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Linear evolution equations of "hyperbolic" type, II

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§ 0. Introduction.

In a previous paper [1] with the same title, we studied linear evolution equations of the form

$$\text{(E)} \qquad du/dt+A(t)u=f(t), \qquad 0\leqq t\leqq T, \qquad u(0)=\phi,$$

in a Banach space X. The main purpose of the present paper is to prove some approximation theorems related to (E). Before doing so, however, we find it useful to strengthen some of the fundamental results of [1] by replacing the strong continuity of certain operator-valued functions with strong measurability.

As regards the approximation theorems, we considered similar problems in another paper [2] under somewhat different assumptions. However, only approximation in the X-norm was considered in [2]. In what follows we are primarily interested in approximation in a stronger norm.

The results of this paper are useful, among others, in applications to nonlinear evolution equations. Such applications will be discussed in other publications.

The paper is self-contained in definitions and statements of theorems, but their proof heavily leans on [1].

§ 1. Quasi-stability.

Let X be a Banach space. We denote by $G(X)$ the set of all negative generators of C_0-semigroups on X. A family $A=\{A(t)\}$ of elements of $G(X)$, defined for a. e. $t\in I=[0, T]$, is said to be *quasi-stable* if

$$\text{(1.1)} \qquad \|\prod_{j=1}^{k}(A(t_j)+\lambda_j)^{-1}\|\leqq M\prod_{j=1}^{k}(\lambda_j-\beta(t_j))^{-1}$$

for every finite family of real numbers $\{t_j, \lambda_j\}$ such that $0\leqq t_1\leqq\cdots\leqq t_k\leqq T$, $\lambda_1>\beta(t_1), \cdots, \lambda_k>\beta(t_k)$, where M is a constant and β is a real-valued, upper-integrable function (in the Lebesgue sense) defined a. e. on I. Of course only

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those t_j where $A(t_j)$ and $\beta(t_j)$ are defined are considered in (1.1). The product on the left member of (1.1) is *time-ordered* (see [1]). We shall call the pair $\{M, \beta\}$ a *stability index* for A.

Quasi-stability is a generalization of the notion of stability introduced in [1], in which β was a constant. Some results of [1] for stable families can be extended immediately to quasi-stable ones. For example, (1.1) is equivalent to

$$\| \Pi \exp[-s_j A(t_j)] \| \leqq M \exp[\Sigma s_j \beta(t_j)], \qquad s_j \geqq 0. \tag{1.2}$$

Also, Propositions 3.4 and 3.5 of [1] have obvious generalizations. Thus if A is quasi-stable as above and if $B: I \rightarrow B(X)$ a.e. is given with $\|B(\cdot)\|$ upper-integrable, then $A+B$ is quasi-stable with a stability index $\{M, \beta + M\|B(\cdot)\|\}$. (Here $B(X)=B(X, X)$, and $B(X, Y)$ is the set of all bounded linear operators on X to Y. $B: I \rightarrow B(X)$ a.e. means that $B(t) \in B(X)$ is defined for a.e. $t \in I$. $\|B(\cdot)\|$ denotes a function $t \mapsto \|B(t)\|$ a.e., where $\| \ \| = \| \ \|_X$ denotes the operator norm.)

Quasi-stability is introduced with the purpose of estimating *Riemann products*

$$\Pi \exp[-(a_j - a_{j-1}) A(t_j)], \tag{1.3}$$

where $0 \leqq a_0 \leqq t_1 \leqq a_1 \leqq \cdots \leqq a_k \leqq T$. (1.2) implies that the norm of (1.3) does not exceed

$$M \exp[\Sigma (a_j - a_{j-1}) \beta(t_j)]. \tag{1.4}$$

We want (1.4) to be uniformly bounded. This is true only when β is bounded, if we admit *all* possible a_j and t_j. It will be shown, however, that there are sufficiently many sums appearing in (1.4) that are uniformly bounded, provided β is upper-integrable. In this restricted sense, quasi-stability implies stability.

§ 2. A new existence theorem for the evolution operator.

