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Linear Estimation: The Kalman-Bucy Filter

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LINEAR ESTIMATION:

THE KALMAN-BUCY FILTER

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ABSTRACT

The problem of linear dynamic estimation, its solution as developed by Kalman and Bucy, and interpretations, properties and illustrations of that solution are discussed. The central problem considered is the estimation of the system state vector X, describing a linear dynamic system governed by

$$\frac{dX}{dt} = F(t)X(t) + G(t)U(t)$$

$$Y(t) = H(t)X(t) + V(t)$$

from observations of Y (system output), where V is a random observation-corrupting process, and U is a random system driving process.

An extension of the Kalman-Bucy filter to estimation in the absence of a priori knowledge of the random processes U and V is developed and illustrated.

CONTENTS

I. INTRODUCTION

I.1 Principal Problem

In this paper we are concerned with situations described by the system of linear differential equations

$$\frac{d\mathbf{X}}{dt} = F(t)\mathbf{X}(t) + G(t)U(t)$$

where $U(t)$ represents a random vector driving process lending a stochastic behavior to the system. Observation of the system will be described by the linear, noise-corrupted observation model

$$Y(t) = H(t)X(t) + V(t)$$

where $V(t)$ represents a vector random "noise" or observation-corrupting process.

Our general problem will involve the estimation of $X(t)$, based upon the corrupted observations $Y(t)$, and with establishing the reliability and behavior of this estimate. Because the convenient means of implementing the resulting estimator is the digital computer, an intrinsically time-discretized device, much attention will be focused on the time-discrete form of the general problem. We shall utilize the state-transition point of view, resulting in the system model

$$X(t+T) = \phi(t+T;t)X(t) + \Gamma(t)U^*(t)$$

where $\phi(t+T;t)$ represents the state transition matrix described by

$$\frac{d}{dT} \phi(t+T;t) = F(t)\phi(t+T;t) \ ; \ \phi(t;t) = I \ ,$$

and $U^*(t)$ is a time-discretized random process driving the system. (See Appendix IX.6)

I.2 Outline of Paper

Toward presenting initial problems and solutions in a simple and perhaps more readily digested form, we begin in Section II with the statement of a basic, time-discretized problem amenable to the theory, and state the form of solution without proof. Section III presents a collection of example systems lending illustrative substance to the statements of Section II; in spite of the simplicity of these examples, a number of the fundamental questions and concepts of the theory are represented.

In Section IV, we present basic derivations and interpretations of the stated solutions of Section II, postponing more general viewpoints to Section VII.

Section V presents computer simulation runs of the previously described example systems, with discussion.

Section VI provides a discussion of additional complications which arise in the application of the theory to practical problems.

In Section VII, we take a slightly more abstract view, moving simultaneously to the continuous problem. General questions raised in preceding sections are answered at this point.

Finally, Section VIII presents an extension of the Kalman-Bucy filter, providing for adaptive estimation in the event of imprecise knowledge of system statistics.

I.3 History

Modern estimation theory and its application to engineering products lead to the study and use of a variety of tools or ideas under the headings of engineering and mathematics. Interest in the study of these topics and the opportunity to combine them in the design of a useful system may in themselves constitute a justification for the study of estimation theory. The study of this topic, carried out to the extent that a useful engineering skill is obtained, may be expected to include aspects of differential and difference equations, numerical analysis, statistics, control system analysis, linear algebra, the use of digital and perhaps analog computers, and that collection of ideas and methods referred to today as "systems analysis." In addition, the application of estimation theory to a specific engineering problem generally carries an implicit requirement that the individual be conversant in the area of application (e.g., navigation, instrumentation).

Interest in the theory of estimation, prediction, smoothing, or filtering of variables or data describing dynamic physical systems has become particularly prominent, in published and applied form, in the years since 1960. Although certain aspects of this subject antedate considerably that period, modern developments have contributed to the recent prominence. The "method of least squares" was known to Gauss, at the beginning of the 19th century, as a scheme for improving estimates by redundant observation. [8] The study of control systems stimulated by World War II led Norbert Wiener to a mathematical theory ("Wiener filtering") which added statistical and spectral aspects to the theory, although his results were limited to statistically stationary, single-variable applications. [18]

In 1960, R. E. Kalman and R. S. Bucy published a synthesis of ideas and solutions to important problems which brought within reach of the practicing engineer the design and evaluation of estimators applicable to a wide variety of dynamic physical systems. These results not only provided solutions to statistically non-stationary and multi-variate problems, but were obtained at a mathematical level directly accessable to the engineering community, in a form leading directly to the synthesis of practical estimators. These attributes were not shared by Wiener's earlier results; one of the early copies of the Wiener solution was known as the "Yellow Peril" in the engineering community, owing to the level of mathematical abstraction and difficulty of filter synthesis which were a part of his treatment.

During the same decade (1960-70), the technology of digital computation has seen numerous and marked advances in the areas of device speed, reliability, physical dimension, cost, fabrication, system complexity, and general applicability to problem solving. The digital computer has invaded with considerable impact the fields of on-board navigation of space and military vehicles, and fire control in military applications.

The resulting coincidence of the theoretical tool and the medium in which it may conveniently take form to solve existing problems has led to the appearance of the numerical estimator as an important component in tangible products of engineering.

I.4 Applications

The applications of the Kalman-Bucy filter which have received the most attention are concerned with navigation and fire control problems in the aerospace industry. Typically, a small on-board digital computer performs real-time processing of sensor inputs within the space or airborne vehicle, to arrive at solutions (estimates) which describe vehicle position and velocity, target range, bearing, and rate, etc. In the navigation problem, for example, an inertial measurement set organized around a set of mutually orthogonal accelerometers mounted on a gyroscopically stabilized platform provides sensed vehicle inertial acceleration signals. These signals are a sensor input to the computer-implemented filter which arrives at updated estimates of position and velocity in real time. Because of the error characteristic of the inertial measurement system, other sensors are also used -- doppler radar for the direct measurement of velocity, radio navigation receivers for direct position measurement, periodic pilot position fixes, and so on. The filter must combine these various data streams in a manner that results in estimates of position and velocity which are superior to the use of single instruments.

On-board navigation systems utilizing the Kalman-Bucy filter have been used on the Ranger VI and VII trans-lunar vehicles, the Apollo program, the proposed American supersonic transport aircraft, the C-5A transport aircraft and is to be found in some form in a variety of aircraft and spacecraft navigation systems. [4]

II. STATEMENT OF A PROBLEM AND THE KALMAN SOLUTION

The time-dependent behavior of a certain dynamic, linear, time-discretized system is described by

SYSTEM MODEL $$X_{N+1} = \Phi_{N+1} \cdot X_N + W_{N+1} \tag{1}$$

where

X_N = vector of n state variables describing total system state at time (stage) N, $N = 0,1,2,\ldots$

Φ_N = $(n \times n)$ system state transition matrix describing the linear time development of the system; the suffix indicates a possible time-dependent character of the system. Φ_N describes transition from stage $(N-1)$ to stage (N), and is sometimes labeled $\Phi_{N-1,N}$. Properties of Φ_N are discussed in Appendix IX.6, which also motivates the form of equation (1).

W_N = Nth member of a vector random process (of n-element vectors) representing the stochastic aspect of the system behavior; a random forcing function. We further require that

ZERO MEAN: $E[W_N] = 0$, $N = 1,2,\ldots$ [1] (2)

WHITENESS: $E[W_N \cdot W_M^T] = \delta_{MN}\, Q_N$, $N = 1,2,\ldots$, $M = 1,2,\ldots$ [2] (3)

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[1] $E[\cdot]$ represents the expectation operator.

[2] δ_{MN} represents the Kronecker delta: $\delta_{MN} = \begin{cases} 0, & M \neq N \\ 1, & M = N \end{cases}$

where Q_N is the $(n \times n)$ covariance matrix describing the second moments of W_N.

We observe the system by monitoring a set of system outputs which are noise-corrupted linear combinations of the system state vector:

<u>OBSERVATION MODEL</u> $$Y_N = H_N X_N + V_N \tag{4}$$

where

Y_N = vector of m observables, representing information obtained by measurements at stage N.

H_N = $(m \times n)$ observation matrix, indicating a linear relation between system state variables and observables, perhaps time-dependent.

V_N = Nth member of a vector random sequence (of m-element vectors), representing additive observation-corrupting effects ("noise"). We further require that

ZERO MEAN: $$E[V_N] = 0, \quad N = 1,2,\ldots \tag{5}$$

WHITENESS: $$E(V_N \cdot V_M^T) = \delta_{NM} R_N, \quad \begin{matrix} N = 1,2,\ldots \\ M = 1,2,\ldots \end{matrix} \tag{6}$$

where R_N is the $(m \times m)$ covariance matrix describing the second moments of $\{V_N\}$.

We also require that $X_o, \hat{X}_o, V_N$, and W_M be mutually independent for all N,M. (7)

THE PROBLEM: Given an estimate of the state of the system at time stage N, $\hat{X}_N$, a measure of the quality of that estimate, trace (P_N), where

$$P_N = E\{(X_N - \hat{X}_N)(X_N - \hat{X}_N)^T\}$$

is the covariance matrix for the error in the given estimate, and given the observation Y_{N+1}, compute $\hat{X}_{N+1}$, an estimate of X_{N+1}, which minimizes trace (P_{N+1}), where

$$P_{N+1} = E\{(X_{N+1} - \hat{X}_{N+1})(X_{N+1} - \hat{X}_{N+1})^T\}$$

is the covariance matrix for the error in the newly-computed estimate.

(Observe that a solution to this problem provides an inductive procedure for computing $\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \ldots$ $P_1, P_2, P_3, P_4, \ldots$ given $\hat{X}_o$ and P_o.)

THE SOLUTION (Kalman)[11]: Under conditions to be discussed, the optimal estimate is given by

$$\hat{X}_{N+1} = \Phi_{N+1}\hat{X}_N + C_{N+1}\{Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\} \quad , \quad (8)$$

where

$$C_{N+1} = P'_{N+1}\; H^T_{N+1}\{H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}\}^{-1} \quad , \quad (9)$$

$$P'_{N+1} = \Phi_{N+1}P_N\Phi^T_{N+1} + Q_{N+1} \qquad (10)$$

$$P_{N+1} = (I - C_{N+1}H_{N+1})P'_{N+1} \quad ; \quad \text{here} \qquad (11)$$

$\hat{X}_{N+1}$ represents the optimal (under the objective function described) estimate of X_{N+1}

C_{N+1} is an (n x m) coefficient matrix controlling the system gain or mixing of propagated and observed signals.

P'_{N+1} is the (n x n) covariance matrix for the error in $\Phi_{N+1}\hat{X}_N$ as an estimate of X_{N+1}.

P_{N+1} is the (n x n) propagated covariance matrix, representing the covariance in the error for $\hat{X}_{N+1}$ as an estimate of X_{N+1}.

In signal flow graph form, the modeled system and estimator (filter) take the form indicated in Figure II.1.

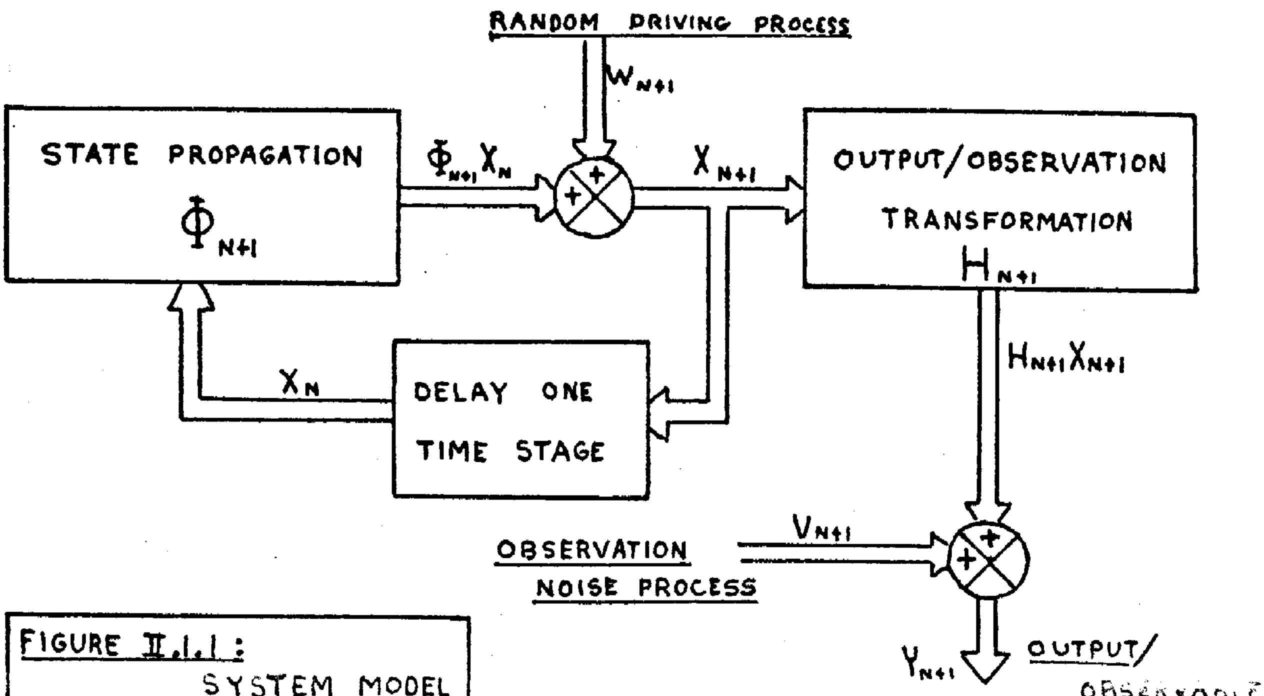


FIGURE II.1.1: SYSTEM MODEL

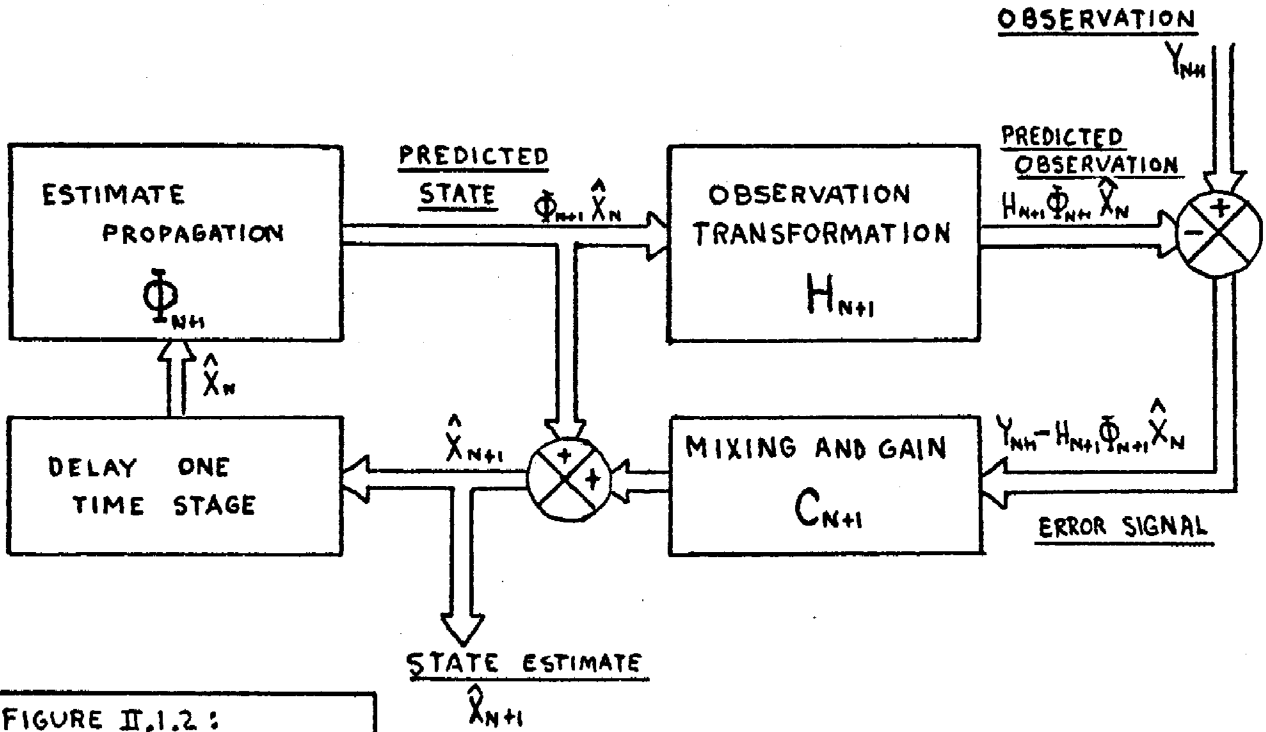


FIGURE II.1.2 : ESTIMATOR

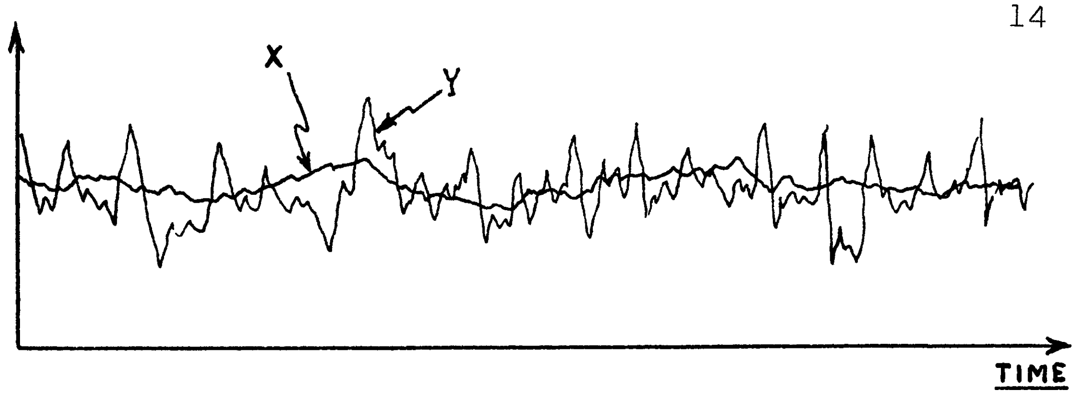
III. EXAMPLE SYSTEMS: MODEL FORM

Prior to deriving and interpreting the results stated in Section II, we consider several example linear systems of the type discussed, illustrating the form taken by the model. The examples cited are particularly simple, in comparison with the typical application problem, but the fundamental considerations of application are represented. We discuss these examples further in Section V.

III.1 Scalar State

Repeated measurements of a physical parameter X, using a "noisy" instrument with stationary error statistics, produces a sequence of scalar observations, $y_1, y_2, y_3, \ldots$ It is known that the instrument errors are additive, distributed with zero mean, and variance σ_R^2. It is known that the actual value of the parameter varies through additive random perturbations of zero mean and known variance σ_Q^2. Letting X_N represent the value of the parameter at stage N, and representing computed estimates by the caret (^),

MODEL:
$$X_{N+1} = X_N + W_{N+1} \quad ; \quad \Phi_{N+1} = 1 \tag{1}$$
$$Y_N = X_N + V_N \quad ; \quad H_N = 1$$
$$E[W_i \cdot W_j] = \delta_{ij}\,\sigma_Q^2$$
$$E[V_i \cdot V_j] = \delta_{ij}\,\sigma_R^2$$



ESTIMATOR: Given an initial estimate $\hat{X}_o$, of state X_o, and the associated error variance $\sigma_o^2 = E[(X_o - \hat{X}_o)^2]$, exercise the following equations as measurements are recorded:

COEFFICIENT GENERATION: $$c_{N+1} = \frac{\sigma_N^2 + \sigma_Q^2}{\sigma_N^2 + \sigma_Q^2 + \sigma_R^2} \quad (2)$$

FILTER: $$\hat{X}_{N+1} = \hat{X}_N + c_{N+1}\{y_{N+1} - \hat{X}_N\}$$

ERROR COVARIANCE PROPAGATION:

$$\sigma_{N+1}^2 = (1 - c_{N+1})\sigma'^2_{N+1}$$

$$= \frac{\sigma_R^2[\sigma_N^2 + \sigma_Q^2]}{\sigma_N^2 + \sigma_Q^2 + \sigma_R^2}, \quad N = 1,2,\ldots$$

We see that the form taken here is that of a first-order filter with time-varying coefficient.

III.2 Rate Estimation (Differentiation)

The position and velocity of a moving target, relative to a radar antenna and in one dimension, are represented as functions of time by $x(t)$ and $\dot{x}(t)$, respectively. Measurements taken by the radar at times $N = 1,2,\ldots$ yield observations of distance to the target, corrupted by errors in the radar system. Further, target velocity is subject to random variations. We wish to estimate both target position and velocity. As in III.1, the random errors and system perturbations are of mean zero and known variance, σ_R^2 and σ_Q^2, respectively.

MODEL: $$X_{N+1} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{N+1} = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_N + \begin{bmatrix} 0 \\ w \end{bmatrix}_{N+1} \quad ;$$

$$\phi_{N+1} = \phi = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \quad ; \quad W_{N+1} = \begin{bmatrix} 0 \\ w_{N+1} \end{bmatrix} \tag{3}$$

where ΔT = length of stage interval, in seconds.

$$y_N = [1 \quad 0] \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_N + [v_N] \quad ; \quad H_N = [1 \quad 0]$$

We again assume stationary statistics:

$$E[W_i W_j] = \delta_{ij} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_Q^2 \end{bmatrix} = \delta_{ij} Q \quad ;$$

$$E[v_i v_j] = \delta_{ij}\, \sigma_R^2 = \delta_{ij}\, R$$

ESTIMATOR:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (\phi P_N \phi^T + Q) H^T (H \phi P_N \phi^T H^T + HQH^T + R)^{-1} \tag{4}$$

$$= \frac{1}{p_{11}+2p_{12}\Delta T+p_{22}\Delta T^2+\sigma_R^{\;2}} \begin{bmatrix} p_{11}+2p_{21}\Delta T+p_{22}\Delta T^2 \\ p_{12} + p_{22}\Delta T \end{bmatrix}$$

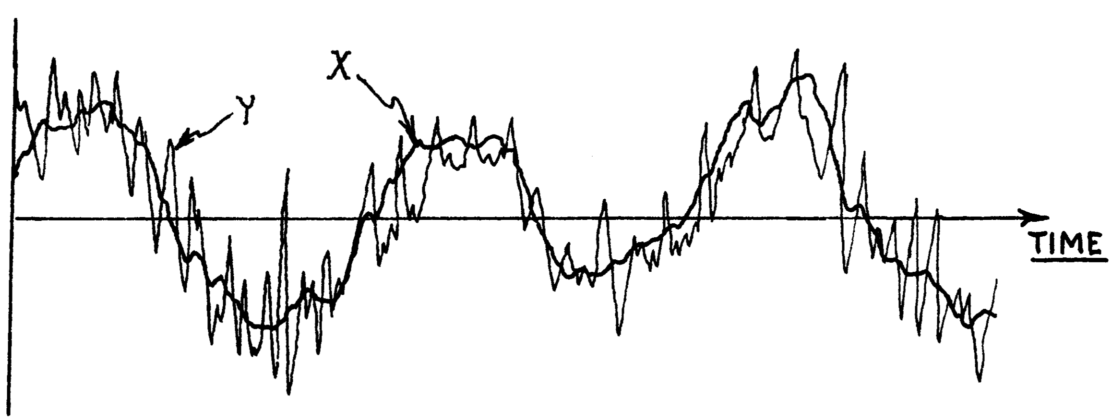
$$\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix}_{N+1} = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix}_N + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{N+1} \times \{y_{N+1} - \hat{x}_N - \hat{\dot{x}}_N \Delta T\}$$

$$P_{N+1} = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} P_N \begin{bmatrix} 1 & 0 \\ \Delta T & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_Q^{\;2} \end{bmatrix}$$

$$P_{N+1} = (I - C_{N+1}\, H)\, P'_{N+1} = \begin{bmatrix} (1-c_1) & 0 \\ (\;-c_2) & 1 \end{bmatrix} \cdot P'_{N+1}$$

III.3 Harmonic Oscillator

A physical parameter is known to vary sinusoidally, with constant angular frequency ω_0 . Noise-corrupted observations are to be used to estimate the actual value of the variable, and implicitly its amplitude and phase. Random perturbations of the oscillator are known to occur.



MODEL: Letting x_{1_N} denote the value of x at time $t = t_N$, and $x_{2_N} = x(t_N + \pi/2\omega_o)$, i.e., shifted by 90^o, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{N+1} = \begin{bmatrix} \cos(\omega_o \Delta T) & \sin(\omega_o \Delta T) \\ -\sin(\omega_o \Delta T) & \cos(\omega_o \Delta T) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_N + \begin{bmatrix} w_{N+1} \\ 0 \end{bmatrix} \qquad (5)$$

where the state transition matrix may be viewed as a rotation transformation through $(\omega_o \Delta T)$ radians.

$$y_N = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_N + [v_N] \; ; \quad H_N = [1 \quad 0]$$

$$Q_N = \begin{bmatrix} \sigma_Q^2 & 0 \\ 0 & 0 \end{bmatrix} \; ; \qquad R_N = \sigma_R^2 \; .$$

III.4 Redundant Observation

The scalar measurement problem, III.1, is modified by the use of two instruments for redundant data-taking; the instruments have independent error statistics, with variances $\sigma_{R_1}^2$ and $\sigma_{R_2}^2$.

MODEL: $$X_{N+1} = X_N + W_{N+1} \quad ; \quad \Phi_{N;N+1} = 1 \qquad (6)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_N = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X_N + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_N \quad ; \quad H_N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E[V_i \; V_j^T] = \delta_{ij} \begin{bmatrix} \sigma_{R_1}^2 & 0 \\ 0 & \sigma_{R_2}^2 \end{bmatrix} = \delta_{ij} R_i$$

$$E[W_i \; W_j] = \delta_{ij} \, \sigma_Q^2 = \delta_{ij} Q_i$$

IV. DERIVATIONS AND INTERPRETATIONS OF THE KALMAN-BUCY OPTIMAL ESTIMATOR

We derive and discuss the results stated in Section II, from several viewpoints.

IV.1 As an Heuristic Use of Observable Error Signals

Given the estimated state vector $\hat{X}_N$, a plausible estimate of X_{N+1}, without further information, is $\phi_{N+1}\hat{X}_N$, since $E[W_{N+1}] = 0$ (see equation II.2); that is, we merely propagate the estimate as the system is known to propagate. If such a state were actually to develop, an expected observation (output) generated would be $\hat{Y}_{N+1} = H_{N+1}\phi_{N+1}\hat{X}_N$, since $E[V_{N+1}] = 0$ (see equation II.5). The facts that V_{N+1} and W_{N+1} will not in general vanish, and that $\hat{X}_N$ is itself not a correct estimate of X_N will be reflected in the actual observation, Y_{N+1}; the error in our estimate of X_{N+1} is therefore manifest in the error signal:

$$\varepsilon_{N+1} = (Y_{N+1} - \hat{Y}_{N+1}) = (Y_{N+1} - H_{N+1}\phi_{N+1}\hat{X}_N) \tag{1}$$

a vector quantity having the dimension (m) of the vector of observables, Y_{N+1}. We propose to utilize these m indicated error signals, in some linear combination, as a correction to the estimate $\phi_{N+1}\hat{X}_N$, to form $\hat{X}_{N+1}$:

$$\hat{X}_{N+1} = \Phi_{N+1}\hat{X}_N + C_{N+1}\{Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\} \tag{2}$$

representing both state (estimate) propagation and observation incorporation. C_{N+1} represents a "gain and mixing" matrix, as yet unspecified. Refer to Figure II.1.1 for a signal flow graph of this process.

Two questions arise at this point:

(i) Need the observable, Y_{N+1}, represent <u>any</u> information concerning the system state?

(ii) Is a <u>linear</u> combination of observations overly restrictive; i.e., would a non-linear estimate be "superior"?

Systems $(\{\Phi_N\},\{H_N\})$ for which the answer to (i) is either obviously "yes" or obviously "no" may readily be constructed. A full answer to this question is postponed until Section VII.4. The answer to (ii) is discussed in Section IV.3.

We now choose the value of C_{N+1} to minimize the objective function stated in SectionII--the trace of the covariance matrix for the error in the estimate $\hat{X}_{N+1}$:

$$\min_{C_{N+1}} [t_r[P_{N+1}]] = \min_{C_{N+1}} [t_r\{E(X_{N+1}-\hat{X}_{N+1})(X_{N+1}-\hat{X}_{N+1})^T\}] \tag{3}$$

$$= \min_{C_{N+1}} [E(X_{N+1}-\hat{X}_{N+1})^T(X_{N+1}-\hat{X}_{N+1})]$$

That is, we minimize the sum of the variances of the errors for the individual variables of X_{N+1} .

$$(\hat{X}_{N+1} - X_{N+1}) = (\Phi_{N+1}\hat{X}_N + C_{N+1}\{Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\}) - (X_{N+1}) \quad (4)$$

$$= (\Phi_{N+1}\hat{X}_N + C_{N+1}\{H_{N+1}X_{N+1} + V_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\})$$

$$- (\Phi_{N+1}X_N + W_{N+1})$$

$$= \Phi_{N+1}\hat{X}_N + C_{N+1}\{H_{N+1}\Phi_{N+1}X_N + H_{N+1}W_{N+1} + V_{N+1}$$

$$- H_{N+1}\Phi_{N+1}\hat{X}_N\} - \Phi_{N+1}X_N - W_{N+1}$$

$$= (I - C_{N+1}H_{N+1})\ \Phi_{N+1}\ (\hat{X}_N - X_N)$$

$$- (I - C_{N+1}H_{N+1})W_{N+1} + C_{N+1}V_{N+1}$$

$$= (I - C_{N+1}H_{N+1})\{\Phi_{N+1}(\hat{X}_N - X_N) - W_{N+1}\}$$

$$+ C_{N+1}V_{N+1}$$

Upon computing $(X_{N+1} - \hat{X}_{N+1})(X_{N+1} - \hat{X}_{N+1})^T$ and calculating expected values, we make use of the independence of V_{N+1}, W_{N+1}, X_N and $\hat{X}_N$, as implied by II.7, to arrive at

$$P_{N+1} = E[(X_{N+1} - \hat{X}_{N+1})(X_{N+1} - \hat{X}_{N+1})^T] \quad (5)$$

$$= (I - C_{N+1}H_{N+1})\big(\Phi_{N+1}\ E[(X_N - \hat{X}_N)(X_N - \hat{X}_N)^T]\ \Phi^T_{N+1}$$

$$+ E[W_{N+1}W^T_{N+1}]\big)\cdot(I - C_{N+1}H_{N+1})^T$$

$$+ C_{N+1}\ E[V_{N+1}V^T_{N+1}]C^T_{N+1}$$

$$= (I - C_{N+1}H_{N+1})(\Phi_{N+1}P_N\Phi^T_{N+1} + Q_{N+1})(I - C_{N+1}H_{N+1})^T$$

$$+ C_{N+1}R_{N+1}C^T_{N+1}$$

$$= (I - C_{N+1}H_{N+1})\ P'_{N+1}(I - C_{N+1}H_{N+1})^T + C_{N+1}R_{N+1}C^T_{N+1},$$

where

P'_{N+1} represents the propagated error covariance matrix P_N, based upon the elapse of one time stage in the absence (i.e., prior to the inclusion of) further observations

R_{N+1} represents the covariance matrix describing observation noise statistics at stage (N+1).

Q_{N+1} represents the covariance matrix describing system random forcing function statistics at stage (N+1).

Expanding equation (5),

$$\begin{aligned} P_{N+1} &= P'_{N+1} - C_{N+1}H_{N+1}P'_{N+1} - P'_{N+1}H^T_{N+1}C^T_{N+1} \\ &\quad + C_{N+1}H_{N+1}P'_{N+1}H^T_{N+1}C^T_{N+1} + C_{N+1}R_{N+1}C^T_{N+1} \\ &= (P'_{N+1}) - C_{N+1}(H_{N+1}P'_{N+1}) - (P'_{N+1}H^T_{N+1})C^T_{N+1} \\ &\quad + C_{N+1}(H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1})\, C^T_{N+1} \end{aligned} \tag{6}$$

In order to minimize the trace of the right side of (6) with respect to C_{N+1}, we rewrite it in factored form, using Lemma 1 of Appendix IX.1:

$$\begin{aligned} P_{N+1} &= P'_{N+1} + (C_{N+1} - P'_{N+1}H^T_{N+1}\{H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}\}^{-1}) \\ &\quad \times (H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1})(C_{N+1} - P'_{N+1}H^T_{N+1} \\ &\quad \times \{H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}\}^{-1})^T \\ &\quad - P'_{N+1}H^T_{N+1}\{H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}\}^{-1} H_{N+1}P'_{N+1}\ , \end{aligned} \tag{7}$$

analagous to a "completion of square." Inspection of the non-negative definite, symmetric matrices of (7) indicates a minimum trace is achieved by

$$C_{N+1} = P'_{N+1}H^T_{N+1}\{H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}\}^{-1}\ , \text{ so that} \tag{8}$$

$$P_{N+1} = P'_{N+1} - P'_{N+1}H^T_{N+1}\{H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}\}^{-1}H_{N+1}P'_{N+1}$$

$$= (I - C_{N+1}H_{N+1})P'_{N+1} \quad , \tag{9}$$

representing in product form the effect of the optimal linear utilization of the observation on the previously propagated covariance term, P'_{N+1}. We have arrived at equations II.8-11.

