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Linear differential systems with singularities and an application to transitive Lie algebras

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Introduction

In this note we will first define linear differential systems *with* singularities, and then give a sufficient condition for their integrability (Theorem 1 below), similar to Frobenius' theorem. As an application we will prove that a compact manifold M is completely determined by the Lie algebra $L(M)$ of all analytic vector fields on M along with an isotropy subalgebra of $L(M)$ (Theorem 2). On this regard we recall that Myers [4] proved that M is completely determined by the ring $F(M)$ of all analytic functions on M.

We shall always be in the analytic category. The Lie algebra $L(M)$ of the vector fields is a module over the function ring $F(M)$.

DEFINITION. *A linear differential system* on M is an $F(M)$-submodule of $L(M)$.

This notion is a generalization of the usual linear differential system (without singularities), i.e., a vector subbundle of the tangent bundle $T(M)$ (indeed any real analytic vector bundle has sufficient ample sections, [5]), or a "distribution" as Chevalley calls [2]. A vector subspace L of $L(M)$ over the reals $\boldsymbol{R}$ generates a unique $F(M)$-module L^F. We shall understand L^F under the linear differential system of L. Given a subspace L of $L(M)$ and a point x of M, we put $L(x)=\{u(x)\mid u\in L\}$ and $L_x=\{u\in L\mid u(x)=0\}$ $(0\rightarrow L_x\rightarrow L\rightarrow L(x)\rightarrow 0$ is exact). $L(x)$ is a subspace of the tangent space $T_x(M)$ to M at x, which we call *the integral element* of L at x. *An integral manifold* N of L is by definition a connected submanifold of M such that each tangent space to N is an integral element of L; $T_x(N)=L(x)$, $x\in N$. The linear differential system L^F of L has the same integral elements and integral manifolds as L. If L is a Lie subalgebra of $L(M)$, then so is L^F and L_x is a subalgebra of L, which we call *the isotropy subalgebra* of L at x.

THEOREM 1. *If L is a Lie subalgebra of $L(M)$, then M has a unique partition $\mathfrak{N}=\{N\}$ by integral manifolds N of L.* (That is, M is the disjoint union

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of the members N of $\mathfrak{N}$.)

This is exactly the Frobenius theorem as described in [2] (Theorem 2, p. 94), in the special case where $\dim L(x)$ is independent of x. Theorem 1 is false in the C^∞-category (see § 1).

An abstract transitive Lie algebra (L, L_0) is a pair of a Lie algebra L and its subalgebra L_0 without containing nonzero ideals of L and having $\dim L/L_0$ finite. We say that another abstract transitive Lie algebra (L', L_0') is isomorphic with (L, L_0) when there exists an isomorphism of L' onto L which sends L_0' onto L_0. When M is connected, $(L(M), L(M)_x)$ gives an example of an abstract transitive Lie algebra for any point x of M.

THEOREM 2. *Let M, N be two compact connected analytic manifolds. If there exists an isomorphism α of $(L(M), L(M)_a)$ onto $(L(N), L(N)_b)$ for some $a \in M$ and $b \in N$, then there exists an isomorphism f of an analytic manifold M onto N which induces α.*

The proofs will be elementary except the usage for Theorem 2 of the fact that $T(M)$ and its jet bundles have sufficiently ample (analytic) sections.

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§ 1. Proof of Theorem 1

Let $\mathfrak{N}_x$ be the set of all the integral manifolds of L which contain the point x of M.

(1.1) $\mathfrak{N}_x$ *is not empty for any point* x.

PROOF. Take m elements $u_1, \cdots, u_m$ of L such that $(u_\alpha(x))_{1 \leqq \alpha \leqq m}$ is a base of $L(x)$. Fix a coordinate system (x^i) around x. Let u_α^i be the components of u_α. We may assume that $\det(u_\alpha^\beta)_{1 \leqq \alpha, \beta \leqq m}$ never vanishes on a neighborhood U of x. For simplicity, throughout the proof of (1.1), we denote by the same symbol L the linear differential system of L considered as a subspace of $L(U)$, so that L is an $F(U)$-module. Then L contains the vector fields v_α, $1 \leqq \alpha \leqq m$, of the form: $v_\alpha = \partial/\partial x^\alpha + \sum_{m < \lambda \leqq n} v_\alpha^\lambda \partial/\partial x^\lambda$ with all $v_\alpha^\lambda \in F(U)$ vanishing at x. Let $L'_{(-1)}$ be the space spanned by these v_α over $\boldsymbol{R}$, and L_0' be the subspace of L defined by $L_0' = \{v \in L \mid v = \sum_{m < \lambda \leqq n} v^\lambda \partial/\partial x^\lambda,\ v^\lambda \in F(U)\}$. We put $L' = L'_{(-1)} + L_0'$. (L_0' might well be denoted by L_x', though it shall not.) L' is a subalgebra of L. In fact we readily have