We now state our fundamental assumptions for the family $A = \{A(t)\}$ in (E). These are generalizations of conditions (i), (ii″), and (iii) of [1], which were not necessarily the most general ones considered in [1] but were the most satisfactory in the case of a *general Banach space.*

(i′) $A: I \rightarrow G(X)$ is quasi-stable, with index $\{M, \beta\}$.

(ii‴) There is a Banach space Y, continuously and densely embedded in X, and a family $S = \{S(t)\}_{t \in I}$ of isomorphisms of Y onto X, such that

$$S(t) A(t) S(t)^{-1} = A(t) + B(t), \quad B(t) \in B(X), \qquad \text{for a.e. } t \in I, \tag{B}$$

where $B: I \to B(X)$ a.e. is strongly measurable with $\|B(\cdot)\|_X$ upper-integrable on I. Furthermore, there is a strongly measurable function $\dot{S}: I \to B(Y, X)$ a.e., with $\|\dot{S}(\cdot)\|_{Y,X}$ upper-integrable on I, such that S is equal to an indefinite strong integral of $\dot{S}$.

(iii) $Y \subset D(A(t))$, $t \in I$, and $A: I \to B(Y, X)$ is norm-continuous.

REMARKS. 1. (i′) is a simple generalization of (i) replacing stability by quasi-stability. (ii‴) is a generalization of (ii″), in which B and $\dot{S}$ were assumed to be strongly continuous. (B) should hold with the strict domain relation implied. (iii) is the same as in [1], but see the remark after Theorem I.

2. Strong measurability of operator-valued functions is taken in the usual sense (see Hille-Phillips [3, p. 74], Yosida [4, p. 130]). Thus B is strongly measurable if $B(\cdot)x$ is strongly measurable (as an X-valued function) for each $x \in X$. This does not necessarily mean that $\|B(\cdot)\|_X$ is Lebesgue-measurable; hence we assume the upper-integrability of $\|B(\cdot)\|_X$ and $\|\dot{S}(\cdot)\|_{Y,X}$. The assumptions on $\dot{S}$ imply that $\dot{S}$ is integrable in the strong sense, and it is assumed that S is one of the strong indefinite integrals. Thus we have $dS(t)y/dt = \dot{S}(t)y$ for a.e. $t \in I$ for each $y \in Y$, the exceptional set depending on y. We note that $S: I \to B(Y, X)$ is norm-continuous under these assumptions.

THEOREM I. *Let* (i′), (ii‴), *and* (iii) *be satisfied. Then there exists a unique evolution operator* $U = \{U(t, s)\}$, *defined on the triangle* $\Delta: T \geqq t \geqq s \geqq 0$, *with the following properties.*

(a) *U is strongly continuous on Δ to $B(X)$, with $U(s, s) = 1$.*

(b) $U(t, s)U(s, r) = U(t, r)$.

(c) *$U(t, s)Y \subset Y$, and U is strongly continuous on Δ to $B(Y)$.*

(d) *$dU(t, s)/dt = -A(t)U(t, s)$, $dU(t, s)/ds = U(t, s)A(s)$, which exist in the strong sense in $B(Y, X)$ and are strongly continuous on Δ to $B(Y, X)$.*

REMARK. This theorem strengthens Theorem 6.1 of [1]; note that the properties (a) to (d) are exactly the same as in that theorem, $U(t, s)$ being strongly continuously differentiable in $B(Y, X)$ *everywhere.* If we are satisfied instead with a.e. differentiability, we can weaken condition (iii) to $A \in L^1(I, B(Y, X))$. A similar generalization was made by Hackman [5] under somewhat different conditions, but he assumes that X is separable and reflexive.

The first half of this paper is devoted to the proof of Theorem I (and the remark above). The proof follows the same line as in [1], but it is rather long. The reason is that we have to approximate certain integrals involving β, B, and other functions by *Riemann sums,* but this is not so easy when these functions are only measurable in the Lebesgue sense.

§ 3. Construction of U.

In what follows we assume (i′), (ii‴), and (iii). First we shall show that Theorem 4.1 of [1] is still true, but we have to modify slightly the construction of U given in [1].