It is instructive to note that $(I - C_{N+1}H_{N+1})$ (or $[1 - c_{N+1}]$ in problem III.1; where $0 < c_{N+1} < 1$) acts as a contraction on the cone of non-negative definite matrices:

By equation IV.65,

$$C_{N+1} = P_{N+1}H^T_{N+1}R^{-1}_{N+1} \quad , \quad \text{so that}$$

$$P_{N+1} = (I - C_{N+1}H_{N+1})P'_{N+1}$$

$$= P'_{N+1} - [P_{N+1}(H^T_{N+1}R^{-1}_{N+1}H_{N+1})]P'_{N+1} \ .$$

Now P_{N+1} is non-negative definite, yet the bracketed symmetric matrix above is non-negative definite, so that

$$0 \leq P_{N+1} = (I - C_{N+1}H_{N+1})P'_{N+1} \leq P'_{N+1} \tag{10}$$

IV.2 <u>As a Conditional Expectation in a Gaussian Environment</u>

The approach taken in the more general problem of estimation for non-linear systems involves the determination of the conditional probability density function $p(x|Y)$, where x represents system state and Y the set of available observations. Given this function, one may proceed to calculate the <u>conditional</u> <u>expectation</u>:

$$E[x|Y] = \int_{x\text{-space}} x \ p(x|Y)dx \quad , \tag{11}$$

an estimate we shall show significant below. Bucy has stated that the wealth of published interpretations of estimation in the linear case which ignore this most general point of view has contributed to the lack of development of the non-linear theory. Following Sorenson [17], we illustrate the importance of the conditional distribution:

<u>Theorem IV.1</u>: The minimum-variance estimate of x is the <u>conditional</u> <u>expectation</u>

$$\hat{x} = E[x|Y] \quad (12)$$

<u>Proof</u>:

$$\mathcal{E}(\hat{x}) = E[(x - \hat{x})^T(x - \hat{x})]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\hat{x})^T(x-\hat{x})p(x,y_1,\ldots,y_m)dxdy_1 \ldots dy_m$$

where $y_1, y_2, \ldots, y_m$ is the set of observation scalars constituting Y. Now $p(x,y_1,\ldots,y_m) = p(x|y_1,\ldots,y_m)$ x $p(y_1,\ldots,y_m)$, so that

$$\mathcal{E}(\hat{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\int_{x=-\infty}^{\infty} (x-\hat{x})^T(x-\hat{x})p(x|y_1,\ldots,y_m)dx \right] \times p(y_1,\ldots,y_m)dy_1,\ldots,dy_m$$

$$= \int_{y_1=-\infty}^{\infty} \cdots \int_{y_m=-\infty}^{\infty} E[(x-\hat{x})^T(x-\hat{x})|Y]p(y_1,\ldots,y_m)dy_1, \ldots dy_m$$

Expanding the integrand, we have

$$\hat{x}^T\hat{x} - \hat{x}^T E[x|Y] - E[x^T|Y]\hat{x} + E[x^T x|Y] \tag{13}$$
$$= (\hat{x} - E[x|Y])^T(\hat{x} - E[x|Y]) + (E[x^T x|Y] - E[x^T|Y]E[x|Y])$$

The integrand being non-negative for all $x, Y, \hat{x}$, the right term of (13) being independent of the choice of $\hat{x}$, and the left term non-negative, we clearly minimize $\mathcal{E}(\hat{x})$ by choosing (12). Q.E.D.

The significance of this theorem is that no assumption on the type of distributions, the relation between x and Y, or the form of the candidate estimate was made. This explains the primacy of this approach for more general problems than those considered in this paper.

Given the system and observation models of equations II.1-7, we now assume that the probability density functions for the system random forcing function, $\{W_n\}$, the observation noise, $\{V_n\}$, and the previous estimate error, $(X_N - \hat{X}_N)$, are <u>Gaussian</u> (normal). We follow Ho [10]:

$$p_{W_{N+1}}(W) = \left[\frac{1}{(2\pi)^{n/2}|Q_{N+1}|^{1/2}}\right] \exp\{-\tfrac{1}{2} W^T Q_{N+1}^{-1} W\} \tag{14}$$

$$p_{V_{N+1}}(V) = \left[\frac{1}{(2\pi)^{m/2}|R_{N+1}|^{1/2}}\right] \exp\{-\tfrac{1}{2} V^T R_{N+1}^{-1} V\} \tag{15}$$

$$p_{\tilde{X}_N}(\tilde{X}) = \left[\frac{1}{(2\pi)^{n/2}|P_N|^{1/2}}\right] \exp\{-\tfrac{1}{2} \tilde{X}^T P_N^{-1} \tilde{X}\} \tag{16}$$

where $\tilde{X}_N = X_N - \hat{X}_N$, and $|\cdot|$ represents the determinant of a matrix.

Each assumed distribution is seen to have zero mean, and covariance matrix labeled as described in Section II. This is as previously required with one exception. The assumption that $E(\tilde{X}_N) = 0$ implies that $\hat{X}_N$ is an unbiased estimate of X_N:

<u>Definition IV.1</u>: $\hat{X}$, an estimate of X, is said to be <u>unbiased</u> if

$$E[\hat{X}] = E[X] \quad ; \tag{17}$$

$$\text{i.e.,} \quad E[\tilde{X}] = E[X - \hat{X}] = 0$$

$$\begin{aligned} \text{Since} \quad E[Y_{N+1}] &= E[H_{N+1}X_{N+1} + V_{N+1}] \qquad (18) \\ &= E[H_{N+1}(\Phi_{N+1}X_N + W_{N+1}) + V_{N+1}] \\ &= H_{N+1}\Phi_{N+1}\hat{X}_N \\ &= H_{N+1}\hat{X}'_N \qquad , \text{ and} \end{aligned}$$

$$\begin{aligned} E[(Y_{N+1} &- H_{N+1}\hat{X}'_N)(Y_{N+1} - H_{N+1}\hat{X}'_N)^T] \qquad (19) \\ &= E[(H_{N+1}[X_{N+1}-\hat{X}'_N]+V_{N+1})(H_{N+1}[X_{N+1}-\hat{X}'_N]+V_{N+1})^T] \\ &= E[(H_{N+1}\{\Phi_{N+1}\tilde{X}_N + W_{N+1}\}+V_{N+1}) \\ &\qquad \cdot(H_{N+1}\{\Phi_{N+1}\tilde{X}_N + W_{N+1}\}+V_{N+1})^T] \\ &= H_{N+1}\Phi_{N+1} \; E[\tilde{X}_N \; \tilde{X}_N^{\;T}]\Phi_{N+1}^T \; H_{N+1}^T \\ &\qquad + H_{N+1} \; E[W_{N+1} \; W_{N+1}^T] \; H_{N+1}^T \\ &\qquad + E[V_{N+1} \; V_{N+1}^T] \end{aligned}$$

$$= H_{N+1}(\Phi_{N+1}P_N\Phi_{N+1}^T + Q_{N+1})H_{N+1}^T + R_{N+1}$$
$$= H_{N+1}P_{N+1}'H_{N+1}^T + R_{N+1}$$

Since Y_{N+1}, a linear combination of Gaussian variables, is itself Gaussian we have

$$p_{Y_{N+1}}(Y) = \frac{1}{(2\pi)^{m/2}|H_{N+1}P_{N+1}'H_{N+1}^T + R_{N+1}|^{1/2}} \exp\{-\tfrac{1}{2}(Y-H_{N+1}\hat{X}_N')^T$$
$$\times\ (H_{N+1}P_{N+1}'H_{N+1}^T + R_{N+1})^{-1}(Y - H_{N+1}\hat{X}_N')\} \qquad (20)$$

Likewise, X_{N+1} is normally distributed, with

$$E[X_{N+1}] = E[\Phi_{N+1}X_N + W_{N+1}] = \Phi_{N+1}\hat{X}_N \qquad (21)$$

$$E[(X_{N+1} - \Phi_{N+1}\hat{X}_N)(X_{N+1}-\Phi_{N+1}\hat{X}_N)^T] = E[(\Phi_{N+1}\tilde{X}_N + W_{N+1}) \times (\Phi_{N+1}\tilde{X}_N + W_{N+1})^T] \qquad (22)$$
$$= \Phi_{N+1}P_N\Phi_{N+1} + Q_{N+1}$$
$$= P_{N+1}'$$

$$p_{X_{N+1}}(X) = \frac{1}{(2\pi)^{n/2}|P_{N+1}'|^{1/2})} \exp\{-\tfrac{1}{2}(X - \Phi_{N+1}\hat{X}_N)^T P_{N+1}'^{-1}(X-\Phi_{N+1}\hat{X}_N)\} \qquad (23)$$

We now seek the conditional density function which describes the statistical situation given the observation Y_{N+1}, using Bayes' rule:

$$p(X_{N+1}|Y_{N+1}) = \frac{p(Y_{N+1}, X_{N+1})}{p(Y_{N+1})} \qquad (24)$$
$$= \frac{p(Y_{N+1}|X_{N+1})p(X_{N+1})}{p(Y_{N+1})}$$

$$= \frac{p_{V_{N+1}}(Y_{N+1} - H_{N+1}X_{N+1})\ p_{X_{N+1}}(X_{N+1})}{p_{Y_{N+1}}(Y_{N+1})}$$

$$= \frac{|H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}|^{1/2}}{(2\pi)^{n/2}|P'_{N+1}|^{1/2}|R_{N+1}|^{1/2}} \cdot \exp \{$$

$$- \frac{1}{2} [(Y_{N+1} - H_{N+1}X_{N+1})^T R^{-1}_{N+1}(Y_{N+1} - H_{N+1}X_{N+1})$$

$$+ (X_{N+1}-\Phi_{N+1}\hat{X}_N)^T P'^{-1}_{N+1}(X_{N+1}-\Phi_{N+1}\hat{X}_N)$$

$$- (Y_{N+1}-H_{N+1}\hat{X}'_N)^T(H_{N+1}P'_{N+1}H_{N+1}+R_{N+1})^{-1}$$

$$\times (Y_{N+1} - H_{N+1}\hat{X}'_N)]\}$$

Lemma 1 of Appendix IX.1 may be used to factor the exponent of (24) into the form $(X_{N+1}-\hat{X}_{N+1})^T P^{-1}_{N+1}(X_{N+1}-\hat{X}_{N+1})$, where

$$A = (P'^{-1}_{N+1} + H^T_{N+1}R^{-1}_{N+1}H_{N+1}) \qquad (25)$$

$$B^T = (P'^{-1}_{N+1}\ \Phi_{N+1}\hat{X}'_N + H^T_{N+1}R^{-1}_{N+1}Y_{N+1})$$

$$C = (\hat{X}^T_N\Phi^T_{N+1}P'^{-1}_{N+1}\Phi_{N+1}\hat{X}_N + Y^T_{N+1}R^{-1}_{N+1}Y_{N+1})$$

$$- (Y_{N+1}-H_{N+1}\Phi_{N+1}\hat{X}_N)^T(H_{N+1}P'_{N+1}H_{N+1}+R_{N+1})^{-1}$$

$$\times (Y_{N+1}-H_{N+1}\Phi_{N+1}\hat{X}_N) ,$$

so that

$$p(X_{N+1}|Y_{N+1}) = \frac{|H_{N+1}P'_{N+1}H^T_{N+1} + R_{N+1}|^{1/2}}{(2\pi)^{n/2}|P'_{N+1}|^{1/2}|R_{N+1}|^{1/2}} \exp \{ \qquad (26)$$

$$- \frac{1}{2} (X_{N+1} - \hat{X}_{N+1})^T P_{N+1}^{-1} (X_{N+1} - \hat{X}_{N+1}) \} \quad ,$$

where, by Lemma 1, and the matrix inversion (Lemma 2) of Appendix IX.2,

$$\hat{X}_{N+1} = A^{-1} B^T \tag{27}$$

$$= (P'^{-1}_{N+1} + H^T_{N+1} R^{-1}_{N+1} H_{N+1})^{-1} (P'^{-1}_{N+1} \phi_{N+1} \hat{X}_N + H^T_{N+1} R^{-1}_{N+1} Y_{N+1})$$

$$= (P'_{N+1} - P'_{N+1} H^T_{N+1} \{H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1}\}^{-1} H_{N+1} P'_{N+1})$$

$$\times \; (P'^{-1}_{N+1} \phi_{N+1} \hat{X}_N + H^T_{N+1} R^{-1}_{N+1} Y_{N+1})$$

$$= \phi_{N+1} \hat{X}_N - P'_{N+1} H^T_{N+1} (H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1})^{-1} H_{N+1} \phi_{N+1} \hat{X}_N$$

$$+ \; P'_{N+1} H^T_{N+1} R^{-1}_{N+1} Y_{N+1} - P'_{N+1} H^T_{N+1} \{H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1}\}^{-1}$$

$$\times \; H_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} Y_{N+1}$$

$$= \phi_{N+1} \hat{X}_N - C_{N+1} H_{N+1} \phi_{N+1} \hat{X}_N + P'_{N+1} H^T_{N+1} R^{-1}_{N+1} Y_{N+1}$$

$$- \; P'_{N+1} H^T_{N+1} (H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1})^{-1}$$

$$\times \; (H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1} - R_{N+1}) R^{-1}_{N+1} Y_{N+1}$$

$$= \phi_{N+1} \hat{X}_N - C_{N+1} H_{N+1} \phi_{N+1} \hat{X}_N + P'_{N+1} H^T_{N+1}$$

$$\times \; (H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1})^{-1} Y_{N+1}$$

$$= \phi_{N+1} \hat{X}_N + C_{N+1} (Y_{N+1} - H_{N+1} \phi_{N+1} \hat{X}_N) \quad ,$$

where

$$C_{N+1} = P'_{N+1} H^T_{N+1} (H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1})^{-1} \tag{28}$$

Also, P_{N+1} is defined by

$$P_{N+1}^{-1} = A \tag{29}$$

$$= P_{N+1}^{'-1} + H_{N+1}^{T} R_{N+1}^{-1} H_{N+1}$$

so that Lemma 2 now implies

$$P_{N+1} = P_{N+1}' - P_{N+1}' H_{N+1}^{T} (H_{N+1} P_{N+1}' H_{N+1}^{T} + R_{N+1})^{-1} H_{N+1} P_{N+1}'$$

$$= P_{N+1}' - C_{N+1} H_{N+1} P_{N+1}' \tag{30}$$

$$= (I - C_{N+1} H_{N+1}) P_{N+1}'$$

Again, we have arrived at equations II.8-11.

The significant observation here is that the linear estimator of section IV.1 has reappeared as a solution to the conditional estimation problem, as optimal out of the class of all unbiased estimators. We summarize these results below.

IV.3 Summary of Optimality Extent

(i) In the class of all unbiased linear estimators, the Kalman filter is optimal under the minimum error variance criterion, with no assumptions on the nature of the distribution functions for the system environment.

(ii) In the class of all unbiased estimators, the Kalman filter is optimal under the minimum error variance criterion, provided the environmental statistics are Gaussian; that is, the optimal unbiased estimator is linear in a Gaussian environment.

Thus, the Kalman filter represents the optimal choice of unbiased estimator for all cases except the non-Gaussian, non-linear estimator case. We also note that the Kalman filter, providing the expected value of system state, provices the maximum likelihood estimate in the Gaussian case.

IV.4 As An Orthogonal Projection in Hilbert Space

Consider the Hilbert space H consisting of all random variables of zero mean and finite variance, and the subspace Y spanned by the observables $y(1), y(2), \ldots$. Note that the elements of H are random variables, and not particular outcomes. H is clearly closed under the natural addition and scalar multiplication; we define an inner product by

$$(x,y) = E(x,y) \qquad , \quad x,y \in H. \tag{31}$$

The properties of the expectation operator render $(\cdot,\cdot)$ a valid inner product.[1] The induced norm on H is thus derived from variance:

$$\|x\| = (x,x)^{1/2} = [E(x^2)]^{1/2} = [E(x-0)^2]^{1/2} \tag{32}$$

---

[1] To be completely correct, we must note that $\|x\| = 0$ does not imply $x = 0$, but only that $\{x|x \neq 0\}$ is a set of events of probability measure zero. We may then form the ideal $M=\{x|p(x\neq 0) = 0\}$, and the quotient space H/M, thence proceeding as above.

Consider the subspace $Y(N)$ spanned by $y(1)$, $y(2)$, $\ldots y(N)$. The following theorem was presented by Kalman [11], and stands in hypothesis analagous to the statements of Section IV.3:

Theorem IV.2: Let $\{x(n)\}$ and $\{y(n)\}$ be random processes with zero mean. We observe $y(1), y(2), \ldots, y(N)$. If either

(a) Candidate estimators are restricted to be linear functions of the observed random variables (i.e., elements of $Y(N)$), and the norm of loss is variance (i.e., the problem is cast naturally in H),

or

(b) The random process $\{x(n)\}$, $\{y(n)\}$ are Gaussian, and the criterion of optimality is variance of error,

then the minimum variance estimate of $x(n_o)$, given $y(1), \ldots, y(N)$, is the orthogonal projection of $x(n_o)$ on $Y(N)$.

Proof: (a) We seek an optimal estimate $\hat{x}(n_o)$ which lies in $Y(N)$ and which minimizes $\|x - \hat{x}\|^2 = (x - \hat{x}, x - \hat{x})$. It is an elementary property of Hilbert space that the orthogonal projection of $x(n_o)$ on $Y(N)$ is the element of $Y(N)$ closest to $x(n_o)$:

Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis for $Y(N)$. The orthogonal projection of $x = x(n_o)$ onto Y is then

$\hat{x} = \sum_{i=1}^{N} (x, e_i)e_i$; the error in $\hat{x}$ is $\tilde{x} = (x - \hat{x})$. Let $\hat{x}'$ be any point in $Y(N)$, serving as an alternative estimate.

Then

$$\|x - \hat{x}'\|^2 = \|(x - \hat{x}) + (\hat{x} - \hat{x}')\|^2 \tag{33}$$
$$= \|\tilde{x} - (\hat{x} - \hat{x}')\|^2$$
$$= \|\tilde{x}\|^2 + \|\hat{x} - \hat{x}'\|^2 \quad ,$$

the last line following from the generalized Pythagorean identity, which is equivalent to $(\hat{x} - \hat{x}') \perp \tilde{x}$.

From (33),

$$\|x - \hat{x}'\|^2 \geq \|\tilde{x}\|^2 = \|x - \hat{x}\|^2 \quad , \tag{34}$$

with equality holding only for $\hat{x}' = \hat{x}$.

(b) We must show that $\hat{x}(n_o)$, the orthogonal projection of $x(n_o)$ on $Y(N)$, is the conditional expectation, in order to invoke Theorem IV.1. If $\{x(n)\}$ and $\{y(n)\}$ are normal random processes with zero mean, then so also is $\hat{x}$, as a linear combination of such processes, and similarly for $\{\tilde{x}(n)\}$.

Then

$0 = E(\tilde{x}(n_o)) = E(\tilde{x}(n_o)|y(1),\ldots,y(N))$,

since $\tilde{x}(n_o)$ is orthogonal to $y(1)\ldots,y(N)$, and orthogonal normally distributed random variables of zero mean are independent. It follows that

$$E[x(n_o) - \hat{x}(n_o) | y(1),\ldots,y(N)] \tag{35}$$
$$= E[x(n_o)|y(1),\ldots,y(N)] - E[\hat{x}(n_o)|y(1),$$
$$\ldots,y(N)] = E[x(n_o)|Y(N)] - \hat{x}(n_o) = 0.$$

Q.E.D.

We are thus justified in considering the minimum-variance estimate of a normally distributed random variable, or the minimum-variance linear estimate of an arbitrary random variable, to be the orthogonal projection of that variable onto the subspace spanned by the observation processes; the symbol $E[x|Y(N)]$ will accordingly be interpreted as a projection, in some cases.

We further observe that Theorem IV.2 holds in the case of vector random processes by adaptation of the Euclidean vector inner product via expectation.

Turning to the derivation of the filter equations, we assume that the observations $y(1),\ldots, y(N)$, spanning $Y(N)$, have been made; the resulting estimate is assumed to be $\hat{x}(N) = E[x_N|Y(N)]$. $y(N+1)$ now becomes available, and we must compute

$$\hat{x}_{N+1} = E[X_{N+1}|Y(N+1)] \tag{36}$$

$$= E[X_{N+1}|Y(N)] + E[X_{N+1}|Z(N+1)] ,$$

where $Z(N+1)$ is that portion of $Y(N+1)$ orthogonal to $Y(N)$, and we have taken a projection viewpoint. (Observe that $Z(N+1) = \{0\}$ implies $\hat{X}_{N+1} = E[X_{N+1}|Y(N)] = \phi_{N+1}\hat{X}_N$, a pure extrapolation, since $Z(N+1) = \{0\}$ implies $y(N+1)$ lies completely in $Y(N)$ implies $E[y(N+1)|Y(N)] = y(N+1)$ implies no new information is contained in $y(N+1)$.)

Then

$$\hat{X}_{N+1} = E[\phi_{N+1}X_N + W_{N+1}|Y(N)] + E[X_{N+1}|Z(N+1)] \tag{37}$$

$$= \phi_{N+1}E[X_N|Y(N)] + E[W_{N+1}|Y(N)] + E[X_{N+1}|Z(N+1)]$$

$$= \phi_{N+1}\hat{X}_N + 0 + E[X_{N+1}|Z(N+1)] \quad ,$$

since W_{N+1} is independent of $Y(N)$. Now under the projection viewpoint, $Z(N+1)$ is spanned by the single vector

$$\tilde{y}_{N+1} = y_{N+1} - E[y_{N+1}|Y(N)] \tag{38}$$

$$= y_{N+1} - E[H_{N+1}X_{N+1} + V_{N+1}|Y(N)]$$

$$= y_{N+1} - E[H_{N+1}\phi_{N+1}X_N + H_{N+1}W_{N+1} + V_{N+1}|Y(N)]$$

$$= y_{N+1} - H_{N+1}\phi_{N+1}E[X_N|Y(N)] + 0$$

$$= y_{N+1} - H_{N+1}\phi_{N+1}\hat{X}_N \quad ,$$

since V_{N+1}, W_{N+1}, and $Y(N)$ are independent. Then by (37) and (38), $\hat{X}_{N+1}$ is of the form

$$\hat{X}_{N+1} = \phi_{N+1}\hat{X}_N + C_{N+1}(y_{N+1} - H_{N+1}\phi_{N+1}\hat{X}_N)$$

with C_{N+1} to be determined. Now X_{N+1}, less its projection onto $Z(N+1)$, is orthogonal to $Z(N+1)$:

$$\left(X_{N+1} - E[X_{N+1}|Z(N+1)]\right) \perp Z(N+1) \ . \tag{39}$$

Then

$$0 = ((X_{N+1} - E[X_{N+1}|Z(N+1)]),\ \tilde{y}_{N+1}) \tag{40}$$

$$= E[(X_{N+1} - E[X_{N+1}|Z(N+1)])^T \cdot (\tilde{y}_{N+1})]$$

$$= t_r\ E[(X_{N+1} - C_{N+1}\tilde{y}_{N+1})(\tilde{y}_{N+1})^T]$$

$$= t_r\ [E(X_{N+1}\ \tilde{y}^T_{N+1}) - C_{N+1}\ E(\tilde{y}_{N+1}\tilde{y}^T_{N+1})]$$

Using

$$\tilde{y}_{N+1} = y_{N+1} - E[y_{N+1}|Y(N)] \tag{41}$$

$$= H_{N+1}X_{N+1}+V_{N+1} - E[H_{N+1}X_{N+1}+V_{N+1}|Y(N)]$$

$$= H_{N+1}(\phi_{N+1}X_N+W_{N+1})+V_{N+1} - H_{N+1}\phi_{N+1}\ E[X_N|Y(N)]$$

$$= H_{N+1}\phi_{N+1}(X_N-\hat{X}_N) + H_{N+1}W_{N+1} + V_{N+1}$$

$$= H_{N+1}\phi_{N+1}\tilde{X}_n + H_{N+1}W_{N+1}+V_{N+1}\quad ,\qquad \text{we have}$$

from equation (40),

$$t_r[E(X_{N+1}\cdot\tilde{y}^T_{N+1}) - C_{N+1}(H_{N+1}\phi_{N+1}P_N\phi^T_{N+1}H^T_{N+1}+H_{N+1}Q_{N+1}H^T_{N+1}+R_{N+1})]$$

$$= 0\quad , \tag{42}$$

where

$$P_{N+1} = E[\tilde{X}_{N+1}\cdot\tilde{X}^T_{N+1}]$$

$$Q_{N+1} = E[W_{N+1}\ W^T_{N+1}]$$

$$R_{N+1} = E[V_{N+1}\ V^T_{N+1}]$$

A sufficient condition for the choice of C_{N+1} to be optimal is thus that the bracketed <u>matrix</u> in (42) be zero; further, by the uniqueness of the orthogonal projection, this condition is necessary. Then letting $P'_{N+1} = \phi_{N+1}P_N\phi^T_{N+1}+Q_{N+1}$, we have

$$E(X_{N+1} \cdot \tilde{y}_{N+1}^T) = C_{N+1}(H_{N+1}P'_{N+1}H_{N+1}^T + R_{N+1}) \text{ , or}$$

$$C_{N+1} = E(X_{N+1} \cdot \tilde{y}_{N+1}^T)(H_{N+1}P'_{N+1}H_{N+1}^T + R_{N+1})^{-1} \text{ ,} \tag{43}$$

since the right matrix is positive-definite. Computing the expectation, using equation (41),

$$\begin{aligned} E[X_{N+1} \cdot \tilde{y}_{N+1}^T] &= E[(X_{N+1})(H_{N+1}\phi_{N+1}\tilde{X}_N + H_{N+1}W_{N+1} + V_{N+1})^T] \qquad (44) \\ &= E[(\phi_{N+1}\hat{X}_N + \phi_{N+1}\tilde{X}_N + W_{N+1})(H_{N+1}\phi_{N+1}\tilde{X}_N + H_{N+1}W_{N+1} + V_{N+1})^T] \\ &= \phi_{N+1}P_N\phi_{N+1}^T H_{N+1}^T + Q_{N+1}H_{N+1}^T \\ &= P'_{N+1}H_{N+1}^T \quad , \end{aligned}$$

since $\hat{X}_N \perp \tilde{X}_N$. We arrive at the expected result:

$$C_{N+1} = P'_{N+1}H_{N+1}^T(H_{N+1}P'_{N+1}H_{N+1}^T + R_{N+1})^{-1} \text{ ;} \tag{45}$$

the remaining filtering equations follow as before.

IV. 5 Relationship to "Least Squares" Estimation

The (possible) choice of the statistical mean square norm on the space of random variables, as discussed in Section IV.4, and the resulting minimum expected mean square error viewpoint of the Kalman filter remind us of the filtering (curve fitting, parameter estimation) scheme referred to as "the method of least squares," as practiced by Gauss.

Within the theory of Kalman filtering, one may discover the traditional method of least squares, as a (more familiar) perspective from which to view the newer filtering equations;

conversely, the newer theory provides a more general setting from which to view the traditional curve-fitting ideas.

It is instructive to consider several schemes for processing observation data, which provide alternate viewpoints of the estimation process and illustrate methods for implementing the computational process in the most efficient manner:

(1) Batch Least Squares: The most familiar form of least squares estimation, in which a complete set ("batch") of observation data is processed at one time, resulting in estimates of selected parameters.

(2) Sequential Least Squares: A scheme providing the same (eventual) parameter estimates as the batch mode, but implemented in a recursive form to process observation data points sequentially, resulting in intermediate, sequentially updated parameter estimates. In particular, this scheme lends itself well to real-time processing of incoming data, to provide simultaneous parameter estimates.

(3) Single-Stage Kalman Filter: The application of the Kalman filter to estimation problems in which the dynamic (i.e., time-developing) nature of the system is either not present or implicitly represented in a one stage viewpoint. The single stage Kalman filter is closely related to batch least squares, and provides a solution to a more general problem

than that method.

(4) Multi-Stage Kalman Filter: The viewpoint represented in the material of this paper as discussed to this point, which may be related to sequential least squares methods.

(5) Stage-Sequential Kalman Filter: A method of practical computational significance to multi-stage processing for systems in which a single time stage involves the incorporation of dimensionally large quantities of observations; by segmenting each time stage into sequential sub-stages, certain practical computation difficulties are reduced. While this device does not relate to the least squares discussion, its consideration is nevertheless appropriate at this point.

We now discuss these topics in detail.

IV.5.1 Batch Least Squares Method

We assume the (static) model

$$Y = HX + \mathcal{E}, \quad \text{where} \tag{46}$$

Y = observation data vector ($m \times 1$).

X = state vector of parameters to be estimated ($n \times 1$).

H = observation matrix relating states to observables ($m \times n$).

$\mathcal{E}$ = vector of observation errors ($m \times 1$).

$R = E(\mathcal{E} \cdot \mathcal{E}^T) = \mathrm{cov}(\mathcal{E})$, ($m \times m$).

As in the case of the Kalman equations, the estimator is identical for two cases:

(i) Measurement Errors Distributed Normally

The maximum likelihood estimate of X, for the case in which ε is a Gaussian random variable of zero mean, is the estimate $\hat{X}$ which minimizes

$$M(\hat{X}) = (Y-H\hat{X})^T R^{-1}(Y-H\hat{X}) \tag{47}$$

Using Lemma 1 of Appendix IX.1,

$$\begin{aligned} M(\hat{X}) &= Y^T R^{-1} Y - \hat{X}^T(H^T R^{-1} Y) - (Y^T R^{-1} H)\hat{X} + \hat{X}^T(H^T R^{-1} H)\hat{X} \qquad (48) \\ &= (\hat{X} - (H^T R^{-1} H)^{-1} H^T R^{-1} Y)^T (H^T R^{-1} H)(\hat{X} - (H^T R^{-1} H)^{-1} \\ &\qquad \times H^T R^{-1} Y) + (Y^T R^{-1} Y - Y^T R^{-1} H[H^T R^{-1} H]^{-1} H^T R^{-1} Y) \end{aligned}$$

The first term of this quantity being positive semi-definite, and the second independent of $\hat{X}$, the optimal estimate becomes

$$\begin{aligned} \hat{X} &= KY \qquad (49) \\ &= [(H^T R^{-1} H)^{-1} H^T R^{-1}]Y \end{aligned}$$

this estimator is also the minimum variance state estimate in the Gaussian case:

$$\begin{aligned} \mathrm{cov}(\tilde{X}) &= E[(X-\hat{X})(X-\hat{X})^T] \qquad (50) \\ &= E[(X-KY)(X-KY)^T] \\ &= E[(X-(H^T R^{-1} H)^{-1} H^T R^{-1} HX - K\varepsilon)(X-(H^T R^{-1} H)^{-1} \\ &\qquad \times H^T R^{-1} HX - K\varepsilon)^T] \\ &= E[(X-X-K\varepsilon)(X-X-K\varepsilon)^T] \\ &= E[K\varepsilon \cdot \varepsilon^T K] \end{aligned}$$

$$= (H^TR^{-1}H)^{-1}H^TR^{-1}R\ R^{-1}H(H^TR^{-1}H)^{-1}$$
$$= (H^TR^{-1}H)^{-1}$$

(ii) Measurement Error Distribution Arbitrary; Minimum Variance Linear Estimator

We seek K to minimize

$$M(K) = E[\tilde{X}^T\cdot\tilde{X}] = \mathrm{tr}\, E[\tilde{X}\ \tilde{X}^T] \quad (51)$$
$$= \mathrm{tr}\, E[(X-KY)(X-KY)^T]$$
$$= \mathrm{tr}\, E[(X-KHX-K\varepsilon)(X-KHX-K\varepsilon)^T]$$
$$= \mathrm{tr}\{E[X\cdot X^T] + KH\cdot E[X\cdot X^T]H^TK^T - KH\cdot E[X\cdot X^T] - E[X\cdot X^T]H^TK^T + KRK^T\}$$
$$= \mathrm{tr}\{P + KHPH^TK^T - KHP - PH^TK^T + KRK^T\}$$
$$= \mathrm{tr}\{P + K(HPH^T+R)K^T - K(HP)-(PH^T)K^T\}$$
$$= \mathrm{tr}\{(K^T - (HPH^T+R)^{-1}HP)^T(HPH^T+R)(K^T-(HPH^T+R)^{-1}HP) + (P - PH^T(HPH^T+R)^{-1}HP)\}$$

where $P = E[X\ X^T] = \mathrm{cov}[X]$ represents initial uncertainty in knowledge of X_1 and we have invoked Lemma 1 of Appendix IX.1.

By the logic of Section IV.4, the optimal choice of K is

$$K = [(HPH^T+R)^{-1}HP]^T \quad (52)$$
$$= PH^T(HPH^T+R)^{-1}$$

It is interesting that this represents an optimal choice of estimator, in the minimum variance sense, given initial knowledge (prior to observing Y) of X, as measured by P. The optimal variance becomes

$$M(K) = \mathrm{tr}[P - PH^T(HPH^T+R)^{-1}HP] \tag{53}$$
$$= \mathrm{tr}[(P^{-1}+H^TR^{-1}H)^{-1}] ,$$

where we have utilized Lemma 2 of Appendix IX.2. In the case of interest, no a priori knowledge of X is available, so that $P^{-1} \longrightarrow 0$, and so

$$M(K) \longrightarrow \mathrm{tr}[(H^TR^{-1}H)^{-1}] , \quad \text{or} \tag{54}$$

$E[\tilde{X}\tilde{X}^T] = (H^TR^{-1}H)^{-1}$, as in equation (50).

We may solve for the optimal value of K, including the $|P| \rightarrow \infty$ case (as opposed to equation (52)) by observing that the above equations may be viewed as Hilbert space (inner product) arguments, and that the optimal estimate is characterized by

$$\begin{aligned} 0 &= (\hat{X},\tilde{X}) \qquad (55)\\ &= (KY,\tilde{X}) \\ &= (KH[\hat{X} + \tilde{X}] + K\varepsilon,\tilde{X}) \\ &= (KH\hat{X},\tilde{X})+(KH\tilde{X},\tilde{X}) + (K\varepsilon ,\tilde{X}) \\ &= \mathrm{tr}\ E[KH\hat{X}\cdot\tilde{X}^T] + \mathrm{tr}\ E[KH\tilde{X}\cdot\tilde{X}^T] + \mathrm{tr}\ E[K\varepsilon\cdot\tilde{X}^T] \\ &= \mathrm{tr}\{KH0 + KHM + E[K\varepsilon(X-KHX-K\varepsilon)^T]\} \\ &= \mathrm{tr}\{KHM - KRK^T\} \\ &= 0 , \end{aligned}$$

where $M = E[\tilde{X}\tilde{X}^T]$.

By the uniqueness of the optimal estimate the optimal K is given by

$$\begin{aligned} HM - RK^T &= 0 \\ K^T &= R^{-1}HM \end{aligned} \tag{56}$$

$$K = M^T H^T R^{-1}$$
$$= [P^{-1} + H^T R^{-1} H]^{-1} H^T R^{-1}$$

where we have used (53). In the case of interest,

$$K \longrightarrow (H^T R^{-1} H)^{-1} H^T R^{-1} \quad , \tag{57}$$

as found in equation (49).

IV.5.2 Sequential Least Squares Method

We assume the "growing" model

$$Y_N = H_N X + \varepsilon_N \quad , \quad \text{where} \tag{58}$$

Y_N = vector of observation data (N x 1) accumulated from stage 1 to N; Y_N increases in dimension with time.

X = state vector of parameters to be estimated (n x 1).

H_N = observation matrix relating states to observables (N x n),

ε_N = vector of observation errors (N x 1).

$R_N = E(\varepsilon_N \cdot \varepsilon_N^T) = \mathrm{cov}(\varepsilon_N)$ (N x N).

