LINEAR CONNECTIONS AND QUASI-CONNECTIONS ON A DIFFERENTIABLE MANIFOLD

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1. Introduction. In the modern theory of linear connections on an n-dimensional differentiable manifold M, an important role is played by the frame bundle B over M, and by the n^2+n fundamental and basic vector fields E^μ_λ, E_α $(1 \leqq \alpha, \lambda, \mu \leqq n)$ on B. While the fundamental vector fields E^μ_λ are determined by the differential structure of M alone, the basic vector fields E_α together with the differential structure determine, and are determined by, a linear connection on M. The vector fields E^μ_λ and E_α are linearly independent everywhere on B and satisfy the following structure equations:

$$
\begin{aligned}
[E^\mu_\lambda,\ E^\sigma_\rho] &= \quad \delta^\mu_\rho E^\sigma_\lambda - \delta^\sigma_\lambda E^\mu_\rho, \\
[E_\alpha,\ E^\mu_\lambda] &= -\ \delta^\mu_\alpha E_\lambda, \\
[E_\alpha,\ E_\beta] &= -\ T^\gamma_{\alpha\beta} E_\gamma - R^\lambda_{\mu\alpha\beta} E^\mu_\lambda,
\end{aligned} \tag{1.1}
$$

where $[\ ,\ ]$ denotes the Lie product (bracket operation), δ^μ_ρ is the Kronecker delta, and $T^\gamma_{\alpha\beta}$, $R^\lambda_{\mu\alpha\beta}$ are functions on B corresponding to the torsion tensor and the curvature tensor on M of the linear connection.

Now equation $(1.1)_1$ merely expresses the Lie product in the Lie algebra of $GL(n, R)$, and equation $(1.1)_3$ determines the torsion and curvature of the linear connection. Therefore, among the equations (1, 1), only $(1.1)_2$ imposes any condition on the basic vector fields E_α. There arises then the natural question: The fundamental vector fields E^μ_λ being known, will any set of n vector fields E_α on B satisfying the condition

$$[E_\alpha,\ E^\mu_\lambda] = -\ \delta^\mu_\alpha E_\lambda \tag{$(1.1)_2$}$$

determine a linear connection on M?

In an attempt to answer this question, we discover a new kind of connections on M, to be called *quasi-connections*, which include the linear connections as particular case. More precisely, we shall obtain in this paper the following results:

i) With any set of n vector fields E_α on B satisfying $(1.1)_2$ there is associated a tensor[1] C of type (1. 1) on M and an assignment ϕ to each coordinate system (U, u^i) in M of a set of n^3 functions ϕ^i_{jk} for which the law of transformation in $U \cap U^*$ is (Theorem 3. 1)

$$\phi^a_{jk}\frac{\partial u^{i*}}{\partial u^a} = C^a_j\frac{\partial^2 u^{i*}}{\partial u^a \partial u^k} + \frac{\partial u^{a*}}{\partial u^j}\frac{\partial u^{b*}}{\partial u^k}\phi^{i*}_{a*b}\,.$$

ii) The field of planes (i.e. tangent subspaces) spanned by the vector fields E_α on B is projectable onto the field of image planes of C on M (Theorem 4. 3).

iii) If E^μ_λ, E_α are linearly independent everywhere on B, then (C, ϕ) determinesa unique linear connection on M (Theorem 5. 1 and § 9).

iv) If E^μ_λ, E_α are not assumed to be everywhere independent on B, then (C, ϕ) may not determine a linear connection on M. However, by means of C and ϕ, a covariant differentiation of tensors on M can be defined. We call this structure (C, ϕ) a *quasi-connection* on M (§ 6).

v) In the general case iv), equation $(1. 1)_3$ imposes a further condition on E_α. If the number $n^2 + m$ $(\leqq n^2 + n)$ of independent vectors among $(E^\mu_\lambda)_z$, $(E_\alpha)_z$ is the same at all points z of B, then the tensor C is of the same rank m $(\leqq n)$ on M, and conversely (Theorem 4. 2). In this case,

a) the condition imposed on E_α by $(1. 1)_3$ is that the field of image $m$planes of C on M is involutive (Theorem 7. 2);

b) certain 'curvature' tensors for the quasi-connection (C, ϕ) exist (Theorem 8. 1).

vi) If C is of rank n everywhere on M, the quasi-connection (C, ϕ) is equivalent to the linear connection with components $\Gamma^i_{jk} = \bar{C}^a_j\phi^i_{ak}$, where $\bar{C}$ is the reciprocal of C (§ 9).

Since quasi-connection is a generalization of linear connection, there are various concepts and problems relating to a quasi-connection similar to those relating to a linear connection. But we shall not consider them in this paper.

2. Linear connections on smooth manifolds (cf. Chern [2], chapter 4; and Wong [5]). In this section we give a summary of the theory of linear connections which is needed in our later work. We assume as known the classical (local) theory of linear connections and the elementary properties of n-dimensional smooth manifolds (i. e. of class C^∞ and with countable base) and their frame bundles. All the functions, vector fields and tensors defined on a smooth manifold or an open submanifold of it are assumed to be smooth (i. e. of class C^∞). Each of the indices $a, b, c, \ldots\ldots, i, j, k, \ldots\ldots, \alpha, \beta, \gamma, \ldots\ldots, \lambda, \mu, \nu, \ldots$ runs from 1

1) For simplicity and following the convention in tensor calculus, we refer to a tensor field T on M simply as a tensor T on M, and we sometimes even call T^i_{jk} (for example) a tensor, meaning by it a tensor T of type (1,2) on M whose components in the coordinate system (U, u^i) are T^i_{jk}.

to n. Summation over repeated indices, Latin or Greek, is implied.

Let M be an n-dimensional smooth manifold, B the frame bundle over M, and $\pi: B \to M$ the natural projection which maps the frame $z(u)$ at $u \in M$ onto the point u.

A covering of M by (local) coordinate systems gives rise to a covering of B by (local) coordinate systems in the following manner. Let (U, u^i) be any coordinate system in M with coordinate neighborhood U and local coordinates u^i. Then the tangent vectors $X_\alpha(u)$ of any frame $z(u)$ in M can be expressed locally as

$$X_\alpha(u) = x^i_\alpha \left(\frac{\partial}{\partial u^i}\right)_u, \tag{2.1}$$

where x^i_α are n^2 real numbers such that $\det(x^i_\alpha) \neq 0$. Thus, $\{\pi^{-1}(U), (u^i, x^i_\alpha)\}$ form a covering of B by coordinate neighborhoods $\pi^{-1}(U)$ and local coordinates (u^i, x^i_α).

