PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 2, 1970

LINEAR EIGENVALUES AND A NONLINEAR BOUNDARY VALUE PROBLEM

E. M. LANDESMAN AND A. C. LAZER

In this paper a nonlinear boundary value problem for elliptic partial differential equations is considered. The principal result generalizes a previous result on a two point boundary value problem for a nonlinear second order ordinary differential equation. The solvability condition obtained for the nonlinear problem is related to the eigenvalues of an associated linear problem.

In [5] the second author and D. E. Leach considered the two point boundary value problem

$$(1.1)\qquad \begin{aligned} u'' + p(x, u, u')u &= h(x, u, u') \\ u(0) = a,\ u(\pi) &= b\ . \end{aligned}$$

It was shown that, if for some integer N there exist numbers γ_N and γ_{N+1} such that

$$N^2 < \gamma_N \leqq p(x, s, r) \leqq \gamma_{N+1} < (N+1)^2\ ,$$

if $p(x, s, r)$ and $h(x, s, r)$ are continuous on $[0, \pi] \times (-\infty, \infty) \times (-\infty, \infty)$, and h is bounded, then the problem (1.1) has at least one solution.

In this paper we consider the n-dimensional analogue of the problem (1.1) which is

$$(1.2)\qquad \begin{gathered} \Delta u + p\Big(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\Big)u = h\Big(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\Big)\ , \\ u(x) = g(x) \text{ on } \partial D\ , \end{gathered}$$

where D is a domain in R^n and Δ is the n-dimensional Laplacian. The corresponding result is that if D is a Dirichlet domain, if there exist numbers γ_N and γ_{N+1} such that

$$\alpha_N < \gamma_N \leqq p(x, t, s_1, \cdots, s_n) \leqq \gamma_{N+1} < \alpha_{N+1}\ ,$$

for $(x, t, s_1, \cdots, s_n) \in \bar{D} \times R^{n+1}$ where

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_K \leqq \alpha_{K+1} \leqq \cdots\ ,$$

are the eigenvalues of the problem

$$(1.3)\qquad \Delta u + \alpha u = 0, \qquad u = 0 \text{ on } \partial D\ ,$$

if $p(x, t, s_1, \cdots, s_n)$, $h(x, t, s_1, \cdots, s_n)$ are continuous and h bounded on

$\bar{D} \times R^n$, then for any continuous $g(x)$ there exists a function v continuously differentiable on $\bar{D}$ with $v(x) = g(x)$ on ∂D which satisfies (1.2) in the distribution sense.

With the exception of the Schauder fixed point theorem the methods used in [5] were very elementary. The methods in this paper are similar to those of [5] but rely strongly on the spectral theory of symmetric completely continuous operators and variational properties of eigenvalues.

In the second section we consider the linear homogeneous problem

$$(1.4) \qquad Lu + p(x)u = h(x), \qquad u = 0 \text{ on } \partial D$$

where L is a strongly elliptic self adjoint operator and $D \subset R^n$ is an arbitrary domain. Under suitable conditions on p and h we obtain an *a priori* bound for solutions of (1.4).

Using this result and Schauder's method we obtain an existence theorem for the problem

$$(1.5) \qquad Lu + p\Big(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\Big)u = h\Big(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\Big),$$

$$u = 0 \text{ on } \partial D .$$

The aforementioned result follows quickly from this theorem.

A special case of our principal result follows in a straight-forward way from a result on Hammerstein integral equations due to C. L. Dolph [4]. Namely, if $F(x, t)$ is continuously differentiable in t and x and continuous for $(x, t) \in \bar{D} \times R$, if $h(x)$ is continuously differentiable on $\bar{D}$, and

$$\alpha_N < \gamma_N \leqq \frac{\partial F}{\partial t}(x, t) \leqq \gamma_{N+1} < \alpha_{N+1}$$

then the problem

$$(1.6) \qquad \Delta u + F(x, u) = h(x), \qquad u = g \text{ on } \partial D ,$$

g continuous, has a *unique* solution in the classical sense.

To see that this problem is included in the problem (1.2) we note that the differential equation can be written

$$\Delta u + p(x, u)u = \hat{h}(x), \text{ where}$$

$$p(x, u) = \int_0^1 \frac{\partial F}{\partial t}(x, su)ds, \quad \hat{h}(x) = h(x) - F(x, 0) .$$

Clearly $\gamma_N \leqq p(x, t) \leqq \gamma_{N+1}$.

2. Preliminaries. In this section we recall briefly results concerning elliptic differential equations, completely continuous operators, and variational properties of eigenvalues in forms applicable to our problem. We also give an auxiliary lemma which is applied in the next section.

Let D be a bounded domain in R^n. In the following L will denote a second-order, self adjoint, strongly elliptic differential operator involving only principal part. That is, a formal expression

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i} a^{ij} \frac{\partial}{\partial x_j}$$

where for $i, j = 1, \cdots, n$, $a^{ij} = a^{ji}$ is a real valued function bounded and measurable on D and there exists a constant $c > 0$ such that for all $x \in D$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}(x)\xi_i\xi_j \geqq c\sum_{i=1}^{n}\xi_i^2 \tag{2.1}$$

for arbitrary real numbers $\xi_1, \cdots, \xi_n$.

Let H_0 denote real $L^2(D)$ and if $f, g \in H_0$ let

$$\langle f, g\rangle_0 = \int_D fg dx .$$

More generally if p is a real measurable function defined on D such that there are numbers $\delta > 0$ and Δ with

$$0 < \delta \leqq p(x) \leqq \Delta \tag{2.2}$$

for all $x \in D$, let

$$\langle f, g\rangle_{0,p} = \int_D pfg dx .$$

Clearly $\langle\ \ \rangle_{0,p}$ defines an inner product on H_0 which induces the same topology on H_0 as $\langle\ ,\ \rangle_0$.

