible state x_0, it follows from the proof of this theorem that even $u^{-1} = 0$. Indeed, the next LCP in the series is

$$y^1 = Cx_0 + CABu^{-1}$$

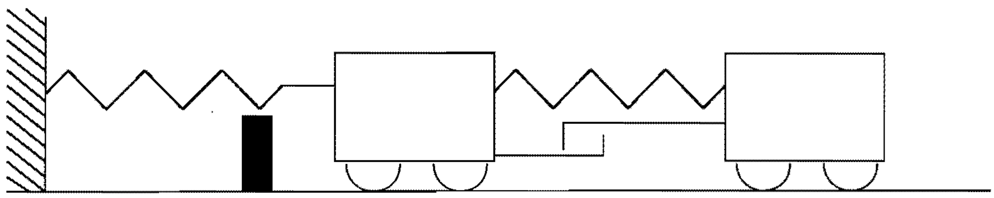

Figure 4: Two-carts system with hook.

with the corresponding complementarity conditions. Since this LCP has a unique solution, the solution must satisfy $u^{-1} = 0$. Hence, $y^{-n+1} = y^{-n+2} = \ldots = y^0 = 0$ and $y^1 = Cx_0$. The next relevant equality in (34) is

$$y^2 = CAx_0 + CABu^0. \tag{68}$$

We define $J := \{i \in \bar{k} \mid C_i x_0 = 0\}$. Since one of the expressions (35) or (36) has to be satisfied for $i \in J$, the conditions

$$y_i^2 \geqslant 0,\ u_i^0 \geqslant 0,\ y_i^2 u_i^0 = 0,\ i \in J$$

have to hold. Because $y_i^1 > 0$ for elements $i \in J^c$, $0 = u_i^0 = u_i^1 = \ldots = u_i^n$ must hold to satisfy (36). Considering only $i \in J$, we can write down the LCP following from (68) and the above complementarity conditions:

$$y_J^2 = C_J A x_0 + C_J A B_J u_J^0 \tag{69}$$

$$y_J^2 \geqslant 0,\ u_J^0 \geqslant 0,\ (y_J^2)^\top u_J^0 = 0. \tag{70}$$

This LCP is identical to the LCP (66) and (67). This shows that the re-initialisation by means of $\mathrm{LDCP}_n(x_0)$ leads to the same result as the minimization problem (8.1). $\square$

From this proof, we see that for feasible initial states only proper rational solutions to RCP occur, i.e. only jumps take place along Im B.

Example 8.3 To illustrate the mode selection and re-initialisation, we consider the two-carts system as in section 2, but this time with an additional hook. See figure 4.

The complementarity description is given by

$$\begin{aligned}
\dot{x}_1(t) &= x_3(t)\\
\dot{x}_2(t) &= x_4(t)\\
\dot{x}_3(t) &= -2x_1(t) + x_2(t) + u_1(t) + u_2(t)\\
\dot{x}_4(t) &= x_1(t) - x_2(t) - u_2(t)\\
y_1(t) &:= x_1(t)\\
y_2(t) &:= x_1(t) - x_2(t)
\end{aligned}$$

where u_1, u_2 denote the reaction forces exerted by the stop and hook, respectively. These equations are completed by the complementarity conditions (18c). Note that

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \; D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \; K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}; \; F = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{71}$$

leads to a description as in the beginning of this section.

Using the minimization problem to determine the re-initialisation and mode selection in case of an initial state $(x_{10}, x_{20}, x_{30}, x_{40})^\top$ with $x_{10} = x_{20} = 0$ results in the alternatives as shown in figure 5. Note that the minimization problem consists of finding the minimal distance to the feasible set (area indicated by "unconstrained.") The arrows denote the re-initialisation directions.

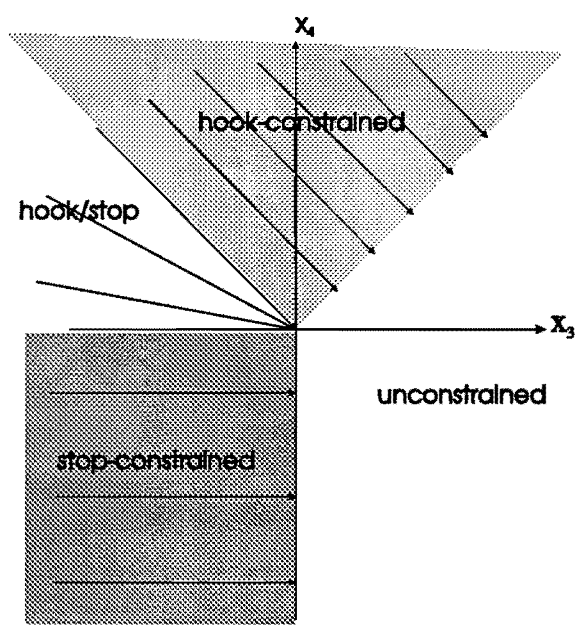

Figure 5: Re-initialisation scheme

To illustrate that RCP gives the same results, we first give the corresponding equations:

$$\begin{aligned} (s^4 + 3s^2 + 1)y_1(s) &= (s^2+1)x_{30} + x_{40} + (s^2+1)u_1(s) + s^2 u_2(s) \\ (s^4 + 3s^2 + 1)y_2(s) &= s^2 x_{30} - (s^2+1)x_{40} + s^2 u_1(s) + (2s^2+1)u_2(s). \end{aligned}$$