Considering the results of Section IV.5.1, the optimal estimate is

$$\hat{X}_N = (H^T R_N^{-1} H_N)^{-1} H_N^T R_N^{-1} Y_N \tag{59}$$

where $\hat{X}_N$ represents the estimate of X at stage N, and we have momentarily taken the view that the entire problem is solved in the "batch" mode at the inclusion of each new observation. We now seek a description of the stage-to-stage progression of the solution to these batch problems. At stage (N+1),

$$\hat{X}_{N+1} = \left(H_{N+1}^T R_{N+1}^{-1} H_{N+1}\right)^{-1} H_{N+1}^T R_{N+1}^{-1} Y_{N+1} \tag{60}$$

$$= \left(\begin{bmatrix} H_N \\ \hline h_{N+1} \end{bmatrix}^T \begin{bmatrix} R_N & \Big| \; 0 \\ \hline 0 & \Big| \; \sigma_{N+1}^2 \end{bmatrix}^{-1} \begin{bmatrix} H_N \\ \hline h_{N+1} \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} H_N \\ \hline h_{N+1} \end{bmatrix}^T \cdot \begin{bmatrix} R_N & \Big| \; 0 \\ \hline 0 & \Big| \; \sigma_{N+1}^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} Y_N \\ \hline y_{N+1} \end{bmatrix}$$

$$= \left(H_N^T R_N^{-1} H_N + \frac{h_{N+1}^T h_{N+1}}{\sigma_{N+1}^2} \right)^{-1} \cdot \left(H_N^T R_N^{-1} Y_N + \frac{h_{N+1}^T y_{N+1}}{\sigma_{N+1}^2} \right)$$

$$= \left(H_N^T R_N^{-1} H_N + \frac{h_{N+1}^T h_{N+1}}{\sigma_{N+1}^2} \right)^{-1} \cdot \left\{ \left(H_N^T R_N^{-1} H_N\right) \hat{X}_N + \frac{h_{N+1}^T h_{N+1}}{\sigma_{N+1}^2} \hat{X}_N - \frac{h_{N+1}^T h_{N+1}}{\sigma_{N+1}^2} \hat{X}_N + \frac{h_{N+1}^T}{\sigma_{N+1}^2} y_{N+1} \right\}$$

$$= \hat{X}_N + \left(H_N^T R_N^{-1} H_N + \frac{h_{N+1}^T h_{N+1}}{\sigma_{N+1}^2} \right)^{-1} \cdot \left(\frac{h_{N+1}^T}{\sigma_{N+1}^2} \right) \cdot \left(y_{N+1} - h_{N+1} \hat{X}_N \right),$$

where we have partitioned H_{N+1}, R_{N+1}, and Y_{N+1}, according to the addition of the new observation, and have restricted ourselves to the case of statistically independent observation errors (i.e., R_N diagonal). Then

$$\hat{X}_{N+1} = \hat{X}_N + P_{N+1} \left(\frac{h_{N+1}^T}{\sigma_{N+1}^2} \right) \left(y_{N+1} - h_{N+1} \hat{X}_N \right), \tag{61}$$

where

$$P_{N+1} = \left(H_{N+1}^T R_{N+1}^{-1} H_{N+1} \right)^{-1} \tag{62}$$

$$= \left(H_N^T R_N^{-1} H_N + \frac{h_{N+1}^T h_{N+1}}{\sigma_{N+1}^2} \right)^{-1}$$

$$= \left(H_N^T R_N^{-1} H_N \right)^{-1} - \left(H_N^T R_N^{-1} H_N \right)^{-1} h_{N+1}^T \left\{ \sigma_{N+1}^2 + h_{N+1} \left(H_N^T R_N^{-1} H_N \right)^{-1} h_{N+1}^T \right\}^{-1} \cdot h_{N+1} \left(H_N^T R_N^{-1} H_N \right)^{-1}$$

$$= P_N - P_N h_{N+1}^T \left\{ h_{N+1} P_N h_{N+1}^T + \sigma_{N+1}^2 \right\}^{-1} h_{N+1} P_N$$

form the recursive equations of the least-squares estimation procedure; we have utilized Lemma 2 of Appendix IX.2. This computational scheme has the advantages of (1) inversion of a 1 x 1 (scalar) matrix at each iteration, and (2) the availability of real-time estimates of X, based upon observations to date. Our principal concern here is with the <u>form</u> of these equations. Since, from Section IV.5.1, $P_N = E[\tilde{X}_N \tilde{X}_N^T]$, we see that the recursive least squares estimator is a special case of the Kalman-Bucy Filter, (equations II.8-11), for the case in which the time-developing nature of the system state is removed (i.e., $\phi = I$), and system randomness is not present (i.e., $Q_N = 0$, so that $P'_{N+1} = \phi_{N+1} P_N \phi^T + Q = P_N$).

IV.5.3 Single-Stage Kalman Filter

We now derive a form of the Kalman filter which is appropriate to problems in which the explicit dynamic aspect of the system is removed; this form provides a specialized estimator for one-stage problems and demonstrates a relationship to batch least-squares estimation.

Using Lemma 2 of Appendix IX.2, we may write the Kalman filter equations (II.9-11) in an alternate form:

$$P_{N+1} = P'_{N+1} - P'_{N+1} H^T_{N+1} \left[H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1} \right]^{-1} H_{N+1} P'_{N+1} \tag{63}$$

$$P^{-1}_{N+1} = P'^{-1}_{N+1} + H^T_{N+1} R^{-1}_{N+1} H_{N+1} \tag{64}$$

$$C_{N+1} = P'_{N+1} H^T_{N+1} \left[H_{N+1} P'_{N+1} H^T_{N+1} + R_{N+1} \right]^{-1} \tag{65}$$

$$= P_{N+1} P^{-1}_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} \left[H_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} + I \right]^{-1}$$

$$= P_{N+1} \left[P'^{-1}_{N+1} + H^T_{N+1} R^{-1}_{N+1} H_{N+1} \right] P'_{N+1} H^T_{N+1} R^{-1}_{N+1}$$

$$\times \left[H_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} + I \right]^{-1}$$

$$= P_{N+1} \left[I + H^T_{N+1} R^{-1}_{N+1} H_{N+1} P'_{N+1} \right] H^T_{N+1} R^{-1}_{N+1}$$

$$\times \left[H_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} + I \right]^{-1}$$

$$= P_{N+1} H^T_{N+1} R^{-1}_{N+1} \left[I + H_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} \right]$$

$$\times \left[H_{N+1} P'_{N+1} H^T_{N+1} R^{-1}_{N+1} + I \right]^{-1}$$

$$= P_{N+1} H_{N+1}^{T} R_{N+1}^{-1}$$

(We have <u>not</u> commuted products here.)

Thus

$$C_{N+1} = P_{N+1} H_{N+1}^{T} R_{N+1}^{-1}$$

$$= \left(P_{N+1}^{\prime -1} + H_{N+1}^{T} R_{N+1} H_{N+1}\right)^{-1} H_{N+1}^{T} R_{N+1}^{-1}$$

The single-stage Kalman filter becomes

$$\hat{X}_1 = \hat{X}_0 + C_1 \left[Y_1 - H_1 \hat{X}_0 \right] \qquad (66)$$

$$C_1 = \left(P_1^{\prime -1} + H_1^T R_1^{-1} H_1\right)^{-1} H_1^T R_1^{-1}$$

$$P_1' = P_0 + Q_1$$

We have utilized the inverse form to allow us to once again observe that, for the total initial uncertainty case, with zero system randomness,

$$Q_1 = 0 \qquad (67)$$

$$P_1^{\prime -1} \to 0$$

$$C_1 \to \left(H_1^T R_1^{-1} H_1\right)^{-1} H_1^T R_1^{-1}$$

$$P_1 \to \left(H_1^T R_1^{-1} H_1\right)^{-1} ,$$

the conventional least-squares estimator result. This should not be surprising; the problems solved by the least squares estimator and Kalman filter are identical for the case described.

We summarize by observing that the discrete Kalman filter may be viewed as the solution to the conventional least-squares problem extended to include explicit system dynamics and randomness (Φ, Q).

IV.5.4 Multi-Stage Kalman Filter

The multi-stage Kalman Filter is the estimator discussed as the principal topic of this paper, to this point.

IV.5.5 Stage-Sequential Kalman Filter

The stage-sequential form provides a means of incorporating dimensionally large (we refer to Y) observations in the filtering equations II.8-11, without the computationally expensive inversion of the observation-dimension matrix of equation II.9. A single time stage is divided into an arbitrary number of pseudo-stages, for which the transition matrix is temporarily equal to the identity matrix. At each pseudo-stage, a single observation or set of observations from within the vector Y_{N+1} is incorporated, by appropriate adjustments to H and R. In order that this scheme represent a time saving, the computation of extra stages must be accomplished more rapidly than the matrix inversion.

V. EXAMPLE SYSTEM COMPUTER SIMULATIONS

We return to the example systems of Section III, the solutions (estimators) stated in that section having been derived in Section IV. Digital computer simulation of the examples was carried out by computer programs playing the roles of

(1) The time-discretized, deterministic portion of the linear system (H,Φ).

(2) The stochastic effects present in the system forcing function $(\{W_n\})$ and the observation model $(\{V_n\})$.

(3) The discrete estimator, including both the time-dependent filter equation (II.8) and the propagation of estimator covariance and coefficients (II.9-11).

By generating (1) and (2) within the computer, we have the luxury of observing the unobservable; that is, the system state vector is available in uncorrupted form (although not, of course, to the estimator) for our comparison to the estimated state, as produced by the filter. This is certainly not to be the case in an actual application of the estimator, but allows us to study the overall behavior in a meaningful way, prior to an application. To avoid confusion, we reiterate the fact that the estimator includes the stage-by-stage calculation of expected error variance, thus maintaining a measure of its own certainty.

The computer program exercises equations and makes available the relevant variables in printed and graphic form. A detailed description will be found in Appendix IX.4; we only add here that the program is designed to accept a description of the system in the general form of equations II.1-6, by the user's specification of the matrices Φ, H, Q, R, and P(0), which may be stationary or time-dependent.

The reader is referred to Section III for review of the example systems.

V.1 Scalar State (refer to Section III.1)

MODEL:

$$x_{N+1} = x_N + w_{N+1}$$

$$y_N = x_N + v_N$$

$$\mathrm{var}[w_N] = \sigma_Q^2 = Q$$

$$\mathrm{var}[v_N] = \sigma_R^2 = R$$

ESTIMATOR:

$$\hat{x}_{N+1} = \hat{x}_N + c_{N+1}[y_{N+1} - \hat{x}_N]$$

$$c_{N+1} = \frac{\sigma_N^2 + \sigma_Q^2}{\sigma_N^2 + \sigma_Q^2 + \sigma_R^2},$$

$$\sigma_{N+1}^2 = (1 - c_{N+1})(\sigma_N^2 + \sigma_Q^2)$$

$$= \frac{\sigma_R^2(\sigma_N^2 + \sigma_Q^2)}{\sigma_N^2 + \sigma_Q^2 + \sigma_R^2}$$

SPECIAL CASE: $\sigma_Q^2 = 0 \Rightarrow$

$$c_{N+1} = \frac{\sigma_0^2}{(N+1)\sigma_0^2 + \sigma_R^2}, \qquad (1)$$

$$\sigma_{N+1}^2 = \frac{\sigma_0^2 \sigma_R^2}{(N+1)\sigma_0^2 + \sigma_R^2}.$$

Figure V.1.1 illustrates both system and estimator behavior for the case

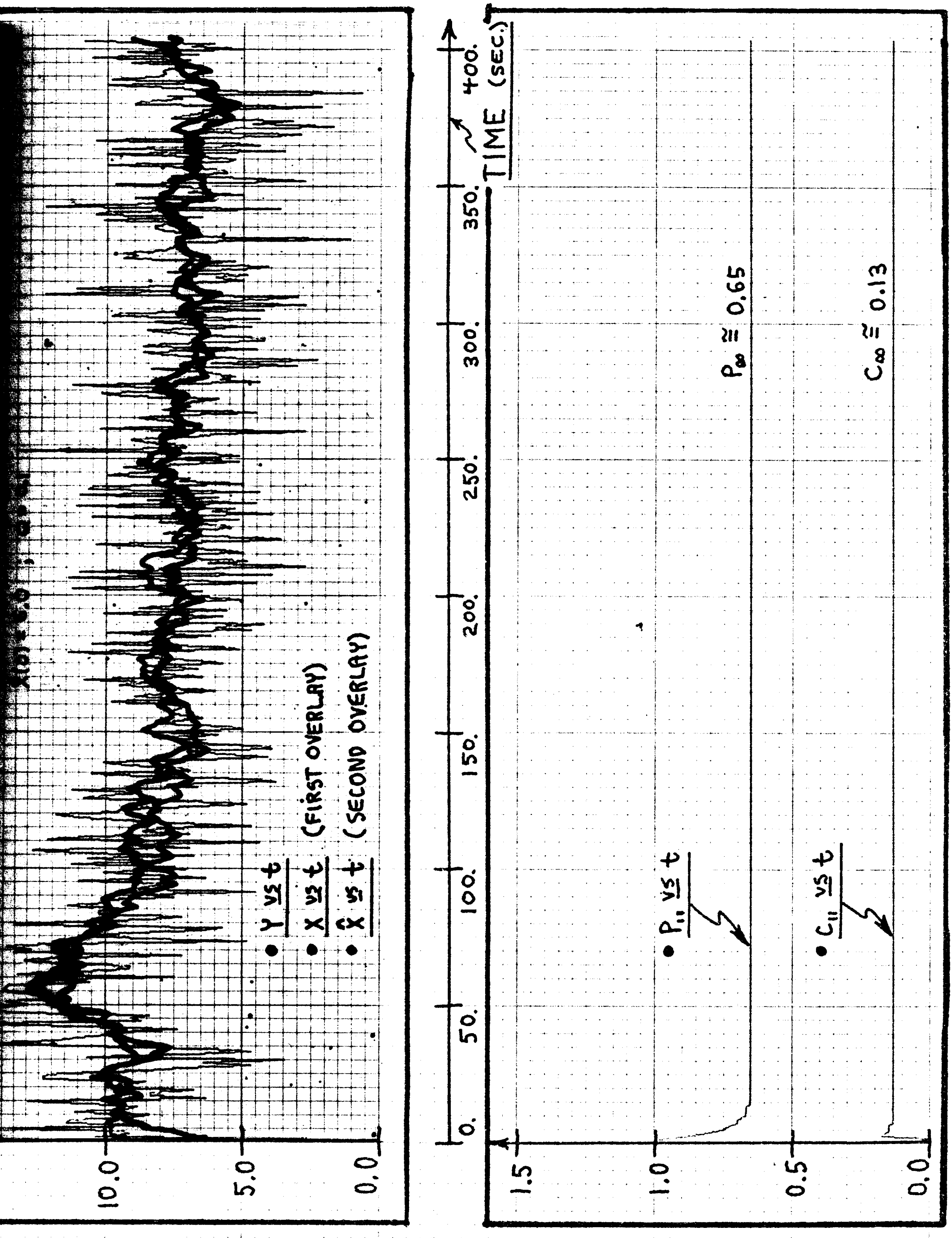
$\sigma_R^2 = 5.0$ $\qquad x(0) = 9.0$

$\sigma_Q^2 = 0.1$ $\qquad \hat{x}(0) = 6.0$

$\sigma_0^2 = 1.0$

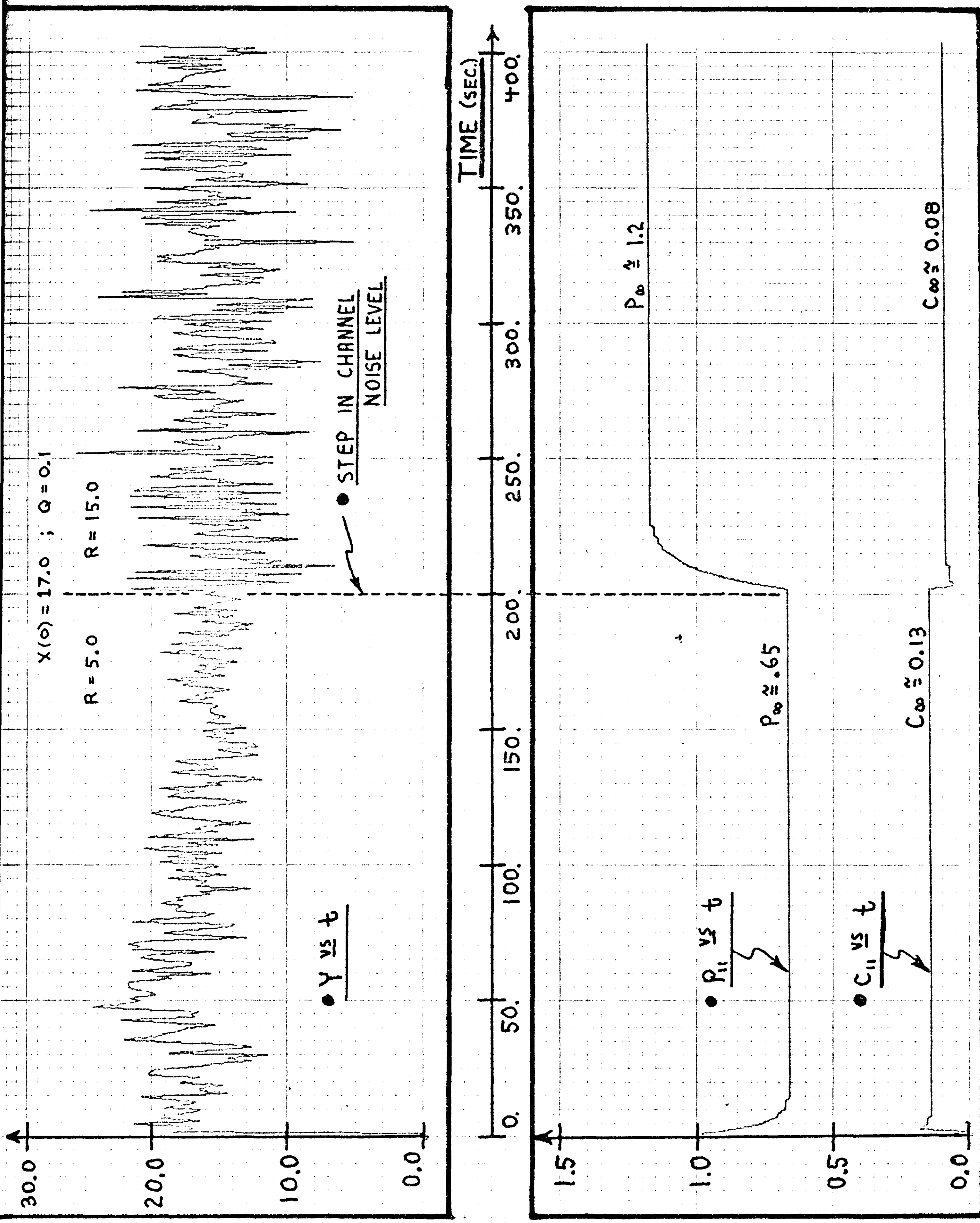
It can be seen that the covariance/coefficient system (the filter) reaches a steady state; that is the filter becomes stationary in the limit. In general, the Kalman-Bucy filter is time-dependent. However, we shall show in Section VII that, under a wide variety of models, a stationary environment (R , Q) leads to a limiting stationary estimator which is independent of the initial uncertainty, P_0 (see also Figure V.1.4). This feature may be advantageous under con-

FIGURE V.1.1



• Y vs t
• X vs t (FIRST OVERLAY)
• X̂ vs t (SECOND OVERLAY)
10.0
5.0
0.0
0.
50.
100.
150.
200.
250.
300.
350.
400.
TIME (SEC.)
• P₁₁ vs t
• C₁₁ vs t
P∞ ≅ 0.65
C∞ ≅ 0.13
1.5
1.0
0.5
0.0

FIGURE V.1.2


X(0) = 17.0 ; Q = 0.1
R = 5.0
R = 15.0
STEP IN CHANNEL NOISE LEVEL
Y vs t
30.0
20.0
10.0
0.0
TIME (SEC.)
0.
50.
100.
150.
200.
250.
300.
350.
400.
P_11 vs t
C_11 vs t
P∞ ≅ .65
P∞ ≅ 1.2
C∞ ≅ 0.13
C∞ ≅ 0.08
1.5
1.0
0.5
0.0

ditions of tight computer time and memory constraints, wherein one may utilize the constant limiting coefficients, eliminating equations II.9-11 from the computer load, at the cost of transient suboptimality. We also note that the limiting filter provides a solution to the Wiener problem, which seeks a stationary, infinite-memory filter.

The power of the K-B filter rests in part in its ability to handle non-stationary statistical environments. Figure V.1.2 illustrates filter response in the presence of a step in observation noise power-- σ_R^2 . Observe that the filter "closes down" at this step, each new estimate being optimal under the current statistical environment $(\sigma_{R_n}^2, \sigma_{Q_n}^2)$. <u>It is important to note that we have supplied the filter with the description of the statistical environment $(\sigma_{R_n}^2, \sigma_{Q_n}^2)$, both level and step, throughout the run</u>. Knowledge of the statistical environment, in the form of specification of the sequences $\{\sigma_{R_n}^2\}$ and $\{\sigma_{Q_n}^2\}$, is a fundamental prerequisite for Kalman filtering. Such knowledge is often not available or precise, but approximate figures make near-optimal filtering feasible in many cases. When no data on environmental statistics are available, the K-B filter is reduced to an elegant statement of what the solution <u>should</u> be, but falls short of a solution which leads to the synthesis of a filter. In Section VIII we shall be concerned with methods for utilizing the Kalman-Bucy filter in the absence of <u>a priori</u> statistical information.

FIGURE V.1.3

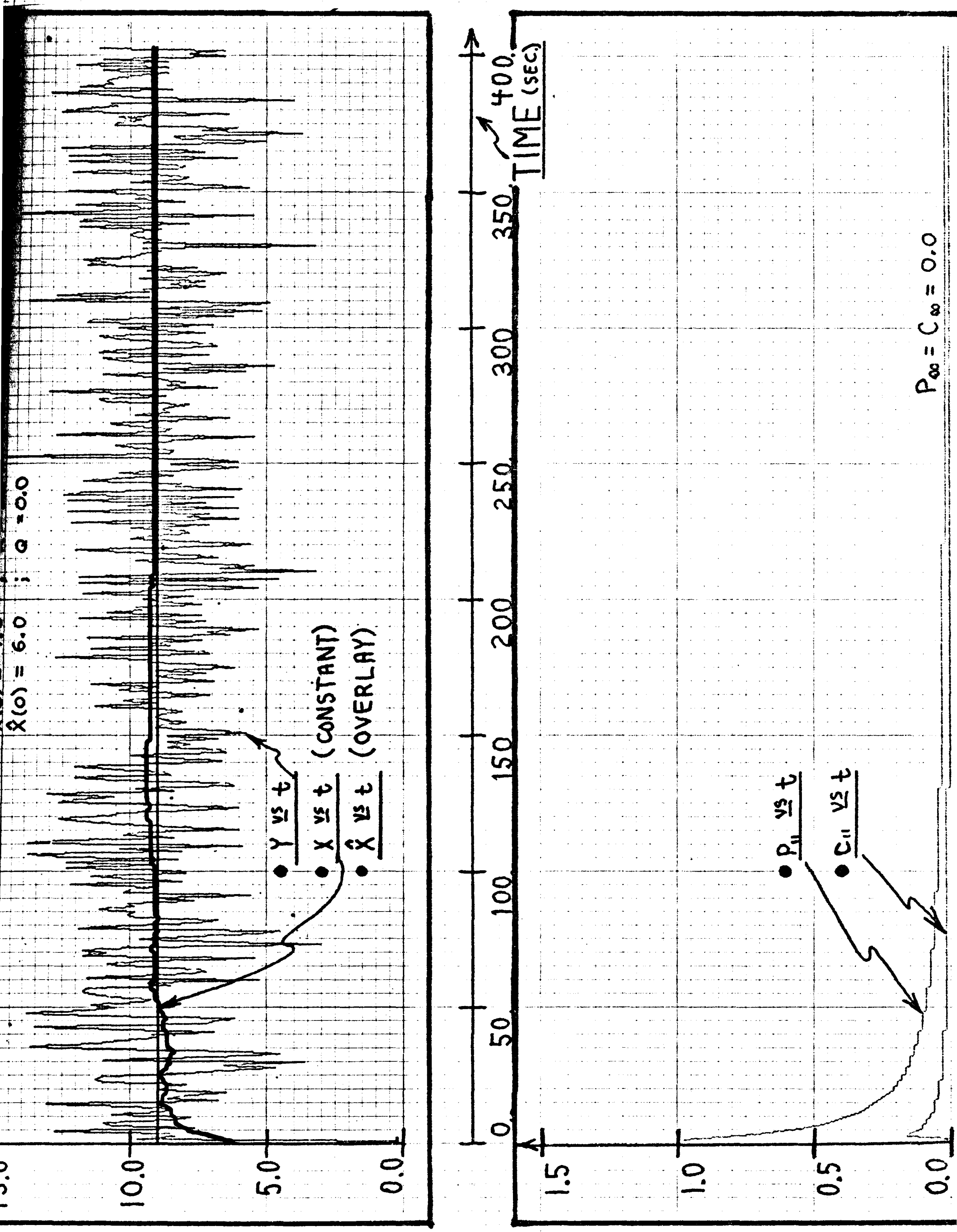
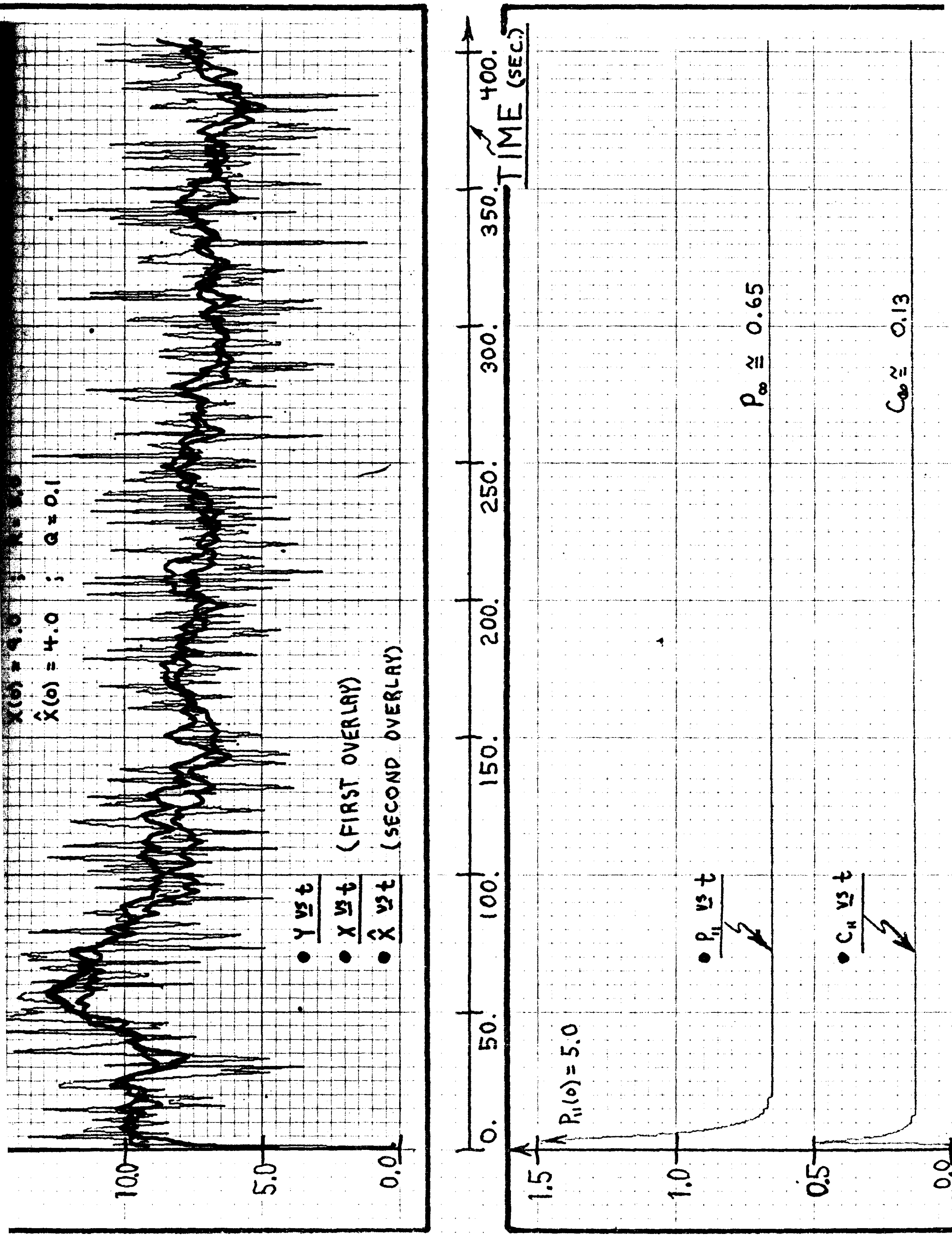

X̂(0) = 6.0 ; Q = 0.0
• Y vs t
• X vs t (CONSTANT)
• X̂ vs t (OVERLAY)
10.0
5.0
0.0
0
50
100
150
200
250
300
350
400.
TIME (SEC)
• P11 vs t
• C11 vs t
P∞ = C∞ = 0.0
1.5
1.0
0.5
0.0

FIGURE V.1.4



$\hat{X}(0) = 4.0$; $Q = 0.1$
• Y vs t
• X vs t (FIRST OVERLAY)
• $\hat{X}$ vs t (SECOND OVERLAY)
10.0
5.0
0.0
0.
50.
100.
150.
200.
250.
300.
350.
400.
TIME (SEC.)
$P_{11}(0) = 5.0$
• P_{11} vs t
• C_{11} vs t
$P_\infty \cong 0.65$
$C_\infty \cong 0.13$
1.5
1.0
0.5
0.0

In the special case of an unforced, deterministic system ($\sigma_Q^2 = 0$), with observations still corrupted ($\sigma_R^2 > 0$), continuing observation of a <u>predictably</u> propagating state leads in the limit to complete knowledge of state (i.e., $P_n \rightarrow 0$; see equations (1)). The appropriate filter relies increasingly on the propagation of $\hat{X}_N$, through Φ, and decreasingly on incoming information, Y_N. Figure V.1.3 illustrates this situation. Note that $C_N \rightarrow 0$ and $P_N \rightarrow 0$. The steady-state form of the estimator is thus

$$\hat{X}_{N+1} = \Phi \hat{X}_N$$

$$P_{N+1} = 0$$

A completely unforced system is an idealization which is never completely correct in physical systems. Pure propagation simply reflects our unwillingness to model microscopic effects which play a small part in the problem. Unfortunately this idealization, representing a "lie" to the filter by the modeller, leads to difficulties. This topic is discussed further in Section VI.3.

V.2 <u>Rate Estimation</u> (refer to Section III.2):

<u>MODEL</u>:

$$X_{N+1} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{N+1} = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_N + \begin{bmatrix} 0 \\ \omega \end{bmatrix}_{N+1} ,$$

$$\Delta T = 0.1 \text{ SEC.},$$

$$y_N = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_N + V_N \quad ,$$

$$\mathrm{var}[W_N] = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_Q^2 \end{bmatrix} = Q = \begin{bmatrix} 0. & 0. \\ 0. & 0.02 \end{bmatrix} \quad ,$$

$$\mathrm{var}[V_N] = \sigma_R^2 = R = 8.0$$

ESTIMATOR:

$$\hat{X}_{N+1} = \begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix}_{N+1} = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix}_N + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{N+1} \left\{ y_{N+1} - \hat{x}_N - \hat{\dot{x}}_N \Delta T \right\}$$

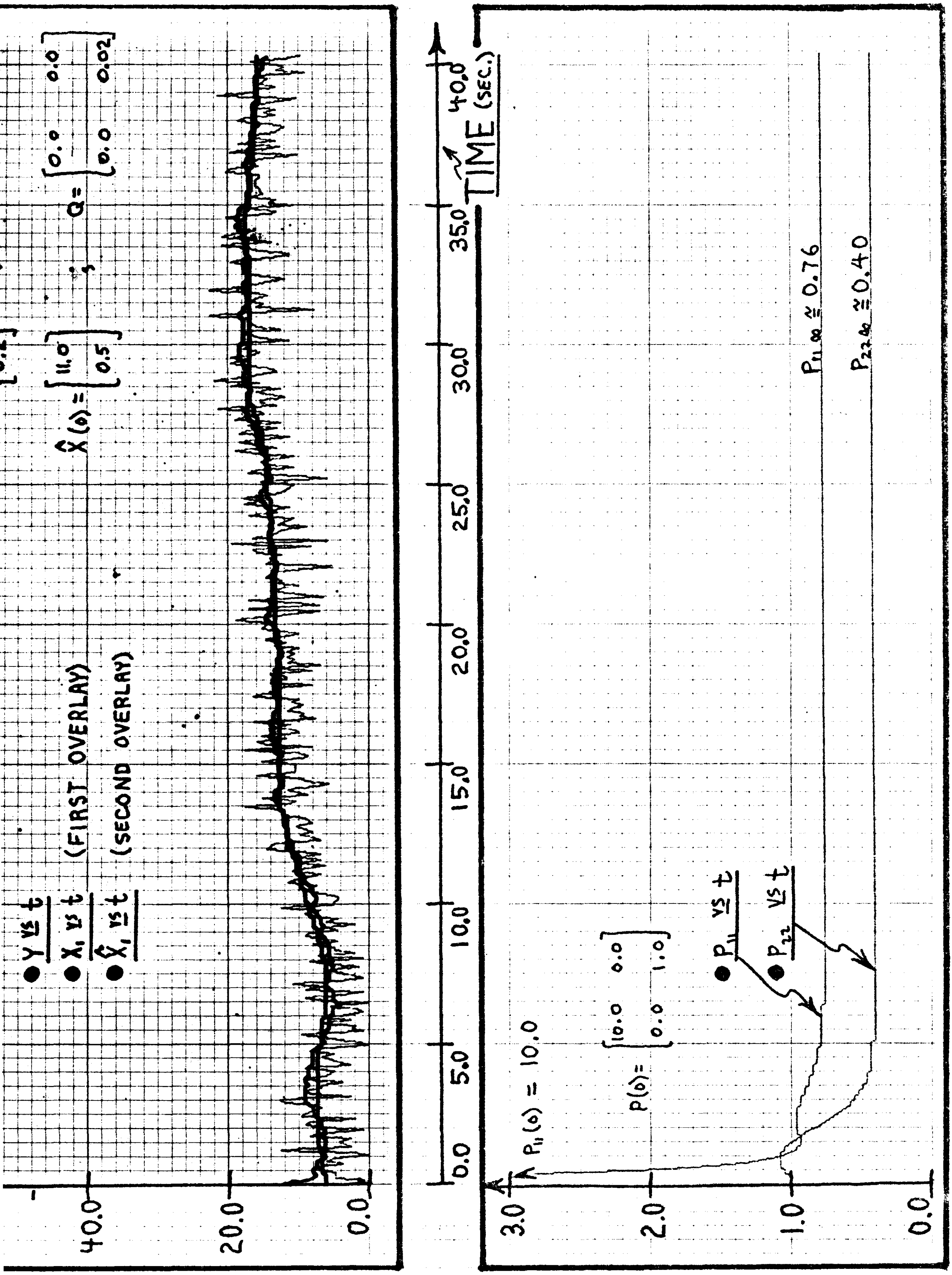
$$C_{N+1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{N+1} = \left[\Phi P_N \Phi^T + Q\right] H^T \left[H \Phi P_N \Phi^T H^T + H Q H^T + \sigma_R^2 \right]^{-1}$$

$$= \left(\frac{1}{p_{11} + 2 p_{12} \Delta T + p_{22} \Delta T^2 + \sigma_R^2} \right) \begin{bmatrix} p_{11} + 2 p_{21} \Delta T + p_{22} \Delta T^2 \\ p_{12} + p_{22} \Delta T \end{bmatrix}$$

$$P_{N+1} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}_{N+1} = \left(I - C_{N+1} H \right) \left(\Phi P_N \Phi^T + Q \right)$$

Figure V.2.1 illustrates the estimation of (x) and $(\dot{x})$.