Let $\dot{C}_1$ denote the inner product space of real continuously differentiable functions defined on R^n and having compact support contained in D with real inner product given by

$$\langle u, v\rangle_1 = \int_D \Big[uv + \sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)\Big(\frac{\partial v}{\partial x_i}\Big)\Big]dx$$

for $u, v \in \dot{C}_1$. Let B be the symmetric bilinear form defined on $\dot{C}_1$ by

$$B(u, v) = \int_D \sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}\Big(\frac{\partial u}{\partial x_i}\Big)\Big(\frac{\partial v}{\partial x_j}\Big)dx .$$

The boundedness of the a^{ij}, the strong ellipticity condition (2.1), and Poincare's inequality [1, p. 73] imply the existence of positive constants K_1, K_2, and K_3 such that for all $u \in \dot{C}_1$

(2.3) $$K_1\langle u, u\rangle_0 \leqq B(u, u) ,$$

(2.4) $$K_2\langle u, u\rangle_1 \leqq B(u, u) \leqq K_3\langle u, u\rangle_1 .$$

Let $\overset{\circ}{H}_1$ denote the real Hilbert space obtained by completing $\dot{C}_1$ with respect to $\langle\ ,\ \rangle_1$. It is known that the underlying space of $\overset{\circ}{H}_1$ may be assumed to be a subset of H_0 ([1, p. 2]). The quadratic form B may be extended by continuity to $\overset{\circ}{H}_1$ and as a consequence of (2.4) defines an inner product on $\overset{\circ}{H}_1$ which induces the same topology on $\overset{\circ}{H}_1$ as $\langle\ ,\ \rangle_1$.

The following definition connects the operator L and the quadratic form B: Let $f \in H_0$. A *weak solution* of the boundary value problem

(2.5) $$Lu = -f, \qquad u = 0 \text{ on } \partial D$$

is a member v of $\overset{\circ}{H}_1$ such that

(2.6) $$B(\varphi, v) = \langle\varphi, f\rangle_0$$

for all $\varphi \in \overset{\circ}{H}_1$. This definition is motivated by multiplying (2.5) by a member φ of $\dot{C}_1$ and formally integrating by parts. The additional condition $v = 0$ on ∂D "in a weak sense," is interpreted simply to mean $v \in \overset{\circ}{H}_1$. Since the linear functional L_f defined on $\overset{\circ}{H}_1$ by $L_f(\varphi) = \langle\varphi, f\rangle_0$ is continuous, by (2.3), it follows by the Riesz-Frechét theorem that there exists a unique $v \in \overset{\circ}{H}_1$ such that $B(\varphi, v) = L_f(\varphi) = \langle\varphi, f\rangle_0$ for all $\varphi \in \overset{\circ}{H}_1$. Hence there exists a unique weak solution of (2.5).

More generally if p is a function which satisfies a condition of the form (2.2) and $f \in \overset{\circ}{H}_1$ then the linear functional $L_{p,f}$ defined on $\overset{\circ}{H}_1$ by $L_{p,f}(\varphi) = \langle\varphi, f\rangle_{0,p}$ is continuous so there exists a unique $T_pf \in \overset{\circ}{H}_1$ such that

(2.7) $$B(\varphi, T_pf) = L_{p,f}(\varphi) = \langle\varphi, f\rangle_{0,p}$$

for all $\varphi \in \overset{\circ}{H}_1$. This defines a linear map T_p: $H_0 \to \overset{\circ}{H}_1$ but since $\overset{\circ}{H}_1 \subset H_0$ we may consider T_p as a linear map from H_0 into H_0. As a consequence of (2.2) and (2.3) it follows that T_p is continuous and maps bounded subsets of H_0 into bounded subsets of $\overset{\circ}{H}_1$. Thus, by Rellich's selection principle [1, p. 30], T_p: $H_0 \to H_0$ is completely continuous. Moreover T_p is symmetric and positive with respect to the $\langle\ ,\ \rangle_{0,p}$ inner product. Indeed, if $f, g \in H_0$, then by taking φ to be T_pf and T_pg in (2.7) we obtain

$$\begin{aligned}\langle T_pf, g\rangle_{0,p} &= B(T_pf, T_pg) = B(T_pg, T_pf)\\ &= \langle T_pg, f\rangle_{0,p} = \langle f, T_pg\rangle_{0,p} .\end{aligned}$$

If for some $f \in H_0$, $T_pf = 0$ then $\langle\varphi, f\rangle_{0,p} = 0$ for all $\varphi \in \dot{C}_1$ and since

$\dot{C}_1$ is dense in H_0, $f = 0$. Thus if $f \in H_0$ it follows from (2.7) by taking $\varphi = T_p f \in \mathring{H}_1$ that $\langle T_p f, f\rangle_{0,p} = B(T_p f, T_p f)$ so

$$\langle T_p f, f\rangle_{0,p} > 0 \text{ if } f \neq 0 . \tag{2.8}$$

Applying the results of §93 and §94 of [7] to T_p, we infer the existence of a sequence of real numbers $\{\lambda_k\}_1^\infty$ and a sequence $\{\varphi_k\}_1^\infty$ in H_0 such that:

$$\varphi_k = \lambda_k T_p \varphi_k \tag{2.9}$$

$$\langle \varphi_k, \varphi_j\rangle_{0,p} = \delta_{kj} = \begin{cases} 0; & k \neq j \\ 1; & k = j \end{cases} \tag{2.10}$$

$$T_p f = \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k\rangle_{0,p}}{\lambda_k} \varphi_k \tag{2.11}$$

for all $f \in H_0$, and if $\mu \neq \lambda_k$ for all $k = 1, 2 \cdots$ the mapping $[I - \mu T_p]$: $H_0 \to H_0$ is bijective and has a continuous inverse defined by

$$[I - \mu T_p]^{-1} g = g + \mu \sum_{k=1}^{\infty} \frac{\langle g, \varphi_k\rangle_{0,p}}{\lambda_k - \mu} \varphi_k . \tag{2.12}$$

Moreover, the sequence $\{\lambda_k\}$ has no finite cluster point so we may assume by (2.8) that

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \lambda_{k+1} \leqq \cdots . \tag{2.13}$$

Using (2.7) and (2.9) we obtain

$$B(\theta, \varphi_k) = \langle \theta, \lambda_k \varphi_k\rangle_{0,p} = \langle \theta, \lambda_k p \varphi_k\rangle_0$$

for all $\theta \in \mathring{H}_1$ Hence, for each $k = 1, 2, \cdots$, φ_k is a nontrivial weak solution of the boundary value problem

$$Lu + \lambda_k p(x) u = 0, \qquad u = 0 \text{ on } \partial D .$$

We therefore call λ_k a *weak eigenvalue* corresponding to p.

In the following we will want to consider different functions which satisfy a condition of the form (2.2) so henceforth we write $\lambda_k = \lambda_k(p)$, $k = 1, 2, \cdots$.

