Linear Demand Systems are Inconsistent with Discrete Choice*

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December 10, 2010

Abstract

We show that with more than two options, a discrete choice model cannot generate linear demand.

JEL Classification: C35, D43
Keywords: Linear Demand Systems, Discrete Choice

Introduction

Economists model demand as a linear system of prices in a variety of contexts.[1] However, linear demand is not easily generated by underlying preferences or market structures. LaFrance (1985) shows that individual linear demand places strong restrictions on the functional form of the underlying utility, requiring it to be quadratic – contradicting non-satiation – or Leontief – requiring perfect complementarity. However, it has remained a question since Anderson et al. (1989) whether linear demand may arise from the aggregation of individual preferences.

A standard paradigm for microfounding aggregate demand in individual preferences is the discrete choice model, where consumers choose to buy at most one good from the options available.[2] This model is appealing for products such as cars and other durable goods where most consumers only buy one product. We show that, with unit demand and more than two options, this model cannot generate a linear system of aggregate demand.[3] We conclude by discussing the breadth and implications of our result.

Intuition

To intuitively see why linear demand is inconsistent with discrete choice, consider a market with two goods, Mercedes and Lexus. With linear demand, the number of consumers who shift away from the Mercedes when both companies increase their price a little is independent of the level of the price of a Lexus. However, when the price of Lexus is higher, there are more consumers buying a Mercedes and therefore more who switch away when both prices increase.

Figure 1 shows a graphical representation of our proof for the two good case: the changes in demand due to raising both prices are different at different starting prices. Firm 1's demand is consumers with valuations above the horizontal and diagonal lines. At (P_2, P_1), if both firms increase their prices by ϵ, firm 1 looses the area A. At a $(P_2 + \epsilon, P_1)$, if both prices increase by ϵ firm 1 loses $A + B$. With linear demand these two would have to be equal, which is only possible if no consumers have valuations in B. As the size of the price change goes to zero, this corresponds to no one having valuations (P_2, P_1).

*Thanks to Carl Shapiro and Joseph Farrell for the inspiration for this work, to Simon Anderson for encouraging us to write it, and to Bruno Biais, Mark Chicu, and Scott Kominers for their helpful suggestions.

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[1] Theoretical models based on linear demand include Singh and Vives (1984) on duopoly competition, Bulow et al. (1985) on strategic complements and substitutes, and Katz and Shapiro (2003) on merger analysis. Empirical work uses linear demand to study topics ranging from recreation sites (Burt and Brewer (1971)) to milk (Ippolito and Masson (1978)).

[2] See Anderson et al. (1992), Berry et al. (1995), and Berry (1994).

[3] Two goods with the option not to buy gives three options. This result is sharp since linear demand is achieved trivially in the case of one good or two goods with no outside option by a uniform distribution of consumer valuations.

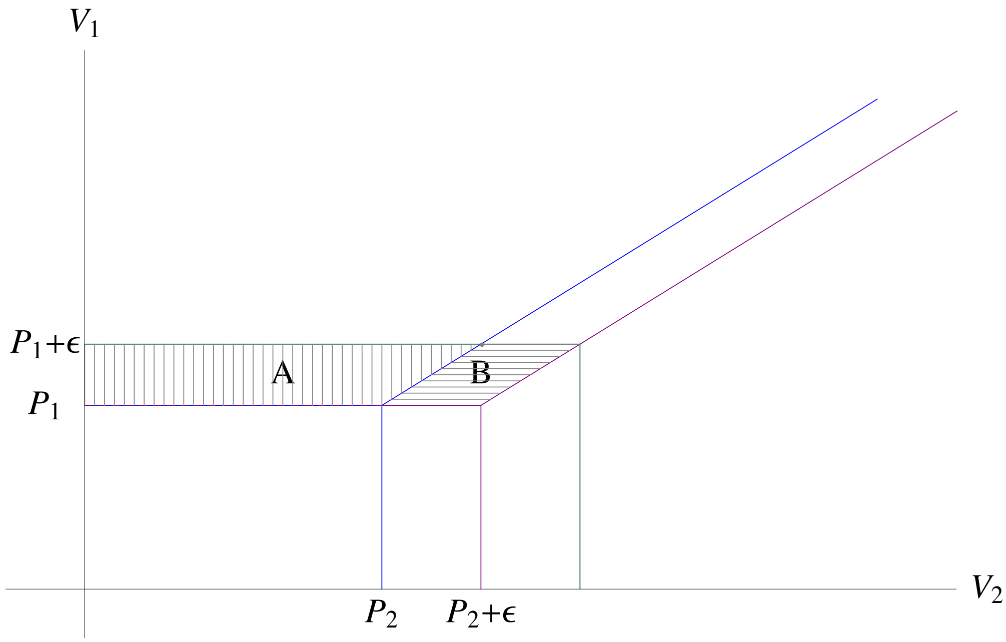

Figure 1: The two good case

Main Result

Formally, consider a market with $N+1>2$ goods; if consumers have to chose one option, only the relative prices and valuations matter so we normalize the price and valuation of good 0 to zero for all consumers. We refer to this as the outside option or not purchasing, but it could also be a product. The remaining N goods are priced at $P=p_1,p_2,\ldots,p_N$ valued by consumers at $V=(v_1,v_2,\ldots,v_N)$. The valuations are distributed according to a smooth density function $f(v_1,v_2,\ldots,v_N)$. Consumers choose the good that maximizes their surplus v_i-p_i, unless that surplus is negative for all goods (in which case they chose good 0). The demand for good i is the number of consumers who chose good i, which is the integral of the density of consumers over the range of valuations where good i is chosen. This is given by