Condition (B) implies that Y is $A(t)$-admissible for a. e. t (see Proposition 2.4 of [1]). Let $\tilde{A}(t)$ be the part of $A(t)$ in Y. On the other hand, $A_1=A+B$ is quasi-stable with a stability index $\{M_1, \beta_1\}$, where $M_1=M$ and $\beta_1=\beta+M\|B(\cdot)\|$ (see § 2). Thus we see, by slightly modifying the proof of Proposition 4.4 of [1], that $\tilde{A}$ is quasi-stable with a stability index $\{\tilde{M}, \tilde{\beta}\}$, where $\tilde{M}=M_1c^2e^{cM_1V}$ and $\tilde{\beta}=\beta_1$ with the notation used there. (V is the total variation of S; it is finite by $V\leqq\int^*\|\dot{S}(t)\|_{Y,X}dt$, where $\int^*$ denotes upper-integral. $c=\max\{\sup_t\|S(t)\|_{Y,X}, \sup_t\|S(t)^{-1}\|_{X,Y}\}$ is also finite.)

Now we define approximate evolution operators U_n as in [1] by approximating A by step functions $A_n=\{I_{nk}, A(t_{nk})\}$, $n=1, 2, \cdots$ (for this notation see Appendix at the end of the paper). But the rather arbitrary partitions $\{I_{nk}\}$ of I used in [1] will not work, since the associated Riemann products (1.3) may not be uniformly bounded. A more careful choice of the partitions is necessary.

In what follows we assume, without loss of generality, that β and $\tilde{\beta}$ are nonnegative and Lebesgue integrable; otherwise we can replace them by dominating integrable functions. We then choose $\{I_{nk}\}$ and the numbers $t_{nk}\in I_{nk}$ in such a way that $\sup_k|I_{nk}|\to 0$, $n\to\infty$, and that the corresponding Riemann step functions $\beta_n=\{I_{nk}, \beta(t_{nk})\}$ and $\tilde{\beta}_n=\{I_{nk}, \tilde{\beta}(t_{nk})\}$ converge to β and $\tilde{\beta}$, respectively, in L^1-norm as well as pointwise a. e. This is possible by Lemma A1 and a remark after Lemma A2 (see Appendix).

With this choice of $\{I_{nk}\}$ and $t_{nk}\in I_{nk}$, we construct the step function $A_n=\{I_{nk}, A(t_{nk})\}$ and the associated evolution operator U_n as in [1], with obvious modifications. (For example, $U_n(t, s)=\exp[-(t-s)A(t_{nk})]$ if $t, s\in I_{nk}$ and $t\geqq s$.) Then we have by (1.4)

$$\|U_n(t, s)\|_X\leqq M\exp|||\beta_n|||_1, \qquad \|U_n(t, s)\|_Y\leqq\tilde{M}\exp|||\tilde{\beta}_n|||_1, \tag{3.1}$$

corresponding to (4.3) of [1], where $|||\ |||_1$ denotes the L^1-norm. (3.1) implies that the $U_n(t, s)$ are uniformly bounded in $B(X)$ as well as in $B(Y)$, since $\beta_n\to\beta$ and $\tilde{\beta}_n\to\tilde{\beta}$ in L^1-norm. Thus the construction given in [1] applies without further modification, yielding $U=\lim U_n$ satisfying Theorem 4.1. (We note in passing that Proposition 4.3 of [1] may no longer be true in general, but we need only a special case of this proposition that is easy to prove.)

To prove the remaining part of Theorem I, we need some preparations.

§ 4. Convolution of operator-valued kernels.

In what follows we consider various vector- and operator-valued functions. For $f: I \to X$ a.e., we write

$$|||f|||_{\infty,X} = \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X, \qquad |||f|||_{1,X} = \int_I^* \|f(t)\|_X dt.$$

We use similar notation for functions on Δ to X. For an operator-valued function $F: I \to B(X, Y)$ a.e. or $\Delta \to B(X, Y)$ a.e., we define $|||F|||_{\infty,X,Y}$ and $|||F|||_{1,X,Y}$ similarly. Here X, Y are arbitrary Banach spaces, and we write $|||F|||_{\infty,X}$ for $|||F|||_{\infty,X,X}$. In all these expressions, subscripts X or X, Y may be omitted if there is no ambiguity.