A result in §93 of [7] and (2.8) implies that the sequence $\{\varphi_k\}_1^\infty$ is complete in H_0 so Parseval's formula

$$(f, f)_{0,p} = \sum_{k=1}^{\infty} \langle f, \varphi_k\rangle_{0,p}^2 \tag{2.14}$$

holds for all $f \in H_0$. We now derive a similar identity involving the space $\mathring{H}_1$ and the inner product B.

Using (2.7), (2.9), and (2.10) we have

$$B(\varphi_k, \varphi_j) = B(\lambda_k(p) T_p \varphi_k, \varphi_j) = \lambda_k(p) \langle \varphi_k, \varphi_j \rangle_{0,p} = \lambda_k(p) \delta_{kj}$$

which shows that the sequence $\{(1/\sqrt{\lambda_k(p)})\varphi_k\}_1^\infty$ in $\mathring{H}_1$ is orthonormal with respect to the inner product B. If for some $\theta \in \mathring{H}_1$, $B(\varphi_k, \theta) = 0$ for all k then by (2.7) and (2.9),

$$\langle \varphi_k, \theta \rangle_{0,p} = 0 \text{ for all } k ,$$

so $\theta = 0$. Thus the sequence $\{(1/\sqrt{\lambda_k(p)})\varphi_k\}$ is complete in $\mathring{H}_1$ and by Parseval's formula

$$B(\theta, \theta) = \sum_{k=1}^{\infty} B\left(\frac{1}{\sqrt{\lambda_k(p)}}\varphi_k, \theta\right)^2 .$$

Using (2.7) and (2.9) we may rewrite this in the more convenient form

$$B(\theta, \theta) = \sum_{k=1}^{\infty} \lambda_k(p) \langle \theta, \varphi_k \rangle_{0,p}^2 . \tag{2.15}$$

The identities (2.14) and (2.15) together with (2.13) now yield the following variational characterization of the weak eigenvalues in terms of the inner products B and $\langle \; \rangle_{0,p}$:

$$\begin{aligned} \lambda_1(p) &= \min \{B(\theta, \theta) \,|\, \theta \in \mathring{H}_1, \langle \theta, \theta \rangle_{0,p} = 1\} \\ \lambda_{k+1}(p) &= \min \begin{Bmatrix} B(\theta, \theta) \,|\, \theta \in \mathring{H}_1, \langle \theta, \theta \rangle_{0,p} = 1, \\ \langle \theta, \varphi \rangle_{0,p} = 0; j = 1, \cdots, k \end{Bmatrix} . \end{aligned} \tag{2.16}$$

Indeed if $\theta \in \mathring{H}_1$, $\langle \theta, \theta \rangle_{0,p} = 1$ and $\langle \theta, \varphi_j \rangle_{0,p} = 0$ for all $j = 1, \cdots, k$ then by (2.13), (2.14) and (2.15)

$$\begin{aligned} B(\theta, \theta) &= \sum_{m=k+1}^{\infty} \lambda_m(p) \langle \theta, \varphi_m \rangle_{0,p}^2 \geqq \lambda_{k+1}(p) \sum_{m=k+1}^{\infty} \langle \theta, \varphi_m \rangle_{0,p}^2 \\ &= \lambda_{k+1}(p) \langle \theta, \theta \rangle_{0,p} = \lambda_{k+1}(p) , \end{aligned}$$

while $B(\varphi_{k+1}, \varphi_{k+1}) = \lambda_{k+1}$. The verification of the first identity in (2.16) is similar.

The proofs of the following two lemmas are essentially the same as the proofs given for similar results in [2, Chapter 6].

LEMMA 2.1 (*Courant*). *If for* $v_1, \cdots, v_k$ *in* H_0 *one defines*

$$\mu_k(p)(v_1, \cdots, v_k) = \inf \begin{Bmatrix} B(\theta, \theta) \,|\, \theta \in \mathring{H}_1, \langle \theta, \theta \rangle_{0,p} = 1, \\ \langle \theta, v_j \rangle_{0,p} = 0; j = 1, \cdots, k \end{Bmatrix}$$

then

$$\lambda_{k+1}(p) = \sup \begin{Bmatrix} \mu_k(p)(v_1, \cdots, v_k) \,|\, v_j \in H_0, \\ j = 1, \cdots, k \end{Bmatrix} .$$

Proof. Since we have established above that $\lambda_{k+1}(p) = \mu_k(p)(\varphi_1, \cdots, \varphi_k)$ we need only show that for arbitrary $v_1, \cdots, v_k \in H_0$

$$\mu_k(p)(v_1, \cdots, v_k) \leqq \lambda_{k+1}(p) . \tag{2.17}$$

Given $v_1, \cdots, v_k \in H_0$ let c_j, $j = 1, \cdots, k+1$ be numbers such that $\sum_{j=1}^{k+1} c_j\langle\varphi, v_i\rangle_{0,p} = 0$ for $i = 1, \cdots, k$ and $\sum_{j=1}^{k+1} c_j^2 = 1$. If $\theta = \sum_{j=1}^{k+1} c_j\varphi_j$ then $\langle\theta, v_i\rangle_{0,p} = 0$; $i = 1, \cdots, k$ and $\langle\theta, \theta\rangle_{0,p} = 1$. Thus from (2.15), $\mu_k(p)(v_1, \cdots, v_k) \leqq B(\theta, \theta) = \sum_{j=1}^{k+1} c_j^2\lambda_j(p) \leqq \lambda_{k+1}(p)\sum_{j=1}^{k+1} c_j^2 = \lambda_{k+1}(p)$.

This proves (2.17) and hence the lemma.