If (U, u^i) and (U^*, u^{i^*}) are two local coordinate systems in M, and $u \in U \cap U^*$, then

$$u^{i^*} = u^{i^*}(u^1, \dots, u^n). \tag{2.2}$$

If $z(u) \in \pi^{-1}(U \cap U^*)$ has the local coordinates (u^i, x^i_α) and $(u^{i^*}, x^{i^*}_\alpha)$, then

$$x^{i^*}_\alpha = x^j_\alpha \frac{\partial u^{i^*}}{\partial u^j}. \tag{2.3}$$

Thus, the transformation of coordinates in $\pi^{-1}(U \cap U^*)$ is expressed by equations (2.2) and (2.3).

In the classical theory, a *linear connection* on M is an assignment Γ to each coordinate system (U, u^i) in M of a set of n^3 functions Γ^i_{jk} such that, in $U \cap U^*$, the two sets of functions Γ^i_{jk} and $\Gamma^{i^*}_{j^*k^*}$ assigned to (U, u^i) and (U^*, u^{i^*}) are related by

$$\Gamma^a_{jk} p^{i^*}_a = p^{i^*}_{jk} + p^{a^*}_j p^{b^*}_k \Gamma^{i^*}_{a^*b^*}, \tag{2.4}$$

where

$$p^{i^*}_j = \frac{\partial u^{i^*}}{\partial u^j}, \quad p^{i^*}_{jk} = \frac{\partial^2 u^{i^*}}{\partial u^j \partial u^k}. \tag{2.5}$$

The transition to the modern theory in terms of differential forms can be described briefly as follows: Let (x^α_i) be the inverse of the matrix (x^i_α). Then,

i) *There exist on B n 1-forms θ^α such that in $\pi^{-1}(U)$,*

$$\theta^\alpha = du^i \cdot x^\alpha_i. \tag{2.6}$$

ii) *If a linear connection Γ on M is given, there exist on B n^2 1-forms ω^λ_μ such that in $\pi^{-1}(U)$*

(2.7) $$\omega_\mu^\lambda = dx_\mu^k \cdot x_k^\lambda + x_\mu^k \Gamma_{jk}^i x_i^\lambda du^j.$$

iii) *The* $n + n^2$ *1-forms* θ^α, ω_μ^λ *are everywhere linearly independent on* B *and satisfy the following structure equations*

(2.8)
$$d\theta^\gamma = \theta^\alpha \wedge \omega_\alpha^\gamma + \frac{1}{2} T_{\alpha\beta}^\gamma \theta^\alpha \wedge \theta^\beta,$$
$$d\omega_\mu^\lambda = \omega_\mu^\rho \wedge \omega_\rho^\lambda + \frac{1}{2} R_{\mu\alpha\beta}^\lambda \theta^\alpha \wedge \theta^\beta.$$

iv) The $n^2 + n$ vector fields E_λ^μ and E_α on B which are dual to the $n + n^2$ 1-forms θ^β and ω_σ^ρ are called the *fundamental vector fields* and the *basic vector fields* respectively. They are characterized by the following equations:

(2.9)
$$<\theta^\beta, E_\alpha> = \delta_\alpha^\beta, \quad <\omega_\sigma^\rho, E_\alpha> = 0,$$
$$<\theta^\beta, E_\lambda^\mu> = 0, \quad <\omega_\sigma^\rho, E_\lambda^\mu> = \delta_\lambda^\rho \delta_\sigma^\mu,$$

and have the following local expressions:

(2.10) $$E_\lambda^\mu = x_\lambda^j \frac{\partial}{\partial x_\mu^j}, \quad E_\alpha = x_\alpha^j \left(\frac{\partial}{\partial u^j} - x_\gamma^k \Gamma_{jk}^i \frac{\partial}{\partial x_\gamma^i} \right).$$

It is easy to see that the fundamental vector fields and the basic vector fields as defined here are essentially those defined by Ambrose and Singer [1] and Nomizu [3, p. 49].

Expressed in terms of the vector fields E_λ^μ and E_α, the structure equations (2.8) take the form (1.1).

3. The equation $(1.1)_2$. Let M be an n-dimensional smooth manifold and E_λ^μ the n^2 fundamental vector fields on the frame bundle B over M. In this section, we shall obtain the local expressions for the most general set of n vector fields E_α not necessarily linearly independent satisfying the equation $[E_\alpha, E_\lambda^\mu] = -\delta_\alpha^\mu E_\lambda$.

We first state without proof the easy

LEMMA 3.1. *In* $\pi^{-1}(U \cap U^*) \subset B$, *the following relations hold*:

(3.1) $$u^{i^*} = u^{i^*}(u^1, \ldots\ldots, u^n), \qquad x_\alpha^{i^*} = x_\alpha^j p_j^{i^*}.$$

(3.2) $$\frac{\partial}{\partial x_\gamma^h} = p_h^{i^*} \frac{\partial}{\partial x_\gamma^{i^*}}.$$

(3.3) $$\frac{\partial}{\partial u^k} = p_k^{i^*} \frac{\partial}{\partial u^{i^*}} + x_\alpha^j p_{jk}^{i^*} \frac{\partial}{\partial x_\alpha^{i^*}}.$$

Here, as before, $p_k^{i^*} = \partial u^{i^*}/\partial u^k$, $p_{jk}^{i^*} = \partial^2 u^{i^*}/\partial u^j \partial u^k$.

We now prove

THEOREM 3.1. *Let* M *be an* n*-dimensional smooth manifold,* B *its frame* *bundle, and* E^μ_λ *the* n^2 *fundamental vector fields on B. Then the most general set of n vector fields* $\overline{E}_\alpha$ *on B satisfying the equation*

$$[\overline{E}_\alpha, E^\mu_\lambda] = -\delta^\mu_\alpha \overline{E}_\lambda \tag{3.4}$$

is given locally by

$$\overline{E}_\alpha = x^j_\alpha\left(C^i_j \frac{\partial}{\partial u^i} - x^k_\gamma \phi^i_{jk} \frac{\partial}{\partial x^i_\gamma}\right), \tag{3.5}$$

where C^i_j, ϕ^i_{jk} *are functions of* u^i *alone such that in* $U \cap U^*$,

$$p^{a*}_j C^{i*}_{a*} = C^a_j p^{i*}_a, \tag{3.6}$$

$$\phi^a_{jk} p^{i*}_a = C^a_j p^{i*}_{ak} + p^{a*}_j p^{b*}_k \phi^{i*}_{a*b*}. \tag{3.7}$$

Equation (3.6) *shows that* C^i_j *are the components of a tensor C of type* (1.1) *on M. Equation* (3.7) *gives the transformation law for the set of functions* ϕ^i_{jk} *defined for each coordinate system* (U, u^i) *in M.*