FIGURE V.2.1


● Y vs t
● X₁ vs t (FIRST OVERLAY)
● X̂₁ vs t (SECOND OVERLAY)
X̂(0) = [11.0 0.5]
Q = [0.0 0.0 / 0.0 0.02]
40.0
20.0
0.0
0.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0
40.0
TIME (SEC.)
● P11 vs t
● P22 vs t
P(0) = [10.0 0.0 / 0.0 1.0]
P11(0) = 10.0
P11∞ ≅ 0.76
P22∞ ≅ 0.40
3.0
2.0
1.0
0.0

V.3 Harmonic Oscillator (refer to Section III.3):

MODEL:

$$X_{N+1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{N+1} = \begin{bmatrix} \cos(\omega_0 \Delta T) & \sin(\omega_0 \Delta T) \\ -\sin(\omega_0 \Delta T) & \cos(\omega_0 \Delta T) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_N + \begin{bmatrix} \omega_{N+1} \\ 0 \end{bmatrix} ,$$

$$y_{N+1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{N+1} + V_{N+1} \quad ; (\omega_0 \Delta T) = \frac{\pi}{20} \text{ rad.} ,$$

$$\text{var}[W_N] = \begin{bmatrix} \sigma_Q^2 & 0 \\ 0 & 0 \end{bmatrix} = Q = \begin{bmatrix} 0.1 & 0. \\ 0. & 0. \end{bmatrix} ,$$

$$\text{var}[V_N] = \sigma_R^2 = R = 1.0$$

ESTIMATOR:

$$\hat{X}_{N+1} = \Phi \hat{X}_N + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{N+1} \cdot \left\{ y_{N+1} - \hat{x}_{1_N} \cos(\omega_0 \Delta T) - \hat{x}_{2_N} \sin(\omega_0 \Delta T) \right\}$$

Figures III.3.1 and III.3.2 illustrate the behavior of this second-order system. Note the dependence on propagation to estimate the second state variable, induced by its deterministic dependence (as described by Q) on the first state variable.

FIGURE V.3.1

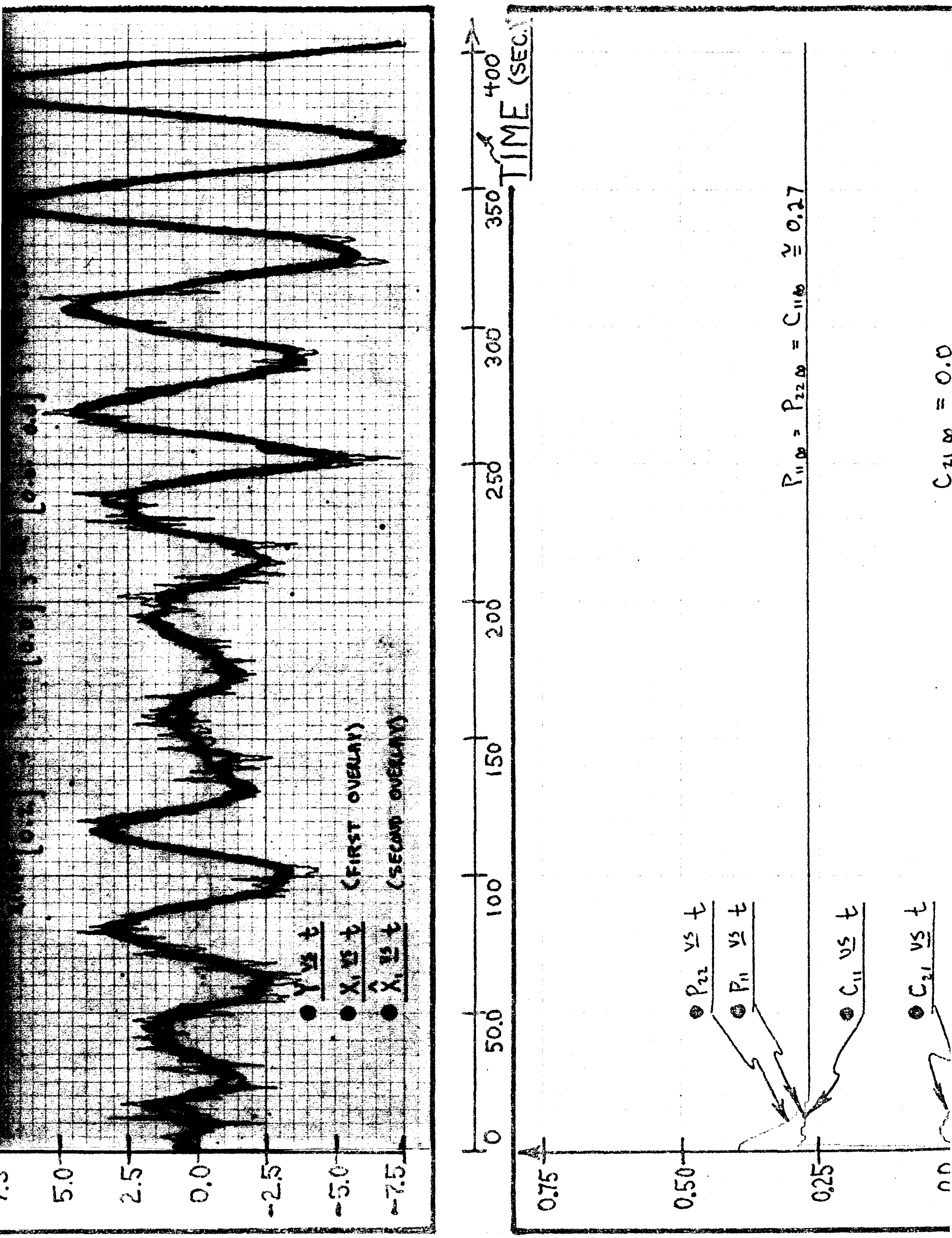
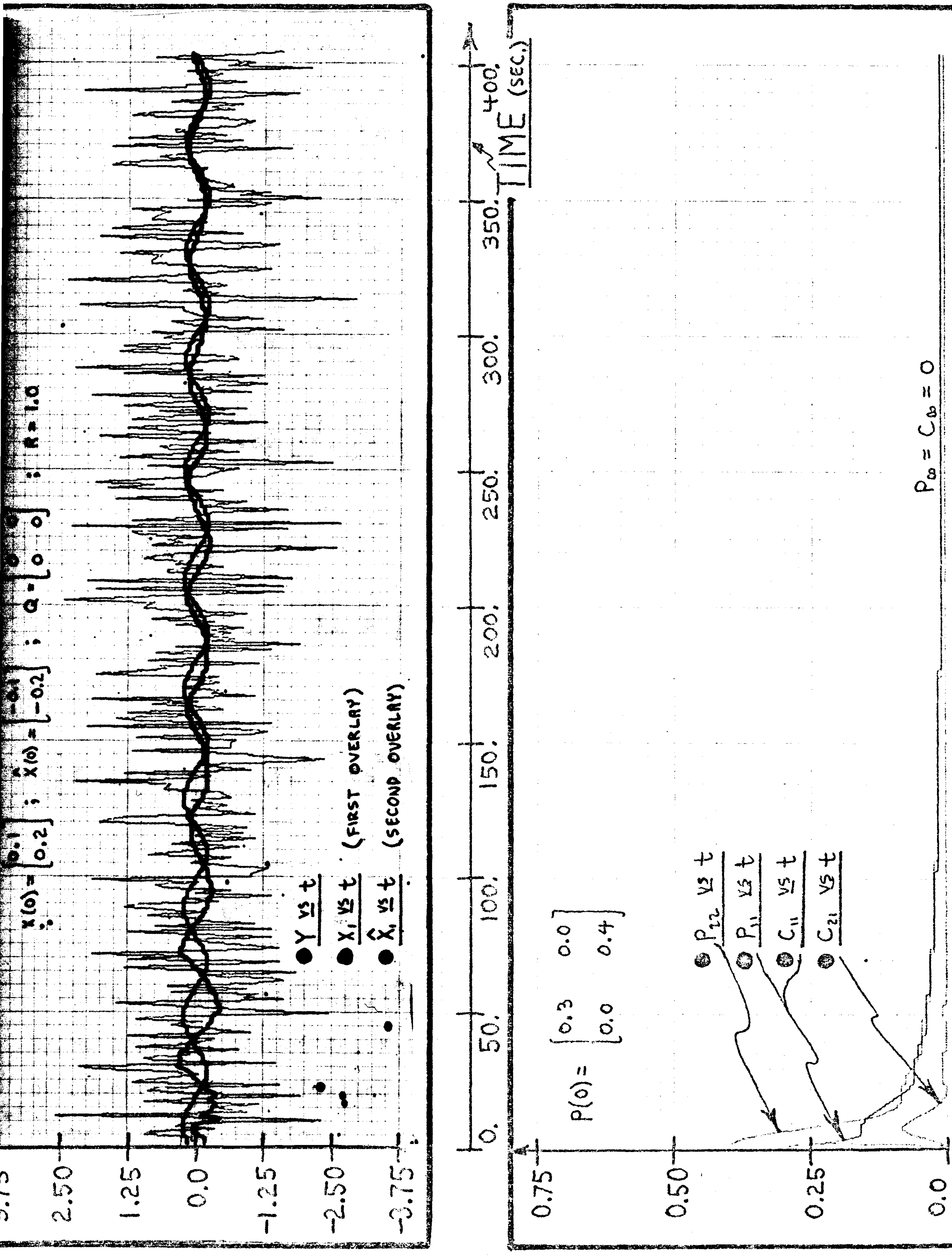

TIME (SEC.)
0
50.0
100
150
200
250
300
350
400
5.0
2.5
0.0
-2.5
-5.0
-7.5
Y vs t
X1 vs t (FIRST OVERLAY)
X̂1 vs t (SECOND OVERLAY)
P22 vs t
P11 vs t
C11 vs t
C21 vs t
P11∞ = P22∞ = C11∞ ≅ 0.27
C21∞ = 0.0
0.75
0.50
0.25

FIGURE V.3.2


X(0) = [0.1 0.2] ; X̂(0) = [-0.1 -0.2] ; Q = [0 0 0 0] ; R = 1.0
Y vs t
X1 vs t (FIRST OVERLAY)
X̂1 vs t (SECOND OVERLAY)
2.50
1.25
0.0
-1.25
-2.50
-3.75
0. 50. 100. 150. 200. 250. 300. 350. 400.
TIME (SEC.)
P(0) = [0.3 0.0 0.0 0.4]
P12 vs t
P11 vs t
C11 vs t
C21 vs t
P∞ = C∞ = 0
0.75
0.50
0.25
0.0

V.4 Redundant Observation (refer to Section III.4):

MODEL:

$$X_{N+1} = X_N + W_{N+1}$$

$$Y_N = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_N = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X_N + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_N$$

$$\operatorname{var}[W_N] = \sigma_Q^2 = Q = 0.1$$

$$\operatorname{var}[V_N] = \begin{bmatrix} \sigma_{R_1}^2 & 0 \\ 0 & \sigma_{R_2}^2 \end{bmatrix} = R = \begin{bmatrix} 5.0 & 5.0 \\ 0.0 & 0.0 \end{bmatrix}$$

ESTIMATOR:

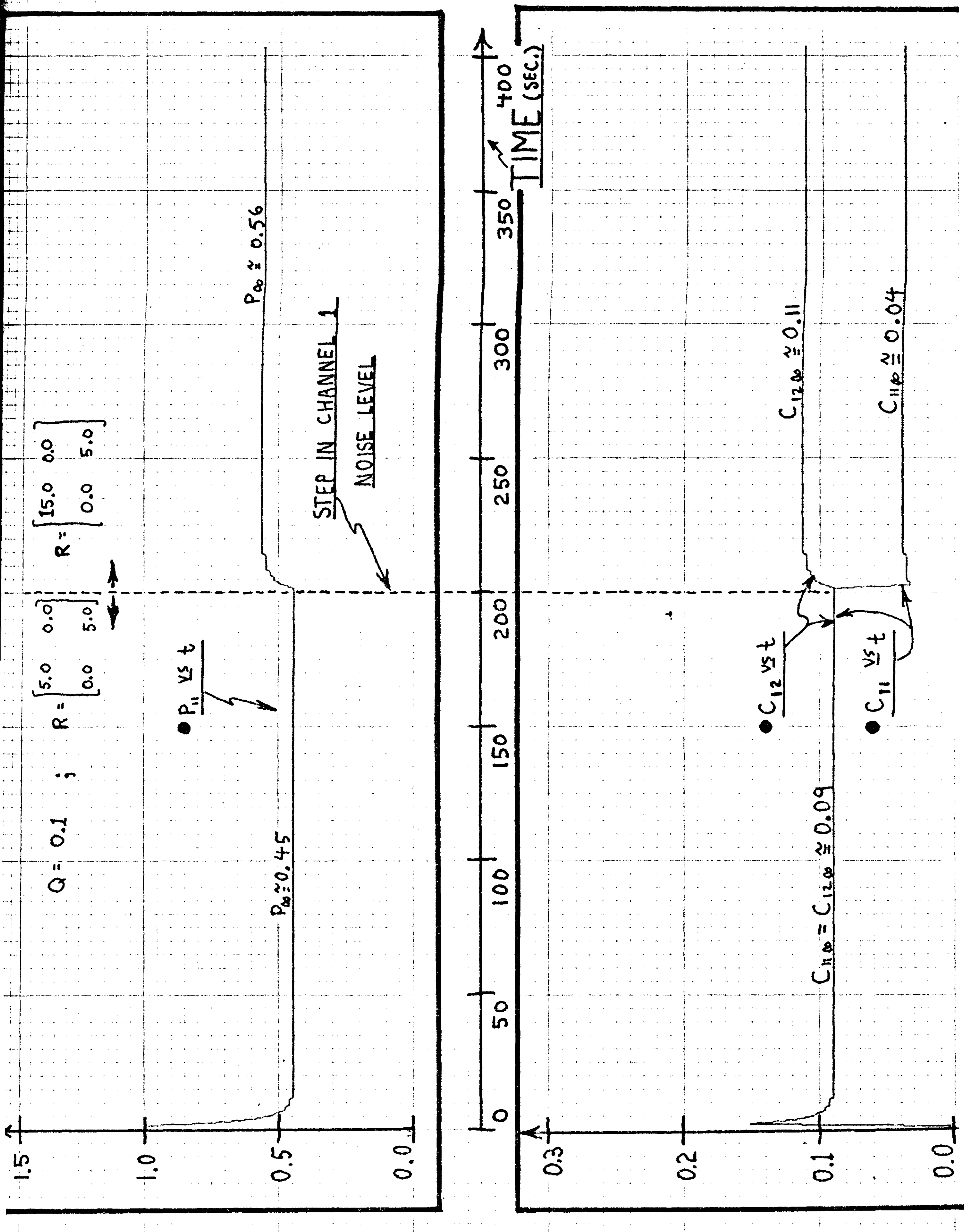
$$\hat{X}_{N+1} = \hat{X}_N + [c_1 \quad c_2]_{N+1} \begin{bmatrix} y_{1_{N+1}} - \hat{X}_N \\ y_{2_{N+1}} - \hat{X}_N \end{bmatrix}$$

$$[c_1 \quad c_2]_{N+1} = (\sigma_N^2 + \sigma_Q^2)(1 \quad 1) \begin{Bmatrix} (\sigma_N^2 + \sigma_Q^2 + \sigma_{R_1}^2) & (\sigma_N^2 + \sigma_Q^2) \\ (\sigma_N^2 + \sigma_Q^2) & (\sigma_N^2 + \sigma_Q^2 + \sigma_{R_2}^2) \end{Bmatrix}^{-1}$$

$$\sigma_N^2 = P_N$$

This example illustrates in simplistic form one of the prime applications of the K-B filter--the optimal combination of observations originating from several instruments. For example, in navigation problems we are faced with combining accelerometer outputs from inertial measurement units,

FIGURE V.4.1



Q = 0.1 ; R = [5.0 0.0 ; 0.0 5.0] R = [15.0 0.0 ; 0.0 5.0]
P11 vs t
P∞ ≈ 0.45
P∞ ≈ 0.56
STEP IN CHANNEL 1
NOISE LEVEL
TIME (SEC.)
0
50
100
150
200
250
300
350
400
1.5
1.0
0.5
0.0
0.3
0.2
0.1
0.0
C12 vs t
C11 vs t
C11∞ = C12∞ ≅ 0.09
C12∞ ≈ 0.11
C11∞ ≅ 0.04

velocity measurements from doppler radar or airspeed instruments, position measurements from radio navigation or pilot visual acquisition methods, and so on. Each measurement source has its own characteristics in terms of bias, random errors, drift, etc. Indeed, the use of instruments of diverse error character is precisely what makes the use of more than one instrument a fruitful pursuit.

Figure V.4.1 illustrates the performance of the estimator, which is seen to be improved over that of Figure V.1.1. Also demonstrated is the effect of variation in the separate instrument statistics on the filter coefficients; as discussed in section V.1, it is important to note that the description of the non-stationary behavior of the instrument errors was provided to the filter.

VI. ADDITIONAL CONSIDERATIONS

VI.1 Deterministic Forcing Functions

We may add slightly to the class of systems which are open to K-B estimation, by the inclusion of a (known) deterministic forcing function, $f(t)$, with time-discretized values $\{f_N\}$:

$$X_{N+1} = \Phi_{N+1} X_N + f_{N+1} + W_{N+1} \tag{1}$$

We utilize Theorem IV.1, and the projection methods of Section IV.4:

$$\begin{aligned} \hat{X}_{N+1} &= E\left[X_{N+1} \middle| Y(N+1)\right] \qquad (2) \\ &= E\left[X_{N+1} \middle| Y(N)\right] + E\left[X_{N+1} \middle| Z(N+1)\right] \\ &= E\left[\Phi_{N+1} X_N + f_{N+1} + W_{N+1} \middle| Y(N)\right] + C_{N+1} \tilde{y}_{N+1} \\ &= \Phi_{N+1} \hat{X}_N + f_{N+1} + C_{N+1}\left[Y_{N+1} - H_{N+1}\left(\Phi_{N+1} \hat{X}_N + f_{N+1}\right)\right] \end{aligned}$$

where we have followed equations (IV.37-38).

We observe that both the choice of optimal coefficient, C_{N+1}, and the resulting filter variance are unaffected by the addition of the deterministic function, by noting that

$$\tilde{X}_{N+1} = X_{N+1} - \hat{X}_{N+1} \tag{3}$$

$$= \left(\Phi_{N+1} X_N + w_{N+1} + f_{N+1}\right) - \Big\{ \Phi_{N+1} \hat{X}_N + f_{N+1} + C_{N+1} \Big(Y_{N+1} - H_{N+1} \Phi_{N+1} \hat{X}_N - H_{N+1} f_{N+1} \Big) \Big\}$$

$$= \Phi_{N+1} \tilde{X}_N + w_{N+1} - C_{N+1} \left(H_{N+1} \Phi_{N+1} \tilde{X}_N + H_{N+1} w_{N+1} + V_{N+1} \right)$$

$$= \left(I - C_{N+1} H_{N+1} \right) \left(\Phi_{N+1} \tilde{X}_N + w_{N+1} \right) - C_{N+1} V_{N+1}$$

as was the case in equation IV.4.

VI.2 Correlated Processes

Random effects on system state and observations are represented by the sequences $\{W_n\}$ and $\{V_n\}$, having their origin in the physical world; these sequences are likely to have passed through "pre-processing" effects prior to their impinging on the subject system. One significant result of this is the "coloring" of these random processes such that our assumptions II.3, II.6 are no longer valid. For example,

$$E\left[v_i v_j^T \right] = R e^{-|i-j|/\tau} \qquad (4)$$

may be more reasonable if the observation noise has passed through an appropriate first-order linear system prior to disturbing observations.

One means of coping with this situation is the inclusion of the noise pre-processing model in the estimator [17]; this scheme suffers from the disadvantage of a vanishing observation error covariance matrix, which will be seen to be undesirable in Section VII.

Bucy [4] has developed a more fundamental scheme for estimation in the presence of colored noise which does not suffer from this disadvantage. (We have not studied this method.)

VI.3 Unforced Systems

In Examples V.1.3 and V.3.2, we observed that estimation in an undriven (free) system ($Q = 0$) leads to vanishing filter coefficents and error variance. As discussed in Section V.1, such a condition is generally an intentional misrepresentation on the part of the modeller. In an idealized error-free computing device, the operation of the filter under such conditions suffers no ill effects. However, real-world, finite-precision computing devices depend directly on the stability properties of the estimation equations. In Section VII.5, we shall see that the matrix Q determines, through the concept of controllability, the stability of the estimator. In a practical computing device, instability due to $Q = 0$ may lead to diverging computing errors in propagated estimates and covariance.

To describe the situation in another fashion, the K-B filter supplies the optimal filter coefficient under the model-supplied conditions. The inclusion of new observation data through a non-vanishing filter coefficient contributes directly to filter stability (see comments at end of Section IV.1). If the model data imply that the optimal coefficient should vanish, the filter relies increasingly on pure propagation (through Φ), and decreasingly on the error-bounding observations Y. Any perturbing effects (e.g., computing errors: round-off, thermal noise) may drive the resulting unstable system away from the desired equilibrium.

One means of dealing with this situation is the inclusion of knowledge of computational or physical perturbing effects in the model supplied the filter: Q is set to reflect the pseudo-random "noise" of computation round-off. The result is a "humble" filter which in equilibrium is not cut off from observation of the physical world (e.g., V.1).

VI.4 Non-linear Systems: Linearization

The linear nature of the model (II.1, II.4) is often itself an idealization. A more general situation is represented by the model

$$
\begin{aligned}
X_{N+1} &= P(X_N, W_{N+1}) \\
Y_{N+1} &= h(X_{N+1}, V_{N+1})
\end{aligned}
\qquad (5)
$$

In some cases, it is possible to treat such a non-linear system with the linear modelling scheme we have discussed, by use of a perturbation approach. The principle of use is that the propagation in time of the difference between two solutions to the non-linear system (5) is nearly linear if the solutions are close to each other. We presume that a solution X^* to (5) is known, and that this solution is nominal in the sense that the actual solution will be close enough to X^* to make expansion about X^* adequately dominated by the linear term:

$$\begin{aligned} \Delta X_{N+1} &= X^*_{N+1} - X_{N+1} \\ &= P(X^*_N, W_{N+1}) - P(X_N, W_{N+1}) \\ &= \left.\frac{\partial P}{\partial X}\right|_{X=X^*} \cdot (X^*_N - X_N) + O(X^*_N - X_N)^2 \\ &\cong \Phi'_{N+1} \Delta X_N + O \end{aligned} \tag{6}$$

The addition of a random driving term to (6), as discussed in the previous section, is a means of keeping the resulting filter sufficiently open to accept corrections to the linearization of (6).

We also linearize the observation process:

$$\Delta Y_N = Y_N^* - Y_N \tag{7}$$

$$= h(X_N^*, V_N) - h(X_N, V_N)$$

$$= \left.\frac{\partial h}{\partial X}\right|_{X = X^*} (X_N^* - X_N) + O(X_N^* - X_N)^2$$

$$\cong H_N' \Delta X_N + 0$$

It is likewise appropriate to add an observation noise term to (7).

The resulting filter will estimate the deviation from a known, nominal trajectory. Such a filter is of use when a nominal solution is available, and we wish to use observational data to correct that solution for deviations not modeled by the nominal case. As an example, consider the orbit of a satellite about the moon, under the influence of lunar gravity:

$$\dot{X} = f(X) \tag{8}$$

$$Y = h(X, V) \quad ,$$

where $X = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$ represents a cartesian frame state vector

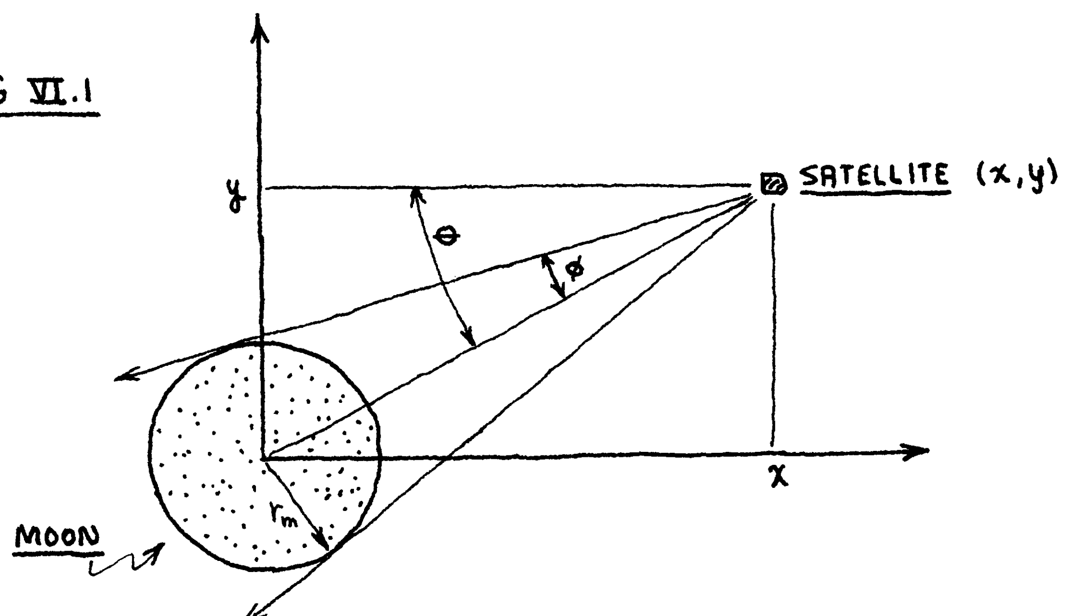
with origin fixed at moon center. Then

$$f(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \left(\frac{F_x}{m}\right) \\ \left(\frac{F_y}{m}\right) \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ -\frac{GM_m x}{(x^2+y^2)^{3/2}} \\ -\frac{GM_m y}{(x^2+y^2)^{3/2}} \end{bmatrix} , \qquad (9)$$

where we have assumed a spherical, homogeneous moon of mass M_m, and are solving only the two dimensional problem.

Instrumentation may include inertial measurement of a_x and a_y, doppler radar measurement of $\dot{x}$ and $\dot{y}$, implicit radar measurement of x and y, or other schemes. We consider the use of a horizon-sensing instrument, which measures the sight angles Θ and ϕ, as shown in Figure VI.1, relative to a stabilized platform:

FIG VI.1



Then

$$Y = \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \sin^{-1} \dfrac{y}{(x^2+y^2)^{1/2}} \\ \sin^{-1} \dfrac{r_m}{(x^2+y^2)^{1/2}} \end{bmatrix} = h(X) \tag{10}$$

Now (8) implies

$$(\Delta \dot{X}) = \left.\frac{df}{dx}\right|_{X^*} \Delta X + O(\Delta X)^2 \tag{11}$$

or

$$\Delta X_{N+1} \cong \Delta X_N + \left.\frac{df}{dx}\right|_{X^*} \Delta X_N \, \Delta T \quad , \tag{12}$$

where, from (9),

$$\frac{df}{dx} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-GM_m(y^2-2x^2)}{(x^2+y^2)^{5/2}} & \dfrac{3GM_m yx}{(x^2+y^2)^{5/2}} & 0 & 0 \\ \dfrac{3GM_m xy}{(x^2+y^2)^{5/2}} & \dfrac{-GM_m(x^2-2y^2)}{(x^2+y^2)^{5/2}} & 0 & 0 \end{bmatrix} . \tag{13}$$

If we assume that the nominal trajectory X^* is a circular orbit of radius R_o, then

$$\left.\frac{df}{dX}\right|_{X^*} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-GM_m(R_o^2-3x^2)}{R_o^5} & \dfrac{3GM_m yx}{R_o^5} & 0 & 0 \\ \dfrac{3GM_m yx}{R_o^5} & \dfrac{-GM_m(R_o^2-3y^2)}{R_o^5} & 0 & 0 \end{bmatrix} . \tag{15}$$

From (6) and (12),

$$\Phi'_{N+1} \cong \left[I + \left.\frac{df}{dx}\right|_{X^*} \Delta T \right] \cong e^{\frac{df}{dx}\Delta T} , \tag{16}$$

or

$$\Phi'_{N+1} \cong \begin{bmatrix} 1 & 0 & \Delta T & 0 \\ 0 & 1 & 0 & \Delta T \\ \left[\dfrac{-GM_m(R_o^2-3x_N^2)\Delta T}{R_o^5}\right] & \left[\dfrac{3GM_m y_N x_N \Delta T}{R_o^5}\right] & 1 & 0 \\ \left[\dfrac{3GM_m y_N x_N \Delta T}{R_o^5}\right] & \left[\dfrac{-GM_m(R_o^2-3y_N^2)\Delta T}{R_o^5}\right] & 0 & 1 \end{bmatrix} . \quad (17)$$

By similar calculation,

$$H_N' = \left.\frac{dh}{dX}\right|_{X^*} = \begin{bmatrix} 0 & 0 & \left[\dfrac{-y_N}{R_0^2}\right] & \left[\dfrac{x_N}{R_0^2}\right] \\ 0 & 0 & \left[\dfrac{-r_m}{R_0^2}\right] & \left[\dfrac{-r_m}{R_0^2}\right] \end{bmatrix} \tag{18}$$

Equations (6), (7), (17) and (18) form the model. At each time stage, an observation is compared to a stored or computed nominal observation, to form ΔY_N . The resulting estimate of ΔX_N is used to correct the stored or computed nominal state, X_N^*.

VI.5 Computer Errors and Stability

The implementation of the filter equations in a digital or analog computer leads to the question of error introduced by the limited precision of computer arithmetic. The principal issue is the stability of computer error thus introduced. As discussed in Section VII.5, the stability of error is determined by the stability of the filtering equations proper.

In the simplest case, we utilize a fixed-point arithmetic computing device, which may be seen to have the following error propagation character:

Letting $\mathcal{E}_x$ = error in the computed value of x, the arithmetic operations lead to error propagation through

$$\text{Addition:} \quad \mathcal{E}_{a+b} = \mathcal{E}_a + \mathcal{E}_b + f$$

$$\text{Multiplication:} \quad \mathcal{E}_{a \cdot b} = b\mathcal{E}_a + a\mathcal{E}_b + \mathcal{E}_a\mathcal{E}_b + f \,, \tag{19}$$

f representing the pseudo-random truncation error. The filter equation (II.8) thus leads to the error equation

$$\mathcal{E}_{\hat{x}_{N+1}} = (I - C_{N+1}H_{N+1})\Phi_{N+1}\mathcal{E}_{\hat{x}_N} + f_{N+1} \,, \tag{20}$$

assuming $\mathcal{E}_{C_{N+1}} = \mathcal{E}_{\Phi_{N+1}} = 0$, which is not unreasonable. In the limiting case discussed in Section VII, (20) leads to

$$\dot{\mathcal{E}}_{\hat{x}} = (F - KH)\mathcal{E}_{\hat{x}} + f(t) \,. \tag{21}$$

Further, the covariance equation (II.9,10,11) leads to the limiting case Ricatti equation

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP \,, \tag{22}$$

as discussed in Section VII. The error equation for (22) is, from (19), the Ricatti equation

$$\dot{\mathcal{E}}_P = (F - \bar{P}H^TR^{-1}H)\mathcal{E}_P + \mathcal{E}_P(F - \bar{P}H^TR^{-1}H)^T - \mathcal{E}_P(H^TR^{-1}H)\mathcal{E}_P + f(t) \,, \tag{23}$$

where $\bar{P}$ represents computed variance.

A major result of Section VII.5 is demonstration of the uniform asymptotic stability of (21) and (23). Other computational problems are thus generally more serious than that of stability. These include:

(1) Dimensionality. The computational load and memory requirement of the filtering/variance equations for practical problems generally lead to efforts toward eliminating some of the state variables from the model, at minimal performance loss, in order to conserve computing time and hardware.

(2) Dynamic Range. The range of variance encountered in transient conditions may lead to scaling problems in fixed-point machines, or expensive floating-point arithmetic.

VII. GENERAL FILTERING DERIVATIONS AND INTERPRETATIONS

We now discuss the filtering problem as first described in the Introduction--the estimation of the continuous function satisfying

$$\dot{X} = FX + GU \qquad (1)$$

through the observation of system "output"

$$Y = HX + V \qquad (2)$$

where

- o F,G, and H are system-determining matrix functions of time, assumed continuous by element.
- o U and V are random processes of zero mean and known second moment, playing the part of system driving and observation perturbing processes, respectfully.
- o We seek $\hat{x}(t)$, the minimum-variance unbiased linear estimate of $x(t)$.

VII.1 A Formal Argument

One method of formally arriving at the continuous estimator is via a limiting argument, based upon the discrete results of previous sections. The disadvantages of formalism are offset by the intuition gained, particularly in the area of the behavior of the random processes involved.

Consider the whiteness requirements of equations II.3

and II.6. The limiting case ($\Delta T \to 0$) of the discrete random process $\{W_n\}$ is the continuous(*) random process $w(t)$, satisfying

$$E\left[w(t)\,w^T(\tau)\right] = Q(t)\,\delta_{t,\tau} \tag{3}$$

where $Q(t)$ is a deterministic function describing the covariance of $w(t)$, and $\delta_{t,\tau}$ the Kronecker delta. This turns out to be undesirable, for such a random process has no effect on a linear system of the form (1). That is, no energy is delivered to the system by the driving process:

Letting Φ be the fundamental matrix of F,

$$E\left[X(t)\right] = \Phi(t,t_0)\,E\left[X(t_0)\right] + \int_{t_0}^{t} \Phi(t,\tau)\,E\left[w(\tau)\right] d\tau \tag{4}$$

$$= \Phi(t,t_0)\,X(t_0) \quad + \; 0$$

$$E\left[X(t)\,X^T(t)\right] = \Phi(t,t_0)\left[X(t_0)\,X^T(t_0)\right]\Phi^T(t,t_0) \tag{5}$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t} \Phi(t,\tau_1)\,E\left[w(\tau_1)\,w^T(\tau_2)\right]\Phi^T(t,\tau_2)\,d\tau_1\,d\tau_2$$

$$= \Phi(t,t_0)\,P_0\,\Phi^T(t,t_0) \quad + \; 0 \;,$$

(*) We refer to the continuum of the index set $\{t\}$ of the variables, and not to continuity in resulting sample functions.

the integral vanishing by (3). At this point it is customary in the literature (in a <u>formal</u> argument; we are not at the foundation level) to introduce the remedial formalism

$$E\left[w(t)\, w^T(\tau)\right] = \delta(t-\tau)\, Q(t) \tag{6}$$

where $\delta(t-\tau)$ is the "Dirac Impulse Function," to replace equation (3). We make the following observations:

(1) The impulsive device may be placed on solid <u>mathematical</u> footing as an entity which is not a function, through the algebraic completion of the ring of continuous functions [6]. $\delta(s)$ becomes the identity in the resulting division ring, under convolution.