(1.2) $$[L', L'] \subset L_0'.$$

Besides L is clearly the linear differential system of L'. Therefore (1.1) follows from

(1.3) *L' has an integral manifold which contains x.*

(1.3) will be proved after the demonstration of (1.4), in which the analyticity will be used. Let N be $\exp L'_{(-1)}(x)$, i.e., the image of the imbedding: $v \to (\exp v)(x)$ of some neighborhood of the zero in $L'_{(-1)}$ ($\approx \boldsymbol{R}^m$) into M, where $(\exp v)(x)$ is as usual the point $x(1)$ on the curve $x(t)$ given by the ordinary differential equation $dx(t)/dt = v(x(t))$ with $x(0) = x$.

(1.4) *The vector fields in L'_0 vanish everywhere on N.*

Take any vector field w in L'_0. We know that w vanishes at x. Let v be any member of $L'_{(-1)}$. It suffices to show that w vanishes on the integral curve $(\exp tv)(x)$ of v with initial point x. The successive derivatives of w (or its components in terms of (x^i)) with respect to the parameter t are obtained by applying $ad(v)$ to w repeatedly; $(d/dt)^k w = [(ad\, v)^k, w]$, which is inductively defined by $[v, [(ad\, v)^{k-1}, w]]$, $k = 1, 2, 3, \cdots$. They vanish at x by (1.2). Since w is analytic, w vanishes on that curve. (1.4) is proved.

We will complete the proof of (1.3) by showing that $(v_\alpha(y))_{1 \leqq \alpha \leqq m}$ span $T_y(N)$ at each point y of N so that N is an integral manifold of L', hence of L. Let u and v be any two members of $L'_{(-1)}$. Then the point $f(s, t) = \exp(t(su+v))(x)$ lies on N whenever this makes sense for $(s, t) \in \boldsymbol{R}^2$. To show the above, we claim that

$$(1.5) \qquad [\partial f(s, t)/\partial s]_{s=0} = tu(f(0, t)).$$

The left hand side of (1.5) is a vector field, u', on the curve $f(0, t)$. u' vanishes at $x = f(0, 0)$ and satisfies the ordinary differential equation $du'/dt = u + \sum(\partial v/\partial x^i)u'^i$ in terms of the fixed coordinate system (x^i), since we have $\partial f/\partial t = (su+v)$ at $f(s, t)$ by the definition of f so that $\partial^2 f/\partial s \partial t = u + \sum(\partial(su+v)/\partial x^i)\cdot(\partial f^i/\partial s)$. On the other hand tu satisfies the same equation with the same initial condition along the curve $f(0, t)$. In fact we have $\partial(tu)/\partial t = u + t(\partial u/\partial t) = u + t\sum(\partial u/\partial x^i)v^i = u - t[u, v] + t\sum(\partial v/\partial x^i)u^i = u + \sum(\partial v/\partial x^i)(tu^i)$, the last equality being due to (1.2) and (1.4). Thus we obtain $u' = tu$ and (1.5) is proved. Hence (1.3) is proved, and so is (1.1).

In passing to the global part, we need the local uniqueness, or more precisely

(1.6) *For two members N_1, N_2 of $\mathfrak{N}_x$, $\mathfrak{N}_x$ contains some $N \subset N_1 \cap N_2$.*

PROOF OF (1.6). Let $L_{(-1)}$ be a subspace of L with dimension $= L_{(-1)}(x) = L(x)$. For any vector field v in $L_{(-1)}$ the restriction of v to N_1, N_2 are vector fields v_1, v_2 on these integral manifolds N_1, N_2 of L. We have three exponential maps of $L_{(-1)}$ into M, N_1 and N_2 respectively, which are of course given by the equations $dx(t)/dt = v(x(t))$, $dx_1/dt = v_1$ and $dx_2/dt = v_2$ with $x(0) = x_1(0) = x_2(0) = x$ respectively. Each of them is an imbedding defined on a neighborhood of 0 in $L_{(-1)}$. We may assume the definition domains are the same,

U. Restricted to U, these imbeddings are the same if they are considered as maps into M, since then they are all given by integral curves of v passing through x. The common image N is what is wanted.