PROOF. Let us substitute in (3.4), $E^\mu_\lambda = x^i_\lambda \frac{\partial}{\partial x^i_\mu}$ and

$$\overline{E}_\alpha = f^i_\alpha \frac{\partial}{\partial u^i} + g^i_{\alpha\lambda} \frac{\partial}{\partial x^i_\gamma}, \tag{3.8}$$

where f^i_α, $g^i_{\alpha\gamma}$ are unknown functions of u^i and x^i_α. Then the left and right sides of (3.4) are respectively

$$\overline{E}_\alpha E^\mu_\lambda - E^\mu_\lambda \overline{E}_\alpha = g^i_{\alpha\lambda} \frac{\partial}{\partial x^i_\mu} - x^j_\lambda \frac{\partial f^i_\alpha}{\partial x^j_\mu} \frac{\partial}{\partial u^i} - x^j_\lambda \frac{\partial g^i_{\alpha\gamma}}{\partial x^j_\mu} \frac{\partial}{\partial x^i_\gamma},$$

$$-\delta^\mu_\alpha \overline{E}_\lambda = -\delta^\mu_\alpha \left(f^i_\lambda \frac{\partial}{\partial u^i} + g^i_{\lambda\gamma} \frac{\partial}{\partial x^i_\gamma}\right).$$

Hence equation (3.4) is equivalent to

$$x^j_\lambda \frac{\partial f^i_\alpha}{\partial x^j_\mu} = \delta^\mu_\alpha f^i_\lambda, \tag{3.9}$$

$$\delta^\mu_\gamma g^i_{\alpha\lambda} - x^j_\lambda \frac{\partial g^i_{\alpha\gamma}}{\partial x^j_\mu} = -\delta^\mu_\alpha g^i_{\lambda\gamma}. \tag{3.10}$$

Consider first equation (3.9). For $\mu \neq \alpha$, it becomes $\partial f^i_\alpha / \partial x^j_\mu = 0$. Therefore,

(3.11) *each* f^i_α *is a function of* u^k, x^k_α $(1 \leq k \leq n)$ *alone.*

For $\mu = \alpha$, equation (3.9) becomes

$$x^j_\lambda \frac{\partial f^i_\alpha}{\partial x^j_\alpha} = f^i_\lambda \qquad (\alpha \text{ not summed}). \tag{3.12}$$

Let $\alpha \neq \lambda$. Differentiation of (3.12) with respect to x^h_λ gives, an account of (3.11),

$$\frac{\partial f^i_\alpha}{\partial x^h_\alpha} = \frac{\partial f^i_\lambda}{\partial x^h_\lambda} \qquad (\alpha, \lambda \text{ not summed}).$$

On account of (3.11), both side of this equation are functions of u^k alone. Therefore, $f^i_\alpha = x^h_\alpha C^i_h + D^i_\alpha$, where C^i_h, D^i_α are functions of u^k alone. Substitution of this in (3.12) gives $D^i_\alpha = 0$. Hence

$$(3.13) \qquad f^i_\alpha = x^j_\alpha C^i_j, \qquad C^i_j = C^i_j(u^k),$$

which is equivalent to (3.9).

Next consider equation (3.10). For $\mu \neq \gamma$, $\mu \neq \alpha$, it becomes $\partial g^i_{\alpha\gamma}/\partial x^j_\mu = 0$. Therefore,

(3.14) *each* $g^i_{\alpha\gamma}$ *is a function of* $u^k, x^k_\alpha, x^k_\gamma$ $(1 \leqq k \leqq n)$ *alone.*

For $\mu \neq \alpha$, $\mu = \gamma$ so that $\alpha \neq \gamma$, equation (3.10) becomes

$$(3.15) \qquad g^i_{\alpha\gamma} = x^j_\lambda \frac{\partial g^i_{\alpha\gamma}}{\partial x^j_\gamma} \qquad (\gamma \text{ not summed}).$$

For any fixed α, conditions (3.14) and (3.15) in $g^i_{\alpha\gamma}$ are of the same form as conditions (3.11) and (3.12). Therefore we may conclude that

(3.16) *each* $g^i_{\alpha\gamma}$ *is homogeneous and linear in* x^k_γ $(1 \leqq k \leqq n)$.

For $\mu = \alpha$, $\mu \neq \gamma$, equation (3.10) becomes

$$(3.17) \qquad x^j_\lambda \frac{\partial g^i_{\alpha\gamma}}{\partial x^j_\alpha} = g^i_{\lambda\gamma} \qquad (\alpha \text{ not summed}).$$

For any fixed γ, conditions (3.14) and (3.17) in $g^i_{\alpha\gamma}$ are again of the same form as conditions (3.11) and (3.12). Therefore,

(3.18) *each* $g^i_{\alpha\gamma}$ *is homogeneous and linear in* x^k_α $(1 \leqq k \leqq n)$.

Combining (3.14), (3.16) and (3.18) we find that

$$(3.19) \qquad g^i_{\alpha\gamma} = - x^j_\alpha x^k_\gamma \phi^i_{jk}, \qquad \phi^i_{jk} = \phi^i_{jk}(u^l),$$

which is equivalent to (3.10).

Equations (3.13), (3.19) and (3.8) now show that any set of n vector fields E_α satisfying (3.4) are locally given by (3.5).

It remains to show that in $U \cap U^*$, the transformation laws for C^i_j and ϕ^i_{jk} are respectively (3.6) and (3.7). In $\pi^{-1}(U \cap U^*)$, we have (3.5) as well as

$$(3.20) \qquad \bar{E}_\alpha = x^{j*}_\alpha \left(C^{i*}_{j*} \frac{\partial}{\partial u^{i*}} - x^{k*}_\gamma \phi^{i*}_{j*k*} \frac{\partial}{\partial x^{i*}_\gamma} \right).$$

Rewrite (3. 20) and (3. 5) by means of Lemma 3. 1, and equate the results. We obtain

$$x_\alpha^h p_h^{j*}\left(C_{j*}^{i*}\frac{\partial}{\partial u^{i*}} - x_\gamma^l p_l^{k*}\phi_{j*k*}^{i*}\frac{\partial}{\partial x_\gamma^{i*}}\right)$$

$$= x_\alpha^h\left\{C_h^j\left(p_j^{i*}\frac{\partial}{\partial u^{i*}} + x_\gamma^k p_{jk}^{i*}\frac{\partial}{\partial x_\gamma^{i*}}\right) - x_\gamma^k p_l^{i*}\phi_{hk}^l\frac{\partial}{\partial x_\gamma^{i*}}\right\}.$$

Comparison of the coefficients of $\partial/\partial u^{i*}$ and $\partial/\partial x_\gamma^{i*}$ then gives (3. 6) and (3. 7). Hence Theorem 3. 1 is completely proved.