(2) Such a mathematical justification adds little insight toward interpreting the <u>physical</u> meaning (if any) of the requirement (6).

(3) Indeed, the mathematical properties of $\delta(t-\tau)\, Q(t)$ do <u>not</u> seem to make physical sense here.

What we seek is a random process, w(t), which preserves some aspects of both the whiteness and energy-delivery requirements which our intuition demands. The solution rests in the use of a random process which <u>appears</u> white to the physical system, but which is fact correlated over a relatively short period. That is, we require that $E\left[w(t)\, w^T(\tau)\right]$ be non-vanishing for $|t-\tau|$ small compared to the typical response time for the physical system, as characterized by

F or ϕ. Letting T represent the "short" time period we have in mind, we are requiring

$$\Phi(t+T, t) \cong I \quad , \text{ or } \quad \int_t^{t+T} F(s)\,ds \cong 0 \,. \tag{7}$$

The requirement on w(t) becomes

$$E\left[w(t)w^T(\tau)\right] = \begin{cases} 0, & |t-\tau| > T/2 \\ \left(\frac{1}{T}\right)Q(t), & |t-\tau| \le T/2 \end{cases} , \tag{8}$$

so that (9)

$$E\left[X(t)X^T(t)\right] = \Phi(t,t_0)P_0\Phi^T(t,t_0) + \int_{\tau_1=t_0}^{t}\int_{s=t_0-\tau_1}^{t-\tau_1} \Phi(t,\tau_1)E\left[w(\tau_1)\cdot w^T(\tau_1+s)\right] \times \Phi^T(t,s+\tau_1)\,ds\,d\tau_1$$

$$= \Phi(t,t_0)P_0\Phi^T(t,t_0) + \int_{\tau_1=t_0}^{t}\int_{s=\max\left[t_0-\tau_1,\,-\frac{T}{2}\right]}^{\min\left[t-\tau_1,\,\frac{T}{2}\right]} \Phi(t,\tau_1)\left(\frac{1}{T}\right)Q(\tau_1) \times \Phi^T(t,s+\tau_1)\,ds\,d\tau_1$$

$$\cong \Phi(t,t_0)P_0\Phi^T(t,t_0) + \int_{\tau_1=t_0}^{t} \Phi(t,\tau_1)Q(\tau_1)\Phi^T(t,\tau_1)\,d\tau_1 \,,$$

where we have used $\Phi(t,\tau_1+T) = \Phi(t,\tau_1)\Phi(\tau_1,T+\tau_1) \cong \Phi(t,\tau_1)$.

The degree of approximation involved is determined by the correlation period T; more significant is the achievement of non-vanishing energy delivered to the physical system by a random process of finite covariance and system-relative whiteness to within any challenge level.

Based on the above discussion, we feel free to revert to the abbreviation (6), which has the formal properties desired, provided this expectation appears only within integrals. We redefine the random processes $\{W_n\}$ and $\{V_n\}$, such that

$$
\begin{aligned}
E[W_N W_M^T] &= \left(\frac{1}{\Delta T}\right) Q_N \delta_{N,M} \\
E[V_N V_M^T] &= \left(\frac{1}{\Delta T}\right) R_N \delta_{N,M}
\end{aligned}
\qquad (10)
$$

We propose to carry out a limiting argument on the resulting discrete system, keeping in mind that the symbols serve as carriers of the above described interpretation.

It is helpful to review the connection between the discretized system II.1-8 and the continuous system (1), as summarized by Appendix IX.6. The solution X(t), to (1) has the property

$$X(t_{N+1}) = \Phi(t_{N+1},t_N)X(t_N) + \int_{t_N}^{t_{N+1}} \Phi(t_{N+1},\tau)G(\tau)U(\tau)\,d\tau, \quad (11)$$

or

$$X_{N+1} = \Phi_{N+1} X_N + I_{N+1} W_{N+1} \quad (12)$$

, where

$$\frac{d}{dt}\Phi(t,t_N) = F(t)\Phi(t,t_N) \quad ; \quad \Phi(t_N,t_N) = I$$

Assuming G is constant or at least continuous over the range of integration, we approximate (12) by

$$X_{N+1} = \Phi_{N+1} X_N + G(t_N)U(t_N)\Delta T \quad , \quad \text{or}$$

$$G(t_N) = \lim_{\Delta T \to 0}\left(\frac{I_{N+1}}{\Delta T}\right) \quad . \tag{13}$$

Following Sorenson [17],

$$X(t_{N+1}) = X_{N+1} = \Phi_{N+1} X_N + I_{N+1} W_{N+1} \tag{14}$$

$$= \left(I + F(t_N)\Delta T\right)X_N + G(t_{N+1})\Delta T\, W_{N+1} + O(\Delta T^2)$$

$$\frac{X(t_{N+1}) - X(t_N)}{\Delta T} = F(t_N)X(t_N) + G(t_{N+1})W_{N+1} + O(\Delta T) \tag{15}$$

having limit case

$$\dot{X}(t) = F(t)X(t) + G(t)U(t) \quad . \tag{16}$$

Similarly, for the optimal estimator,

$$\hat{x}(t_{N+1}) = \hat{X}_{N+1} = \Phi_{N+1}\hat{X}_N + C_{N+1}\left(Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\right) \tag{17}$$

$$= \left(I + F(t_N)\Delta T\right)\hat{X}_N + C_{N+1}\left(Y_{N+1} - H_{N+1}\{I + F(t_N)\Delta T\}\hat{X}_N\right) + O(\Delta T^2)$$

$$\left[\frac{\hat{x}(t_{N+1}) - \hat{x}(t_N)}{\Delta T}\right] = F(t_N)\hat{X}_N + \left(\frac{C_{N+1}}{\Delta T}\right)\left(Y(t_{N+1}) - H(t_{N+1})\hat{x}(t_N)\right) + O(\Delta T) \tag{18}$$

having limit case

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + K(t)\left[Y(t) - H(t)\hat{x}(t)\right] , \tag{19}$$

where

$$K(t_N) = \lim_{\Delta T \to 0}\left[\frac{C(t_{N+1})}{\Delta T}\right]$$

exists for the optimal choice of C_{N+1}:

$$\frac{C_{N+1}}{\Delta T} = \left(\frac{1}{\Delta T}\right)P'_{N+1}H^T_{N+1}\left(H_{N+1}P'_{N+1}H^T_{N+1} + R'_{N+1}\right)^{-1} \tag{20}$$

$$= \left(\frac{1}{\Delta T}\right)\left(\Phi_{N+1}P_N\Phi^T_{N+1} + \Gamma_{N+1}\left\{\frac{Q_{N+1}}{\Delta T}\right\}\Gamma^T_{N+1}\right)H^T_{N+1}\left(H_{N+1}\Phi_{N+1}P_{N+1}\Phi^T_{N+1}H^T_{N+1}\right.$$

$$\left. + H_{N+1}\Gamma_{N+1}\left\{\frac{Q_{N+1}}{\Delta T}\right\}\Gamma^T_{N+1} + \frac{R_{N+1}}{\Delta T}\right)^{-1}$$

$$= \left[P(t_N) + o(\Delta T) \right] H^T(t_{N+1}) \left[R(t_{N+1}) + o(\Delta T) \right]^{-1}$$

$$K(t_{N+1}) = \lim_{\Delta T \to 0} \left(\frac{C_{N+1}}{\Delta T} \right) = P(t_{N+1}) H^T(t_{N+1}) R^{-1}(t_{N+1}) \tag{21}$$

(note the similarity to equation IV.65). The variance equation becomes

$$P(t_{N+1}) = P_{N+1} = (I - C_{N+1} H_{N+1}) P_{N+1}' \tag{22}$$

$$= \left[P_N + F_N P_N \Delta T + P_N F_N^T \Delta T + \Gamma_{N+1} \left\{ \frac{Q_{N+1}}{\Delta T} \right\} \Gamma_{N+1}^T + o(\Delta T^2) \right]$$

$$- \left[C_{N+1} H_{N+1} \left(P_N + o(\Delta T) \right) \right]$$

$$\frac{P(t_{N+1}) - P(t_N)}{\Delta T} = F(t_N) P(t_N) + P(t_N) F^T(t_N) + G(t_{N+1}) Q(t_{N+1}) G^T(t_{N+1}) \tag{23}$$

$$- \left(\frac{C(t_{N+1})}{\Delta T} \right) H(t_{N+1}) \left[P(t_N) + o(\Delta T) \right] + o(\Delta T),$$

having limit case (24)

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - K(t)H(t)P(t)$$

$$= F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t)$$

This non-linear matrix differential equation is the Riccati Equation; its solution, given the initial uncertainty P(0), characterizes the performance of the continuous estimator. Section VII.5 is devoted to the study of properties of solutions to this equation. To summarize,

DISCRETE CASE

$$X_{N+1} = \Phi_{N+1}X_N + \Gamma_{N+1}W_{N+1} \quad (25)$$

$$Y_N = H_N X_N + V_N \quad (26)$$

$$\hat{X}_{N+1} = \Phi_{N+1}\hat{X}_N + C_{N+1}(Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N) \quad (27)$$

$$P'_{N+1} = \Phi_{N+1}P_N\Phi^T_{N+1} + \Gamma_{N+1}Q'_{N+1}\Gamma^T_{N+1} \quad (28)$$

$$C_{N+1} = P'_{N+1}H^T_{N+1}(H_{N+1}P'_{N+1}H^T_{N+1} + R'_{N+1})^{-1} = P_{N+1}H^T_{N+1}R'^{-1}_{N+1} \quad (29)$$

$$P_{N+1} = (I - C_{N+1}H_{N+1})P'_{N+1}, \quad \text{where} \quad (30)$$

$$\Phi_{N+1} = \Phi(t_{N+1}, t_N)$$

$$\frac{d}{dt}\Phi(t,t_N) = F(t)\Phi(t,t_N) \; ; \; \Phi(t_N,t_N) = I$$

$$\Gamma_{N+1}W_{N+1} = \int_{t_N}^{t_{N+1}} \Phi(t_{N+1},\tau)\,G(\tau)\,U(\tau)\,d\tau$$

$$E[W_N W_N^T] = \left(\frac{1}{\Delta T}\right)Q_N = Q'_N; \quad E[V_N V_N^T] = \left(\frac{1}{\Delta T}\right)R_N = R'_N$$

LIMIT CASE

$$\dot{X} = FX + GU \tag{31}$$

$$Y = HX + V \tag{32}$$

$$\dot{\hat{X}} = F\hat{X} + K[Y - H\hat{X}] \tag{33}$$

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP \tag{34}$$

$$K = PH^TR^{-1} \quad , \tag{35}$$

where

$$F(t_N) = \lim_{\Delta T \to 0} \left[\frac{\Phi(t_N + \Delta T, t_N) - I}{\Delta T} \right]$$

$$G(t_N) = \lim_{\Delta T \to 0} \left[\frac{I_N}{\Delta T} \right]$$

$$K(t_N) = \lim_{\Delta T \to 0} \left[\frac{C_N}{\Delta T} \right]$$

$$E[U(t)U^T(\tau)] = \delta(t-\tau)\, Q(t) \; ; \quad E[V(t)V^T(\tau)] = \delta(t-\tau)\, R(t)$$

We now approach the continuous case from an alternate viewpoint, in line with the history of filtering theory's development.

VII.2 Continuous Filtering and the Wiener Problem

In Section IV, we considered the selection of a coefficient sequence for an optimal time-discretized filter. The resulting filter might be viewed thusly:

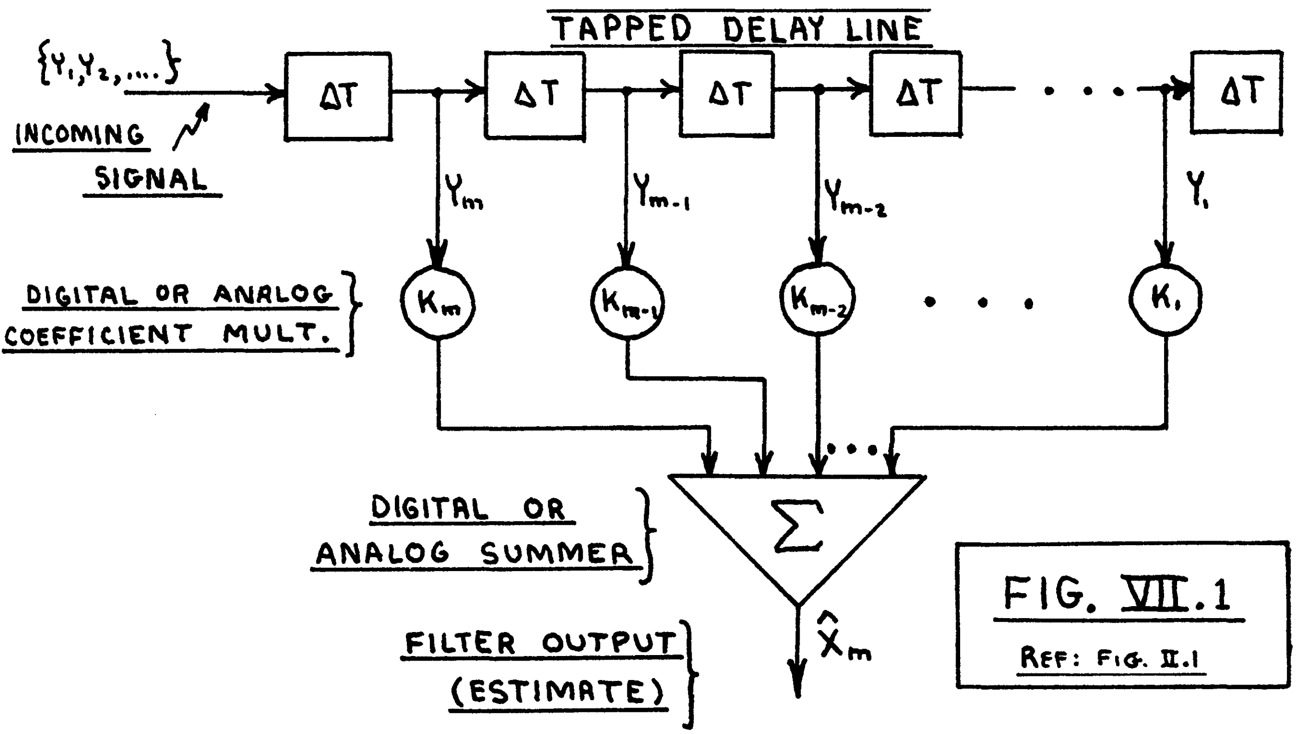


FIG. VII.1

REF: FIG. II.1

We have emphasized here the viewpoint that the estimate (in the steady state, or ignoring initial estimates) is formed from a linear combination of the incoming observations. It is interesting that recent developments in digital electronic technology have made practical the construction of filters in precisely this form [5]. These filters often take the (non-Kalman) configuration of a "finite-memory" delay line which combines the M most recent observations in a <u>fixed</u> linear combination, resulting in a sub-optimal but highly flexible and useful filter. It would appear that an increased use of digital signal processing on ever-lower levels of component structure is the current trend.

However, the historical origin of the Kalman-Bucy Filter is the Wiener Problem, predating the availability of such technology. During the years during and following World War II, Norbert Wiener was concerned with the problem of extracting from a noise-corrupted time-continuous signal a system state or message signal, using electronic filters constructed of "classical" analog components. In the language of convolution, impulse response, and transfer function, Wiener stated and to a degree solved this problem [18]. The continuous Kalman-Bucy filter solves a problem broader than the Wiener problem; in the special case of a stationary statistical environment (R,Q), the limiting K-B filter is the solution to the Wiener problem, and presents directly the data necessary for actual synthesis of the filter, whereas the Wiener solution is in fact only a step toward synthesis.

Consider the formation of an estimate, pertaining to the model (1), by

$$\hat{X}_A(t) = \int_{t_0}^{t} A(t,s)Y(s)\,ds \qquad (36)$$

$\hat{X}_A(t)$ may thus be considered the output of a non-stationary linear filter having impulse response $A(t,s)$, to input $\delta(t-s)$:

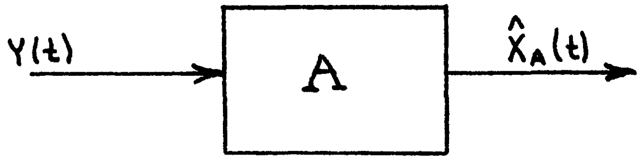


Fig. VII.2

Observe that the operation of convolution has replaced the discrete operations of Fig. VII.1.

We now seek $A(t,s)$ such that $E[(X(t)-\hat{X}_A(t))^T \times (X(t)-\hat{X}_A(t))]$ is minimized. Following Kalman and Bucy [13], we characterize the optimal filter impulse response function $A(t,s)$; the following refers principally to equations (1), (2), and (36):

Theorem VII.1: $\hat{X}_A(t)$ optimal $\Leftrightarrow$ $\mathrm{cov}[\tilde{X}_A(t), Y(\tau)] = 0$, $\forall\ \tau \in (t_0, t)$. (37)

Proof: $\hat{X}_A(t)$ optimal $\Leftrightarrow$ $(\tilde{X}_A(t), \hat{X}_B(t)) = 0$, $\forall\ B(t,t_0)$,

using the Hilbert space view of Section IV.4. Now

$$(\tilde{X}_A(t), \hat{X}_B(t)) = \mathrm{tr}\,E\left[\tilde{X}_A(t)\int_{t_0}^{t} Y^T(s)\,B^T(t,s)\,ds\right] \tag{38}$$

$$= \mathrm{tr}\int_{t_0}^{t} \mathrm{cov}[\tilde{X}_A(t), Y(s)]\,B^T(t,s)\,ds \quad .$$

Then (i) $\operatorname{cov}[\tilde{X}_A(t), Y(s)] = 0 \ \forall \ s \in (t_0, t) \Rightarrow \hat{X}_A(t)$ optimal.

(ii) $\hat{X}_A(t)$ optimal $\Rightarrow \operatorname{tr}\int_{t_0}^{t} \operatorname{cov}[\tilde{X}_A(t), Y(s)]\, B^T(t,s)\, ds = 0$

$\forall \ B(t,s)$. Let $\bar{B}(t,s) = \operatorname{cov}[\tilde{X}_A(t), Y(s)]$.

Then

$$\operatorname{tr}\int_{t_0}^{t} \bar{B}(t,s)\bar{B}^T(t,s)\, ds = 0 \quad , \text{ so} \tag{39}$$

$$\bar{B}(t,s) = \operatorname{cov}[\tilde{X}_A(t), Y(s)] = 0 \ , \ \forall \ s \in (t_0, t) .$$

Q.E.D.

We note that (37) is a form of the <u>Wiener-Hopf Equation</u>, which characterizes the optimal filter impulse response. Wiener discovered this equation in <u>integral</u> form, and a solution was thus difficult to obtain. One of Kalman/Bucy's principal contributions was the expression of the problem in <u>differential</u> form:

<u>Theorem VII</u>.2: $\hat{X}_A(t)$ optimal $\Leftrightarrow \frac{\partial}{\partial t} A(t,s) = F(t)A(t,s) - A(t,t)H(t)A(t,s),$

$$\forall \ s \in (t_0, t) \ , \text{ provided } R(s) > 0 . \tag{40}$$

<u>Proof</u>: By theorem VII.1,

$$\hat{X}_A(t) \text{ optimal} \Leftrightarrow \operatorname{cov}[\tilde{X}_A(t), Y(s)] = 0 \ \forall \ s \in (t_0, t)$$

$$\Leftrightarrow \quad \operatorname{cov}[X(t), Y(s)] = \operatorname{cov}[\hat{X}_A(t), Y(s)], \quad \forall\, s \in (t_0, t) \tag{41}$$

$$\Leftrightarrow \quad \operatorname{cov}[X(t), Y(s)] = \int_{t_0}^{t} A(t,\tau) \operatorname{cov}[Y(\tau), Y(s)]\, d\tau, \quad \forall\, s \in (t_0, t).$$

Differentiating each side of (41) with respect to t, using (1),

$$F(t)\operatorname{cov}[X(t), Y(s)] + G(t)\operatorname{cov}[\cancelto{0}{U(t), Y(s)}] = \frac{\partial}{\partial t}\int_{t_0}^{t} A(t,\tau)\operatorname{cov}[Y(\tau), Y(s)]\, d\tau \tag{42}$$

$\forall\, s \in (t_0, t)$. But

$$\frac{\partial}{\partial t}\int_{t_0}^{t} A(t,\tau)\operatorname{cov}[Y(\tau), Y(s)]\, d\tau = \int_{t_0}^{t} \frac{\partial}{\partial t} A(t,\tau)\operatorname{cov}[Y(\tau), Y(s)]\, d\tau + A(t,t)\operatorname{cov}[Y(t), Y(s)] \tag{43}$$

$$= \int_{t_0}^{t} \frac{\partial}{\partial t} A(t,\tau)\operatorname{cov}[Y(\tau), Y(s)]\, d\tau + A(t,t)\operatorname{cov}[H(t)X(t), Y(s)], \quad \forall\, s \in (t_0, t).$$

Equating (42) and (43), and using (41),

$$F(t)\operatorname{cov}[X(t), Y(s)] = F(t)\operatorname{cov}[\hat{X}_A(t), Y(s)]$$

$$= \int_{t_0}^{t} F(t) A(t,\tau)\operatorname{cov}[Y(\tau), Y(s)]\, d\tau \tag{44}$$

$$= \int_{t_0}^{t} \frac{\partial}{\partial t} A(t,\tau)\, \mathrm{cov}[Y(\tau), Y(s)]\, d\tau + A(t,t) H(t)\, \mathrm{cov}[X(t), Y(s)]$$

$$= \int_{t_0}^{t} \left[\frac{\partial}{\partial t} A(t,\tau) + A(t,t) H(t) A(t,\tau) \right] \mathrm{cov}[Y(\tau), Y(s)]\, d\tau$$

$$= 0 \quad , \quad \forall\ s \in (t_0, t),$$

or,
$$\int_{t_0}^{t} \left[F(t) A(t,\tau) - \frac{\partial}{\partial t} A(t,\tau) - A(t,t) H(t) A(t,\tau) \right] \mathrm{cov}[Y(\tau), Y(s)]\, d\tau = 0, \quad (45)$$

$\forall\ s \in (t_0, t) \Leftrightarrow \hat{X}_A(t)$ optimal. Then

(i) (40) is a sufficient condition for optimality.

(ii) To establish the necessity of (40), we assume optimality and observe that $\hat{X}_{A+B}$ satisfies the Wiener-Hopf equation (37), where $B(t,\tau)$ is the bracketed quantity of (45):

$$\begin{aligned} \mathrm{cov}[\tilde{X}_{A+B}(t), Y(s)] &= \mathrm{cov}[\tilde{X}_A(t), Y(s)] - \mathrm{cov}[\hat{X}_B(t), Y(s)] \\ &= 0 - \int_{t_0}^{t} B(t,\tau)\, \mathrm{cov}[Y(\tau), Y(s)]\, d\tau \\ &= 0 \quad , \quad \forall\ s \in (t_0, t), \end{aligned}$$

from (37) and (45).

Then by the uniqueness(*) of the (optimal) orthogonal projection,

$$\|\hat{X}_B\|^2 = 0 \quad , \text{ or}$$

$$\int_{t_0}^{t}\int_{t_0}^{t} B(t,\tau_1)\,\mathrm{cov}[Y(\tau_1),Y(\tau_2)]\,B^T(t,\tau_2)\,d\tau_1\,d\tau_2 = 0,$$

or, using (2),

$$\int_{t_0}^{t}\left[\int_{t_0}^{t} B(t,\tau_1)\,\mathrm{cov}[H(\tau_1)X(\tau_1),H(\tau_2)X(\tau_2)]\,B^T(t,\tau_2)\,d\tau_1 + B(t,\tau_2)R(\tau_2)B(t,\tau_2)\right]d\tau_2 = 0 \tag{48}$$

Then since $R(\tau_2) > 0$, $\tau_2 \in (t_0,t)$, we have $B(t,\tau_2) = 0$, $\forall\, \tau_2 \in (t_0,t)$. Q.E.D.

<u>Theorem VII.3</u>: $\hat{X}_A(t)$ optimal $\Rightarrow A(t,t) = P(t)H(t)R^{-1}(t)$ (5

<u>Proof</u>: $\hat{X}_A(t)$ optimal $\Rightarrow \mathrm{cov}[\tilde{X}_A(t), Y(s)] = 0 \quad \forall\, s \in (t_0,t)$

$$\Rightarrow \mathrm{cov}[X(t),Y(s)] = \int_{t_0}^{t} A(t,\tau)\,\mathrm{cov}[Y(\tau),Y(s)]\,d\tau \qquad (51$$

$$= \int_{t_0}^{t} A(t,\tau)\,\mathrm{cov}\left[Y(\tau), H(s)X(s)\right] d\tau + A(t,s)\,R(s)$$

$$\Rightarrow \mathrm{cov}\left[X(t), H(s)X(s)\right] = \mathrm{cov}\left[\hat{X}_A(t), H(s)X(s)\right] + A(t,s)\,R(s)$$

$$\Rightarrow A(t,s) = \mathrm{cov}\left[\tilde{X}_A(t), H(s)X(s)\right] R^{-1}(s), \quad \forall s \in (t_0, t). \quad (52)$$

Now (52) is continuous in s on both sides, implying that equality holds for $s = t$:

$$A(t,t) = \mathrm{cov}\left[\tilde{X}_A(t), H(t)X(t)\right] R^{-1}(t)$$

$$= \mathrm{cov}\left[\tilde{X}_A(t), H(t)\{X(t) - \hat{X}_A(t)\}\right] R^{-1}(t) \quad (54)$$

$$= \mathrm{cov}\left[\tilde{X}_A(t), \tilde{X}_A(t)\right] H^{T}(t)\, R^{-1}(t),$$

since $(\tilde{X}_A(t), \hat{X}_A(t)) = 0$. Q.E.D.

The similarity between this conclusion and equation (35) leads us to observe that the filter integral form (36) is related to the differential form (33) by $K(t) = A(t,t)$. (55)

Theorem VII.4 The optimal estimator is the solution to

$$\dot{\hat{X}}(t) = F(t)\hat{X}(t) + K(t)\left[Y(t) - H(t)\hat{X}(t)\right]$$

$$= \left[F - KH\right]\hat{X} + KY \quad ,$$

where $K(t) = A(t,t)$.

Proof: Let $\hat{X}_A(t)$ represent the optimal estimate of $X(t)$:

$$\hat{X}_A(t) = \int_{t_0}^{t} A(t,s)\, Y(s)\, ds$$

Then by theorem VII.2,

$$\dot{\hat{X}}_A(t) = \int_{t_0}^{t} \frac{\partial}{\partial t} A(t,s)\, Y(s)\, ds + A(t,t)\, Y(t) \qquad (56)$$

$$= \int_{t_0}^{t} \left[F(t)A(t,s) - A(t,t)H(t)A(t,s) \right] Y(s)\, ds + A(t,t)Y(t)$$

$$= \left[F(t) - A(t,t)H(t)\right] \int_{t_0}^{t} A(t,s)\, Y(s)\, ds + A(t,t)\, Y(t)$$

$$= \left[F - KH\right]\hat{X}_A(t) + KY \quad ,$$

where $K(t) = A(t,t)$. Q.E.D.

We have arrived at equations (33) and (35).

Theorem VII.5 $\hat{X}(t)$ optimal implies

$$(i)\ \dot{P}(t) = F(t)P(t) - P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) \quad (57)$$

and

$$(ii)\ \dot{\tilde{X}} = (F-KH)\tilde{X} + GU - KV \quad (58)$$

Proof: $(ii)\ \dot{\tilde{X}} = (\dot{X} - \dot{\hat{X}}) = (FX + GU) - (F\hat{X} + K\{Y - H\hat{X}\})$

$$= F(X - \hat{X}) + GU - K\{HX + V - H\hat{X}\}$$

$$= (F-KH)\tilde{X} + GU - KV \quad (59)$$

$$(i)\quad P(t) = E\left[\tilde{X}(t)\cdot\tilde{X}^T(t)\right].$$

But by (59),

$$\tilde{X}(t) = \Phi_{F-KH}(t,t_0)\tilde{X}(t_0) + \int_{t_0}^{t} \Phi_{F-KH}(t,s)\left[G(s)U(s) - K(s)V(s)\right]ds, \quad (60)$$

so that

$$P(t) = \Phi_{F-KH}(t,t_0)P(t_0)\Phi^T_{F-KH}(t,t_0)$$
$$+ \int_{t_0}^{t} \Phi_{F-KH}(t,s)\left[G(s)Q(s)G^T(s) + K(s)R(s)K^T(s)\right]\Phi^T_{F-KH}(t,s)\,ds \quad (61)$$

Then differentiating (61),

$$\dot{P}(t) = (F-KH)P(t) + P(t)(F-KH)^T + GQG^T + KRK^T \tag{62}$$

We observe that this differential equation is a valid description of filter error variance in the general (non-optimal) case, analagous to equation IV.5. In the optimal estimator case, we choose $K = PH^TR^{-1}$, so that

$$\dot{P}(t) = FP + PF^T + GQG^T - PH^TR^{-1}HP \quad . \tag{63}$$

We have arrived at equation (34).

VII.3 The Wiener-Hopf Equation as an Euler-Lagrange Equation

We show in this section that the Wiener-Hopf equation, (37), may be reached by means other than those of Section VII.2; it is the necessary and sufficient condition for the minimization of the variance integral loss function, known in the calculus of variations as the Euler-Lagrange equation.

We seek the linear filter, specified by $A(t,s)$ in (36), which minimizes the variance in estimator error; we consider a simple scalar system:

Signal: $y(t) = x(t) + v(t)$

Estimate: $\hat{x}(t) = \int_{t_0}^{t} A(t,s)\,y(s)\,ds$

$$\underline{\text{Error}}: \quad \tilde{x}(t) = x(t) - \int_{t_0}^{t} A(t,s)\left[x(s) + v(s)\right] ds$$

$$\underline{\text{Variance}}: \quad P(t) = E\left[\tilde{x}^2(t)\right]$$

$$= E\left[x^2(t)\right] - 2\int_{t_0}^{t} A(t,s)E\left[x(s)\,x(t)\right] ds \tag{64}$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t} A(t,s_1)A(t,s_2)\,E\left[x(s_1)x(s_2) + v(s_1)v(s_2)\right] ds_1\, ds_2$$

$$= E\left[x^2(t)\right] + \int_{t_0}^{t}\left\{\int_{t_0}^{t} A(t,s_1)A(t,s_2)\left[\psi_x(s_1,s_2) + \psi_v(s_1,s_2)\right] ds_1 - 2A(t,s_2)\psi_x(s_2,t)\right\} ds_2 \quad ,$$

where ψ denotes the covariance or autocorrelation function. The necessary (and in this case sufficient) condition for the minimization of (64) with respect to $A(t,s_2)$ is the Euler-Lagrange equation

$$\int_{t_0}^{t} A(t,s_1)\left[\psi_x(s_1,s_2) + \psi_v(s_1,s_2)\right] ds_1 - \psi_x(s_2,t) = 0, \tag{65}$$

for all $s_2 \in (t_0,t)$. This is a more familiar form of the Wiener-Hopf equation. However, it is equivalent to (37), by reversing the order of integration and expectation:

$$E\left[\int_{t_0}^{t} A(t,s_1)\left[X(s_1)+v(s_1)\right]ds_1 \cdot \left\{X(s_2)+v(s_2)\right\} - X(s_2)X(t)\right] \qquad (66)$$

$$= E\left[\hat{X}_A(t)y(s_2) - X(t)X(s_2)\right]$$

$$= \left(\tilde{X}_A(t), y(s_2)\right)$$

$$= 0$$

VII.4 Observability and Controllability

We return to a question posed in Section IV.1: Under what conditions does observation of system output, $Y(t)$, provide information of use in estimating system state, $X(t)$?

In the noise-free, undriven case,

$$\begin{aligned} \dot{X}(t) &= F(t)X(t) \\ Y(t) &= H(t)X(t) \end{aligned} \qquad (67)$$

(1) The observation matrix, H, is relevant through its singularity (or regularity).

(2) The system matrix, F, is relevant in that the time development of state, as characterized by F, can "pull apart" the kernel of H.

Example VII.1:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{68}$$

$$Y = [1 \quad 1] X = x_1 + x_2$$

Clearly x_1 and x_2 cannot be distinguished via Y.

Example VII.2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{69}$$

$$Y = [1 \quad 1] X = x_1 + x_2$$

Since Y will be of the form $X_1(0)e^{-t} + X_2(0)e^{-2t}$ observation of Y over a period of time (2 points!) provides exact knowledge of $X(t)$. Note that H is identical in Examples 1 and 2.

We quantify these with

Defn. VII.1: The system (F,H) (i.e., (67)) is completely observable iff for each t, $x(t)$ can be determined exactly when $F(s)$, $H(s)$, $Y(s)$, are known for all $s \in (t_o, t)$, for some $t_o(t) < t$.

Theorem VII.6 (Bucy): The system (F,H) is completely observable iff for every t there exists $t_o(t) < t$ such that

$$M(t,t_o) = \int_{t_o}^{t} \Phi^T(s,t) H^T(s) H(s) \Phi(s,t)\, ds > 0 \qquad (70)$$

(i.e., is positive definite); Φ is the fundamental matrix of F.

Proof: (i) $M(t,t_o) > 0 \Rightarrow M^{-1}(t,t_o)$ exists. But then

$$Y(s) = H(s)\Phi(s,t)X(t) \qquad (71)$$

$$\int_{t_o}^{t} \Phi^T(s,t) H^T(s) Y(s)\, ds = \int_{t_o}^{t} \Phi^T(s,t) H^T(s) H(s) \Phi(s,t)\, ds \cdot X(t)$$

$$= M(t,t_o)X(t) \text{ , so that} \qquad (72)$$

$$X(t) = M^{-1}(t,t_o) \int_{t_o}^{t} \Phi^T(s,t) H^T(s) Y(s)\, ds \qquad (73)$$

describes $X(t)$ in terms of observations.