LEMMA 2.2. *If p and q are two real measurable functions defined on D each of which satisfies a condition of the form* (2.2) *and if for all $x \in D$*

$$p(x) \leqq q(x) \tag{2.18}$$

then

$$\lambda_j(q) \leqq \lambda_j(p);\ j = 1, 2, \cdots . \tag{2.19}$$

Proof. If $v \in H_0$ let

$$\hat{v}(x) = (q(x)/p(x))v(x) . \tag{2.20}$$

Let $v_1, \cdots, v_k \in H_0$ be arbitrary, $k \geqq 1$. We assert that

$$\mu_k(q)(v_1, \cdots, v_k) \leqq \mu_k(p)(\hat{v}_1, \cdots, \hat{v}_k) . \tag{2.21}$$

To prove this inequality we note that if $\varepsilon > 0$ it follows from the definition of $\mu_k(p)(\hat{v}_1, \cdots, \hat{v}_k)$ that there exists $\theta \in \mathring{H}_1$ such that $\langle\theta, \theta\rangle_{0,p} = 1, \langle\theta, \hat{v}_j\rangle_{0,p} = 0; j = 1, \cdots, k$, and

$$B(\theta, \theta) \leqq \mu_k(p)(\hat{v}_1, \cdots, \hat{v}_k) + \varepsilon . \tag{2.22}$$

If $\theta^* = (1/\sqrt{\langle\theta, \theta\rangle_{0,q}})\theta$ then using (2.20) we obtain

$$\langle\theta^*, v_j\rangle_{0,q} = \langle\theta^*, \hat{v}_j\rangle_{0,p} = 0;\ j = 1, \cdots, k .$$

Consequently, since $\langle\theta^*, \theta^*\rangle_{0,q} = 1$, we have the inequality

$$\mu_k(q)(v_1, \cdots, v_k) \leqq B(\theta^*, \theta^*) . \tag{2.23}$$

Since $B(\theta^*, \theta^*) = (1/\langle\theta, \theta\rangle_{0,q})B(\theta, \theta)$, and

$$1 \leqq \langle\theta, \theta\rangle_{0,p} = \int_D p\theta^2 dx \leqq \int_D q\theta^2 dx = \langle\theta, \theta\rangle_{0,q} ,$$
$$B(\theta^*, \theta^*) \leqq B(\theta, \theta) .$$

Combining this inequality with (2.22) and (2.23) we obtain

$$\mu_k(q)(v_1, \cdots, v_k) \leqq \mu_k(p)(\hat{v}_1, \cdots, \hat{v}_k) + \varepsilon ,$$

and since $\varepsilon > 0$ is arbitrary (2.21) follows.

Now as v ranges over all elements of H_0, $\hat{v}$ ranges over all elements of H_0 and conversely. Hence for $k \geqq 1$, Lemma 2.1 implies

$$\begin{aligned}\lambda_{k+1}(q) &= \sup\begin{Bmatrix}\mu_k(q)(v_1, \cdots, v_k) \mid v_j \in H_0, \\ j = 1, \cdots, k\end{Bmatrix} \\ &\leqq \sup\begin{Bmatrix}\mu_k(p)(\hat{v}_1, \cdots, \hat{v}_k) \mid v_j \in H_0, \\ j = 1, \cdots, k\end{Bmatrix} \\ &= \sup\begin{Bmatrix}\mu_k(p)(v_1, \cdots, v_k) \mid v_j \in H_0, \\ j = 1, \cdots, k\end{Bmatrix} = \lambda_{k+1}(p)\end{aligned}$$

and this proves (2.19) for $j \geqq 2$. The proof for $j = 1$ follows from the first identity of (2.16) and an argument similar to that given above.

In the following lemma the sequence $\{\alpha_k\}_1^\infty$ will be defined by

$$\alpha_k = \lambda_k(1) \qquad k = 1, 2, \cdots \tag{2.24}$$

so that each α_k is a weak eigenvalue for the problem

$$Lu + \lambda u = 0, \qquad u = 0 \text{ on } \partial D .$$

We let γ_N and γ_{N+1} denote fixed numbers such that for a fixed integer N,

$$\alpha_N < \gamma_N < \gamma_{N+1} < \alpha_{N+1} . \tag{2.25}$$

$\mathscr{P}(\gamma_N, \gamma_{N+1})$ will denote the set of functions p, measurable on D, such that

$$\gamma_N \leqq p(x) \leqq \gamma_{N+1} \text{ for all } x \in D . \tag{2.26}$$

LEMMA 2.3. *If $h \in H_0$ and $p \in \mathscr{P}(\gamma_N, \gamma_{N+1})$ there exists a unique weak solution of the boundary value problem*

$$Lu + pu = h, \qquad u = 0 \textit{ on } \partial D . \tag{2.27}$$

Moreover there exists a number M, independent of $p \in \mathscr{P}(\gamma_N, \gamma_{N+1})$ such that if v denotes this weak solution then

$$B(v, v) \leqq M\langle h, h\rangle_0 . \tag{2.28}$$

Proof. The condition that v be a weak solution of (2.27) is equivalent to the condition that for all $\varphi \in \mathring{H}_1$

$$B(\varphi, v) = \langle \varphi, pv - h\rangle_0 = \langle \varphi, v - h/p\rangle_{0,p}$$

or by (2.7) that

$$[I - T_p]v = T_p[-h/p] \,. \tag{2.29}$$

By (2.12) this equation can be solved uniquely for v provided that

$$\lambda_k(p) \neq 1 \text{ for all } k = 1, 2, \cdots \,. \tag{2.30}$$

From the inequality (2.26) and Lemma 2.2 we have

$$\lambda_k(\gamma_{N+1}) \leqq \lambda_k(p) \leqq \lambda_k(\gamma_N),\ k = 1, 2, \cdots \,. \tag{2.31}$$

Clearly, for all $k = 1, 2, \cdots$,

$$\lambda_k(\gamma_N) = \frac{1}{\gamma_N}\lambda_k(1) = \frac{\alpha_k}{\gamma_N} \,,$$

$$\lambda_k(\gamma_{N+1}) = \frac{1}{\gamma_{N+1}}\lambda_k(1) = \frac{\alpha_k}{\gamma_{N+1}} \,,$$

and hence from (2.25),

$$\begin{aligned}\lambda_1(p) \leqq \lambda_2(p) \leqq \cdots \leqq \lambda_N(p) \leqq \lambda_N(\gamma_N) = \frac{\alpha_N}{\gamma_N} \\ < 1 < \frac{\alpha_{N+1}}{\gamma_{N+1}} = \lambda_{N+1}(\gamma_{N+1}) \leqq \lambda_{N+1}(p) \leqq \lambda_{N+2}(p) \leqq \cdots \,.\end{aligned} \tag{2.32}$$

Thus, if

$$\delta = \min\left[1 - \frac{\alpha_N}{\gamma_N},\ \frac{\alpha_{N+1}}{\gamma_{N+1}} - 1\right],$$

then for all $p \in \mathscr{P}(\mu_N, \mu_{N+1})$,

$$|\lambda_k(p) - 1| \geqq \delta \text{ for all } k = 1, 2, \cdots \,. \tag{2.33}$$