4. Some properties of the tensor C on M. Using (2. 6) and (3. 5), we obtain

$$<\theta^\beta, \bar{E}_\alpha> = \left\langle du^i\cdot x_i^\beta,\ x_\alpha^j\left(C_j^i\frac{\partial}{\partial u^i} - \phi_{jk}^i x_\gamma^k\frac{\partial}{\partial x_\gamma^i}\right)\right\rangle$$

$$= x_\alpha^j C_j^i x_i^\beta.$$

Hence

THEOREM 4. 1. *$C_\alpha^\beta = <\theta^\beta, \bar{E}_\alpha>$ are the n^2 functions on B corresponding*[2] *to the tensor C on M.*

It follows from (3. 5) that in $\pi^{-1}(U)$,

$$\bar{E}_\alpha = x_\alpha^j C_j^i\frac{\partial}{\partial u^i} + (\text{linear combination of } E_\lambda^\mu).$$

But the $n + n^2$ vector fields $\partial/\partial u^i$ and E_λ^μ on $\pi^{-1}(U)$ are everywhere independent and the matrix (x_α^j) is of rank n. Hence,

THEOREM 4. 2. *If $z \in B$ and $\pi z = u \in M$, the number of linearly independent ones among the vectors $(\bar{E}_\alpha)_z$ and $(E_\lambda^\mu)_z$ at z is*

$$n^2 + (\textit{rank of the tensor } C \textit{ at } u).$$

Now let us regard the tensor $C(u)$ at u as an endomorphism of the tangent n-plane T_u to M at u. If the rank of $C(u)$ is m ($\leqq n$), then the image of T_u under $C(u)$ is an m-plane spanned by the vectors $C(u)X_\alpha$, where X_α are any n linearly independent vectors at u. We call this m-plane the *image m-plane* of C at u. As the tensor C may not have the same rank at all the points u of M, the image plane of C need not be of the same dimension at all the points of M. In any case, however, the tensor C determines a field of image-planes on M.

2) For a discussion of a natural correspondence between tensors of type (r, s) on M and certain sets of n^{r+s} functions on B, see Wong [5].

Let z be a frame in M consisting of the linearly independent vectors $X_\alpha = x^i_\alpha \partial/\partial u^i$ at $u \in U \subset M$ so that $z \in B$ and $u = \pi z$. Consider the vectors $(E^\mu_\lambda)_z$, $(\overline{E}_\alpha)_z$ in B at z. Denoting also by π the differential of the natural projection $\pi: B \to M$, we have easily from (3.5) that

$$\pi(E^\mu_\lambda)_z = 0, \qquad \pi(\overline{E}_\alpha)_z = \left(x^j_\alpha C^i_j \frac{\partial}{\partial u^i}\right)_u.$$

Thus, $\pi(\overline{E}_\alpha)_z$ is a vector in M at u with components

$$x^j_\alpha C^i_j(u). \tag{4.1}$$

In other words, if Q_z is the (n^2+m)-plane at z spanned by the vectors $(E^\mu_\lambda)_z$ and $(\overline{E}_\alpha)_z$, then πQ_z is the m-plane at $u \in M$ spanned by the vectors (4.1). Consequently, πQ_z is the image m-plane of C at u. Similarly, if z' is any other point in $\pi^{-1}(u)$, $\pi Q_{z'}$ is also the image m-plane of C at u. Hence we have proved

THEOREM 4.3. *The field of planes on B spanned by the vector fields $\overline{E}_\alpha$ is projectable under π onto the field of image planes of the tensor C on M.*

5. The case when the tensor C is of full rank everywhere on M. In this case, Theorem 4.3 becomes trivial. Let $\overline{C}$ be the reciprocal of the tensor C (so that $\overline{C}^j_i C^i_k = \delta^j_k = \overline{C}^i_k C^j_i$) and put

$$\Gamma^i_{jk} = \overline{C}^a_j \phi^i_{ak}. \tag{5.1}$$

Then the local expression (3.5) for $\overline{E}_\alpha$ can be written as

$$\overline{E}_\alpha = x^h_\alpha C^j_h \left(\frac{\partial}{\partial u^j} - x^k_\gamma \Gamma^i_{jk} \frac{\partial}{\partial x^i_\gamma}\right). \tag{5.2}$$

Putting $\Gamma^{i*}_{j*k*} = \overline{C}^{a*}_{j*} \phi^{i*}_{a*k*}$, we can also rewrite (3.7) as

$$\Gamma^a_{jk} p^{i*}_a = p^{i*}_{jk} + p^{a*}_j p^{b*}_k \Gamma^{i*}_{a*b*}.$$

This is of the form (2.4). Therefore, the n^3 functions Γ^i_{jk} defined by (5.1) are components of a linear connection Γ.

For the linear connection Γ, the basic vector fields

$$E_\alpha = x^j_\alpha \left(\frac{\partial}{\partial u^j} - x^k_\gamma \Gamma^i_{jk} \frac{\partial}{\partial x^i_\gamma}\right) \tag{5.3}$$

are such that $<\theta^\beta, E_\alpha> = \delta^\beta_\alpha$. Let $C^\beta_\alpha = x^h_\alpha C^l_h x^\beta_l$. Then it follows from (5.2) and (5.3) that

$$\overline{E}_\alpha = C^\beta_\alpha E_\beta.$$

Summing up, we have

THROREM 5. 1. *Let M be an n-dimensional smooth manifold, B its frame bundle, and E^{μ}_{λ} the n^2 fundamental vector fields on B. If $\overline{E}_{\alpha}$ are any n vector fields on B satisfying the equations*

$$[\overline{E}_{\alpha}, E^{\mu}_{\lambda}] = -\delta^{\mu}_{\alpha}\overline{E}_{\lambda}$$

such that $\overline{E}_{\alpha}$, E^{μ}_{λ} are linearly independent everywhere on B, then $\overline{E}_{\alpha}$ determine a unique linear connection Γ on M.

Let $<\theta^{\beta}, \overline{E}_{\alpha}> = C^{\beta}_{\alpha}$. Then the n^2 functions C^{β}_{α} on B correspond[3] *to a tensor C of type $(1,1)$ on M which is of rank n everywhere on M. Furthermore, if $(\overline{C}^{\beta}_{\alpha})$ is the inverse of the matrix (C^{α}_{β}), then $\overline{C}^{\beta}_{\alpha}\ \overline{E}_{\beta}$ are the n basic vector fields of the linear connection.*

When a linear connection on an n-dimensional smooth manifold M is defined by means of a suitable field of n-planes on the frame bundle B over M, the action of the real general linear group $GL(n, R)$ on B is an essential part of the definition (See Chern [2] and Nomizu [3]). It is interesting to observe that as a consequence of Theorem 5. 1, equation $(1.1)_2$ may be regarded as giving a global definition of linear connection on M which does not explicitly involve the action of $GL(n, R)$ on the frame bundle B.