Using these equations and the corresponding complementarity conditions, we check when to enter the stop-constrained mode ($I = \{1\}$) which has $y_1(s) \equiv 0$ and $u_2(s) \equiv 0$. Inserting these equations in the above and solving for $u_1(s)$ and $y_2(s)$ leads to

$$\begin{aligned} u_1(s) &= -x_{30} - \frac{1}{s^2+1}x_{40} \\ y_2(s) &= \frac{1}{s^4+3s^2+1}[-s^2 - 1 - \frac{-s^2}{s^2+1}]x_{40}. \end{aligned}$$

Entering the stop-constrained mode is only valid if for sufficiently large s the above two expressions are nonnegative. This requires $x_{30} \leqslant 0$ and $x_{40} \leqslant 0$. This indeed corresponds to the indicated area for the stop-constrained mode in figure 5. Note that the polynomial parts of u_1 and u_2 equal $-x_{30}$ and 0 respectively. Hence, $u_{imp} = (-x_{30},\ 0)^{\top}\delta$. According to (9), the state jump equals $B(-x_{30}\ 0)^{\top} = (0,\ 0,\ -x_{30},\ 0)^{\top}$. This agrees with the direction of the arrows in figure 5. In the same way, the other modes can be verified.

This example shows also that the mode selection procedure as mentioned in [20] does not agree with Moreau's sweeping process. It is proposed there that if I is the current mode and violation of (22) occurs at time τ in state $x(\tau)$, the new mode is given by

$$J := (I \setminus \Gamma_2) \cup \Gamma_1,$$

where

$$\Gamma_1 := \{i \in I^c \mid y_i(t, x(\tau), I) < 0,\ t \in (\tau, \tau + \varepsilon) \text{ for some } \varepsilon > 0\}$$

$$\Gamma_2 := \{i \in I \mid u_i(t, x(\tau), I) < 0,\ t \in (\tau, \tau + \varepsilon) \text{ for some } \varepsilon > 0\}$$

In the example, this means that if we are in the unconstrained mode ($I = \varnothing$) and we arrive in $x(\tau) = (0, 0, -1, 2)^{\top}$, the selected mode should be $J = \{1, 2\}$, the hook/stop constrained mode. This does not agree with the minimization problem illustrated in figure 5, which indicates the hook-constrained mode. A physical argument against the proposal of [20] might be that removing the stop does not lead to violation of $y_1(t) \geqslant 0$.

Another phenomenon that may be illustrated in the above example is that the solutions of linear complementarity systems do not always depend continuously on the initial state. This discontinuous dependence is caused by the sensitivity of the solutions to the order in which constraints become active. Consider the initial states $x_0(\varepsilon) = (\varepsilon, \varepsilon, -2, 1)^{\top}, \varepsilon \geqslant 0$. For $\varepsilon = 0$ the solution is a jump to $(0, 0, 0, 0)^{\top}$, after which the system stays in its equilibrium position. For $\varepsilon > 0$, first the hook becomes active, resulting in a jump to$(\varepsilon, \varepsilon, -\frac{1}{2}, -\frac{1}{2})^{\top}$. This is followed by a regular continuation in the hook-constrained mode until the left cart hits the stop. The state just before the impact is $(0, 0, -\frac{1}{2} + g(\varepsilon), -\frac{1}{2} + g(\varepsilon))^{\top}$ for some continuous function $g(\varepsilon)$ with $g(0) = 0$. Re-initialisation yields the new state $(0, 0, 0, -\frac{1}{2} + g(\varepsilon))^{\top}$, which converges to $(0, 0, 0, \frac{1}{2})^{\top}$ if $\varepsilon \downarrow 0$. Obviously, the system has a discontinuity in $(0, 0, -2, 1)^{\top}$. One may also note that the sequence of initial states $x_0(\varepsilon) = (0, -\varepsilon, -2, 1)$, $\varepsilon \geqslant 0$ leads after two re-initialisations for $\varepsilon \downarrow 0$ to the limit state $(0, 0, \frac{1}{2}, \frac{1}{2})$. This alternative limit corresponds to a situation where first the stop-constrained and then the hook-constrained mode occurs.

9 Conclusions

In this paper we studied linear complementarity systems. Constraints allowing a complementarity description occur in a natural way in many physical systems. The basic characteristic of these systems is the interconnection of continuous dynamics and discrete transition rules. As such, these systems can be seen as "hybrid systems" involving both continuous and discrete dynamics. A description of the complete dynamics of linear complementarity systems has been proposed. Our description is based on an explicit notion of "mode" or "discrete state." We have shown the equivalence of several methods of carrying out the crucial mode selection step, which connects continuous states to discrete states. We focused on questions of existence and uniqueness of solutions of linear complementarity systems. A notion of well-posedness has been introduced, which formalizes the idea that from all states smooth continuation is possible after at most a finite number of jumps. Well-posedness is guaranteed whenever the leading column coefficient matrix and the leading row coefficient matrix associated with the state space representation of the system are P-matrices. In particular, this result implies that linear mechanical system with unilateral constraints are well-posed. We showed that the description of solutions produces the same jump rule as in Moreau's sweeping process. The framework proposed here is well-suited for the numerical computation of trajectories of complementarity systems, as is illustrated by various examples.

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