Consequently, (2.29) has a unique solution which by (2.11) and (2.12) is given by

$$\begin{aligned} v &= [I - T_p]^{-1}T_p\left[-\frac{h}{p}\right] = T_p[I - T_p]^{-1}\left[-\frac{h}{p}\right] \\ &= T_p\left[-\frac{h}{p} + \sum_{k=1}^{\infty}\frac{\langle -h/p, \varphi_k\rangle_{0,p}\varphi_k}{\lambda_k(p) - 1}\right] \\ &= \sum_{k=1}^{\infty}\frac{\langle -h/p, \varphi_k\rangle_{0,p}\varphi_k}{\lambda_k(p)} + \sum_{k=1}^{\infty}\frac{\langle -h/p, \varphi_k\rangle_{0,p}\varphi_k}{\lambda_k(p)(\lambda_k(p) - 1)} \\ &= \sum_{k=1}^{\infty}\frac{\langle -h/p, \varphi_k\rangle_{0,p}\varphi_k}{\lambda_k(p) - 1} \,. \end{aligned}$$

Hence, from (2.15) and (2.33) we have

$$B(v, v) = \sum_{k=1}^{\infty} \lambda_k(p)\langle v, \varphi_k\rangle_{0,p}^2 = \sum_{k=1}^{\infty} \frac{\lambda_k(p)\langle -h/p, \varphi_k\rangle_{0,p}^2}{[\lambda_k(p) - 1]^2}$$
$$\leqq \frac{1}{\delta} \sup_k \left|\frac{\lambda_k(p)}{\lambda_k(p) - 1}\right| \sum_{k=1}^{\infty} \langle -h/p, \varphi_k\rangle_{0,p}^2 .$$

Since the function $t/(t-1)$ is increasing for $t < 1$ and decreasing for $t > 1$, the inequalities (2.32) together with the last inequality and Parseval's identity (2.14) yield

$$B(v, v) \leqq (L/\delta)\langle -h/p, -h/p\rangle_{0,p} ,$$

where

$$L = \max\left[\frac{\alpha_N/\gamma_N}{1 - \alpha_N/\gamma_N}, \frac{\alpha_{N+1}/\gamma_{N+1}}{\alpha_{N+1}/\gamma_{N+1} - 1}\right].$$

Thus if $M = L/\gamma_N\delta$, then M is independent of $p \in \mathscr{P}(\gamma_N, \gamma_{N+1})$, and since

$$\langle -h/p, -h/p\rangle_{0,p} = \int_D p\frac{h^2}{p^2}dx \leqq \frac{1}{\gamma_N}\int_D h^2 dx$$

we obtain (2.28). This proves the lemma.

3. A nonlinear problem. In this section γ_N and γ_{N+1} will have the same meaning as in Lemma 2.3. We will assume that $p(x, r, s_1, \cdots, s_n)$ and $h(x, r, s_1, \cdots, s_n)$ are real valued functions defined and continuous on $\bar{D} \times R^{n+1}$,

$$\gamma_N \leqq p(x, r, s_1, \cdots, s_n) \leqq \gamma_{N+1} \tag{3.1}$$

for all $(x, r, s_1, \cdots, s_n) \in \bar{D} \times R^{n+1}$, and for some constant L

$$|h(x, r, s_1, \cdots, s_n)| \leqq L \tag{3.2}$$

on $\bar{D} \times R^{n+1}$.

THEOREM 3.1. *Under conditions* (3.1) *and* (3.2) *there exists a weak solution of the boundary value problem*

$$Lu + p\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)u = h\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right), \tag{3.3}$$
$$u = 0 \text{ on } \partial D .$$

(Here $(\partial u/\partial x_k)$, $k = 1, \cdots, n$ denote the strong $L^2(D)$ derivatives of u, i.e., there exists a sequence $\{u_m\}$ in $\mathring{C}_1$ such that

$$|u - u_m|_0 \to 0, \left|\frac{\partial u}{\partial x_k} - \frac{\partial u_m}{\partial x_k}\right|_0 \to 0, k = 1, \cdots, n$$

as $m \to \infty$.)

To prove Theorem 3.1 we use the well-known Schauder method and two auxiliary lemmas.

If $w \in \mathring{H}_1$, then by (3.1) and (3.2)

$$p\left(x, w, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\right) \in \mathscr{P}(\gamma_N, \gamma_{N+1}),\ h\left(x, w, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\right) \in H_0 ,$$

and

$$\left| h\left(x, w, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\right)\right|_0^2 \leqq R \equiv L^2 \operatorname{meas} D .$$

Therefore by Lemma 2.3 there exists a unique $w^* \in \mathring{H}_1$ such that w^* is a weak solution of the problem

$$(3.5) \qquad Lu + p\left(x, w, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\right)u = h\left(x, w, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\right) ,$$
$$u = 0 \text{ on } \partial D .$$

Furthermore, we have by (2.28)

$$(3.6) \qquad B(w^*, w^*) \leqq MR \text{ for all } w \in \mathring{H}_1 .$$

We define a mapping $G: \mathring{H}_1 \to \mathring{H}_1$ such that for $w \in \mathring{H}_1$, $G(w) = w^*$ is the unique $\mathring{H}_1$ weak solution of (3.5). If $S = \{u \in \mathring{H}_1 \mid B(u, u) \leqq MR\}$ then since B is an inner product on $\mathring{H}_1$ which induces the same topology on $\mathring{H}_1$ as $\langle\ ,\ \rangle_1$, S is a closed, bounded, and convex subset of $\mathring{H}_1$. Now according to (3.6), $G(S) \subseteqq S$, so if it can be shown that G is a compact mapping and that G is continuous then by Schauder's theorem ([3, p. 131]) there exists a $v \in \mathring{H}_1$ such that $G(v) = v$. Consequently v is a weak $\mathring{H}_1$ solution of (3.5). Accordingly, Theorem (3.1) will follow from the next two lemmas.