6. A quasi-connection on M. When no assumption is made on the rank of the tensor C which appears in Theorem 3. 1, the vector fields $\overline{E}_{\alpha}$ may not define a linear connection on M. But by means of the tensor C and the sets of local functions ϕ^{i}_{jk} for which the law of transformation is (3. 7), a covariant differentiation of tensors on M can be defined. In fact, we shall prove

THEOREM 6. 1. *Let C be any tensor of type $(1,1)$ on M, and ϕ an assignment to each coordinate system (U, u^i) in M of a set of n^3 functions ϕ^{i}_{jk} for which the law of transformation is*

$$\phi^{a}_{jk}p^{i*}_{a} = C^{a}_{j}p^{i*}_{ak} + p^{a*}_{j}p^{b*}_{k}\phi^{i*}_{a*b*}. \tag{6.1}$$

Then for any tensors X, Y, Z of type $(1,0)$, $(0,1)$, $(1,1)$ respectively on M,

$$\overline{\nabla}_{l}X^{i} = C^{a}_{l}\partial_{a}X^{i} + X^{a}\phi^{i}_{la},\quad \overline{\nabla}_{l}Y_{j} = C^{a}_{l}\partial_{a}Y_{j} - \phi^{a}_{lj}Y_{a}, \tag{6.2}$$

$$\overline{\nabla}_{l}Z^{i}_{j} = C^{a}_{l}\partial_{a}Z^{i}_{j} + Z^{a}_{j}\phi^{i}_{la} - \phi^{a}_{lj}Z^{i}_{a}, \tag{6.3}$$

where $\partial_{a} = \partial/\partial u^{a}$, are components in (U, u^i) of tensors of type $(1,1)$, $(0,2)$, $(1,2)$ respectively on M. Furthermore, the following equations hold:

3) See Footnote 2).

$$(6.4)\qquad \bar{\nabla}_l(X^iY_j) = (\bar{\nabla}_lX^i)Y_j + X^i(\bar{\nabla}_lY_j),$$

$$(6.5)\qquad \bar{\nabla}_l(X^aY_a) = C_l^b\partial_b(X^aY_a).$$

We call the structure on M defined by C and ϕ a *quasi-connection* (C, ϕ) on M, and call the tensors $\bar{\nabla}X$, $\bar{\nabla}Y$, $\bar{\nabla}Z$, as defined locally by (6.2) and (6.3), the *covariant derivatives* of the tensors X, Y, and Z with respect to the quasi-connection (C, ϕ). The covariant derivative $\bar{\nabla}Z$ of a tensor of any other type can be defined in a similar manner such that for any two tensors X and Y, the equation

$$\bar{\nabla}(X\otimes Y) = (\bar{\nabla}X)\otimes Y + X\otimes(\bar{\nabla}Y)$$

holds, where $\otimes$ denotes the tensor product. Obviously, if $C_j^i = \delta_j^i$, the quasi-connection (C, ϕ) becomes a linear connection (see also § 9).

The proof of Theorem 6.1 follows familiar lines. Differentiate $X^{i*} = X^a p_a^{i*}$ with respect to u^b, contract the result by C_l^b and then eliminate the second derivative p_{ab}^{i*} by means of (6.1). We obtain, after rearrangement of terms,

$$p_{i*}^{b*}(C_{b*}^{a*}\partial_{a*}X^i + X^{c*}\phi_{b*c*}^{i*}) = (C_l^b\partial_bX^a + X^b\phi_{lb}^a)p_a^{i*}.$$

Thus, $\bar{\nabla}_lX^i$ is a tensor of type (1, 1). Similarly we can prove that $\bar{\nabla}_lY_j$ is a tensor of type (0, 2).

Now $(\bar{\nabla}_lX^i)Y_j + X^i(\bar{\nabla}_lY_j)$ is a tensor, and on account of (6.2) and (6.3),

$$(6.6)\qquad (\bar{\nabla}_lX^i)Y_j + X^i(\bar{\nabla}_lY_j) = C_l^a\partial_a(X^iY_j) + (X^aY_j)\phi_{la}^i - \phi_{lj}^a(X^iY_a).$$

Since a tensor Z_j^i of type (1, 1) has the same law of transformation as the tensor X^iY_j, comparison of formula (6.6) with the definition (6.3) of $\bar{\nabla}_lZ_j^i$ shows that $\bar{\nabla}_lZ_j^i$ is a tensor and, moreover, equation (6.4) holds. Finally, (6.5) is a direct consequence of (6.4) and (6.2). Thus, our theorem is completely proved.

We now proceed to construct a few tensors from C and ϕ.

i) First of all, there is the *Nijenhuis tensor* N for C, defined locally by

$$N_{kl}^h = C_k^a\partial_aC_l^h - C_l^a\partial_aC_k^h - C_a^h(\partial_kC_l^a - \partial_lC_k^a).$$

We shall write this as

$$(6.7)\qquad N_{kl}^h = C_{[k}^a\partial_aC_{l]}^h - C_a^h\partial_{[k}C_{l]}^a,$$

where $[k\ldots\ldots l]$ or $[k\,l]$ indicates that alternation is to be taken with respect to the two indices k and l.

ii) From (6.1) and the law of transformation for C, it follows that

$$C_k^c\phi_{lc}^a p_a^{i*} = C_k^cC_l^a p_{ca}^{i*} + p_k^{c*}p_l^{a*}C_{c*}^{b*}\phi_{a*b*}^{i*},$$

which gives

$$C^{c}_{[k}\phi^{a}_{l]c}p^{i^*}_{a} = p^{c'}_{k}p^{a}_{l}\, C^{b^*}_{[c^*}\phi^{i^*}_{a^*]b^*}.$$

Therefore,

(6.8) $\qquad C^{b}_{[k}\phi^{h}_{l]b}$ *is a tensor.*

iii) From the formula for the components of the tensor $\bar{\nabla}C$, we have

$$\bar{\nabla}_{[k}C^{h}_{l]} = C^{b}_{[k}\partial_b C^{h}_{l]} - C^{b}_{[k}\phi^{h}_{l]b} - \phi^{b}_{[kl]}C^{h}_{b}.$$