(ii) $M(t,t_o) \not> 0 \Rightarrow$ there exists $X_o \neq 0$ such that $X_o^T M(t,t_o) X_o = 0$. Then

$$X_o^T M(t,t_o) X_o = \int_{t_o}^{t} \| H(s)\Phi(s,t)X_o \|^2_{M(t,t_o)}\, ds = 0, \qquad (74)$$

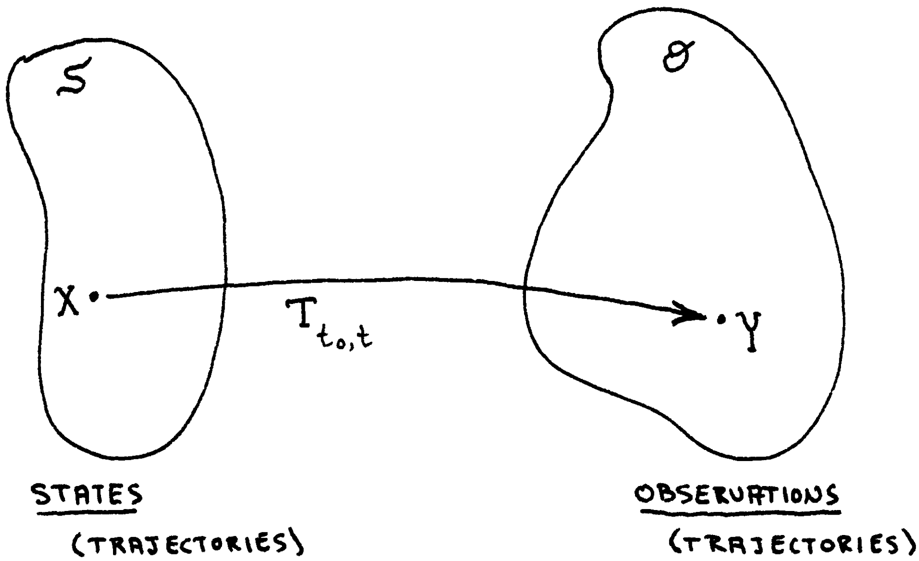
so that $$H(s)\Phi(s,t)X_o = 0 \quad \forall\ s \in (t_o,t). \qquad (75)$$

Then $X = X_0$ and $X = 0$ produce identical observation functions over (t_0,t), contradicting complete observability of (F,H).

Q.E.D.

Comment: We have been concerned here with the mapping $T_{t_0,t}$: $\mathcal{S} \rightarrow \mathcal{O}$, which carries a state function, $X(S)$, into an observation function, $Y(S)$, $S \in (t_0,t)$:

Fig. VII.3



Theorem VII.6 establishes $\text{Ker}(T_{t,t_0}) = \{0\} \Leftrightarrow M(t,t_0) > 0$; that is, we demonstrated that the map is 1:1 $\Leftrightarrow M(t,t_0) > 0$. Note that the form of Theorem VII.6 makes it immediately applicable to time-discrete systems.

Given that perfect observations imply unique states, we inquire as to the effect of imperfect observation (V) and random system inputs (U) on this separation; in Section VII.5, we consider this problem, where the following definitions will become useful:

<u>Defn VII.2</u>: (F,H) is <u>uniformly R observable</u> iff there exist positive reals α_1, β_1, Δ_1, such that, for all t,

$$\beta_1 I > \int_t^{t+\Delta_1} \Phi^T(s, t_0+\Delta_1) H^T(s) R^{-1}(s) H(s) \Phi(s, t_0+\Delta_1)\, ds > \alpha_1 I > 0. \quad (7$$

(This integral will be abbreviated $W_R(t+\Delta_1, t)$; Δ_1 is referred to as the interval of observability.)

A theory of duality between the stochastic filtering problem as stated at the beginning of VII and a deterministic optimal regulator control system problem exists [11]. Although we shall not discuss this topic, we mention it in justification of the terminology of the following definition:

<u>Defn VII.3</u>: (F,G) is <u>uniformly Q controllable</u> iff there exist positive reals α_2, β_2, Δ_2, such that, for all t,

$$\beta_2 I > \int_t^{t+\Delta_2} \Phi(t_0+\Delta_2, s) G(s) Q(s) G^T(s) \Phi^T(t_0+\Delta_2, s)\, ds > \alpha_2 I > 0. \quad (7$$

(This integral will be abbreviated $C_Q(t+\Delta_2, t)$; Δ_2 is referred to as the interval of controllability.)

In the next section, we shall see that W_R and C_Q determine uniform bounds on the uncertainty (P) in optimal estimates based upon observation.

VII.5 The Riccati Equation: Solution and Stability

A prime target for study in filtering theory is the matrix Riccati equation, which has been shown to be the description of filtering error covariance:

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) \quad . \qquad (78)$$

Although existence and uniqueness of solutions to (78) is a valid mathematical question, the origin of the differential equation in this case dictates both answers.

A closed form solution to various cases of (78) was discovered by Levin, Reid, Radon, and others [14]; this solution transfers the computational problem to that of solving a time-dependent first order system of linear differential equations (this system is stationary whenever F,H,G,Q, and R are stationary).

Stability theory is of interest in filtering in the following senses:

(1) The stability of solutions to (78) describes the stability of the performance (quality, certainty) of the estimator; this stability is in the "classic"

direction of deterministic stability theory, as developed by Lyapunov.

(2) The stability of solutions to the filtering equation (33) and the error equation (58) describes the assurance of the occurrence of sample function solutions to these stochastic equations of desirable stability; this is undertaken through stochastic stability theory, as developed by Bucy [4]. We do not pursue this topic.

(3) The stability of (1) and (2) in turn dictate the stability of computational error in exercising the filtering equations (33) and (34) on a limited-precision computing device, digital or analog.

The results discussed in this section are primarily due to Bucy [4]. We have supplemented or discussed further his proofs where appropriate. We utilize the definitions of the previous section to establish uniform bounds on solutions to (78), and these lead to stability descriptions of the Riccati and unforced filtering equations.

<u>Theorem VII.7</u> (A) (Levin et al): The solution to

$$\dot{P} = G_1 P + P G_2 + G_3 + P G_4 P \qquad (79)$$
$$P(t_0) = P_0$$

is given by

$$P(t) = \left[M_{11}(t-t_0)P_0 + M_{12}(t-t_0)\right]\cdot\left[M_{21}(t-t_0)P_0 + M_{22}(t-t_0)\right]^{-1}, \qquad (80)$$

(provided the inverse exists), where

$$\mathcal{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{81}$$

is the fundamental matrix of the system

$$\dot{X} = \begin{bmatrix} G_1 & G_3 \\ -G_4 & -G_2 \end{bmatrix} X . \tag{82}$$

(B) (Bucy) For $(G_1, G_2, G_3, G_4) = (F, F^T, GQG^T, -H^T R^{-1} H)$,

the above inverse exists for all $P_o \geq 0$, $t > t_o$.

(C) $P_o > 0 \Rightarrow P(t) > 0$ for all $t \geq t_o$.

Theorem VII.8 (Solution uniform upper bound)(*): (F,H) uniformly R observable with interval of observability Δ_1 implies that for all $t > t_o + \Delta_1$, $P_o \geq 0$,

$$P(t) \leq W_R^{-1}(t, t-\Delta_1) + C_Q(t, t-\Delta_1) , \tag{83}$$

where C_Q and W_R are defined in VII.4.

Theorem VII.9 (Solution uniform lower bound): (F,G) uniformly Q controllable with interval of controllability Δ_2 implies that for all $t > t_o + \Delta_2$, $P_o \geq 0$,

(*) Uniformly in t.

$$\left[C_Q^{-1}(t,t-\Delta_2) + W_R(t,t-\Delta_2)\right]^{-1} \leq P(t) \quad . \tag{84}$$

Theorem VII.10 (Stability of variance and unforced filter):
(F,G,H) uniformly Q controllable and uniformly R observable, and the existence of $\alpha_0 > 0$ such that

$$GQG^T > \alpha_0 I > 0 \quad \text{for all } t > t_0 \tag{85}$$

imply

(a) The unforced filter

$$\dot{\hat{x}} = \left[F - PH^T R^{-1} H\right]\hat{x} \tag{86}$$

is uniformly asymptotically stable, and

(b) A steady-state solution to the Riccati equation exists, independent of initial condition, such that, for $t > t_o$, $P_1(t)$, $P_2(t)$ solutions to (78),

$$\| P_1(t) - P_2(t) \| \leq A e^{-c(t-t_0)}, \tag{87}$$

where A and C are positive reals dependent only on $P_1(t_o)$ and $P_2(t_o)$.

These theorems complete a line of thought, and are proved below. In the case of an autonomous system, we have a further development of Bucy:

Theorem VII.11 (Direction of solution change): For $P_o \geq 0$, and $t \geq t_o$, and in the autonomous case,

$$\dot{P}(t) = D^{T^{-1}}(t-t_o)\left[FP_o + P_oF^T + GQG^T - P_oH^TR^{-1}HP_o\right]D^{-1}(t-t_o), \quad (88)$$

where

$$D(t-t_o) = [M_{21}(t-t_o)P_o + M_{22}(t-t_o)] \quad . \quad (89)$$

Theorem VII.12 (Character of zero-start solution): In the autonomous case, and for $P_o = 0$,

(A): $P(t)$ is monotone non-decreasing

$$t_1 > t_2 \Rightarrow P(t_1) \geq P(t_2) \quad . \quad (90)$$

(B): If, further, (F,H) is completely observable, then

$$P_\infty = \lim_{t \to \infty} P(t) \quad (91)$$

exists, satisfies

$$0 = FP_\infty + P_\infty F^T + GQG^T - P_\infty H^TR^{-1}HP_\infty , \quad (92)$$

and $P_\infty \geq 0$ is constant.

Theorem VII.13 (Characterization of all autonomous steady state solutions): In the autonomous case, if (F,G,H) is completely observable and controllable, then for all $P_o \geq 0$, P_∞ exists and is the unique constant

$$P_\infty = \lim_{t\to\infty} \overline{P}(t), \tag{93}$$

where $\dot{\overline{P}}(t) = 0$.

We remark that Theorem VII.13 substantiates the empirically observed unique steady state filter of Example V.1, Figure V.1.1 and V.1.4.

Proof of Theorem VII.7:

(A) By direct substitution of (80) in (79), using (82),

$$\begin{aligned}
\dot{P}(t) = {} & [\dot{M}_{11}P_0 + \dot{M}_{12}][M_{21}P_0 + M_{22}]^{-1} \\
& - [M_{11}P_0 + M_{12}][M_{21}P_0 + M_{22}]^{-1}[\dot{M}_{21}P_0 + \dot{M}_{22}][M_{21}P_0 + M_{22}]^{-1} \\
= {} & G_1[M_{11}P_0 + M_{12}][M_{21}P_0 + M_{22}]^{-1} \\
& + [M_{11}P_0 + M_{12}][M_{21}P_0 + M_{22}]^{-1}G_2[M_{21}P_0 + M_{22}][M_{21}P_0 + M_{22}]^{-1} \\
& + G_3[M_{21}P_0 + M_{22}][M_{21}P_0 + M_{22}]^{-1} + [M_{11}P_0 + M_{12}][M_{21}P_0 + M_{22}]^{-1} \\
& \times G_4[M_{11}P_0 + M_{12}][M_{21}P_0 + M_{22}]^{-1} \\
= {} & G_1 P + P G_2 + G_3 + P G_4 P,
\end{aligned} \tag{94}$$

provided the inverse exists.

(B) Suppose the inverse does not exist for some $t = t_1$. Then by the continuity of the fundamental matrix, there exists $\eta \neq 0$ such that

$$\left[M_{21}(t_1 - t_0)P_0 + M_{22}(t_1 - t_0)\right]\eta = 0 . \tag{95}$$

Define $X(t)$, $Y(t)$ by

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} F & GQG^T \\ H^T R^{-1} H & -F^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad ; \quad \begin{bmatrix} X \\ Y \end{bmatrix}(t_0) = \begin{bmatrix} P_0 \eta \\ \eta \end{bmatrix} . \tag{96}$$

Then

$$\begin{bmatrix} X \\ Y \end{bmatrix}(t_1) = M(t_1 - t_0)\begin{bmatrix} P_0 \eta \\ \eta \end{bmatrix} = \begin{bmatrix} \overline{} \\ 0 \end{bmatrix} , \tag{97}$$

and from (96),

$$\begin{aligned} \frac{d}{dt}(Y^T X) &= \dot{Y}^T X + Y^T \dot{X} \\ &= X^T F^T Y + Y^T GQG^T Y + X^T H^T R^{-1} H X - X^T F^T Y \\ &= \| G Y \|_Q^2 + \| H X \|_{R^{-1}}^2 , \end{aligned} \tag{98}$$

so that

$$Y^T(t_1) X(t_1) = Y^T(t_0) X(t_0) + \int_{t_0}^{t} \| G Y \|_Q^2 + \| H X \|_{R^{-1}}^2 \, ds . \tag{99}$$

From (96) and (97),

$$0 = \|\eta\|^2_{P_0} + \int_{t_0}^{t} \|GY\|^2_Q + \|\mathcal{H}X\|^2_{R^{-1}} \, ds \quad . \tag{100}$$

(If $P_0 > 0$, this implies $\eta = 0$, and Q.E.D.; however, we must consider $P_0 = 0$). From (100), $H(s)X(s) = 0$ for all $s \in (t_0, t_1)$, so that, from (96),

$$\dot{Y} = -F^T Y \quad , \text{on} \quad (t_0, t_1) . \tag{101}$$

Then $Y(t_1) = 0 \Rightarrow Y(0) = 0 \Rightarrow \eta = 0$, and Q.E.D.

(C) Suppose $P_0 > 0$, and there exists t_1, η such that $\|\eta\|^2_{P(t_1)} = 0$. Then let X, Y be defined by (96), so that

$$\begin{bmatrix} X \\ Y \end{bmatrix}(t_1) = M \begin{bmatrix} X \\ Y \end{bmatrix}(t_0) = \begin{bmatrix} \mathcal{L}(t_1 - t_0)\eta \\ \mathcal{D}(t_1 - t_0)\eta \end{bmatrix} , \tag{102}$$

where

$$\begin{aligned} \mathcal{L} &= M_{11} P_0 + M_{12} \\ \mathcal{D} &= M_{21} P_0 + M_{22} \quad . \end{aligned} \tag{103}$$

Then by (80), (99), and (100),

$$Y^T(t_1) X(t_1) = \eta^T \mathcal{D}^T(t_1 - t_0) \mathcal{L}(t_1 - t_0) \eta \tag{104}$$

$$= (\mathcal{D}\eta)^T(\mathcal{L}\mathcal{D}^{-1})(\mathcal{D}\eta)$$

$$= \|\mathcal{D}\eta\|^2_{P(t_1)}$$

$$= \|\eta\|^2_{P_0} + \int_{t_0}^{t_1} \|GY\|^2_Q + \|HX\|^2_{R^{-1}}\, ds .$$

But $\|\eta\|^2_{P(t_1)} = 0 \Rightarrow \|\mathcal{D}\eta\|^2_{P(t_1)} = 0 \Rightarrow$
$\|\eta\|^2_{P_0} = 0 \Rightarrow \eta = 0$, and Q.E.D.

Example VII.3: Scalar State Estimation (Refer to Section III.1)

We apply Theorem VII.7 to solve the Riccati variance equation arising from the system

$$\begin{aligned} \dot{x} &= 0 \cdot x + u(t) \quad &; E[u(s)u(t)] = \delta(s-t)\,\sigma_Q^2 \\ y &= x + v(t) \quad &; E[v(s)v(t)] = \delta(s-t)\,\sigma_R^2 \end{aligned} \tag{105}$$

(Note that (105) is the continuous form of Example III.1)

(78) becomes

$$\dot{p}(t) = \sigma_Q^2 - \frac{p^2(t)}{\sigma_R^2} \tag{106}$$

Following Theorem VII.7,

$$\dot{M} = \begin{bmatrix} 0 & \sigma_Q^2 \\ \frac{1}{\sigma_R^2} & 0 \end{bmatrix} M = AM \quad . \tag{107}$$

The eigenvalues of A are $\lambda_{1,2} = \pm\left(\frac{\sigma_Q}{\sigma_R}\right)$, assuming $\sigma_R, \sigma_Q > 0$; the corresponding solutions to (107) are

$$M_{1,2} = \begin{bmatrix} 1 & \pm\,\sigma_R\sigma_Q \\ \pm\,\frac{1}{\sigma_R\sigma_Q} & 1 \end{bmatrix} e^{\pm\left(\frac{\sigma_Q}{\sigma_R}\right)(t-t_o)} \quad . \tag{108}$$

To satisfy $M(0) = I$ (letting $t_o = 0$),

$$M(t) = \tfrac{1}{2}M_1 + \tfrac{1}{2}M_2$$

$$= \begin{bmatrix} \cosh\left(\frac{\sigma_Q}{\sigma_R}t\right) & (\sigma_R\sigma_Q)\sinh\left(\frac{\sigma_Q}{\sigma_R}t\right) \\ \left(\frac{1}{\sigma_R\sigma_Q}\right)\sinh\left(\frac{\sigma_Q}{\sigma_R}t\right) & \cosh\left(\frac{\sigma_Q}{\sigma_R}t\right) \end{bmatrix} . \tag{109}$$

Then from (80),

$$p(t) = (\sigma_R \sigma_Q) \left[\frac{p(0)\cosh\left(\frac{\sigma_Q}{\sigma_R}t\right) + (\sigma_R\sigma_Q)\sinh\left(\frac{\sigma_Q}{\sigma_R}t\right)}{p(0)\sinh\left(\frac{\sigma_Q}{\sigma_R}t\right) + (\sigma_R\sigma_Q)\cosh\left(\frac{\sigma_Q}{\sigma_R}t\right)} \right] , \tag{110}$$

and $$K(t) = p(t) / \sigma_R^2 , \quad \text{from } (35).$$

Observe that

$$p_\infty = \lim_{t \to \infty} \left[p(t) \right] = \sigma_R \sigma_Q . \tag{111}$$

The solution in the case $\sigma_Q = 0$ $(\lambda_{1,2} = 0)$ may be calculated by inspection of (106), or by direct calculation (as above), or by the formal calculation

$$\lim_{\sigma_Q \to 0} p(t) = \lim_{\sigma_Q \to 0} (\sigma_R \sigma_Q) \left[\frac{p(0)}{p(0)\left(\frac{\sigma_Q}{\sigma_R}t\right) + \sigma_R \sigma_Q} \right]$$

$$= \left[\frac{p(0)\,\sigma_R^2}{p(0)\,t + \sigma_R^2} \right] , \tag{112}$$

and

$$K(t) = \left[\frac{p(0)}{p(0)\,t + \sigma_R^2} \right] . \tag{113}$$

Observe that (112) and (113) coincide with the discrete solutions V.1.

Example VII.4: Redundant Observation (Refer to Section III.4)

Example III.4, in the limiting case, becomes

$$\dot{x} = 0x + u(t) \quad ; \quad E[u(t)u(s)] = \delta(t-s)\sigma_Q^2 \tag{114}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad ; \quad E[v(t)v^T(s)] = \delta(t-s)\begin{bmatrix} \sigma_{R_1}^2 & 0 \\ 0 & \sigma_{R_2}^2 \end{bmatrix} .$$

The variance equation is

$$\dot{p}(t) = \sigma_Q^2 - \frac{p^2(t)}{(\bar{\sigma}_R)^2} , \tag{115}$$

where

$$\frac{1}{(\bar{\sigma}_R)^2} = H^T R^{-1} H = \frac{1}{\sigma_{R_1}^2} + \frac{1}{\sigma_{R_2}^2} . \tag{116}$$

Thus (110) provides a solution, changing σ_R to $\bar{\sigma}_R$.

Finally,

$$K(t) = p(t)H^T R^{-1} = p(t)\left[\frac{1}{\sigma_{R_1}^2} , \frac{1}{\sigma_{R_2}^2} \right] . \tag{117}$$

This situation is basically unchanged for other observation situations.

Proof of Theorem VII.8

Motivation: (i) For the case $Q = 0$, the solution to (78) is

$$P(t) = A\left(I + A^T W_R A\right)^{-1} A^T, \tag{118}$$

provided the inverse exists, where

$$\begin{aligned} A &= \Phi_F(t, t_0) P_0^{1/2} \\ W_R &= W_R(t, t_0), \end{aligned} \tag{119}$$

since

$$\dot{P} = FP + PF^T - PH^T R^{-1} H P \quad ; \quad P(t_0) = P_0 \tag{120}$$

implies that $L = P^{-1}$ satisfies

$$\dot{L} = (-F^T)L + L(-F^T)^T + H^T R^{-1} H \quad ; \quad L(t_0) = P_0^{-1}. \tag{121}$$

(Note that this is a general means for converting the free Riccati equation to a linear system.) But (121) has solution

$$L(t) = \Phi_{-F^T}(t, t_0) L_0 \Phi_{-F^T}^T(t, t_0) + \int_{t_0}^{t} \Phi_{-F^T}(t, s) H^T R^{-1} H \Phi_{-F^T}^T(t, s)\, ds \tag{122}$$

$$= \left(\Phi_F(t,t_0) P_0 \Phi_F^T(t,t_0)\right)^{-1}$$

$$+ \int_{t_0}^{t} \Phi_F^T(s,t) H^T R^{-1} H \Phi_F(s,t)\, ds ,$$

so that

$$P(t) = L^{-1}(t)$$

$$= A\left[I + A^T W_R A\right]^{-1} A^T , \tag{123}$$

provided the inverse exists.

(ii) For the case $Q = 0$, and (F,H) completely R observable(*), $P(t)$ is bounded from above by $W_R^{-1}(t, t-\Delta(t))$, since

$$L(t) = (AA^T)^{-1} + W_R(t,t_0) \tag{124}$$

$$\geq (AA^T)^{-1} + W_R(t, t-\Delta(t))$$

$$P(t) = L^{-1}(t) \tag{125}$$

$$\leq \left[(AA^T)^{-1} + W_R(t, t-\Delta(t))\right]^{-1}$$

$$= A\left[I + A^T W_R(t, t-\Delta(t)) A\right]^{-1} A^T$$

(*) By complete R observability, we imply that $\forall\, t\ \exists$ $\Delta(t) > 0 \ni W_R(t, t-\Delta(t)) > 0$.

(We note that the indicated inverse exists; see Simmons [16], Theorem 57.F; formally, $(I+B)^{-1} = \sum_0^{\infty}(-1)^n B^n$, if $0 < B < I$, analagous to $\frac{1}{1+r}$.)

We are reminded by (125) of the scalar case

$$p(t) \leq \frac{a^2}{1+a^2 W_R} = \frac{1}{\left(\frac{1}{a^2}\right) + W_R} < \frac{1}{W_R} . \tag{126}$$

This may be established by

$$\begin{aligned} W_R &< W_R + (AA^T)^{-1} \\ W_R^{-1} &> \left(W_R + (AA^T)^{-1}\right)^{-1} \\ &= A\left[I + A^T W_R A\right]^{-1} A^T \\ &\geq P(t) , \end{aligned} \tag{127}$$

these following from the properties of positive-definite matrices [16].

With these thoughts in mind, we turn to Bucy's proof, for the case $Q \neq 0$. Intuition suggests that the variance of error in estimate for the system driven by random input $U(t)$ should be greater than or equal to the variance for the undriven $(Q = 0)$ system provided initial variance is unchanged. Let $K(t)$ be the solution to the free case

$$\begin{aligned} \dot{K} &= FK + KF^T - KH^T R^{-1} H K \\ K(t_0) &= P_0 \geq 0 , \end{aligned} \tag{128}$$

and $P(t)$ the solution to the driven system (78), with $P(t_o) = P_o$. Let $\delta(t) = P(t) - K(t)$. Then

$$\dot{\delta}(t) = F\delta + \delta F^T - PH^TR^{-1}HP + KH^TR^{-1}HK + GQG^T$$

$$= (F - KH^TR^{-1}H)\delta + \delta(F - KH^TR^{-1}H)^T - \delta(H^TR^{-1}H)\delta + GQG^T$$

$$= \bar{F}\delta + \delta\bar{F}^T - \delta(H^TR^{-1}H)\delta + GQG^T \; . \qquad (129)$$

We now have $P(t) \geq K(t)$, as suspected, since $\delta(t) \geq 0$, because

$$f(t) = \int_{t_0}^{t} \Phi(t,s)(GQG^T + \delta H^TR^{-1}H\delta)\Phi^T(t,s)\,ds \; , \qquad (130)$$

Φ the fundamental matrix of $(\bar{F} - \delta H^TR^{-1}H)$, satisfies

$\dot{f}(t) = \dot{\delta}(t)$ and $f(t_0) = \delta(t_0) = 0$, so that

$$\Delta(t) = f(t) \geq 0 \; . \qquad (131)$$

(We have altered Bucy's proof, which appears to be based upon an ill-defined function.)

We have bounded $P(t)$ from the "wrong side," but evade this difficulty through

$$P(t) = \Phi(t,t-\Delta)P(t-\Delta)\Phi^T(t,t-\Delta) + \int_{t-\Delta}^{t} \Phi(t,s)\left[GQG^T - PH^TR^{-1}HP\right]\Phi^T(t,s)\,ds \tag{132}$$

$$\leq \Phi(t,t-\Delta)P(t-\Delta)\Phi^T(t,t-\Delta) + C_Q(t,t-\Delta) - \int_{t-\Delta}^{t} \Phi(t,s)\bar{K}(s)H^TR^{-1}H\bar{K}(s)\Phi^T(t,s)\,ds$$

$$= \bar{K}(t) + C_Q(t,t-\Delta),$$

where $\bar{K}$ satisfies (128) but $\bar{K}(t-\Delta) = P(t-\Delta)$ is the initial condition. Then from (127),

$$P(t) \leq W_R^{-1}(t,t-\Delta) + C_Q(t,t-\Delta). \tag{133}$$

Recalling Definitions VII.2,3, we have in fact shown

$\exists\ \beta_2, \alpha_1 > 0\ ,\ \ni$

$$P(t) \leq \left[\frac{1}{\alpha_1} + \beta_2\right] I, \quad \text{for all } t. \tag{134}$$

Q.E.D.

<u>Proof of Theorem VII.9</u>: From Theorem VII.7, $P_0 > 0 \Rightarrow P^{-1}(t)$ exists for all $t \geq t_o$. But $P^{-1}(t)$ satisfies

$$\begin{aligned} \dot{K} &= (-F^T)K + K(-F^T)^T - KGQG^TK + H^TR^{-1}H \\ K(t_0) &= P_0^{-1} > 0 \end{aligned} \tag{135}$$

Then by Theorem VII.8, exchanging the roles of terms,

$$K(t) \leq \left\{ \int_{t-\Delta}^{t} \Phi_{-F^T}^T(s,t)\, GQG^T \Phi_{-F^T}(s,t)\, ds \right\}^{-1} \tag{136}$$

$$+ \left\{ \int_{t-\Delta}^{t} \Phi_{-F^T}(t,s) H^T R^{-1} H \Phi_{-F^T}^T(t,s)\, ds \right\}$$

$$= \left\{ \int_{t-\Delta}^{t} \Phi_F(t,s) GQG^T \Phi^T(t,s)\, ds \right\}^{-1}$$

$$+ \left\{ \int_{t-\Delta}^{t} \Phi_F^T(s,t) H^T R^{-1} H \Phi_F(s,t)\, ds \right\}$$

$$= C_Q^{-1}(t,t-\Delta) + W_R(t,t-\Delta)$$

Recalling Definitions VII.2,3, we have in fact shown $\exists\ \alpha_2, \beta_1 > 0 \ni$

$$P(t) = K^{-1}(t) \geq \left[\frac{1}{\alpha_2} + \beta_1 \right] I \ , \quad \forall t \ . \tag{137}$$

Q.E.D.

<u>Proof of Theorem VII.10</u>:

(A) We provide a Lyapunov function for $\hat{X}$:

$$V(\hat{X},t) = \|\hat{X}\|^2_{P^{-1}(t)} = \hat{X}^T \left(P^{-1}(t)\right) \hat{X} \tag{138}$$

(see Appendix IX.10). V satisfies the Lyapunov criteria:

(i) $\mathcal{V}(0,t) \equiv 0$.

(ii) By Theorems VII.8 and VII.9, if Δ = max of intervals of observability and controllability, there exist reals $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\gamma_1 I = \left[\frac{1}{\alpha_1} + \beta_2\right]^{-1} I \leq P^{-1}(t) \leq \left[\frac{1}{\alpha_2} + \beta_1\right]^{-1} I = \gamma_2 I, \tag{139}$$

for all $t > t_o + \Delta$. Then $\exists\ \delta_1, \delta_2 > 0 \ni$ $\hat{X} \neq 0$, $t > t_o + \Delta$ implies

$$0 < \delta_1 \|\hat{X}\|^2 < \mathcal{V}(\hat{X}, t) < \delta_2 \|\hat{X}\|^2 . \tag{140}$$

(ii) The rate of change of $\mathcal{V}$ along motions of (86) is

$$\dot{\mathcal{V}}(\hat{X},t) = \dot{\hat{X}}^T P^{-1} \hat{X} + \hat{X}^T P^{-1} \dot{\hat{X}} - \hat{X}^T P^{-1} \dot{P} P^{-1} \hat{X} \qquad (141)$$

$$= \hat{X}^T [F - PH^T R^{-1} H]^T P^{-1} \hat{X} + \hat{X}^T P^{-1} [F - PH^T R^{-1} H] \hat{X}$$

$$- \hat{X}^T P^{-1} [FP + PF^T + GQG^T - PH^T R^{-1} HP] P^{-1} \hat{X}$$

$$= -\hat{X}^T H^T R^{-1} H \hat{X} - \hat{X}^T P^{-1} G Q G^T P^{-1} \hat{X}$$

$$\leq -\|\hat{X}\|^2_{P^{-1}GQG^T P^{-1}} \qquad (141)$$

By (85), $\exists\ \alpha_0 > 0$ such that

$$P^{-1}GQG^TP^{-1} \geq \alpha_0 P^{-1}P^{-1} \tag{142}$$

$$\geq \alpha_0 \gamma_1^2 I$$

$$= \gamma_3 I$$

$$> 0 .$$

Then (141) becomes

$$\dot{v}(\hat{x},t) \leq -\gamma_3 \|\hat{x}\|^2 < 0 . \tag{143}$$

Then (86) is uniformly asymptotically stable about $\hat{X} = 0$.

(B) By the Lyapunov-Perron-Malkin theorem (Appendix IX.10), $\exists$ reals $A, C > 0 \ni$

$$\|\Phi_{F-PH^TR^{-1}H}(t,s)\| < A e^{-c(t-s)} . \tag{144}$$

Using $P_1(t)$, $P_2(t)$ defined by hypothesis, and $\Delta = P_1 - P_2$,

$$\dot{\Delta} = \dot{P}_1 - \dot{P}_2 \tag{145}$$

$$= (FP_1 + P_1F^T + GQG^T - P_1H^TR^{-1}HP_1)$$
$$-(FP_2 + P_2F^T + GQG^T - P_2H^TR^{-1}HP_2)$$

$$= (F - P_1H^TR^{-1}H)\Delta + \Delta(F - P_1H^TR^{-1}H)^T ,$$

So that

$$\|\Delta\| = \|\Phi_{F-P_1H^TR^{-1}H}(t,t_0)\{P_1(t_0) - P_2(t_0)\}\Phi^T_{F-P_1H^TR^{-1}H}(t,t_0)\| \qquad (146)$$

$$\leq A\|P_{10} - P_{20}\|e^{-2c(t-s)}.$$

Q.E.D.

Proof of Theorem VII.11

Motivation: Consider the scalar case of Example VII.3; specifically, equation (110), which leads to

$$\dot{p}(t) = \frac{\sigma_Q^2 - \left(\frac{P_0^2}{\sigma_R^2}\right)}{\left[p(0)\frac{1}{\sigma_R\sigma_Q}\cdot\sinh\left(\frac{\sigma_Q}{\sigma_R}t\right) + \cosh\left(\frac{\sigma_Q}{\sigma_R}t\right)\right]^2}, \qquad (147)$$

in agreement with (88).

Proof: From Theorem VII.7,

$$P(t) = L(t)D^{-1}(t), \qquad (148)$$

where

$$L(t) = M_{11}(t)P_0 + M_{12}(t) \qquad (149)$$

$$D(t) = M_{21}(t)P_0 + M_{22}(t),$$

and

$$t_0 = 0, \quad \text{w.l.o.g.}$$

Then

$$\dot{P}(t) = \dot{L}D^{-1} - LD^{-1}\dot{D}D^{-1}$$

$$= D^{T^{-1}} \left[D^T \dot{L} - D^T L D^{-1} \dot{D} \right] D^{-1}$$

$$= D^{T^{-1}} \left[D^T \dot{L} - L^T \dot{D} \right] D^{-1}, \tag{150}$$

the last line following from

$$P^T = D^{T^{-1}} L^T = P = L D^{-1}. \tag{151}$$

Now from (149) and (82),

$$\begin{aligned} \dot{L} &= FL + GQG^T D \\ \dot{D} &= H^T R^{-1} H L - F^T D, \end{aligned} \tag{152}$$

so that

$$\left[D^T \dot{L} - L^T \dot{D} \right] = D^T F L + D^T G Q G^T D - L^T H^T R^{-1} H L + L^T F^T D \tag{153}$$

$$= \left[M_{22}^T F M_{11} + M_{22}^T G Q G^T M_{21} - M_{12}^T H^T R^{-1} H M_{11} + M_{12}^T F^T M_{21} \right] P_0$$

$$+ P_0 \left[M_{22}^T F M_{11} + M_{22}^T G Q G^T M_{21} - M_{12}^T H^T R^{-1} H M_{11} + M_{12}^T F^T M_{21} \right]^T$$

$$+ \left[M_{22}^T F M_{12} + M_{22}^T G Q G^T M_{22} - M_{12}^T H^T R^{-1} H M_{12} + M_{12}^T F^T M_{22} \right]$$

$$- P_0 \left[M_{11}^T H^T R^{-1} H M_{11} - M_{21}^T F M_{11} - M_{21}^T G Q G^T M_{21} - M_{11}^T F^T M_{21} \right] P_0$$

$$= F P_0 + P_0 F^T + G Q G^T - P_0 H^T R^{-1} H P_0 .$$

The last line of (153) follows from this reasoning:

$$\text{Let} \quad A = \begin{bmatrix} F & GQG^T \\ H^T R^{-1} H & -F^T \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \tag{154}$$

$$R(t) = M^T J A M \ . \tag{155}$$

Then

$$\begin{aligned} \dot{R}(t) &= M^T A^T J A M + M^T J A A M \\ &= M^T [A^T J + J A] A M \\ &= M^T [\,0\,] A M = 0 , \end{aligned} \tag{156}$$

and $R(0) = JA$, so

$$M^T J A M \equiv J A \ . \tag{157}$$

The four matrix components of (157) lead to (153).

Q.E.D.

<u>Comment</u>: Equation (88) makes the direction of solution change clear, in the event $\dot{P}_o$ is sign definite. Such consideration leads us to

<u>Proof of Theorem VII.12</u>

(A) From the previous theorem, for $P_o = 0$,

$$\dot{P}(t) = D^{T^{-1}} G Q G^T D^{-1}, \tag{158}$$

so that, for all X,

$$\frac{d}{dt}\left(\|X\|^2_{P(t)}\right) = \|D^{-1}X\|^2_{GQG^T} \geq 0, \tag{159}$$

and so $t_1 > t_2$ implies

$$P(t_1) \geq P(t_2) \tag{160}$$

Comment: It is intuitively reasonable that a filter would exhibit monotone non-decreasing variance under these conditions.

(B) In the autonomous case, complete observability implies uniform observability, so that, by Theorem VII.8, P is bounded from above. It follows from part (A) that there exists a constant, non-negative definite matrix P_∞, such that

$$P_\infty = \lim_{t\to\infty} P(t), \tag{161}$$

and (92) follows from this definition.

Q.E.D.

Proof of Theorem VII.13:

This result follows directly from Theorems VII.10 and VII.12. A consequence is the fact that every completely observable, completely controllable autonomous system has an associated steady-state stationary filter, which is the solution to the Wiener problem.