LEMMA 3.1. *The mapping G is compact.*

LEMMA 3.2. *The mapping G is continuous.*

To prove Lemma 3.1 we show that if $\{u_m\}$ is any sequence in $\mathring{H}_1$ then there exists a subsequence $\{G(u_{m_k})\}$ of $\{G(u_m)\}$ which converges in the $\mathring{H}_1$ norm defined by the B inner product. Suppose then that $\{u_m\}$ is such a sequence. For convenience we set

$$f_m(x) = h\left(x, u_m, \frac{\partial u_m}{\partial x_1}, \cdots, \frac{\partial u_m}{\partial x_n}\right)$$
$$- p\left(x, u_m, \frac{\partial u_m}{\partial x_1}, \cdots, \frac{\partial u_m}{\partial x_n}\right)u_m^*(x) .$$

Using (2.3), (3.1), (3.2) and (3.6) and setting $r = L\sqrt{\text{meas } D} + \gamma_{N+1}\sqrt{MR/K_1}$ we obtain the inequality

$$|f_m|_0 \leqq r \tag{3.7}$$

valid for all $m = 1, 2, \cdots$. By the way f_m is defined, $u_m^* = G(u_m)$ is the weak solution of the problem

$$Lu = -f_m, \; u = 0 \text{ on } \partial D ,$$

and hence

$$B(\varphi, u_m^*) = \langle \varphi, f_m \rangle_0 \text{ for all } \varphi \in \mathring{H}_1 . \tag{3.8}$$

By (3.6), $B(u_m^*, u_m^*) \leqq MR$ so by (2.4) the sequence $\{u_m^*\}$ is bounded in the $\mathring{H}_1$ norm. Using Rellich's selection theorem ([1, p. 30]) we infer the existence of a subsequence $\{u_{m_k}^*\}$ of $\{u_m^*\}$ which converges in the H_0 norm.

From (3.8) it follows that for arbitrary integers p and q and arbitrary $\varphi \in \mathring{H}_1$,

$$B(\varphi, u_{m_q}^* - u_{m_p}^*) = \langle \varphi, f_{m_q} - f_{m_p} \rangle_0 .$$

Thus, taking $\varphi = u_{m_q}^* - u_{m_p}^*$ in the above and using the Schwartz inequality and (3.7)

$$B(u_{m_q}^* - u_{m_p}^*, u_{m_q}^* - u_{m_p}^*) \leqq 2r |u_{m_q}^* - u_{m_p}^*|_0 .$$

Thus, since the sequence $\{u_{m_k}^*\}$ is Cauchy with respect to the $|\ |_0$ norm, it follows from (2.4) that $\{u_{m_k}^*\}$ is Cauchy with respect to the $\mathring{H}_1$ norm and hence converges to a member of $\mathring{H}_1$. This proves Lemma 3.1.

The proof of the continuity of the mapping G is less straightforward. We will first show that regarded as a map from $\mathring{H}_1 \to H_0$, G is continuous and then apply an argument similar to that given above.

PROPOSITION 3.1. *The mappings*

$$u \to p\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) , \tag{3.9}$$

and

$$u \to h\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \tag{3.10}$$

from $\mathring{H}_1 \to H_0$ *are continuous.*

Since the proofs of both assertions are similar, we only prove one. Let $\{u_m\}$ be a sequence in $\mathring{H}_1$ and u a member of $\mathring{H}_1$ such that

$$|u - u_m|_1^2 = \int_D \left[(u - u_m)^2 + \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k} - \frac{\partial u_m}{\partial x_k}\right)^2\right] dx \longrightarrow 0$$

as $m \longrightarrow \infty$. Choose $\varepsilon > 0$ and define

$$R(\varepsilon) = \left\{(x, t, s_1, \cdots, s_n) \in \bar{D} \times R^{n+1} \mid t^2 + \sum_{k=1}^n s_k^2 \leqq \frac{1}{\varepsilon^2}\right\}.$$

By the compactness of $R(\varepsilon)$ there exists a number $\delta > 0$ such that

$$|p(x, t, s_1, \cdots, s_n) - p(x, t', s_1', \cdots, s_n')| \leqq \varepsilon$$

if $(x, t, s_1, \cdots s_n) \in R(\varepsilon)$ and

$$(t - t')^2 + \sum_{k=1}^n (s_k - s_k')^2 \leqq \delta^2 .$$

Let

$$A(\varepsilon) = \left\{x \in D \mid u(x)^2 + \sum_{k=1}^n \frac{\partial u}{\partial x_k}(x)^2 > \frac{1}{\varepsilon^2}\right\}$$

and for each integer m define

$$B_m(\delta) = \left\{x \in D \mid (u(x) - u_m(x))^2 + \sum_{k=1}^m \left(\frac{\partial u(x)}{\partial x_k} - \frac{\partial u_m(x)}{\partial x_k}\right)^2 > \delta^2\right\}.$$

Since,

$$\frac{1}{\varepsilon^2} \text{ meas } A(\varepsilon) \leqq \int_{A(\varepsilon)} \left[u(x)^2 + \sum_{k=1}^m \frac{\partial u(x)}{\partial x_k}\right]^2,$$

$$\text{meas } A(\varepsilon) \leqq \varepsilon^2 |u|_1^2 ,$$

and in a similar manner we obtain the estimate

$$\text{meas } B_m(\delta) \leqq (1/\delta^2) |u_m - u|_1^2 .$$

For convenience we set

$$\hat{p}(x) = p\left(x, u(x), \frac{\partial u(x)}{\partial x_1}, \cdots, \frac{\partial u(x)}{\partial x_n}\right) \tag{3.11}$$

$$\hat{p}_m(x) = p_m\left(x, u_m(x), \frac{\partial u_m(x)}{\partial x_1}, \cdots, \frac{\partial u_m(x)}{\partial x_n}\right). \tag{3.12}$$

If $x \in D - (A(\varepsilon) \cup B_m(\delta))$ then $|\hat{p}(x) - \hat{p}_m(x)|^2 \leqq \varepsilon^2$, so by the above

$$\begin{aligned} |\hat{p} - \hat{p}_m|_0^2 &\leqq \int_{D-(A(\varepsilon)\cup B_m(\delta))} |\hat{p} - \hat{p}_m|^2 dx + \int_{A(\varepsilon)\cup B_m(\delta)} |\hat{p} - \hat{p}_m|^2 dx \\ &\leqq \varepsilon^2 \text{ meas } [D - (A(\varepsilon) \cup B_m(\delta)] + 4\gamma_{N+1}^2 \text{ meas } [A(\varepsilon) \cup B_m(\delta)] \\ &\leqq \varepsilon^2 \text{ meas } D + 4\gamma_{N+1}^2 \left[\frac{|u_m - u|_1^2}{\delta^2} + \varepsilon^2 |u|_1^2\right]. \end{aligned}$$

This shows that

$$\overline{\lim_{m\to\infty}} |\hat{p} - \hat{p}_m|_0^2 \leqq \varepsilon^2[\text{meas } D + 4\gamma_{N+1}^2 |u_1|^2]$$

and since $\varepsilon > 0$ is arbitrary, $\lim_{m\to\infty}|\hat{p} - \hat{p}_m|_0^2 = 0$. This proves the continuity of the mapping defined in (3.9) and the proof for (3.10) is similar. In a similar manner one proves that the mapping from $\mathring{H}_1 \to H_0$ defined by

$$u \to \frac{h\left(x, u, \dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_n}\right)}{p\left(x, u, \dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_n}\right)} \tag{3.13}$$

is continuous.