But the middle term on the right side is the tensor (6.8). Therefore,

(6.9) $\qquad -S^{h}_{kl} \equiv C^{b}_{[k}\partial_b C^{h}_{l]} - \phi^{b}_{[kl]}C^{h}_{b} = \bar{\nabla}_{[k}C^{h}_{l]} + C^{b}_{[k}\phi^{h}_{l]b}$ *is a tensor.*

iv) *If W is any tensor of type (r, s), $r \geqq 1$, satisfying the equation* $W^{\cdots a \cdots}_{\cdots\cdots}C^{h}_{a} = 0$ in every (U, u^i), then $W^{\cdots a \cdots}_{\cdots\cdots}\phi^{h}_{al}$ *is a tensor of type* $(r, s+1)$. This can be proved by first verifying it directly for the special case when W is of type $(1, 0)$, and then applying the quotient law in tensor calculus.

v) Although we can define a covariant differentiation for the quasi-connection (C, ϕ), there does not exist a tensor which corresponds exactly to the curvature tensor for a linear connection. In fact, a simple computation will show that

$$\begin{aligned}(6.10)\qquad \bar{\nabla}_{[k}\bar{\nabla}_{l]}X^i = {} & (C^{a}_{[k}\partial_a C^{b}_{l]} - \phi^{a}_{[kl]}C^{b}_{a})\partial_b X^i \\ & + X^b(C^{a}_{[k}\partial_a\phi^{i}_{l]b} - \phi^{a}_{[kb}\phi^{i}_{l]a} - \phi^{a}_{[kl]}\phi^{i}_{ab}).\end{aligned}$$

The right side of (6.10) is a tensor. But $\partial_b X^i$ is not a tensor although its coefficient is. Therefore, the coefficient of X^b is in general not a tensor. It is obvious however that if the tensor $-S^{i}_{kl} \equiv C^{a}_{[k}\partial_a C^{i}_{l]} - \phi^{a}_{[kl]}C^{i}_{a}$ is a zero tensor, then the coefficient

$$C^{a}_{[k}\partial_a\phi^{i}_{l]j} - \phi^{a}_{[kj}\phi^{i}_{l]a} - \phi^{a}_{[kl]}\phi^{i}_{aj}$$

of X^b in (6.10) is a tensor, and conversely.

We shall prove in § 8 that under a weaker condition than the above, tensors resembling the curvature tensor for a linear connection can be constructed by using formula (6.10).

7. Consequences of the Structure Equation $(1.1)_3$. We continue the discussion of the general case where no assumption is made on the rank of the tensor C in Theorem 3.1. In this case, equation $(1.1)_3$ may impose a further condition on the vector fields $\bar{E}_\alpha$. We shall give a geometric interpretation to this condition.

We first prove

THEOREM 7.1. *The condition imposed on $\bar{E}_\alpha$ by $(1.1)_3$ is equivalent to that in every coordinate system (U, u^i),*

$$(7.1) \qquad C^a_{[k}\partial_a C^i_{l]} = \lambda^a_{kl} C^i_a,$$

where λ^a_{kl} *are functions in* (U, u^i).

PROOF. Substitute in $(1.1)_3$ the local expressions for E^μ_λ and $\overline{E}_\alpha$ given by $(2.10)_1$ and (3.5). We obtain

$$[\overline{E}_\alpha, \overline{E}_\beta] = \overline{E}_{[\alpha}\overline{E}_{\beta]}$$
$$= x^k_\alpha x^l_\beta (C^a_{[k}\partial_a C^i_{l]} - \phi^a_{[kl]} C^i_a)\frac{\partial}{\partial u^i}$$
$$- x^k_\alpha x^l_\beta x^j_\mu (C^a_{[k}\partial_a \phi^i_{l]j} - \phi^a_{[kj}\phi^i_{l]a} - \phi^a_{[kl]}\phi^i_{aj})\frac{\partial}{\partial x^i_\mu},$$

$$- \overline{T}^\gamma_{\alpha\beta}\overline{E}_\gamma - \overline{R}^\lambda_{\mu\alpha\beta}E^\mu_\lambda = - \overline{T}^\gamma_{\alpha\beta} x^a_\gamma \left(C^i_a \frac{\partial}{\partial u^i} - \phi^i_{ak} x^k_\mu \frac{\partial}{\partial x^i_\mu}\right) - \overline{R}^\lambda_{\mu\alpha\beta} x^i_\lambda \frac{\partial}{\partial x^i_\mu}.$$

On account of these, equation $(1.1)_3$ is equivalent to

$$x^k_\alpha x^l_\beta (C^a_{[k}\partial_a C^i_{l]} - \phi^a_{[kl]} C^i_a) = - \overline{T}^\gamma_{\alpha\beta} x^a_\gamma C^i_a,$$
$$x^k_\alpha x^l_\beta x^j_\mu (C^a_{[k}\partial_a \phi^i_{l]j} - \phi^a_{[kj}\phi^i_{l]a} - \phi^a_{[kl]}\phi^i_{aj}) = \overline{R}^\lambda_{\mu\alpha\beta} x^i_\lambda - \overline{T}^\gamma_{\alpha\beta} x^a_\gamma \phi^i_{aj} x^j_\mu,$$

i. e.

$$(7.2) \qquad C^a_{[k}\partial_a C^i_{l]} - \phi^a_{[kl]} C^i_a = - \overline{T}^a_{kl} C^i_a,$$

$$(7.3) \qquad C^a_{[k}\partial_a \phi^i_{l]j} - \phi^a_{[kj}\phi^i_{l]a} - \phi^a_{[kl]}\phi^i_{aj} = \overline{R}^i_{jkl} - \overline{T}^a_{kl}\phi^i_{aj},$$

where

$$\overline{T}^a_{kl} = \overline{T}^\gamma_{\alpha\beta} x^a_\gamma x^\alpha_j x^\beta_k, \qquad \overline{R}^i_{jkl} = \overline{R}^\lambda_{\mu\alpha\beta} x^i_\lambda x^\mu_j x^\alpha_k x^\beta_l.$$

Now (7.2) is a condition on the tensor C which is equivalent to the condition that $C^a_{[k}\partial_a C^i_{l]}$ is of the form $\lambda^a_{kl} C^i_a$. On the other hand, (7.3) merely determines the functions $\overline{R}^i_{jkl}$ in terms of C, ϕ and $\overline{T}$. Hence Theorem 7.1 is proved.

Next we prove

LEMMA 7.1. *Let C be any tensor of type* (1, 1) *with constant rank m on M. Then the field of image m-planes of C is involutive iff in every coordinate system* (U, u^i)

$$C^a_{[k}\partial_a C^i_{l]} = \lambda^a_{kl} C^i_a,$$

where λ^a_{kl} *are functions in* (U, u^i).