Example VII.4 (Refer to Section V.1)

We may solve for the steady-state filter of Section V.1 by setting the variance difference equation to its steady-state configuration and solving the resulting quadratic for P_∞:

From (30),

$$P_\infty = (I - P_\infty H^T R^{-1} H)(\Phi P_\infty \Phi^T + Q)$$

$$= (1 - P_\infty/\sigma_R^2)(P_\infty + \sigma_Q^2) \tag{162}$$

$$P_\infty = \frac{-\sigma_Q^2 + \sqrt{\sigma_Q^4 + 4\sigma_Q^2\sigma_R^2}}{2} \cong 0.65$$

$$K_\infty = P_\infty/\sigma_R^2 \cong 0.13 ,$$

in agreement with Figure V.1.

We may likewise utilize Theorem VII.13 to calculate the limiting variance for the analagous continuous filter:

$$0 = FP_\infty + P_\infty F^T + GQG^T - P_\infty H^T R^{-1} H P_\infty \tag{163}$$

$$= \sigma_Q^2 - P_\infty^2/\sigma_R^2$$

$$P_\infty = \sigma_R \sigma_Q \cong 0.71$$

$$K_\infty = \frac{\sigma_Q}{\sigma_R} \cong 0.14 \qquad \text{(see also (111))}.$$

This difference between (162) and (163) is accounted for by the non-linearity of the Riccati Equation: the difference equation is not an exact sampling of the differential form, as would be the case for linear equations.

VIII. ADAPTIVE FILTERING

VIII.1 Introduction

The ability of the K-B filter to perform minimum variance estimation is dependent on our ability to provide an a priori description of the statistical environment $(R(t), Q(t))$. Given such a description, the filter is an optimal estimator even in non-stationary environments, but this optimality is lost if we are unable to provide such a description. Two cases of this sort come to mind:

(1) Data on the filter environment are inadequate or unavailable (a familiar situation in real life undertakings).

(2) The statistical environment is known to be non-stationary to the extent that a fixed environment assumption produces a filter whose performance is unsatisfactory, and a prediction of the variation in statistics is not available.

In such situations, we are led to consider the construction of an <u>adaptive</u> filter--one which adjusts its coefficients (passband, transfer function, etc.) based upon observations of the environment implicit in the ("usual") observation process. The design of adaptive filters has been considered in numerous cases [9].

Adaptive filters often operate by computing a sampled inner product (cross-correlation) between observations and estimator error, perhaps made available through the transmission of a known test signal through the channel. This leads to gradient adjustments in the filter coefficients, toward minimizing variance.

In this section, we consider a new approach--the design of an adaptive Kalman-Bucy form filter, based on the following ideas:

(A) The error variance of an arbitrary (i.e., sub-optimal) linear filter is governed by a system of <u>linear</u> differential equations, (VII.62). The non-linear Riccati equation arises through the optimal connection of coefficient $K(t)$ to variance $P(t)$; however, the fact that $K(t)$ will be a function of known <u>history</u> allows us to replace VII.63 with a <u>linear</u> differential equation.

(B) Uncertainty in environment $(R(t),Q(t))$ renders filter error variance uncertain. However, the error signal $(Y(t) - H(t)\hat{X}(t))$ leads to a corrupted observation of filter error variance and observation noise variance. Stated another way, we may learn about the statistical environment by observing the response of a linear system (the sub-optimal filter) to that environment.

VIII.2 An Adaptive Kalman-Bucy Filter

The adaptive filtering equations we propose for the general case are

$$\dot{\hat{X}} = F\hat{X} + K(Y - H\hat{X}) \tag{1}$$

$$K = \hat{P}H^T\hat{R}^{-1} \tag{2}$$

$$\mathcal{E} = (Y - H\hat{X})(Y - H\hat{X})^T \tag{3}$$

$$\alpha = \mathcal{E} - \left[H\hat{P}H^T + \hat{R} + \tfrac{1}{2}HK\hat{R} + \tfrac{1}{2}\hat{R}K^TH^T\right] \tag{4}$$

$$\dot{\hat{P}} = F\hat{P} + \hat{P}F^T + G\hat{Q}G^T - \hat{P}H^T\hat{R}^{-1}H\hat{P} + C_P\alpha C_P^T \tag{5}$$

$$\dot{\hat{R}} = C_R\alpha C_R^T \tag{6}$$

$$\dot{\hat{Q}} = C_Q\alpha C_Q^T \tag{7}$$

We restrict our discussion to the scalar case $n = m = 1$, and assume the $Q(t)$ is known. That is, we are interested in the case of adaptation to unknown channel noise power. Then $\hat{Q} = Q$ is known precisely, $C_Q = 0$, and (7) is eliminated. It will be seen that the treatment below could be applied equally well to the case $Q(t)$ unknown.

In the case described, (3) and (5) become

$$\mathcal{E} = (Y - H\hat{X})^2 \tag{8}$$

$$\dot{\hat{P}} = 2(F - C_P^2H^2)\hat{P} + GQG^T - \hat{P}H^T\hat{R}^{-1}H\hat{P} + C_P^2\left[\mathcal{E} - (I + HK)\hat{R}\right] \tag{9}$$

$C_{P,R,Q}$ are filter coefficients pertaining to the estimation of P, R, and Q, respectfully.

Recalling the observations (A) and (B) above, we justify these equations. The filter equation (1) defines a sub-optimal unbiased linear estimator. Its associated <u>actual</u> error variance equation is (VII.62), which becomes

$$\dot{P} = 2(F-KH)P + GQG^T + K^2(t)R \tag{10}$$

Assuming R is subject only to random variation, the resulting statistical situation is described by the linear system

$$\begin{bmatrix} \dot{P} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} 2(F-KH) & K^2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P \\ R \end{bmatrix} + \begin{bmatrix} GQG^T \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_R \end{bmatrix}, \tag{11}$$

or

$$\dot{Z} = \mathfrak{F} Z + U_z . \tag{12}$$

(Recall that K(t), although sub-optimal, will be a function of known past history.) The observation model associated with (12) is

$$\begin{aligned} \mathcal{E} &= (Y - H\hat{X})(Y - H\hat{X})^T \\ &= (H\tilde{X} + V)(H\tilde{X} + V)^T \\ &= E\left[H\tilde{X}\tilde{X}^T H^T + VV^T + H\tilde{X}V^T + V\tilde{X}^T H^T\right] + V_{\mathcal{E}} \end{aligned}$$

$$= \left[HPH^T + R + \tfrac{1}{2} HKR + \tfrac{1}{2} RK^T H^T \right] + V_\varepsilon$$

$$= \mathcal{H} Z + V_\varepsilon \tag{13}$$

where V_ε is a mean-zero matrix random process; our new observable $\mathcal{E}$ is a known linear combination of the states of (12), corrupted by observation noise. Then (12) and (13) constitute a filtering problem in the canonical form (VII.1, 2), as promised by (A) and (B). (We note that the usual assumptions about the system driving and observation corrupting processes are no longer valid, and that the deterministic forcing function is handled within the existing framework by the technique of Section VI.1).

The <u>form</u> of an estimator for P and R is immediately suggested; however, the filter coefficients in this estimator cannot be determined as before:

$$\dot{\hat{Z}} = (\mathcal{F} - C\mathcal{H})\hat{Z} + C\mathcal{E}, \tag{14}$$

or

$$\dot{\hat{P}} = 2(F-KH)\hat{P} + K^2\hat{R} + GQG^T + C_P^2\{\mathcal{E} - H\hat{P}H^T - \hat{R} - HK\hat{R}\}$$

$$\dot{\hat{R}} = C_R^2\{\mathcal{E} - H\hat{P}H^T - \hat{R} - HK\hat{R}\}$$

as described by (5,6).

The coefficient submatrices C_P *and* C_R *of* C *cannot* be determined by minimum-variance arguments, for the statistics of $\mathbf{V}_\varepsilon$ and U_Z are dependent upon the higher moments of $V(t)$ and $U(t)$, of the physical system; these are by hypothesis at least in part unknown. Other criteria, such as transient response or stability, may be used to select C.

Other questions which arise are:

(1) What is the relationship between the performance of (1-7) and that of the ideal adaptive estimator, the conventional Kalman-Bucy filter informed of environmental statistics?

(2) What are the stability properties of (1-7)?

We content ourselves here with answering the <u>deterministic</u> form of the above questions; that is, the <u>stochastic</u> stability of (1-7) will not be discussed.

VIII.3 <u>Deterministic Stability</u>

Kalman and Bucy observe in their fundamental article [13] that the solution to the Riccati variance equation (VII.34) establishes a basis for comparison between actual and ideal adaptive behavior. That is, an ideal adaptive estimator would immediately be aware of any changes in its statistical environment $(R(t),Q(t))$, and thereby function precisely as the "informed" Kalman-Bucy filter. The solution to (VII.63) is thus suitable for comparison to a proposed adaptive estimator, as an idealized performance standard.

We have found no evidence of pursuit of this idea in the literature, but propose to follow it here. We shall be concerned with three variance variables:

$P(t)$ the actual error variance associated with the adaptive estimator (1-7), which cannot be precisely known by the system.

$\hat{P}(t)$ the estimated error variance computed by (5).

$P^*(t)$ the idealized error variance, which would be the actual variance for the conventional (informed) K-B filter operating in the actual environment $(R(t), Q(t))$ present.

Our principal result is the demonstration of the uniform asymptotic stability of both P and $\hat{P}$ about P^* for the scalar case described above. This establishes the local convergence (1) of $\hat{P}$ to the actual error variance, providing a reliable computed measure of estimate quality, and (2) of the actual variance to the idealized variance, demonstrating a form of limiting optimal behavior.

We consider the deterministic case $V_\epsilon = 0$; the equations governing P, $\hat{P}$, and P^* are then

$$\dot{P}(t) = 2(F - \hat{P}H^T\hat{R}^{-1}H)P + GQG^T + \hat{P}H^T\hat{R}^{-1}R\hat{R}^{-1}H\hat{P} \tag{17}$$

$$\dot{\hat{P}}(t) = 2F\hat{P} + GQG^T - \hat{P}H^T\hat{R}^{-1}H\hat{P} + C_P^2\left\{H[P-\hat{P}]H^T + (HK+I)(R-\hat{R})\right\} \tag{18}$$

$$\dot{P}^*(t) = 2FP^* + GQG^T - P^*H^TR^{-1}HP^* \tag{19}$$

Defining deviations by

$$\begin{aligned} \Delta P &= P^* - \underline{P} \\ \delta P &= P^* - \hat{P} \\ \delta R &= R - \hat{R} \\ \Delta R &= R^{-1} - \hat{R}^{-1}, \end{aligned} \tag{20}$$

we have, from (17), (18), (19),

$$(\dot{\delta P}) = 2(F - P^* H^T R^{-1} H)(\delta P) - \hat{P} H^T (\Delta R) H \hat{P} + \delta P (H^T R^{-1} H) \delta P - C_P^2 (H\{\delta P - \Delta P\} H^T + \{HK + 1\}\delta R) \tag{21}$$

$$(\dot{\Delta P}) = (2F)(\Delta P) - P^* H^T R^{-1} H P^* + 2(P^* - \delta P) H^T (R^{-1} - \Delta R) H (P^* - \Delta P) - (P^* - \delta P) H^T (R^{-1} - \Delta R) R (R^{-1} - \Delta R) H (P^* - \delta P) \tag{22}$$

$$(\dot{\delta R}) = -C_R^2 (H\{\delta P - \Delta P\} H^T + \{HK + 1\}\delta R) \tag{23}$$

We shall demonstrate that (δP, ΔP, δR) is u.a.s. (uniformly asymptotically stable) about (o,o,o). Inspection of (21), (22), (23) reveals that only terms containing deviations to first or higher power survive. Further, if we linearize in the deviation variables about (o,o,o), these equations become

$$\begin{bmatrix} \delta\dot{P} \\ \\ \Delta\dot{P} \\ \\ \delta\dot{R} \end{bmatrix} \cong \begin{bmatrix} 2\left(F - P^{*}H^{T}R^{-1}H - \frac{C_P^2}{2}HH^{T}\right) & \left(C_P^2 HH^{T}\right) & \left(P^{*}H^{T}R^{-2}HP^{*} - C_P^2\{HP^{*}H^{T}R^{-1}+1\}\right) \\ \\ (0) & 2\left(F - P^{*}H^{T}R^{-1}H\right) & (0) \\ \\ \left(-C_R^2 HH^{T}\right) & \left(C_R^2 HH^{T}\right) & \left(-C_R^2\right)\left(HP^{*}H^{T}R^{-1}+1\right) \end{bmatrix} \cdot \begin{bmatrix} \delta P \\ \\ \Delta P \\ \\ \delta R \end{bmatrix},$$

or

$$\dot{\Delta} = A(t)\,\Delta \tag{25}$$

Now in the autonomous case (R,Q,F,H constant), Theorem VII.13 implies that

$$P_{\infty}^{*} = \lim_{t\to\infty} P^{*}(t) \tag{26}$$

exists as a constant, provided (F,G,H) is completely observable and controllable. Then a limiting form of (24) exists, namely, the autonomous linear system obtained by replacing P^{*} with P_{∞}^{*} in (24). Letting A denote the limiting form of $A(t)$, we find that

$$\begin{aligned} |A - \lambda I| = & \left\{2\left[F - P_{\infty}^{*}H^{T}R^{-1}H\right] - \lambda\right\} \cdot \Big\{\Big(2\left[F - P_{\infty}^{*}H^{T}R^{-1}H - \tfrac{1}{2}C_P^2 HH^{T}\right] - \lambda\Big)\cdot\Big(-\lambda \\ & - C_R^2\left[HP_{\infty}^{*}H^{T}R^{-1}+1\right]\Big) + C_R^2 HH^{T}\Big(P_{\infty}^{*}H^{T}R^{-2}HP_{\infty}^{*} \\ & - C_P^2\left[HP_{\infty}^{*}H^{T}R^{-1}+1\right]\Big)\Big\} \end{aligned} \tag{27}$$

$$= \left\{2\left[F - P_\infty^* H^T R^{-1} H\right] - \lambda\right\}\left\{\lambda^2 + \lambda\left[C_R^2\left(HP_\infty^* H^T R^{-1} + 1\right) - 2\left(F - P_\infty^* H^T R^{-1} H - \tfrac{1}{2} C_P^2 HH^T\right)\right] + C_R^2 HH^T\left[P_\infty^* H^T R^{-2} H P_\infty^* - C_P^2\left(H \times P_\infty^* H^T R^{-1} + 1\right) - \left(HP_\infty^* H^T R^{-1} + 1\right)(2)\left(F - P_\infty^* H^T R^{-1} H - \tfrac{1}{2} C_P^2 HH^T\right)\right]\right\}$$

$$= \left\{2\left[F - P_\infty^* H^T R^{-1} H\right] - \lambda\right\}\left\{\lambda^2 + \lambda\left[C_R^2\left(HP_\infty^* H^T R^{-1} + 1\right) - 2\left(F - P_\infty^* H^T R^{-1} H - \tfrac{1}{2} C_P^2 HH^T\right)\right] + C_R^2 HH^T\left[P_\infty^* H^T R^{-2} H P_\infty^* - 2\left(F - P_\infty^* H^T R^{-1} H\right) \times \left(HP_\infty^* H^T R^{-1} + 1\right) + \left(HP_\infty^* H^T R^{-1} + 1\right) C_P^2 \left(-1 + HH^T\right)\right]\right\}$$

$$= (\lambda_1 - \lambda)(\lambda^2 + a\lambda + b)$$

Now in the case, for example, that $F \leq 0$, $C_R > 0$, $H \geq 1$, and $C_P \geq 0$, we have $a > 0$, $b > 0$, so that the real parts of all three eigenvalues are negative, and the limiting linearized differential equation is a.s.

Now we invoke two theorems:

(1) Theorem IX.10.3: The asymptotic stability of the limiting linearized equation implies that the linearized system (24) is u.a.s.

(2) Theorem IX.10.4: The uniform asymptotic stability of the linearized system (24) implies that the non-linear system (21-23) is u.a.s.

Then from (20), we find that a stable, completely observable, controllable, autonomous system of dimension 1 induces an adaptive estimator for which $\hat{P}(t)$ converges uniformly to P(t), and P(t) to $P^*(t)$, for $\hat{P}(0)$ and $\hat{R}(0)$ sufficiently accurate.

The time-discrete form of the adaptive estimator ((1)-(7)) is as follows:

$$\hat{X}_{N+1} = \Phi_{N+1}\hat{X}_N + C_{N+1}\{Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\} \tag{28}$$

$$C_{N+1} = \hat{P}_{N+1} H^T \hat{R}^{-1}_{N+1} \tag{29}$$

$$\mathcal{E}_{N+1} = \{Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\}\{Y_{N+1} - H_{N+1}\Phi_{N+1}\hat{X}_N\}^T \tag{30}$$

$$\hat{P}'_{N+1} = \Phi_{N+1}\hat{P}_N\Phi^T_{N+1} + \hat{Q}_{N+1} \tag{31}$$

$$\alpha = \mathcal{E} - [H_{N+1}\hat{P}'_{N+1}H^T_{N+1} + \hat{R}_{N+1}] \tag{32}$$

$$\hat{P}_{N+1} = (I - C_{N+1}H_{N+1})\hat{P}'_{N+1} + C_P\alpha C_P^T \tag{33}$$

$$\hat{R}_{N+1} = \hat{R}_N + C_R\alpha C_R^T \tag{34}$$

$$\hat{Q}_{N+1} = \hat{Q}_N + C_Q\alpha C_Q^T \tag{35}$$

VIII.4 Example Simulation

A digital computer simulation program for the adaptive filter equations in time-discrete form was written, and is described in Appendix IX.4.

We provide here an example computer simulation in the case $F = 0$, $H = 1$. Such a system was treated in Example V.1; Figure V.1.2 illustrates the response of the <u>idealized</u> (conventional Kalman) filter to a step in the channel noise power input. Figure VIII.1 demonstrates the response of the proposed adaptive estimator to the input data stream of V.1.2. The variance/coefficient behavior of the idealized filter is superimposed in this figure, labeled $P^*_{11}(t)$, $C^*_{11}(t)$. It can be seen that $\hat{R}$ is an estimator of R; moreover, the input data stream (pictured in Figure V.1.2) shows that $\hat{R}$ responds to local sample function variations in noise power.

The result, in equation (27) that deterministic convergence is not lost by $C_p = 0$, is illustrated in Figure VIII.2, which represents the same model and input data stream, with $C_p = 0$.

Intuition suggests that the time constant of the adaptive system, determined by $C_{P,Q,R}$, should be longer than that of the filter itself, determined by C. Further, the importance of stochastic stability study is illustrated by the apparent possibility of the estimates $\hat{P},\hat{Q},\hat{R}$ losing their positive definite character, for appropriate sample function inputs. A time-dependent aspect for the adaptive filter coefficients has been omitted in the absence of a stochastic study.

Bucy and Follin [3] have used equation VII.62 to design an adaptive estimator, but only to the extent of deriving the steady-state filter from the differential equation,

FIGURE VIII.1

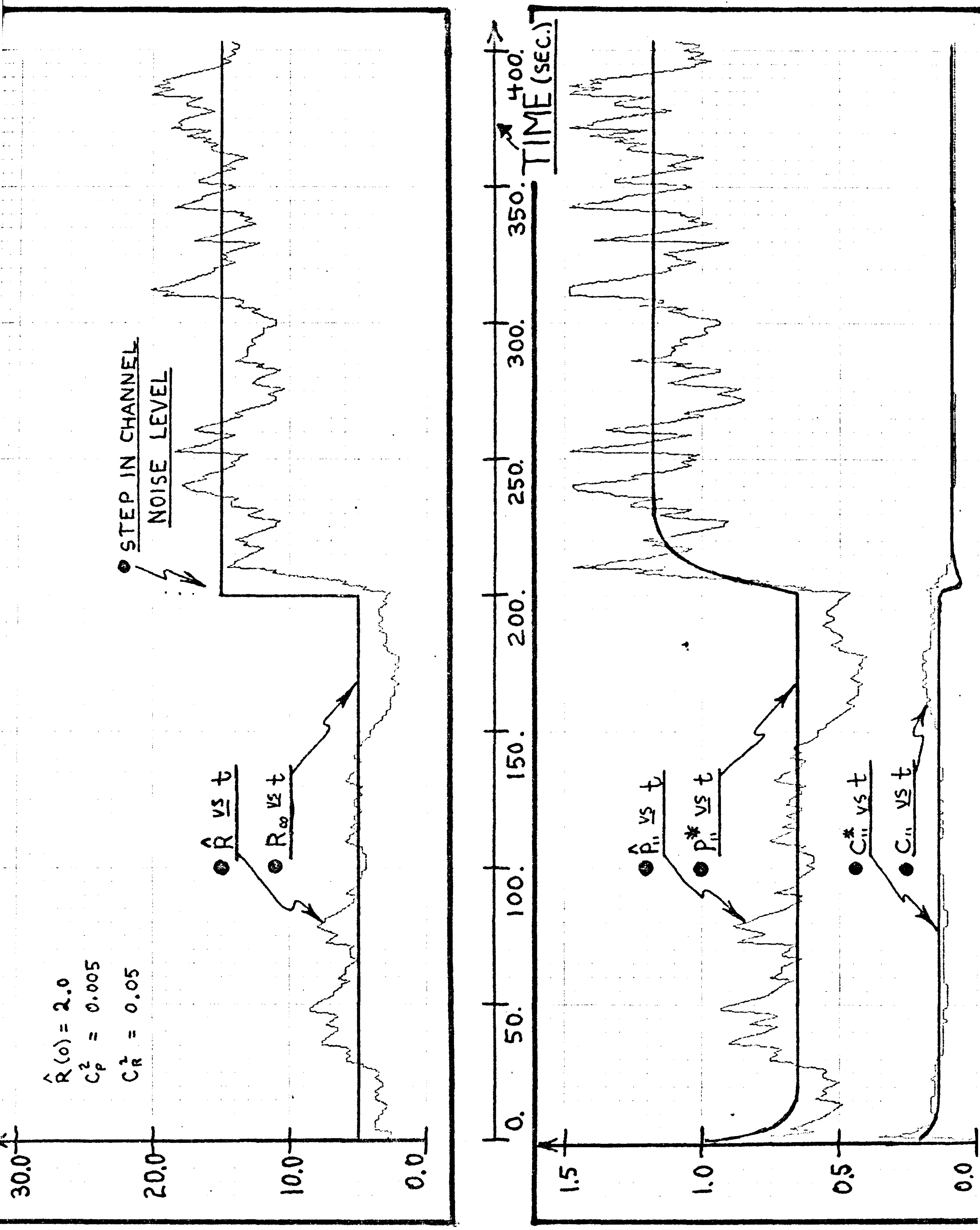
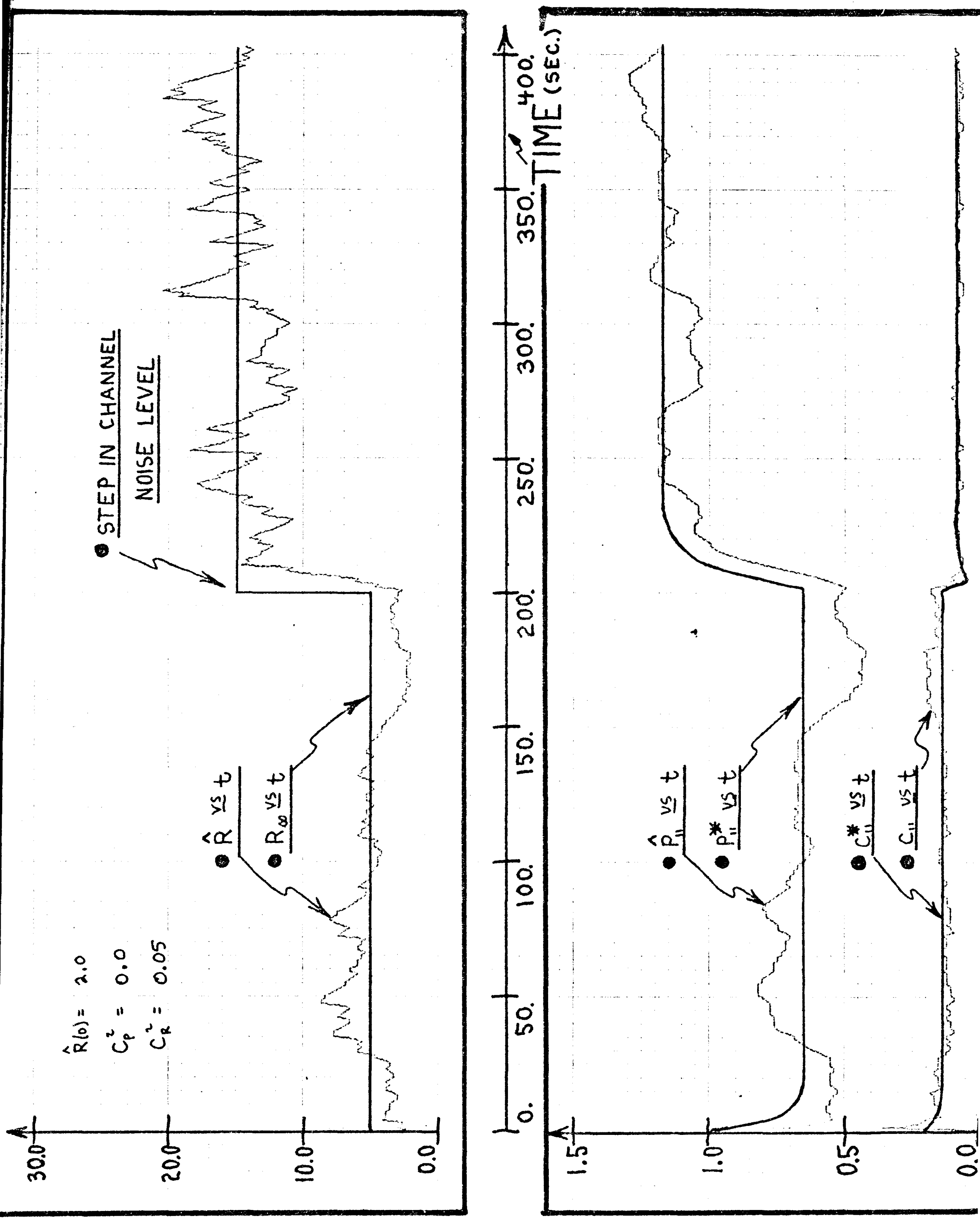


FIGURE VIII.2



and thereafter, without the use of the differential equation, deriving a scheme for adapting from one steady state to another, as environmental statistics vary. Our results indicate convergence from one steady state to another. However, we believe that this result is weak, and that the use of the dynamically descriptive equation VII.62 actually opens the way toward adaptation which includes the transient behavior of the physical system (F,H,G) and the estimator (P,C).

Under the heading of deterministic stability, the following topics merit investigation:

(1) Higher Dimensionality--establishing the uniform asymptotic stability of (21-23) for systems of higher (n,m) dimension.

(2) Global or Regional Asymptotic Stability--while the global stability of (21-23) is doubtful, the local result established should be replaced by calculated bounds on a region of initial error guaranteeing stable behavior.

(3) Time-Dependent Systems--the nature of $P^*(t)$ is pivotal to the arguments of stability above. A broader result in the same spirit would utilize the character of $P^*(t)$ for uniformly observable and controllable time-dependent physical systems, as discussed in Section VII, to establish convergence of $P(t)$ and $\hat{P}(t)$ to or near a nonconstant $P^*(t)$.

APPENDICES

IX.1 Completion of Square

Lemma 1: (i) A a symmetric, regular matrix of dimension n_1.

(ii) B and X $(n_2 \times n_1)$ matrices.

(iii) C an $(n_2 \times n_2)$ matrix.

Then

$$XAX^T - XB^T - BX^T + C$$

$$= (X - BA^{-1})A(X - BA^{-1})^T + (C - BA^{-1}B^T)$$

Proof: By calculation.

IX.2 Matrix Inversion

Lemma 2: (i) A and B positive-definite, symmetric matrices, of dimension n.

(ii) H an $(m \times n)$ matrix.

Then

The inverse of $(A + HBH^T)$ exists, and

$$(A + HBH^T)^{-1} = A^{-1} - A^{-1}H(H^TA^{-1}H+B^{-1})H^TA^{-1} .$$

Proof: Both indicated inverses exist, as they are inverses of positive-definite matrices. The proof follows by calculation of the product of the matrix and its inverse.

IX.3 Equivalence of Projection and Completing the Square

That certain arguments of the text may be proved by both geometric projection and algebraic completion of square arguments is not coincidental; we argue here the equivalence of these viewpoints.

Suppose $f(X) = X^T AX - B^T X - X^T B$ is to be minimized, where X and B are n-vectors, and A an $(n \times n)$ positive definite matrix. The algebraic solution is the completion of square argument, in which $\hat{X}$, an n-vector, and C, a scalar, are determined such that

$$f(X) = (X - \hat{X})^T A(X - \hat{X}) + C,$$

so that $X = \hat{X}$ clearly minimizes f. We argue that this procedure is equivalent to a projection:

Let $(X,Y) = X^T Y$. Then if we can write f in the form

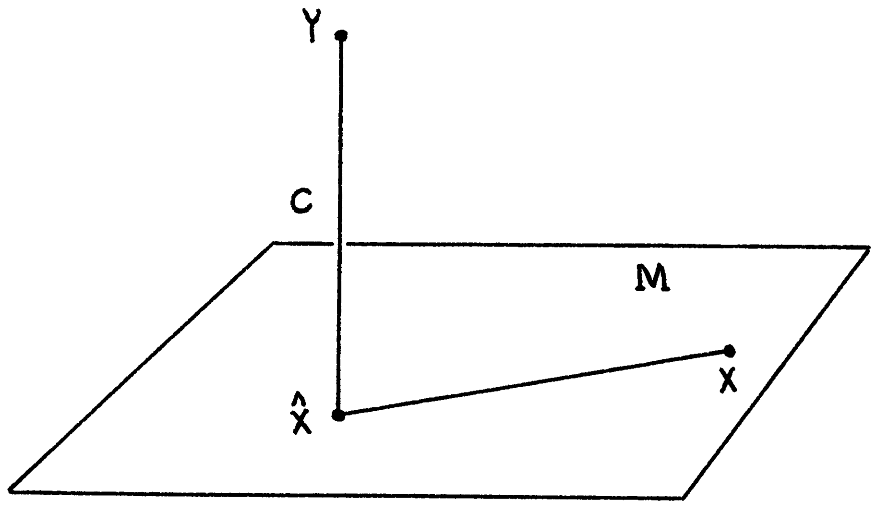
$$\begin{aligned} f(X) &= (X - \hat{X}, A(X - \hat{X})) + C \\ &= \| X - \hat{X} \|_A^2 + C , \end{aligned}$$

we see that, from the theory of Hilbert space, $f(X)$ will correspond to the square of the A-distance from X to Y, where

(i) Y is any vector such that $\|Y - \hat{X}\|_A^2 = C$.

(ii) M is the orthogonal complement of the space spanned by $(Y - \hat{X})$.

(iii) X is free to vary over (i.e., be chosen from) M.



A completion of square problem is derived from a projection problem in a similar fashion.

IX.4 Computer Programs and Listings

IX.4.1 Conventional Filtering

Conventional Kalman-Bucy filtering as demonstrated in Section V was performed by a set of IBM 1130 FORTRAN programs which were written as a part of this work. The system is capable of simulating, in time-discrete form, a given system model, including random inputs, and the corresponding Kalman filter equations. The relevant equations are II.1,4,8,9,10, 11. The problem is described by the user in canonical form by input of program data describing the matrices Φ,H,Q,R,P(0), X(0), and $\hat{X}$(0). The user may also specify non-stationary system behavior by supplying an appropriate subroutine.

Length of the simulation run, data sampling print interval, and selection of variables for plotting are also described by user data input.

Data input is in the form of punched cards, and output of selected data is in printed or plotted form.

We have completely avoided in this work the problem of dimensionality that is a part of many practical filtering problems, by using small systems as examples. We merely note in passing that large navigational filters are necessarily implemented as assembly-language programs, and then with great effort to meet time, memory, and numerical error budgets.