LEMMA 1. *Let $G': \Delta \to B(X', Y')$ and $G'': \Delta \to B(X'', Y'')$ be strongly continuous. Let $F: I \to B(Y', X'')$ a.e. be strongly measurable with $|||F|||_1 < \infty$. Then there is $G: \Delta \to B(X', Y'')$, denoted by $G = G''FG'$, such that*

$$G(t, r)x = \int_r^t G''(t, s)F(s)G'(s, r)x\, ds, \qquad t, r \in \Delta, \tag{4.1}$$

for each $x \in X'$. G is strongly continuous on Δ to $B(X', Y'')$, and

$$|||Gx|||_\infty \leqq |||G''|||_\infty |||F|||_1 |||G'x|||_\infty, \tag{4.2}$$

$$|||G|||_\infty \leqq |||G''|||_\infty |||F|||_1 |||G'|||_\infty,$$

where Gx denotes the vector-valued function $t, s \mapsto G(t, s)x \in Y''$.

PROOF. For fixed t, r and x, the integrand in (4.1) is strongly measurable by Lemma A4 of Appendix. Since it is strongly integrable by $|||F|||_1 < \infty$, the existence of $G(t, r) \in B(X', Y'')$ and the estimates (4.2) follow easily.

It remains to show that (4.1) is continuous in Y''-norm in $t, r \in \Delta$ for fixed x. To this end it is convenient to extend G' and G'' to the square $I \times I$, setting $G' = 0$, $G'' = 0$ outside Δ. Then the range of integration in (4.1) may be taken to be I, and we have

$$\|G(t_n, r_n)x - G(t, r)x\| \leqq \int_I \|G''(t_n, s)F(s)[G'(s, r_n) - G'(s, r)]x\| ds$$

$$+ \int_I \|[G''(t_n, s) - G''(t, s)]F(s)G'(s, r)x\| ds.$$

If we let $t_n \to t$ and $r_n \to r$, the right member tends to 0 by the dominated convergence theorem; note that the extended kernels $G''(t, s)$ and $G'(s, r)$ are strongly continuous except possibly at $s = r$ and $s = t$.

§ 5. Kernels V and W.

From U constructed in § 3 and B given by (B), we construct a new kernel V by

(5.1) $$V=\sum_{p=0}^{\infty}(-UB)^{p}U\,,$$

where $(-UB)^{p}U$ are defined successively using Lemma 1. The series converges uniformly on Δ in $B(X)$-norm, as is seen from the Volterra-type estimates

(5.2) $$|||(UB)^{p}U|||_{\infty}\leqq|||U|||_{\infty}^{p+1}|||B|||_{1}^{p}/p\,!\,,\qquad p=0,1,2,\cdots.$$

Thus $V:\Delta\rightarrow B(X)$ is well defined and is strongly continuous. Similarly, we define a strongly continuous kernel $W:\Delta\rightarrow B(X)$ by

(5.3) $$W=\sum_{p=0}^{\infty}(VC)^{p}V\,,$$

where

(5.4) $$C(t)=\dot{S}(t)S(t)^{-1}\in B(X)\qquad\text{for a. e. } t\in I\,.$$

By (ii'''), $C:I\rightarrow B(X)$ a. e. is strongly measurable with $|||C|||_{1}<\infty$.

LEMMA 2. *We have*

(5.5) $$S(t)U(t,s)S(s)^{-1}=W(t,s)\,,\qquad t,\ s\in\Delta\,.$$

This is our key lemma, from which the remaining part of the assertions of Theorem I follows easily as in [**1**]. (5.5) shows directly that U is strongly continuous on Δ to $B(Y)$. The proof of the lemma will be given in the following sections.

We note in passing that

(5.6) $$W+UDW=U\,,\qquad W=\sum_{p=0}^{\infty}(-UD)^{p}U\,,\qquad D=B-C\,.$$

This follows easily from

(5.7) $$V+UBV=U\,,\qquad W-VCW=V\,,$$

implied by (5.1) and (5.3). Thus W could have been constructed directly from U, without goint through V. But we need V in the proof of Lemma 2.

§ 6. Step function approximation readjusted.