Now we conclude the proof of Theorem 1. For any point x of M, we put $N_x = \bigcup_{N \in \mathfrak{N}_x} N$. N_x is non-empty by (1.1). The topology and the analytic structures are given by the charts of all N in $\mathfrak{N}_x$ by virtue of (1.6). The topology of N_x is stronger than the one induced from M. Hence N_x is a Hausdorff space. Thus N_x is a submanifold of M. When M is paracompact, M has a Riemannian metric, which can be induced on N_x, so that N_x is metrizable and therefore N_x is paracompact. Finally $\mathfrak{R} = \{N_x \mid x \in M\}$ is a partition of M. In fact the union of $\mathfrak{R}$ is clearly M, and the distinct members of $\mathfrak{R}$ are disjoint. We will prove this. Suppose $N_x \cap N_y$ contains a point z. Then N_x and N_y belong to $\mathfrak{N}_z$ so that N_x, N_y are contained in N_z. This argument also gives that N_z is contained in N_x and N_y. Thus we have $N_x = N_z = N_y$. The uniqueness of $\mathfrak{R}$ is obvious by (1.6). The proof of Theorem 1 is complete.

Counter-example for the C^∞-category. Let M be $\boldsymbol{R}^2$ with the usual coordinate system (x, y). Let L be the space of C^∞ vector fields on M spanned by $\partial/\partial x$ and $\{f(x)\partial/\partial y \mid f$ is a C^∞ function on M with all the derivatives $f^{(k)}$ vanishes at $(0, 0)$, $k = 0, 1, 2, \cdots\}$. If a point p of M lies on the y-axis, then we have $\dim L(p) = 1$ and $L(p)$ is parallel to the x-axis, while we have $\dim L(p) = 2$ otherwise. Therefore, there does not exist an integral manifold through a point on the y-axis. This example shows that Theorem 1 is false in the C^∞-category. Theorem 1 holds good in a special case where M is compact and L is finite dimensional, as is well known.

QUESTION. Does there exist "a prolongation" so that L gets free of singularities? In other words, does there exist an analytic manifold $\tilde{M}$ with an analytic surjection $\pi : \tilde{M} \to M$ which has a linear differential system $\tilde{L}$ without singularities such that $\tilde{L}$ is isomorphic with a given Lie algebra $L \subset L(M)$ in some natural way? When $\dim L$ is finite, the answer is affirmative. Put $\tilde{M} = G \times M$ to see this, where G is the simply connected Lie group with Lie algebra L. L is isomorphic with the right-invariant vector fields on G. From this, it is obvious how to define $\tilde{L}$ on $\tilde{M}$.

ANOTHER QUESTION. The converse of Theorem 1 is false. The partition $\mathfrak{R}$ should have some further property in order to be given by the maximal integral manifolds of a linear differential system.

§ 2. Proof of Theorem 2

Our method is basically the one used by E. Cartan frequently. Indeed we ill construct in $M \times N$ the graph of the mapping $f : M \to N$ by means of a suitable linear differential system. We lift up the vector fields $L(M)$ and $L(N)$ onto $M\times N$ via the identification $T(M\times N)=T(M)\times T(N)$, obtaining the Lie algebra $L^*=\{$the vector fields $u^*\,|\,u^*(x, y)=(u(x), \alpha(u)(y))$ for $(x, y)\in M\times N$, $u\in L(M)\}$ where α is the given isomorphism: $(L(M), L(M)_a)\to(L(N), L(N)_b)$. By Theorem 1, there exists a unique maximal integral manifold Γ of the linear differential system L passing through (a, b). Γ will be the graph of f whose existence we have to show. Given a positive integer p, we denote by $J^l(T(M))$ the p-jet bundle $\{j_x^p u\,|\,x\in M,\ u\in L(U)$, and U is some neighborhood of x in $M\}$ where $j_x^p u$ denotes as usual the equivalence class of the local vector fields v around x which has the same derivatives as u of degrees up to p (including 0) at x with respect to some (and so every) coordinate system.

(2.1) *If two jets $j_x, j_y\in J^p(T(M))$ have the distinct sources x, y then there exists some u in $L(M)$ such that one has $j_x^p u=j_x$ and $j_y^p u=j_y$.*

This is an immediate consequence of H. Cartan's theorem [**1**] combined with Grauert's [**3**] or of Shiga's [**5**] and which might be more or less obvious if one admits that an analytic manifold can be analytically and regularly imbedded into some $\boldsymbol{R}^m$.

We say that a subalgebra L of $L(M)$ is *transitive* at a point x of M when $L(x)$ equals $T_x(M)$. Hereafter we will write L for $L(M)$. The following (2.2) and (2.3) are corollaries to (2.1).