The following is a listing of the FORTRAN mainline program and subroutines used in the conventional filter simulation, followed by a brief description of each.

```
// JOB T        067          SCHINDEL                         KALMAN-BUCY FILTER
// GO ERASE FILTR
// FORTRAN MAINLINE PROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
*IOCS(1403 PRINTER,2501 READER)
C-----KALMAN-BUCY FILTER MAINLINE.
C-----LARGER SYSTEMS, SHORTER RUNS ACCOMODATED BY DIMENSIONS OF
C----- NPLOT(5,100),X(10),XH(10),Y(5),,H(50),HT(50),PHI(100),PHIT(100)
C-----    P(100),PPR(100),Q(100),R(25),C(50),W(10),V(5),WORK(100),
C-----WORK2(100),WORK3(100),WORK4(50),WORK5(50),CP(50),CR(25),CQ(50),
C-----    QH(100),RH(25),             THEN NMAX=10,MMAX=5,NCYCLMAX = 100.
      DIMENSION X(2 ),XH(2 ),Y(2),H(2 ),HT(2 ),PHI(4   ),PHIT(4   ),P(4
     X  , PPR(4  ),Q(4  ),R(4 ),C(4 ),W(2 ),V(2), WORK(4  ),WORK2(4  )
     X WORK3(2 ),NTITL(40),NVAR(5),INDX(5)
      COMMON NPLOT(5,400),NCYCL,GMIN(5),SCALE(5),NKEEP
C----- NMAX = 2
C ----- MMAX = 2
C-----RANDOM SEED, IX.
      IX = 10371
      NPIC = 0
      C(1) = 0
      C(2) = 0
C-----READ SYSTEM DESCRIPTION.
      READ(8,107) NTITL
107   FORMAT(40A2)
      READ(8,101) NCYCL,NPI,NSR,NPR
101   FORMAT(8I10)
      IF(NPR) 313,314,313
313   READ(8,1080) NKEEP,(NVAR(K),INDX(K),K=1,NKEEP)
1080  FORMAT(8I10)
      READ(8,108) (GMIN(I),SCALE(I),I = 1,NKEEP)
108   FORMAT(10F8.3)
314   READ(8,101) N,M
      NSQ = N*N
      MSQ = M*M
      NM = N*M
      READ (8,102)(PHI(I),I = 1,NSQ)
102   FORMAT(10F10.3)
      READ (8,102) (H(I),I=1,NM)
      READ (8,102) (P(I),I = 1,NSQ)
      READ (8,102)(Q(I),I= 1,NSQ)
      READ(8,102) (R(I),I=1,MSQ)
      READ(8,109) NRST,RFACT,NQST,QFACT
109   FORMAT(I10,F10.3,I10,F10.3)
C-----PRINT SYSTEM DESCRIPTION.
      WRITE(5,104) NTITL
104   FORMAT('1',1X,40A2,//,' PHI, H, P(0), Q, R  ------',//)
      CALL MTXPT(PHI,N,N)
      CALL MTXPT(H,M,N)
      CALL MTXPT(P,N,N)
      CALL MTXPT(Q,N,N)
      CALL MTXPT(R,M,M)
      IF(NSR)309,308,309
309   READ(8,102)(X( I),I=1,N)
      READ(8,102) (XH(I),I=1,N)
      WRITE(5,105)
105   FORMAT(//,' X(0),XH(0) ----'//)
      CALL MTXPT(X,N,1)
      CALL MTXPT(XH,N,1)
```

```
308    CALL MTXTR(H,HT,M,N)
       CALL MTXTR(PHI,PHIT,N,N)
       WRITE (5,106)
106    FORMAT('1')
C-----PRINCIPAL FILTER LOOP.
       DO 307 NCTR = 1,NCYCL
C-----NON-STATIONARY ADJUSTMENTS.
       CALL NSTAT(Q,R,N,M,NCTR,NRST,RFACT,NQST,QFACT)
301    IF(NPIC) 302,303,302
303    NPIC = -NPI
C-----PRINT OUTPUT DATA.
       WRITE (5,103) NCTR
103    FORMAT (////'*** STAGE ',I5)
       CALL MTXPT(P,N,N)
       CALL MTXPT(C,N,M)
       IF(NSR) 304,302,304
304    CALL MTXPT(X,N,1)
       CALL MTXPT(XH,N,1)
       CALL MTXPT(Y,M,1)
302    NPIC = NPIC+1
       IF(NPR) 305,306,305
C-----SAVE REQUESTED PLOT DATA.
305    CALL SAVE(X,XH,Y,P,C,PHI,H,Q,R,NCTR,NPLOT,GMIN,SCALE,NKEEP,NVAR,
      X  INDX)
C-----PROPAGATE FILTER COVARIANCE, COEFFICIENT.
306    CALL MTXML(PHI,P,WORK,N,N,N)
       CALL MTXML(WORK,PHIT,PPR,N,N,N)
       CALL MTXAD(Q,PPR,PPR,N,N)
       CALL MTXML(H,PPR,WORK,M,N,N)
       CALL MTXML(WORK,HT,WORK2,M,N,M)
       CALL MTXAD(WORK2,R,WORK,M,M)
       CALL MINVS(WORK,WORK2,M)
       CALL MTXML(HT,WORK2,WORK,N,M,M)
       CALL MTXML(PPR,WORK,C,N,N,M)
       CALL MTXML(C,H,WORK,N,M,N)
       CALL MTXML(WORK,PPR,WORK2,N,N,N)
       CALL MTXSB(PPR,WORK2,P,N,N)
       IF(NSR) 310,307,310
C-----GENERATE RANDOM INPUTS.
310    DO 201 I = 1,N
       J = (I-1)*N+I
       SIG = SQRT(Q(J))
201    CALL GAUS(IX,SIG,0.0,W(I))
       DO 202 I = 1,M
       J = (I-1)*M+I
       SIG = SQRT(R(J))
202    CALL GAUS(IX,SIG,0.0,V(I))
C-----UPDATA SYSTEM MODEL.
       CALL MTXML(PHI,X,WORK,N,N,1)
       CALL MTXAD(WORK,W,X,N,1)
       CALL MTXML(H,X,Y,M,N,1)
       CALL MTXAD(Y,V,Y,M,1)
C-----EXERCISE FILTER.
       CALL MTXML(PHI,XH,WORK,N,N,1)
       CALL MTXML(H,WORK,WORK2,M,N,1)
       CALL MTXSB(Y,WORK2,WORK2,M,1)
       CALL MTXML(C,WORK2,WORK3,N,M,1)
       CALL MTXAD(WORK,WORK3,XH,N,1)
307    CONTINUE
```

```
      IF(NPR) 311,312,311
C-----PLOT REQUESTED DATA.
311   CALL LINK(GRAF)
312   STOP
      END
// GO PUT FILTR
// GO ERASE MTXML
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----MATRIX NULTIPLIER, MINIMUM INDEX ARITHMETIC TIME.
      SUBROUTINE MTXML(X,Y,Z,NR,NC,NC2)
      INTEGER XCOL,XROW,YCOL
      DIMENSION X(1),Y(1),Z(1)
      DO 201 YCOL = 1,NC2
      IJ = 0
      II = YCOL-NC2
      DO 201 XROW = 1,NR
      IK = YCOL - NC2
      II = II + NC2
      Z(II) = 0.0
      DO 201 XCOL = 1,NC
      IJ = IJ + 1
      IK = IK + NC2
201   Z(II) = Z(II) + X(IJ)*Y(IK)
      RETURN
      END
// GO PUT MTXML
// GO ERASE MTXSB
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----MATRIX SUBTRACTOR
      SUBROUTINE MTXSB(X,Y,Z,NR,NC)
      DIMENSION X(1),Y(1),Z(1)
      K = NR*NC
      DO 201 I = 1,K
201   Z(I) = X(I) - Y(I)
      RETURN
      END
// GO PUT MTXSB
// GO ERASE MTXAD
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----MATRIX ADDER.
      SUBROUTINE MTXAD(X,Y,Z,NR,NC)
      DIMENSION X(1),Y(1),Z(1)
      K = NR*NC
      DO 201 I = 1,K
201   Z(I) = X(I) + Y(I)
      RETURN
      END
// GO PUT MTXAD
// GO ERASE MTXTR
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----MATRIX TRANSPOSER.
      SUBROUTINE MTXTR(X,Y,NR,NC)
      DIMENSION X(1),Y(1)
      DO 201 I = 1,NC
      DO 201 J = 1,NR
```

```
      II = J + NR*(I-1)
      JJ = I + NC*(J-1)
201   Y(II) = X(JJ)
      RETURN
      END
// GO PUT MTXTR
```

```
// GO ERASE MINVS
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----MATRIX INVERSION, FAST.  DIM(X) LESS THAN OR EQUAL TO 2.
      SUBROUTINE MINVS(X,Y,N)
      DIMENSION X(4),Y(4)
      IF(N-2) 301,302,303
301   Y(1) = 1.0/X(1)
      RETURN
302   R = X(1)*X(4)-X(2)*X(3)
      Y(1)= X(4)/R
      Y(2) = -X(2)/R
      Y(3) = -X(3)/R
      Y(4) =  X(1)/R
303   RETURN
      END
// GO PUT MINVS
```

```
// GO ERASE MTXPT
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----MATRIX PRINTER.
      SUBROUTINE MTXPT(X,NR,NC)
      DIMENSION X(1)
      DO 201 I = 1,NR
      J = (I-1)*NC+1
      JJ = J + NC - 1
201   WRITE(5,101) (X(K),K=J,JJ)
101   FORMAT (6E16.8)
      WRITE (5,102)
102   FORMAT (/)
      RETURN
      END
// GO PUT MTXPT
```

```
// GO ERASE RAND
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----UNIFORM RANDOM NO. GEN (IBM SCI SUB).
      SUBROUTINE RAND(IX,IY,YFL)
      IY = IX*899
      IF(IY) 5,6,6
5     IY = IY + 32767 + 1
6     YFL = IY
      YFL = YFL/32767
      RETURN
      END
// GO PUT RAND
```

```
// GO ERASE GAUS
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----NORMAL RANDOM NO. GEN (IBM SCI SUB).
      SUBROUTINE GAUS(IX,S,AM,V)
      A = 0.
      DO 50 I = 1,12
```

```
      CALL RAND(IX,IY,Y)
      IX = IY
50    A = A + Y
      V = (A-6.0)*S + AM
      RETURN
      END
// GO PUT GAUS
// GO ERASE SAVE
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C---- PLOT DATA SAMPLING ROUTINE.
      SUBROUTINE SAVE(X,XH,Y,P,C,PHI,H,Q,R,NCTR,NPLOT,GMIN,SCALE,NKEEP,
     X NVAR,INDX)
      DIMENSION X(2),XH(1),Y(1),P(1),C(1),PHI(1),H(1),Q(1),R(1),NPLOT(5,
     X 400),GMIN(5),SCALE(5),NVAR(1),INDX(1)
C---- SAVE REQUESTED PLOT DATA ITEM.
      DO 202 I = 1,NKEEP
      II = NVAR(I)
      J = INDX(I)
      GO TO (301,302,303,304,305,306,307,308,309),II
301   S = X(J)
      GO TO 202
302   S = XH(J)
      GO TO 202
303   S = Y(J)
      GO TO 202
304   S = P(J)
      GO TO 202
305   S = C(J)
      GO TO 202
306   S = SQRT(P(J))
      GO TO 202
307   S = (X(J) - XH(J))**2
      GO TO 202
308   S = R(J)
      GO TO 202
309   S = Q(J)
C----- SCALE DATA PER REQUEST.
202   NPLOT(I,NCTR)= 0.5 + (S-GMIN(I))*SCALE(I)
      RETURN
      END
// GO PUT SAVE
// GO ERASE NSTAT
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----NON-STATIONARY ADJUSTMENT ROUTINE, USER-SPECIFIED.
      SUBROUTINE NSTAT(Q,R,N,M,NCTR,NRST,RFACT,NQST,QFACT)
      DIMENSION R(1),Q(1)
      IF(NCTR-NRST)301,302,301
302   R(1) = R(1) + RFACT
301   IF(NCTR-NQST )303,304,303
304   Q(1) = Q(1) + QFACT
303   RETURN
      END
// GO PUT NSTAT
// GO ERASE GRAF
// FORTRAN MAINLINE PROGRAM  *LIST SOURCE PROGRAM  *IOCS(1403 PRINTER,PLOTTE
*ONE WORD INTEGERS
C-----DATA PLOTTER, CALL LINK SAVES REAL STORAGE WHEN PRINTER
```

```
C-----       PLOTTING (RHIT PLIT PROGRAM) IS USED.
      COMMON NPLOT(5,400),NCYCL,GMIN(5),SCALE(5),NKEEP
      DO 201 I = 1,NKEEP
      PAUSE 1111
C----- PRODUCE 3 INCH BY 8 INCH PLOTS.
      TSCAL = 8.0/NCYCL
      YSCAL = 3.0/100.0
      CALL SCALF(TSCAL,YSCAL,0.0,GMIN(I))
      CALL FPLOT(1,0,GMIN(I))
      DO 202 J = 1,NCYCL
C-----ABORT PLOT TEST.
      CALL DATSW(0,K)
      GO TO (201,2020),K
2020  T = J
      Y = NPLOT(I,J)
C-----CLAMP DATA TO PLOT SIZE.
      IF(GMIN(I)-Y) 301,301,302
302   Y = GMIN(I)
301   IF(Y-GMIN(I)-100)202,202,304
304   Y = GMIN(I) + 100
202   CALL FPLOT(-2,T,Y)
201   CALL FPLOT(1,0,GMIN(I))
C-----PRINT SCALING DATA.
      WRITE(5,101) GMIN,SCALE
101   FORMAT(//,15X,'A',13X,'B',13X,'C',13X,'D',13X,'E',//,' GMIN',4X,
     X   5E14.4,/,' SCALE',3X,5E14.4,///)
      STOP
      END
// GO PUT GRAF
```

Description

(1) FILTR Mainline program performing system model and filter equation computation. Variables are named in correspondence with the conventions of this paper, and computations are therefore explained by the listing.

Description of the problem to be solved is read from user-supplied data cards by this mainline. A data deck is organized as follows:

(a) Title Card: Card Columns 2-80 hold title information identifying the run, to be printed with output data.

(b) Run Description Card:

Cols 1-10: Number of time stages to be simulated.

Cols 11-20: Data output print interval, in time stages.

Column 30: 1 ⇒ Conventional filtering run.
0 ⇒ Propagate Covariance and Coefficients only.

Column 40: 1 ⇒ Save data selected below for plotting.
0 ⇒ No plotting.

(c) Plot Data Description Card: If no data plotting is desired, this card omitted; otherwise, Cols 1-10: Number of variables to be saved for plotting.

Cols 11-20: Code number selecting data item to be saved:

1 - X

2 - $\hat{X}$

3 - Y

4 - P

5 - C

6 - $P^{1/2}$ (by element)

7 - $(X-\hat{X})$

8 - R

9 - Q

Cols 21-30: Subscript indicating element of selected data item to be saved, based on generalized matrix storage indexing.

Cols 31-40: As 11-20, second data item.

Cols 41-50: As 21-30, second data item.

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Limit 5 data items, with overflow cards permitted.

(d) Plot Data Scaling Card:

Cols 1-8: Value of bottom ordinate level on graph, for first data item graph.

Cols 9-16: (100/M), where M = (top graph ordinate - bottom graph ordinate), for first data item graph.

Cols 17-24: As 1-8, for second data item graph.

Cols 25-32: As 9-16, for second data item graph.

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(e) System Dimension Card:

Cols 1-10: n = number of system state variables.

Cols 11-20: m = number of system observables.

(f) System Description Cards:

Card 1: Φ, listed by row, 10 card columns per element; overflow cards permitted.

Card 2: H, formatted as Φ.

Card 3: P(0), formatted as Φ.

Card 4: Q, formatted as Φ.

Card 5: R, formatted as Φ.

(g) Environment Step Card: (The non-stationary aspects of a system are simulated by the subroutine NSTAT, described below, which is user-supplied. One scheme is illustrated here, to create a step function in environment statistics.)

Cols 1-10: Cycle number on which step in R_{11} is desired.

Cols 11-20: Size of R_{11} step.

Cols 21-30: Cycle number on which step in Q_{11} is desired.

Cols 31-40: Size of Q_{11} step.

(h) State/Estimate Cards: (Omit if Run Description Card requests non-state run.)

Card 1: X(0), element-wise, 10 card columns per element, overflow cards permitted.

Card 2: $\hat{X}(0)$, formatted as X(0).

As an example, the following data deck was utilized to produce the run of Figure V.1.1:

```
EXAMPLE 1.1 ---- SCALAR STATE ESTIMATION.
      400        398         1          1
        5          1         1          2         1          3        1
        1          5         1     PLOT X(1),XH(1),Y(1),P(1,1),C(1)
.0      6.667      0.0      6.667      0.0     6.667      0.0     66.667     0.0     66.667
        1          1                  N,M
      1.0       PHI
      1.0       H
      1.0       P(0)
      0.1       Q
      5.0       R
                                                      STATIONARY
      9.0       X(0)
      6.0       XH(0)
```

(2) Matrix Algebra Subroutines: Perform basic matrix algebraic operations, using generalized matrix storage techniques. Listed below by call description:

(a) CALL MTXML(X,Y,Z,NR,NC): Computes $Z = XY$, where $\dim(Z) = (NR \times NC)$.

(b) CALL MTXSB(X,Y,Z,NR,NC): Computes $Z = X - Y$, where $\dim(Z) = (NR \times NC)$.

(c) CALL MTXAD(X,Y,Z,NR,NC): Computes $Z = X + Y$, where $\dim(Z) = (NR \times NC)$.

(d) CALL MTXTR(X,Y,NR,NC): Computes $Y = X^T$, where $\dim(X) = (NR \times NC)$.

(e) CALL MINVS(X,Y,N): Computes $Y = X^{-1}$, where $\dim(X) = N \leq 2$, restricted for speed.

(f) CALL MTXPT(X,NR,NC): Prints X, on line printer, where dim(X) = (NR x NC).

(3) Random Number Generators: Generate sample functions for channel and drive noise inputs:

(a) CALL RAND(IX,IY,YFL): Computes limiting uniformly distributed random number YFL, between 0 and 1; IX an odd positive integer random seed; an IBM Scientific Subroutine

(b) CALL GAUS(IX,S,AM,V): Computes limiting normally distributed V, with mean AM and std. deviation S; IX a random seed used by RAND; an IBM Scientific Subroutine.

(4) SAVE: Subroutine which scales and saves for graphic output all selected plot variables. Data is saved in integer form, to conserve storage.

(5) NSTAT: User-defined subroutine specifying non-stationary behavior in Φ,H,R, or Q. Listing indicates example form for step in $R(t)$ and $Q(t)$.

IX.4.2 Adaptive Filtering

Adaptive filtering as demonstrated in Section VIII was performed by a set of IBM 1130 FORTRAN programs, written as a part of this work, which are similar to and overlap those described in Section IX.4.1. This program set is more modular than that of IX.4.1, and is in fact the forerunner of a modularized system of simulation routines capable of filtering of either type.

The use of the adaptive system is similar to that of the conventional system. Listings and descriptions of programs not identical with those of Section IX.4.1 follow:

```
// JOB T      067          SCHINDEL          ADAPTIVE FILTER SOURCE
// GO ERASE MFIL2
// FORTRAN MAINLINE PROGRAM
*IOCS(1403 PRINTER,2501 READER)
*ONE WORD INTEGERS
*LIST SOURCE PROGRAM
C-----ADAPTIVE FILTER MAINLINE PROGRAM.
C-----        2501 READER USE SAVES CORE.
C-----LARGER SYSTEMS, SHORTER RUNS ACCOMODATED BY DIMENSIONS OF
C----- NPLOT(5,100),X(10),XH(10),Y(5),,H(50),HT(50),PHI(100),PHIT(1C
C-----   P(100),PPR(100),Q(100),R(25),C(50),W(10),V(5),WORK(100),
C-----WORK2(100),WORK3(100),WORK4(50),WORK5(50),CP(50),CR(25),CQ(50)
C-----   QH(100),RH(25),            THEN NMAX=10,MMAX=5,NCYCLMAX = 10C
      DIMENSION X(2),XH(2),Y(2),H(2),HT(2),PHI(4),PHIT(4),P(4)
     X  ,PPR(4),Q(4),R(4),C(4),W(2),V(2),WORK(4),WORK2(4),WORK3(2)
     X    ,WORK4(4),WORK5(4),NTITL(40),CP(16),CR(4),CQ(16),QH(4)
     X  ,RH(4),NVAR(5),INDX(5)
      COMMON NPLOT(5,400),NCYCL,GMIN(5),SCALE(5),NKEEP
C-----NCYCLMAX = 400
C------NMAX = 2
C-----MMAX = 2
C-----RANDOM SEED, IX.
      IX = 10371
      C(1) = 0
      C(2) = 0
      NPIC = 0
C-----READ SYSTEM DESCRIPTION.
      CALL SPEC(X,XH,PHI,H,HT,PHIT,Q,R,P,N,M,NTITL,NCYCL,NPI,NSR,NPR
     X  GMIN,SCALE,NSQ,MSQ,NVAR,INDX,NKEEP,NRST,RFACT,NQST,QFACT)
C-----READ ADAPTIVE ESTIMATOR DATA.
      J = NSQ*MSQ
      READ(8,100) (CP(I),I = 1,J)
100   FORMAT(8F10.3)
      READ(8,100) (CQ(I), I = 1,J)
      J = MSQ*MSQ
      READ(8,100) (CR(I), I = 1,J)
      READ(8,100) (QH(I),I = 1,NSQ)
      READ(8,100) (RH(I),I = 1,MSQ)
      CALL MTXPT(CP,N,M)
      CALL MTXPT(CQ,N,M)
      CALL MTXPT(CR,M,M)
      CALL MTXPT(QH,N,N)
      CALL MTXPT(RH,M,M)
C-----MAIN COMPUTATION LOOP.
      DO 307 NCTR = 1,NCYCL
      IF(NPIC) 302,303,302
303   NPIC = -NPI
C-----PRINT OUTPUT DATA.
      CALL OUTPT(X,XH,Y,P,C,N,M,NCTR,NSR)
      CALL MTXPT(RH,M,M)
      CALL MTXPT(QH,N,N)
302   NPIC = NPIC + 1
      IF (NPR) 305,306,305
C-----SAVE PLOT DATA REQUESTED.
305   CALL SAVE (X,XH,Y,P,C,PHI,H,QH,RH,NCTR,NPLOT,GMIN,SCALE,NKEEP,
     X  ,INDX)
C-----ADJUST NON-STATIONARY ENVIRONMENT.
306   CALL NSTAT(Q,R,N,M,NCTR,NRST,RFACT,NQST,QFACT)
C-----UPDATA SYSTEM MODEL.
```

```
      CALL SYSTM(X,Y,PHI,H,Q,R,W,V,N,M,WORK,IX)
C-----ADAPTIVE FILTER EQUATIONS.
      CALL MTXML(PHI,XH,WORK2,N,N,1)
      CALL MTXML(H,WORK2,WORK3,M,N,1)
      CALL MTXSB(Y,WORK3,WORK5,M,1)
      CALL MTXTR(WORK5,WORK2,M,1)
      CALL MTXML(WORK5,WORK2,WORK4,M,1,M)
      CALL MTXML(PHI,P,WORK,N,N,N)
      CALL MTXML(WORK,PHIT,PPR,N,N,N)
      CALL MTXAD(QH,PPR,PPR,N,N)
      CALL MTXML(H,PPR,WORK,M,N,N)
      CALL MTXML(WORK,HT,WORK2,M,N,M)
      CALL MTXAD(WORK2,RH,WORK2,M,M)
      CALL MTXSB(WORK4,WORK2,WORK4,M,M)
      CALL MINVS(WORK2,WORK,M)
      CALL MTXML(HT,WORK,WORK2,N,M,M)
      CALL MTXML(PPR,WORK2,C,N,N,M)
      CALL MTXML(C,H,WORK,N,M,N)
      CALL MTXML(WORK,PPR,P,N,N,N)
      CALL MTXSB(PPR,P,P,N,N)
      CALL MTXML(CP,WORK4,WORK,NSQ,MSQ,1)
C-----CALL MTXML(WORK,CPT,)...)
      CALL MTXAD(WORK,P,P,N,N)
      CALL MTXML(CR,WORK4,WORK,MSQ,MSQ,1)
C-----CALL MTXML(WORK,CRT,)...)
      CALL MTXAD(WORK,RH,RH,M,M)
      CALL MTXML(CQ,WORK4,WORK,NSQ,MSQ,1)
C-----CALL MTXML(WORK,CQT,)...)
      CALL MTXAD(WORK,QH,QH,N,N)
      CALL MTXML(PHI,XH,WORK,N,N,1)
      CALL MTXML(C,WORK5,WORK2,N,M,1)
      CALL MTXAD(WORK,WORK2,XH,N,1)
307   CONTINUE
      IF(NPR) 311,312,311
C-----  PLOT OUTPUT IF REQUESTED.
311   CALL LINK(GRAF)
312   STOP
      END
// GO PUT MFIL2
```

```
// GO ERASE SPEC
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C----- SYSTEM DESCRIPTION INPUT SUBROUTINE.
      SUBROUTINE SPEC(X,XH,PHI,H,HT,PHIT,Q,R,P,N,M,NTITL,NCYCL,NPI,
     X NPR,GMIN,SCALE,NSQ,MSQ,NVAR,INDX,NKEEP,NRST,RFACT,NQST,QFACT
      DIMENSION X(1),XH(1),R(1),Q(1),P(1),H(1),PHI(1),GMIN(1),SCALE
     X  ,NTITL(40),NVAR(5),INDX(5)
C-----READ IN SYSTEM DESCRIPTION.
      READ(8,107) NTITL
107   FORMAT(40A2)
      READ(8,101) NCYCL,NPI,NSR,NPR
101   FORMAT(8I10)
      IF(NPR) 313,314,313
313   READ(8,101) NKEEP,(NVAR(K),INDX(K),K = 1,NKEEP)
      READ(8,108) (GMIN(I),SCALE(I),I= 1,NKEEP)
108   FORMAT(10F8.3)
314   READ(8,101) N,M
      NSQ = N*N
      MSQ = M*M
```

```
      NM = N*M
      READ (8,102) (PHI(I),I = 1,NSQ)
102   FORMAT(8F10.3)
      READ(8,102) (H(I),I = 1,NM)
      READ(8,102) (P(I), I = 1,NSQ)
      READ(8,102) (Q(I),I = 1,NSQ)
      READ(8,102) (R(I),I = 1,MSQ)
      READ(8,109) NRST,RFACT,NQST,QFACT
109   FORMAT(I10,F10.3,I10,F10.3)
      CALL MTXTR(H,HT,M,N)
      CALL MTXTR(PHI,PHIT,N,N)
C-----PRINT SYSTEM DESCRIPTION.
      WRITE(5,104) NTITL
104   FORMAT('1',1X,40A2,//,' PHI, H, P(0), Q, R - - - - - -',//)
      CALL MTXPT(PHI,N,N)
      CALL MTXPT(H,M,N)
      CALL MTXPT(P,N,N)
      CALL MTXPT(Q,N,N)
      CALL MTXPT(R,M,M)
      IF(NSR) 309,308,309
309   READ(8,102) (X(I),I = 1,N)
      READ(8,102)  (XH(I),I = 1,N)
      WRITE(5,105)
105   FORMAT(//,' X(0), XH(0) - - - - - -'//)
      CALL MTXPT(X,N,1)
      CALL MTXPT(XH,N,1)
308   WRITE(5,106)
106   FORMAT('1')
      RETURN
      END
// GO PUT SPEC
```

```
// GO ERASE OUTPT
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----OUTPUT PRINTING ROUTINE.
      SUBROUTINE OUTPT(X,XH,Y,P,C,N,M,NCTR,NSR)
      WRITE(5,103) NCTR
103   FORMAT(////' *** STAGE ',I5)
      CALL MTXPT(P,N,N)
      CALL MTXPT(C,N,M)
      IF(NSR) 304,302,304
304   CALL MTXPT(X,N,1)
      CALL MTXPT(XH,N,1)
      CALL MTXPT(Y,M,1)
302   RETURN
      END
// GO PUT OUTPT
```

```
// GO ERASE SYSTM
// FORTRAN SUBPROGRAM   *LIST SOURCE PROGRAM
*ONE WORD INTEGERS
C-----SYSTEM MODEL ROUTINE.
      SUBROUTINE SYSTM(X,Y,PHI,H,Q,R,W,V,N,M,WORK,IX)
      DIMENSION Q(1),R(1),W(1),V(1)
C-----GENERATE RANDOM INPUTS.
      DO 201 I = 1,N
      J = (I-1)*N + I
      SIG = SQRT(Q(J))
201   CALL GAUS(IX,SIG,0.0,W(I))
      DO 202 I = 1,M
```

```
      J = (I-1)*M + I
      SIG = SQRT(R(J))
202   CALL GAUS(IX,SIG,0.0,V(I))
C-----UPDATE SYSTEM STATE.
      CALL MTXML(PHI,X,WORK,N,N,1)
      CALL MTXAD(WORK,W,X,N,1)
      CALL MTXML(H,X,Y,M,N,1)
      CALL MTXAD(Y,V,Y,M,1)
      RETURN
      END
// GO PUT SYSTM
```

Description

(1) MFIL2: Mainline program which serves as executive over simulation, and performs the adaptive filtering computations of equations VIII.28-35.

Variables are named in correspondence with the computations of this paper, and computations are therefore explained by the listing. Description of the problem to be solved is read from user-supplied data cards, and the format description of Section IX.4.1 should be followed. In addition, the conventional data deck is followed by the following adaptive filter data deck, read by MFIL2:

(a) Adaptation Coefficients:

Card 1: C_P, listed by row, 10 card columns per element, with overflow cards.

Card 2: C_Q, formatted as C_P.

Card 3: C_R, formatted as C_P.

(b) Initial Environment Estimates:

Card 1: $\hat{Q}(0)$, formatted as C_P. (Any precisely known values supplied.)

Card 2: $\hat{R}(0)$, formatted as $\hat{Q}(0)$.

(2) SPEC: System specification data input subroutine, which reads system description portion common

to all filtering problems. (i.e., SPEC does not read adaptive filter data.)

(3) OUTPT: Data output subroutine which prints system/ estimator data at selected intervals.

(4) SYSTM: Subroutine which performs system model simulation, as described by equations II.1 and II.4.

IX.6 The State Transition Approach to Linear Dynamic Systems

We state only definitions and theorems of use in the text:

Definition IX.6.1: Assuming a solution to

$$\dot{x} = F(t)x \qquad (1)$$
$$x(t_0) = x_0$$

exists and is unique, where x is an n-vector, and F an $(n \times n)$ matrix function, the state transition matrix (characteristic matrix, fundamental matrix) of $F(t)$ is the unique solution of

$$\frac{d}{dt}\Phi(t,t_0) = F(t)\Phi(t,t_0)$$

$$\Phi(t_0,t_0) = I$$

Theorem IX.6.1: The unique solution to

$$\dot{X} = F(t)X + G(t)$$
$$X(t_0) = X_0 \tag{2}$$

is

$$x(t) = \Phi(t,t_0)X_0 + \int_{t_0}^{t} \Phi(t,s)\,G(s)\,ds\,. \tag{3}$$

Proof: By differentiation of (3).

Theorem IX.6.2: If $F(t)$ is commutative with $\int_{t_0}^{t} F(s)\,ds$, then

$$\Phi(t,t_0) = e^{\int_{t_0}^{t} F(s)\,ds} \overset{(\Delta)}{=} \sum_{n=0}^{\infty} \left\{ \int_{t_0}^{t} F(s)\,ds \right\}^n \cdot \frac{1}{n!}\,,$$

the series converging uniformly on all finite closed intervals. In the special case F = constant, we often write $\Phi = \Phi(t - t_0)$, as only a single parameter is required.

Proof: See Bellman [2].

Theorem IX.6.3: (1) $\Phi(t,t_0)$ is non-singular for all t.

(2) $\Phi^{-1}(t_1,t_2) = \Phi(t_2,t_1)$, for all t_1,t_2.

Proof: (1) See Bellman [1].

(2) $x(t_2) = \Phi(t_2,t_1)x(t_1) = \Phi(t_2,t_1)\Phi(t_1,t_0)x_0$

$= \Phi(t_2,t_0)x_0$, for all x_0.

Then (2) follows, by the case $t_2 = t_0$.

IX.9 Hilbert Space

Only the most elementary notions of Hilbert space are used in this paper. Refer to [16] for a more complete discussion. We recall the definitions leading to the structure of Hilbert space:

Definition IX.9.1:

(1) A real linear space is a triple $(L,+,\cdot)$ consisting of

(i) A set L,

(ii) An operation (addition) $+: L \times L \to L$, such that

(a) $x+(y+z) = (x+y)+z \quad , \forall\ x,y,z \in L.$

(b) $\exists\ 0 \in L \ni x+0 = x \ , \forall\ x \in L.$

(c) $x+y = y+x, \quad \forall\ x,y \in L.$

(d) $\forall\ x \in L, \exists\ (-x) \in L \ni x+(-x) = 0.$

(iii) An operation (scalar multiplication) $\cdot: R \times L \to L$, such that

(a) $a\cdot(x+y) = a\cdot x + a\cdot y \ , \forall\ a \in R,\ x,y \in L.$

(b) $(a+b)\cdot x = a\cdot x + b\cdot x, \ \forall\ a,b \in R,\ x \in L.$

(c) $(ab)\cdot x = a(b\cdot x), \ \forall\ a,b \in R,\ x \in L.$

(d) $1\cdot x = x, \quad \forall\ x \in L.$

(2) A <u>complex linear space</u> is a linear space with scalar multiplication defined over the field of complex numbers.

(3) A <u>normed linear space</u> is a linear space combined with a <u>norm</u> $\|\cdot\| : L \longrightarrow R$, such that

(i) $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0, \quad \forall x \in L.$

(ii) $\|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in L.$

(iii) $\|\alpha \cdot x\| = |\alpha| \, \|x\|, \quad \forall \alpha \in C, \ x \in L.$

(4) A <u>Banach space</u> is a normed linear space which is complete as a metrix space under the norm-induced metric:

$$d(x,y) = \|x-y\| \quad .$$

(5) A <u>Hilbert space</u> is a complex Banach space with norm induced from an <u>inner product</u> $(\cdot,\cdot) : L \times L \longrightarrow C$, such that

(i) $(ax+by, z) = a(x,z) + b(y,z),$

(ii) $(x,y) = \overline{(y,x)},$

(iii) $(x,x) = \|x\|^2,$

$$\forall \ x, y, z \in L; \ a, b \in C.$$

IX.10 Stability and the Method of Lyapunov

We list only principal definitions and theorems used in the text.

Definition IX.10.1:

(i) x_e is an equilibrium point of

$$\dot{x} = f(x,t) \tag{1}$$

if $f(x_e,t) = 0$.

(ii) $x(t) = \phi(t,x_o,t_o)$ denotes a solution to (1) with $x(t_o) = x_o$.

(iii) x_e, an equilibrium point of (1), is stable if for every t_o, $\epsilon > 0$, there exists $\delta(\epsilon,t_o) > 0$, such that, for every $t \geq t_o$,

$$\|x_o - x_e\| < \delta \quad \Rightarrow \quad \|\phi(t,x_o,t_o) - x_e\| < \epsilon$$

(iv) x_e, an equilibrium point of (1), is uniformly stable if it is stable and δ may be chosen in (iii) independent of t_o.

(v) x_e, an equilibrium point of (1), is asymptotically stable if it is stable and there exists $r(t_o) > 0$ such that

$$\|x - x_e\| < r \quad \Rightarrow \quad \lim_{t\to\infty} \phi(t,x_o,t_o) = x_e .$$

(vi) x_e, an equilibrium point of (1) is uniformly asymptotically stable if it is asymptotically stable, and r, δ, may be chosen independently of t_o in (iii), (v), and convergence is uniform in x_o.

Theorem IX.10.1 (Lyapunov) If (1) has equilibrium point x_e, and there exists a scalar function $V(x,t)$, with continuous first partial derivatives, such that

(i) $V(x_e,t) \equiv 0$, and

(ii) There exist non-decreasing, continuous scalar functions α and β, with

$$\alpha(0) = \beta(0) = 0 \text{ , and}$$

$$x \neq x_e \Rightarrow 0 < \alpha(\|x_e - x\|) \leq V(x,t) < \beta(\|x_e - x\|) \text{, and}$$

(iii) There exists a continuous scalar function γ, such that

$$\gamma(0) = 0, \quad \text{and}$$

$$\frac{d}{dt}\left(V(\phi(t,x_o,t_o),t)\right) \leq \gamma(\|x_e - x_o\|) < 0 .$$

Then, x_e is uniformly asymptotically stable.

Proof: See Kalman and Bertram [12].

Theorem IX.10.2 (Lyapunov-Perron-Malkin) If $x_e = 0$ is uniformly asymptotically stable for

$$\dot{x} = F(t)x, \tag{2}$$

and $\Phi(t,t_o)$ is the state transition matrix for $F(t)$, then there exist $\alpha, \beta > 0$ such that, for all $t \geq t_o$,

$$\| \Phi(t,t_o) \| \leq \alpha e^{-\beta(t-t_o)}$$

Proof: See Kalman and Bertram [12].

Theorem IX.10.3 If

(i) $A = \lim_{t\to\infty} A(t)$, and

(ii) $A(t)$ is continuous, and

(iii) $\dot{x} = Ax$ is a.s. about $x_e = 0$,

Then

$$\dot{x} = A(t)x$$

is u.a.s. about $x_e = 0$.

Proof: See Bellman [2], Theorem 2.2.2.

Theorem IX.10.4 If

(i) (2) is the linearized form of (1), and

(ii) F is a uniformly good approximation to f, near x_e, and

(iii) (2) is u.a.s. about x_e,

Then

(1) is u.a.s.

Proof: See Kalman and Bertram [12].

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X. BIBLIOGRAPHY

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