PROOF. This is an easy consequence of the definition of the field of image m-planes of C and the following condition for a field D of m-planes on M to be *involutive*: If in any neighborhood, Y_ξ $(1 \leqq \xi, \eta, \zeta \leqq r)$ are a set of r $(r \geqq m)$ vector fields which locally span the field D, then $[Y_\xi, Y_\eta] = \mu^\zeta_{\xi\eta} Y_\zeta$, or, in local coordinates, $Y^b_{[\xi}\partial_b Y^i_{\eta]} = \mu^\zeta_{\xi\eta} Y^i_\zeta$, where $\mu^\zeta_{\xi\eta}$ are functions.

Combining Theorem 7.1 with Lemma 7.1, we have

THEOREM 7.2. *Let $\overline{E}_\alpha$, given by (3.5), be any set of n vector fields on B satisfying equation $(1.1)_2$. If the tensor C is of constant rank $m(\leqq n)$ on M, or, what amounts to the same thing, if at every point z of B, exactly the same number n^2+m of the vectors $(E^\mu_\lambda)_z$, $(E_\alpha)_z$ are linearly independent, then equation $(1.1)_3$ expresses the following two equivalent conditions:*

a) The field of (n^2+m)-planes on B spanned by E^i_λ and $\overline{E}_\alpha$ is involutive.

b) The field of image m-planes of the tensor C on M is involutive.

Theorem 7.2 becomes trivial if C is of full rank everywhere on M.

8. Curvature tensors for a quasi-connection. Let us now return to §6 and prove that if the tensor C is of constant rank m on M and if the field of image m-planes of C is involutive, then with respect to the quasi-connection (C, ϕ) there exist 'curvature' tensors on M resembling the curvature tensor for a linear connection.

For this purpose, we need the following key lemma:

LEMMA 8.1. *Let C, S be respectively tensors of type $(1,1)$ and $(1,2)$ on M. If C is of constant rank m on M and if in every coordinate system (U, u^i), there exist n^3 functions ψ^i_{kl} such that*

$$C^i_h \psi^h_{kl} = S^i_{kl}, \tag{8.1}$$

then there exists on M a globally defined tensor T of type $(1,2)$ such that in every (U, u^i)

$$C^i_h T^h_{kl} = S^i_{kl}. \tag{8.2}$$

PROOF. Let u be an arbitrary but fixed point in $U \subset M$. Then the system of n linear equations

$$C^i_h(u)\tau^h_{kl} = S^i_{kl}(u) \qquad (k, l \text{ fixed};\ i = 1, \ldots\ldots, n) \tag{8.3}$$

admits a solution $\tau^h_{kl} = \psi^h_{kl}(u)$. Consequently, since $C^i_j(u)$ is of rank $m(\leqq n)$, the solutions of (8.3) for τ^h_{kl} $(h = 1, \ldots\ldots, n)$ span a linear space R^{n-m} of dimension $(n-m)$. Thus, the solutions of

$$C^i_h(u)\tau^h_{kl} = S^i_{kl}(u) \qquad (1 \leqq i, k, l \leqq n) \tag{8.4}$$

for τ^h_{kl} span a linear space isomorphic to the product space $R^{n-m} \times \ldots\ldots \times R^{n-m}$ (n^2 times), i.e. to $R^{n^2(n-m)}$. Now for any solution τ^h_{kl} of (8.4), we can define a tensor of type $(1,2)$ at u by putting $T^h_{kl}(u) = \tau^h_{kl}$.

Let B^T be the bundle of tensors of type $(1,2)$ at all the points of M. The fiber F_u over each point $u \in M$ is isomorphic to R^{n^3}. The set of tensors $T^h_{kl}(u)$

of type (1, 2) which arise from the solutions of (8. 4) forms a linear subspace $\widetilde{F}_u$ of dimension $n^2(n-m)$ of F_u. Moreover, $\widetilde{F}_u$ is stationary in F_u, i. e., if $u \in U \cap U^*$, the $\widetilde{F}_u$ defined for $u \in U$ coincides with the $\widetilde{F}^*$ defined for $u \in U^*$. In fact, if $u \in U \cap U^*$ and $T^h_{kl}(u) \in \widetilde{F}_u$, then since

$$T^{h^*}_{k^*l^*}(u) = p^{h^*}_h p^k_{k^*} p^l_{l^*} T^h_{kl}(u)$$

satisfies

$$C^{i^*}_{h^*}(u) T^{h^*}_{k^*l^*}(u) = S^{i^*}_{k^*l^*}(u),$$

we have that $T^{h^*}_{k^*l^*}(u) \in \widetilde{F}^*_u$. Thus, the totality of tensors of type (1, 2) at all the points of M which are constructed from the solutions of (8. 4) form a subbundle $\widetilde{B}^T$ of B^T. Since the fiber $\widetilde{F}$ of $\widetilde{B}^T$ being isomorphic to $R^{n^2(n-m)}$ is solid, differentiable cross-sections of $\widetilde{B}^T$ exist (Steenrod [4] p. 55). Any such cross-section is a tensor of type (1, 2) on M satisfying the conditions of Lemma 8. 1.

We note that the tensor T^h_{kl} which satisfies the condition of Lemma 8. 1 need not be skew-symmetric with respect to the indices k and l even when ψ^h_{kl}, S^i_{kl} are. But, it is easy to see that the proof of Lemma 8. 1 can be slightly modified to furnish a proof of

LEMMA 8. 2. *If, in Lemma* 8. 1, ψ^h_{kl} *and* S^i_{kl} *are both skew-symmetric with respect to the indices k and l, then there exists on M a globally defined tensor T of type* (1, 2) *such that in every coordinate system* (U, u^i),

$$C^i_h T^h_{kl} = S^i_{kl} = C^i_h \psi^h_{kl}, \qquad T^h_{kl} + T^h_{lk} = 0.$$

We are now ready to prove the following

THEOREM 8. 1. *Let* (C, ϕ) *be any quasi-connection on M. Assume that the tensor C is of constant rank m on M and its field of image m-planes is involutive, so that (by Lemma* 7. 1) $C^a_{[k}\partial_a C^i_{l]} = \lambda^a_{kl} C^i_a$ *in every coordinate system* (U, u^i). *Then there exists on M a tensor* $\overline{T}$ *of type* (1, 2) *satisfying the equation*

$$\overline{T}^h_{kl} C^i_h = (\phi^h_{[kl]} - \lambda^h_{kl}) C^i_h$$

in every (U, u^i). *Moreover, for any such tensor* $\overline{T}$,

$$\overline{R}^i_{jkl} = C^a_{[k}\partial_a \phi^i_{l]j} - \phi^a_{[kj}\phi^i_{l]a} - \phi^a_{[kl]}\phi^i_{aj} + \overline{T}^a_{kl}\phi^i_{aj}$$

are the components in (U, u^i) *of a tensor* $\overline{R}$ *of type* (1, 3) *on M.*