It appears that the only method available for the proof of Lemma 2 is to approximate U and W by sequences U_n and W_n based on step function approximations for A and B. In view of the delicacy of such approximations and the appearance of B in addition to A, we have to readjust the partitions

$\{I_{nk}\}$ and the numbers t_{nk} involved. There is another complication due to the fact that step function approximation for B, which is only strongly measurable, seems to work only on a separable subspace. (The readjustment is not necessary if X is separable. In this case it suffices to choose the partitions used in §3 adapted to β, $\tilde{\beta}$, *and* B in the sense of the remark to Lemma A2.)

Let $x_0 \in X$ be arbitrary but fixed; we want to show that (5.5) is true when applied to x_0. Let X_0 be the (closed) subspace of X spanned by the $W(t, s)x_0$ for all t, $s \in \Delta$; X_0 is separable since W is strongly continuous.

Since $C: I \to B(X)$ a.e. is strongly measurable, there is a separable subspace X_1 of X containing x_0 and the $C(t)X_0$ for a.e. $t \in I$ (see Lemma A3).

Finally, let X_2 be the subspace of X spanned by the $V(t, s)X_1$ for all t, $s \in \Delta$. X_2 is separable since V is strongly continuous.

We now introduce step functions B_n that approximate B in a convenient way. Let $\hat{B}(t)$ be the restriction of $B(t)$ on X_2, so that $\hat{B}(t) \in B(X_2, X)$ for a.e. t. Then $\hat{B}: I \to B(X_2, X)$ a.e. is strongly measurable with $\|\hat{B}(\cdot)\| \leqq \|B(\cdot)\|$ integrable. Since X_2 is separable, Lemma A2 shows that $\hat{B}$ can be approximated by a sequence of Riemann step functions $\hat{B}_n = \{I_{nk}, \hat{B}(t_{nk})\}$ in the following sense: $\sup_k |I_{nk}| \to 0$ as $n \to \infty$, $\hat{B}_n(t) \to \hat{B}(t)$ strongly in $B(X_2, X)$ for a.e. t, and $\hat{B}_n(\cdot)x \to \hat{B}(\cdot)x$ in $L^1(I, X)$ for every $x \in X_2$. Moreover, we can achieve that $\beta_n = \{I_{nk}, \beta(t_{nk})\}$, $\tilde{\beta}_n = \{I_{nk}, \tilde{\beta}(t_{nk})\}$, and $b_n = \{I_{nk}, b(t_{nk})\}$ approximate β, $\tilde{\beta}$, and b, respectively, pointwise a.e. as well as in the L^1-norm (see the remark after Lemma A2); here b is an integrable function such that $\|B(t)\| \leqq b(t)$ a.e. Then we define also $B_n = \{I_{nk}, B(t_{nk})\}$; B_n is a Riemann step function for B and $\hat{B}_n(t)$ is obviously the restriction of $B_n(t)$ on X_2. Note that the I_{nk} and t_{nk} are in general different from the ones used in §3; this is why we talk about readjustment.

With the new I_{nk} and t_{nk}, we define a new step function $A_n = \{I_{nk}, A(t_{nk})\}$ and the associated evolution operator U_n. These are different from the ones used in §3 to construct U as their limit, but we claim that the new U_n converge to the same U; we have namely the following lemma.

LEMMA 3. *We have*

$$\|\|(U_n - U)x\|\|_\infty \longrightarrow 0, \qquad n \to \infty, \qquad \textit{for each } x \in X. \tag{6.1}$$

PROOF. As in §3, $\lim U_n(t, s) = \bar{U}(t, s)$ exists strongly in $B(X)$, uniformly in t, s, and defines an evolution operator for A satisfying Theorem 4.1 of [**1**]. But since such an operator is unique, we must have $\bar{U} = U$. This proves (6.1).

Next we define new kernels $V_n: \Delta \to B(X)$ by

(6.2) $$V_n=\sum_{p=0}^{\infty}(-U_nB_n)^pU_n$$

corresponding to (5.1); note that U_n and $V_n: \varDelta\rightarrow B(X)$ are strongly continuous.