(2.2) *L is transitive at each point of M.*

(2.3) *If x and y are distinct points of M, then L_x is transitive at y.*

(2.4) *If L' is a subalgebra of L with* $\dim L/L'=n=\dim M$, *then, for any point x of M, L' is either transitive at x or equal to L_x.*

PROOF. (The compactness of M is not necessary.) Suppose neither is the case. Then $q=\dim L'/L_x'$ differs from 0 and n. Let L_1 be the ideal $\{u\in L_x\,|\,[u, L]\subset L_x\}$ of L_x. Then the Lie algebra $L_{(0)}=L_x/L_1$ naturally acts on $L_{(-1)}=L/L_x$ as the general linear Lie algebra on $L_{(-1)}$, $\dim L_{(0)}=n^2$, by (2.1). Its subalgebra $L_x'/L_x'\cap L_1$ leaves invariant the subspace L'/L_x' of L/L_x. It follows that $\dim L'/L_1\cap L'\leqq(n+n^2)-((n-q)+(n-q)q)=\dim L/L_1-(n-q)(q+1)$. Hence we have $q=n-1$ by the assumption $\dim L/L'=n$. Thus $L'\supset L_1$. It is not hard to see that this leads to a contradiction by showing that $[L', L_1]$ must necessarily contain L_x in view of (2.1). (2.4) is proved.

Now we begin with the proof of Theorem 2. (It might be worth noting that M and N are not assumed to be simply connected. By this reason, $L(M)$ or $L(N)$ cannot be replaced by an arbitrary transitive subalgebra.) We have $\dim M=\dim N$ by (2.2). Let π^M, π^N be the projections of $M\times N$ onto M, N respectively. The differential $d\pi^M$ sends $L^*(x, y)\subset T_{(x,y)}(M\times N)$ onto $T_x(M)$

for any point (x, y) of $M\times N$ by (2.2), and the analogous for $d\pi^N$. The kernel of the restriction $d\pi^M | L^*(x, y)$ equals $\{u^*\in L^* | u\in L(M)_x\}$. The analogous for $d\pi^N$ is $\{u^*\in L^* | \alpha(u)\in L(N)_y\}$. Hence, in case $\alpha(L(M)_x)=L(N)_y$, $L^*(x, y)$ is isomorphic with $T_x(M)$ through $d\pi^M$. And, in the contrary case, $\alpha(L(M)_x)$ is transitive at y by (2.4) (applied to N), which shows that L^* is transitive at (x, y). Thus we have proved

(2.5) *The following four conditions are equivalent for a point* (x, y) *of* $M\times N$:

(1) $\alpha(L(M)_x)=L(N)_y$,

(2) L^* *is not transitive at* (x, y),

(3) $d\pi^M$ induces an isomorphism of $L^*(x, y)$ onto $T_x(M)$, and

(4) $d\pi^N$ induces an isomorphism of $L^*(x, y)$ onto $T_y(N)$.

To conclude the proof of Theorem 2, we denote by Γ the maximal integral manifold passing through (a, b) of the linear differential system L^*. By (2.5) and the assumption that $\alpha(L(M)_a)=L(N)_b$, Γ is immersed as an open submanifold into M (resp. N) by π^M (resp. π^N). Hence the image $\pi^M(\Gamma)$ (resp. $\pi^N(\Gamma)$) will coincide with M (resp. N) if Γ is proved to be compact. And Γ will be compact if it is closed. Suppose (x, y) adheres to Γ. Clearly we have $\dim L^*(x, y)=n$ by (2.5). By (2.5) and the argument above, there exists a point x' of M such that (x', y) belongs to Γ. Again by (2.5) we have $\alpha(L(M)_{x'})=L(N)_y=\alpha(L(M)_x)$, hence $L(M)_{x'}=L(M)_x$. We thus have $x'=x$, by (2.3) and (2.5). Thus, (x, y) belongs to Γ. Finally it remains to show that π^M (resp. π^N) restricted to Γ is injective. (Then Γ will be the graph of the required map $f: M\to N$. That f induces α will be obvious.) But the proof of the injectiveness is implicit in that of the closedness of Γ. Theorem 2 is proved.

QUESTION 3. Can $L(M)$, $L(N)$ in Theorem 2 be replaced by transitive Lie algebras when M, N are simply connected?

QUESTION 4. Is Theorem 2 true in the C^∞-category? Moreover can such a question be discussed in terms of some relation between Lie algebras in the two categories?

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