PROPOSITION 3.2. *Let $\{\hat{p}_m\}$ be a sequence in $\mathscr{P}(\gamma_N, \gamma_{N+1})$ and suppose $|\hat{p}_m - \hat{p}|_0 \to 0$ as $m \to \infty$ for some $\hat{p} \in \mathscr{P}(\gamma_N, \gamma_{N+1})$. If $\{T_{\hat{p}_m}\}$ and $T_{\hat{p}}$ are the operators defined by (2.7) then $T_{\hat{p}_m}$ converges strongly to $T_{\hat{p}}$,* i.e., *$|T_{\hat{p}_m}w - T_{\hat{p}}w|_0 \to 0$ as $m \to \infty$ for each $w \in H_0$.*

Proof. According to (2.7) if $w \in H_0$ and $\varphi \in \mathring{H}_1$ then

$$B(\varphi, T_{\hat{p}}w - T_{\hat{p}_m}w) = \int_D (\hat{p}_m - \hat{p})w\varphi dx .$$

Now $\lim_{m\to\infty}|\hat{p}_m - \hat{p}|_0^2 = 0$ implies that $\hat{p}_m$ converges to $\hat{p}$ in measure. Thus, since $|(p_m(x) - \hat{p}(x))w(x)\varphi(x)| \leqq 2\gamma_{N+1}|w(x)\varphi(x)|$ and $w\varphi \in L^1(D)$, by the strong form of Lebesgues' dominated convergence theorem [6, p. 149], $\lim_{m\to\infty}\int_D (\hat{p}_m - \hat{p})w\varphi dx = 0$. This shows that $T_{\hat{p}}w \to T_{\hat{p}}w$ as $m \to \infty$ weakly in $\mathring{H}_1$ and hence by Rellich's theorem $\lim_{m\to\infty} T_{\hat{p}_m}w = T_{\hat{p}}w$ strongly in H_0.

PROPOSITION 3.3. *The mapping $u \to G(u)$ is continuous from $H_0 \to \mathring{H}_1$.*

Proof. If $p \in \mathscr{P}(\gamma_N, \gamma_{N+1})$ let $\|T_p\|_0$ and $\|T_p\|_{0,p}$ denote the norms of T_p: $H_0 \to H_0$ relative to the inner products $\langle\ ,\ \rangle_0$ and $\langle\ ,\ \rangle_{0,p}$ respectively. The identity (2.11) and Lemma 2.2 gives the inequality $\|T_p\|_{0,p} \leqq 1/\lambda_1(p) \leqq 1/\lambda_1(\gamma_{N+1})$. Similarly (2.12) and (2.33) give

$$\|[I - T_p]^{-1}\|_{0,p} \leqq 1 + 1/\delta .$$

Therefore from the inequality $\langle w, w\rangle_0^2 \leqq 1/\gamma_N\langle w, w\rangle_{0,p}$ valid for any $w \in H_0$, we have the estimates

$$\|T_p\|_0 \leqq (1/\sqrt{\gamma_N})(1/\lambda_1(\gamma_{N+1})) \equiv A_1 , \tag{3.14}$$

$$\| [I - T_p]^{-1} \|_0 \leqq (1/\sqrt{\gamma_N})(1 + 1/\delta) \equiv A_2 . \tag{3.15}$$

Let $u \in \mathring{H}_1$ and suppose $\{u_m\}$ is a sequence in $\mathring{H}_1$ such that $|u - u_m|_1 \to 0$ as $m \to \infty$. Let $\hat{p}$ and $\hat{p}_m$ be defined as in (3.11), (3.12) and set

$$\hat{h}_m = h\left(x, u_m, \frac{\partial u_m}{\partial x_1}, \cdots, \frac{\partial u_m}{\partial x_k}\right), \quad \hat{h} = h\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

Let $u^* = G(u)$ and $u_m^* = G(u_m)$. For each m, u_m^* is the weak solution of the boundary value problem

$$Lv = -(\hat{p}_m u_m^* - h_m), \qquad v = 0 \text{ on } \partial D$$

so for arbitrary $\varphi \in \mathring{H}_1$,

$$B(\varphi, u_m^*) = \langle \varphi, \hat{p}_m u_m^* - h_m \rangle_0 = \langle \varphi, u_m^* - h_m/\hat{p}_m \rangle_{0,\hat{p}_m}$$

and hence

$$u_m^* = T_{\hat{p}_m}[u_m^* - \hat{h}_m/\hat{p}_m] ,$$
$$u^* = T_{\hat{p}}[u^* - \hat{h}/\hat{p}] .$$

From the equation

$$u^* - u_m^* = T_{\hat{p}_m}[u^* - u_m^*] + [T_{\hat{p}} - T_{\hat{p}_m}]u^* + T_{\hat{p}_m}[\hat{h}_m/\hat{p}_m - \hat{h}/\hat{p}] + [T_{\hat{p}_m} - T_{\hat{p}}]\hat{h}/\hat{p} ,$$

we obtain

$$u^* - u_m^* = [I - T_{\hat{p}_m}]^{-1}(T_{\hat{p}} - T_{\hat{p}_m})[u^* - \hat{h}/\hat{p}] + [I - T_{\hat{p}_m}]^{-1} T_{\hat{p}_m}[\hat{h}_m/\hat{p}_m - \hat{h}/\hat{p}] .$$

Therefore by the estimates (3.14), (3.15) we have

$$|u^* - u_m^*|_0 \leqq A_2 |(T_{\hat{p}} - T_{\hat{p}_m})(u^* - \hat{h}/\hat{p})|_0 + A_2 A_1 |\hat{h}_m/\hat{p}_m - \hat{h}/\hat{p}|_0 .$$

By Propositions 3.1 and 3.2,

$$|(T_{\hat{p}} - T_{\hat{p}_m})(u^* - \hat{h}/\hat{p})|_0 \to 0 \text{ as } m \to \infty ,$$

and by the remark following the proof of Proposition 3.1,

$$|\hat{h}_m/\hat{p}_m - \hat{h}/\hat{p}|_0 \to 0 \quad \text{as} \quad m \to \infty .$$

This concludes the proof of Proposition 3.