LEMMA 4. *We have*

(6.3) $$|||(V_n-V)x|||_\infty\longrightarrow 0\,,\qquad n\rightarrow\infty\,,\qquad for\ x\in X_1\,.$$

PROOF. We use the identity

(6.4) $$V_n-V=(1+U_nB_n)^{-1}[(U_n-U)-(U_n-U)BV-U_n(B_n-B)V]\,,$$

which follows easily from (5.1) and (6.2). Here the operator $(1+U_nB_n)^{-1}$ is best defined by Neumann series expansion, which makes (6.4) meaningful by the estimates similar to (5.2); note that $|||B_n|||_1\leqq|||b_n|||_1\leqq$ const. Thus it is uniformly bounded when regarded as an operator on $C(\varDelta, X)$ into itself, and (6.4) gives

(6.5) $$|||(V_n-V)x|||_\infty\leqq \text{const.}\,[|||(U_n-U)x|||_\infty+|||(U_n-U)BVx|||_\infty$$
$$+|||U_n(B_n-B)Vx|||_\infty]\,.$$

The lemma will be proved if we show that each term on the right of (6.5) tends to 0.

For the first term, this is obvious by (6.1). The second term is majorized by

(6.6) $$\sup_{t,r}\int_I\|(U_n-U)(t, s)B(s)V(s, r)x\|ds\,,$$

where we suppose that U_n, U, V are extended to $I\times I$ as in the proof of Lemma 1. By Lemma A3, there is a closed subset E of I, with arbitrarily small $|I-E|$, such that B is strongly continuous on E to $B(X_2, X)$. Then BVx is piecewise continuous on $E\times I$ to X (note that $V(s, r)x\in X_2$ because $x\in X_1$). Hence there is a compact set $S'\subset X$ containing the image of $E\times I$ under the map BVx. But (6.1) implies that $(U_n-U)(t, s)\phi\rightarrow 0$ in X uniformly for t, $s\in E\times I$ and $\phi\in S'$. Hence the integrand of (6.6) tends to 0 uniformly for t, $r\in\varDelta$ and $s\in E$. By first choosing $|I-E|$ small and then n large, we see that (6.6) tends to 0 as required (note that $\int_{I-E}\|B(s)\|ds$ is uniformly small if $|I-E|$ is small).

The third term on the right of (6.5) is majorized by

(6.7) $$\sup_r\int_I\|(B_n-B)(s)V(s, r)x\|ds$$

because $|||U_n|||_\infty\leqq$ const. (again extend V to $I\times I$). Since Vx: $\varDelta\rightarrow X_2$ is continuous, there is a compact set $S''\subset X_2$ such that $V(s, r)x\in S''$ for $s, r\in I\times I$. Since $B_n(s)\rightarrow B(s)$ strongly in $B(X_2, X)$ for $s\in I-N$, where N is a null set, we have $(B_n-B)(s)\phi\rightarrow 0$ uniformly for $\phi\in S''$ for each fixed $s\in I-N$. In other words, $\varepsilon_n(s)=\sup_{\phi\in S''}\|(B_n-B)(s)\phi\|\rightarrow 0$, $s\in I-N$. Then (6.7) is majorized by $\int_I \varepsilon_n(s)ds$, which tends to 0 by the Vitali convergence theorem. Note that $\varepsilon_n(\cdot)\leqq \text{const.}(\|B_n(\cdot)\|+\|B(\cdot)\|)\leqq \text{const.}(b_n(\cdot)+b(\cdot))$ satisfies the condition for Vitali's theorem because $b_n\rightarrow b\in L^1$ in L^1-norm (the Vitali-Hahn-Saks theorem, see [**4**, p. 70]). This completes the proof of Lemma 4.

§ 7. Completion of the proof of Theorem 1.

We proceed to the construction of approximating kernels for W. To this end, we have to modify the U_n and V_n slightly.

For each n, let $a_n(t)$ denote the upper end of I_{nk} if $t\in(t_{nk}, t_{n,k+1}]$. We set $a_n(t)=0$ if $t\in[0, t_{n1}]$ and $a_n(t)=T$ if $t\in(t_{nm}, T]$ with m the largest of the indices k. Thus $a_n: I\rightarrow I$ is a monotone nondecreasing step function, and $a_n(t)\rightarrow t$ as $n\rightarrow\infty$ uniformly in t.