$$D^i(P)=\int_{p_i}^{\infty}\int_{\bigtimes_{j\neq i}S_{ji}} f(V)\,dV_{-i}\,dv_i, \tag{1}$$

where S_{ji} denotes $(-\infty,p_j+v_i-p_i]$, the interval of valuations v_j on which $v_i-p_i>v_j-p_j$. Demand has an own-price partial derivative of

$$\frac{\partial D^i}{\partial p_i}=\underbrace{-\int_{\bigtimes_{j\neq i}S_j} f(V_{-i},p_i)\,dV_{-i}}_{\text{Exiters}}\underbrace{-\int_{p_i}^{\infty}\sum_{k\neq i}\int_{\bigtimes_{j\neq i,j\neq k}S_{ji}} f(V_{-k},p_k+v_i-p_i)\,dV_{-k}\,dv_i}_{\text{Switchers}}, \tag{2}$$

where $S_j=(-\infty,p_j]$. The cross price partial derivative ($k\neq i$) is

$$\frac{\partial D^i}{\partial p_k}=\int_{p_i}^{\infty}\int_{\bigtimes_{j\neq i,j\neq k}S_{ji}} f(V_{-k},p_k+v_i-p_i)\,dV_{-k}\,dv_i. \tag{3}$$

Summing across prices, (2) and (3) give

$$\sum_{k=1}^{N}\frac{\partial D^i}{\partial p_k}=-\int_{\bigtimes_{j\neq i}S_j} f(V_{-i},p_i)\,dV_{-i}, \tag{4}$$

which is just the "exiters" from $\frac{\partial D^i}{\partial p_i}$ in (2) since when other prices also increase, the switchers do not switch. With linear demand, $D^i(P)=\beta_0+\sum\beta_k P_k$, the left side of (4) must be constant and have a zero derivative with respect to each price. However, the derivative of the right side with respect to p_k is

$$-\int_{\bigtimes_{j\neq i, j\neq k} S_j} f(V_{-i-k}, p_i, p_k)\, dV_{-i-k}. \tag{5}$$

Since f is a density function and is everywhere non-negative, the only way for the integral of f to be zero over $\bigtimes_{j\neq i, j\neq k}(-\infty, p_j]$ is for f to vanish on that domain. Since equation (5) holds for all levels of p_i and p_k and for all i and k, this would require f to be everywhere zero. Therefore the sum of the derivatives of D^i cannot be constant in prices so demand cannot be linear in all prices.

Implications

Previous models that have linear demand under discrete choice arise in narrow contexts. With two goods and no outside option, as in Hotelling (1929), there are no exiters (demand is just a single integral on the distribution of the difference between the two valuations) so the sum of derivatives (as in equation (4)) is zero and therefore constant. The Salop (1979) model has kinked-linear demand. It can be locally linear because the setup generates large areas of the valuation space with no mass; in particular, it rules out indifference between more than two products.

While our result only strictly rules out globally linear demand, it does not so do trivially because demand would be negative at certain prices. Equation (5) holds for all prices. For demand to be even locally linear at prices P, it must be that there are no consumers who are indifferent between their top three choices at P (in this case products i and k and the outside option). If the distribution of valuations has full support, as is typical in discrete choice modeling, even locally linear demand is ruled out.

Our result is also stronger than saying that linear demand is non-generic; it also rules out any demand system for which there is a $j \neq i$ such that

$$\sum_{k=1}^{N} \frac{\partial^2 D^i}{\partial p_k \partial p_j} > 0$$

anywhere. Thus, linear demand is not special, rather it lies on the boundary of a broad class of demand functions with curvature properties inconsistent with discrete choice. This provides a simple, non-parametric necessary condition for demand patterns to be consistent with discrete choice which, to our knowledge, has never been suggested or employed.

Linear demand may provide a valid local approximation to a discrete choice demand system. However, the typical goal of employing an explicit functional form (like linear demand) instead of a direct first-order approximation around the equilibrium is to analyze issues, such as comparative statics under market power or normative properties, which inherently rely on higher-order properties of demand. Any such analysis based on linear demand will not just be potentially noisy but *systematically biased* whenever true demand patterns are generated by discrete choice.

For a simple illustration of these biases, consider a symmetric industry. If we denote by Q the total demand for the products, by q_i the (symmetric) demand for any individual product, by P the aggregate price and by p_i the price of any particular product then

$$\frac{d^2 Q}{dP^2} = N\frac{\partial^2 q_i}{\partial P \partial p_i} + N(N-1)\frac{\partial^2 q_i}{\partial P \partial p_{-i}}$$

If demand for products is linear in own price, the first term on the right hand side is 0; if demand is derived from a full-support discrete choice model the second term is strictly negative. Thus, in a symmetric industry, if individual product demands are linear in their own prices, the industry-wide demand must be strictly concave in the aggregate price. Therefore linear demand will tend to overstate the consumer surplus created by a whole industry relative to that created by an individual product (Malueg (1994)). Similarly, in this scenario, a monopolist controlling the industry would have a lower pass-through rate of cost to price than an individual firm, but linear demand would make the rates equal (Weyl and Fabinger (2009)). In a more general setting the biases introduced by linear demand are more complex and thus harder to pinpoint, but they are likely to be at least as important as under symmetry.

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