Lemma 3.2 now follows easily. Let u and the sequence u_m be as above. Define $f_m = \hat{h}_m - \hat{p}_m u_m^*$, $f = \hat{h} - \hat{p}u^*$. From (3.7) $|f_m|_0 \leqq r$, $|f|_0 \leqq r$. Referring to the proof of Lemma 3.1 we see that for any $\varphi \in \mathring{H}_1$

$$B(\varphi, u^* - u_m^*) = \langle \varphi, f - f_m \rangle_0 ,$$

so by taking $\varphi = u^* - u_m^*$ we have

$$\begin{aligned} B(u^* - u_m^*, u^* - u_m^*) &= \langle u^* - u_m^*, f - f_m\rangle_0 \\ &\leqq 2|u^* - u_m|_0 r \, . \end{aligned}$$

By Proposition 3.3, $B(u^* - u_m^*, u^* - u_m^*) \to 0$ as $m \to \infty$. This proves the continuity of $G: \mathring{H}_1 \to \mathring{H}_1$ and concludes the proof of Theorem 3.1.

4. Smooth solutions of an inhomogeneous problem. In this section we will assume that $D \subset R^n$ is a Dirichlet domain and $L = \Delta$ where Δ is the n-dimensional Laplacian.

If f is continuous on $\bar{D}$ and has continuous partial derivatives on D, then the weak solution of the problem

$$\Delta u = -f(x), \qquad u = 0 \text{ on } \partial D \tag{4.1}$$

is actually a solution in the classical sense and can be represented in the form

$$u(y) = \int_D G(x, y) f(x) dx, \qquad y \in \bar{D} \, , \tag{4.2}$$

where G is the Green's function for the problem (4.1).

THEOREM 4.1. *If p and h satisfy the conditions* (3.1) *and* (3.2) *of Theorem* 3.1 *and g is continuous on ∂D, then there exists a weak solution v of*

$$\Delta u + p\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)u = h\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \tag{4.3}$$

such that v has continuous derivatives on $\bar{D}$ and

$$v(x) = g(x), \qquad x \in \partial D \, . \tag{4.4}$$

Proof. Since D is a Dirichlet domain there exists a function w such that w is continuous on $\bar{D}$, $\Delta w = 0$ on D, and $w(x) = g(x)$ on ∂D. If

$$\begin{aligned} &P(x, t, s_1, \cdots, s_n) \\ &\quad = p\left(x, t + w(x), s_1 + \frac{\partial w(x)}{\partial x_1}, \cdots, s_n + \frac{\partial w(x)}{\partial x_n}\right) \end{aligned} \tag{4.5}$$

$$\begin{aligned} &H(x, t, s_1, \cdots, s_n) \\ &\quad = h\left(x, t + w(x), s_1 + \frac{\partial w(x)}{\partial x_1}, \cdots, s_n + \frac{\partial w(x)}{\partial x_n}\right) \\ &\qquad - P(x, t, s_1, \cdots, s_n) w(x) \, , \end{aligned} \tag{4.6}$$

then P and H will satisfy conditions of the form (3.1) and (3.2).

Consequently, by Theorem 3.1 there exists a weak $\mathring{H}_1$ solution V of

$$\Delta u + P\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)u = H\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

We assert that if $y \in \bar{D}$,

$$V(y) = \int_D G(x, y)F(x)dx \tag{4.7}$$

where as in (4.2), G is the Green's function and

$$F = P\left(x, V, \frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\right)V - H\left(x, V, \frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\right).$$

Indeed, V is the weak $\mathring{H}_1$ solution of $\Delta u = -F$ so $V = T_1F$ where T_1 is defined by (2.7). If f is continuously differentiable on $\bar{D}$ then by (4.2) for $y \in \bar{D}$

$$(T_1f)(y) = \int_D G(x, y)f(x)dx .$$

Now if f is merely in $L^2(D)$, the operator $S: \mathring{H}_1 \to \mathring{H}_1$ defined by

$$(Sf)(y) = \int_D G(x, y)f(x)dx$$

is continuous. Therefore, since S and T_1 agree on a dense subspace of $L^2(D)$, $T_1 = S$, whence (4.7) holds.

From the representation (4.7) and the fact that F is in $L^\infty(D)$, it follows by standard arguments of potential theory that V has continuous derivatives and vanishes on the boundary of D. Setting $v = V + w$ we see that v satisfies the assertion of Theorem 4.1.

REFERENCES

1. S. Agmon, *Lectures on elliptic boundary value problems*, D. Van Nostrand Company, Inc., New York, 1965.

2. R. Courant and D. Hilbert, *Methods of mathematical physics*, Val. I, Interscience Publishers, Inc., New York, 1953.

3. J. Cronin, *Fixed points and topological degree in nonlinear analysis*, Mathematical Surveys, No. 11, American Mathematical Society, Providence. 1964.

4. C. L. Dolph, *Nonlinear integral equations of the Hammerstein type*, Trans. Amer. Math. Soc. **66** (1949), 289-307.

5. A. C. Lazer and D. E. Leach, *On a nonlinear two-point boundary value problem*, J. Math. Anal. and Appl. **26** (1969), 20-27.

6. I. P. Natanson, *Theory of functions of a real variable*, Vol. I, (transl. from the Russian), Ungar, New York, 1961.

7. F. Riesz and B. Sz.-Nagy, *Functional analysis*, (translated from the 2'nd. French edition), Ungar, New York, 1955.

Received June 27, 1969. The preparation of this paper was sponsored in part by the Office of Naval Research under Contract NONR 233(76) and the U. S. Army Research Office under Grant DA-31-124-ARO(D)-355. Reproduction in whole or in part is permitted for any purpose of the United States Government.

UNIVERSITY OF CALIFORNIA, LOS ANGELES, AND
UNIVERSITY OF CALIFORNIA, SANTA CRUZ

UNIVERSITY OF CALIFORNIA, LOS ANGELES, AND
CASE WESTERN RESERVE UNIVERSITY