PROOF. We have shown in § 6 that $S^i_{kl} = \phi^a_{[kl]} C^i_a - C^a_{[k}\partial_a C^i_{l]}$ is a tensor. Now because $C^a_{[k}\partial_a C^i_{l]} = \lambda^a_{kl} C^i_a$,

$$S^i_{kl} = (\phi^a_{[kl]} - \lambda^a_{kl}) C^i_a.$$

Application of Lemma 8.1 to the above equation shows that there exists on M a tensor $\overline{T}$ of type (1, 2) such that in every (U, u^i)

$$\overline{T}^a_{kl}C^i_a = (\phi^a_{[kl]} - \lambda^a_{kl})C^i_a = \phi^a_{[kl]}C^i_a - C^a_{[k}\partial_a C^i_{l]}.$$

On account of this, equation (6.10) can be written

$$\begin{aligned}\overline{\nabla}_{[k}\overline{\nabla}_{l]}X^i &= -\overline{T}^a_{kl}C^b_a\partial_b X^i + X^b(C^a_{[k}\partial_a\phi^i_{l]b} - \ldots\ldots)\\ &= -\overline{T}^a_{kl}(C^b_a\partial_b X^i + X^b\phi^i_{ab}) + X^b(C^a_{[k}\partial_a\phi^i_{l]b} - \ldots\ldots + \overline{T}^a_{kl}\phi^i_{ab})\\ &= -\overline{T}^a_{kl}\overline{\nabla}_a X^i + X^b(C^a_{[k}\partial_a\phi^i_{l]b} - \phi^a_{[kb}\phi^i_{l]a} - \phi^a_{[kl]}\phi^i_{ab} + \overline{T}^a_{kl}\phi^i_{ab})\\ &= -\overline{T}^a_{kl}\overline{\nabla}_a X^i + X^b\overline{R}^i_{bkl}.\end{aligned}$$

Since X^i is an arbitrary vector, it follows from this that $\overline{R}^i_{jkl}$ is a tensor. Hence Theorem 8.1 is proved.

If $\widetilde{T}$ is another tensor on M satisfying $\widetilde{T}^a_{kl}C^i_a = (\phi^a_{[k,l]} - \lambda^a_{kl})C^i_a$, and $\widetilde{R}$ is the tensor on M arising from it, then

$$\widetilde{R}^i_{jkl} - \overline{R}^i_{jkl} = (\widetilde{T}^a_{kl} - \overline{T}^a_{kl})\phi^i_{aj}$$

is a tensor. But $\widetilde{T}^a_{kl} - \overline{T}^a_{kl} = W^a_{kl}$ may be any tensor satisfying the condition $W^a_{kl}C^i_a = 0$. Hence, *if W is any tensor of type* (1, 2) *on M satisfying the condition $W^a_{kl}C^i_a = 0$, then $W^a_{kl}\phi^i_{aj}$ is a tensor of type* (1, 3). This is a special case of (iv) in § 6.

We remark that the tensors $\overline{T}$ and $\overline{R}$ in Theorem 8.1 need not be skew-symmetric with respect to the indices k and l. However, on account of Lemma 8.2, tensors $\overline{T}$ exist on M which saitsfy the condition stated in Theroem 8.1 and the additional condition that $\overline{T}^h_{kl} + \overline{T}^h_{lk} = 0$. For any such tensor $\overline{T}$, the corresponding tensor $\overline{R}$ is also skew-symmetric with respect to the indices k and l.

9. Linear connection as a particular case of quasi-connection. Let us consider the case when the tensor C is of rank n everywhere on M. Then on the one hand, we have the quasi-connection (C, ϕ) studied in §§ 6 and 8; on the other hand, we have the linear connection studied in § 5. The link between the two is (cf. (5.1))

$$\Gamma^i_{jk} = \overline{C}^a_j\phi^i_{ak}, \tag{9.1}$$

where $\overline{C}$ is the reciprocal of the tensor C. We note that in this case, the tensor $\overline{T}^h_{kl}$ is uniquely determined:

$$\overline{T}^h_{kl} = \phi^h_{[kl]} - C^a_{[k}\partial_a C^b_{l]}\overline{C}^h_b \tag{9.2}$$

and is skew-symmetric with respect to the indices k, l. Consequently, the tensor

$\overline{R}^{i}_{jkl}$ is also unique and is skew-symmetric with respect to the indices k, l.

Let us denote by ∇, T, R the covariant differentiation, the torsion tensor and the curvature tensor with respect to the linear connection Γ. Then we easily find that

$$\nabla_l X^i = \overline{C}^a_l \overline{\nabla}_a X^i, \tag{9.3}$$

$$\begin{aligned} \nabla_{[k}\nabla_{l]} X^i &= \overline{C}^b_{[k} \overline{\nabla}_b (\overline{C}^a_{l]} \overline{\nabla}_a X^i) \\ &= \overline{C}^b_{[k} (\overline{\nabla}_b \overline{C}^a_{l]} - \overline{C}^c_{l]} \overline{T}^a_{bc}) \overline{\nabla}_a X^i + X^b \overline{C}^c_{[k} \overline{C}^d_{l]} \overline{R}^i_{bcd}. \end{aligned} \tag{9.4}$$

But we also have

$$\nabla_{[k}\nabla_{l]} X^i = - T^h_{kl} \nabla_h X^i + X^j R^i_{jkl}, \tag{9.5}$$

where

$$T^h_{kl} = \Gamma^h_{[kl]}, \quad R^i_{jkl} = \partial_{[k} \Gamma^i_{l]j} - \Gamma^a_{[kj} \Gamma^i_{l]a}. \tag{9.6}$$

Comparison of (9.5) with (9.4) gives

$$\begin{cases} T^h_{kl} = \overline{C}^b_k \overline{C}^c_l \overline{T}^a_{bc} C^h_a - \overline{C}^b_{[k} \overline{\nabla}_b \overline{C}^a_{l]} C^h_a, \\ R^i_{jkl} = \overline{C}^b_k \overline{C}^c_l \overline{R}^i_{jbc}, \end{cases} \tag{9.7}$$

which can also be verified diriectly.

On account of (9.1) and $(9.6)_2$, the tensor defined by (6.8) now reduces to $- C^a_k C^b_l T^h_{ab}$, so that equation (6.9) is equivalent to the relation between T and $\overline{T}$ given in $(9.7)_1$. There is no tensor of the kind described in iv) of § 6.

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