For t, $s\in\Delta$ let

(7.1) $$U'_n(t, s)=U_n(a_n(t), a_n(s))\,,\qquad V'_n(t, s)=V_n(a_n(t), a_n(s))\,.$$

These are "step functions" in t, s, and are in general discontinuous.

LEMMA 5. *We have as* $n\rightarrow\infty$

(7.2) $$|||(U'_n-U)x|||_\infty\rightarrow 0 \quad \textit{for } x\in X\,,\qquad |||(V'_n-V)x|||_\infty\rightarrow 0 \quad \textit{for } x\in X_1\,.$$

PROOF. Obvious from Lemmas 3, 4 and the uniform convergence $a_n(t)\rightarrow t$; note that Ux and Vx: $\Delta\rightarrow X$ are continuous.

We now define

(7.3) $$W'_n(t, s)=S(t)U'_n(t, s)S(s)^{-1}\,,\qquad t,\ s\in\Delta\,.$$

$W'_n(t, s)\in B(X)$ since $U_n(t, s)$ maps Y into Y (see (3.1)). W'_n: $\Delta\rightarrow B(X)$ is *piecewise* strongly continuous.

LEMMA 6. *We have*

(7.4) $$|||(W'_n-W)x_0|||_\infty\longrightarrow 0\,,\qquad n\rightarrow\infty\,.$$

PROOF. We use the identity

(7.5) $$W'_n=V'_n+V'_nCW'_n\,,$$

which is an analog of the second identity of (5.7). Here both V'_n and W'_n are piecewise continuous so that there is no difficulty in the meaning of $V'_nCW'_n$. In fact the integral involved can be evaluated on each subinterval $(t_{n,k-1}, t_{nk})$, leading to an algebraic identity which is easy to verify. Since similar identities are deduced in [**4**] and [**6**], we shall omit the rather straightforward proof of (7.5).

(7.4) can now be proved using (7.5) in the same way as (6.3) was proved using (6.2). Since V'_n and W'_n are piecewise continuous, there is no difficulty in applying the argument used in the proof of Lemma 4. It suffices to add the following remarks. We have $|||(V'_n - V)x_0|||_\infty \to 0$ by (7.2) because $x_0 \in X_1$, and $C(s)W(s, r)x_0 \in X_1$ for a. e. s and all r by the definition of X_1.

PROOF OF LEMMA 2. Fix $t, s \in \Delta$. We have $S(t)^{-1}W'_n(t, s)x_0 = U'_n(t, s)S(s)^{-1}x_0$ by (7.3). Letting $n \to \infty$ and using Lemmas 5 and 6, we obtain $S(t)^{-1}W(t, s)x_0 = U(t, s)S(s)^{-1}x_0$ (note that $S(t)^{-1}$ is bounded on X to X). Since $x_0 \in X$ was arbitrary, we have proved Lemma 2.

As remarked after Lemma 2, this completes the proof of Theorem I.

PROOF OF THE REMARK AFTER THEOREM I. Only a slight change of the above argument is required to prove the assertion made in the remark. In the construction of U given in § 3, the partitions $\{I_{nk}\}$ and the numbers t_{nk} should be chosen in such a way that $A_n = \{I_{nk}, A(t_{nk})\}$ approximates A pointwise a. e. as well as in $L^1(I, B(Y, X))$, in addition to satisfying $\beta_n \to \beta$ and $\tilde{\beta}_n \to \tilde{\beta}$; this is possible by Lemma A1 and the remark after Lemma A2. Since it implies $\int_I \|A_n(t) - A_m(t)\|_{Y,X} dt \to 0$ as $m, n \to \infty$, the construction of [1] can be carried out, yielding a unique U satisfying Theorem 4.1 of [1] except for the obvious modification of the part (d) contained in that theorem.

A similar change is required in the readjustment of the I_{nk} and t_{nk} given in § 6. In the final result, $dU(t, s)/dt$ will not exist everywhere but it exists at each point t where $(d/dt)\int^t A(s)ds = A(t)$ holds, and similarly for $dU(t, s)/ds$. Since $A \in L^1(I, B(Y, X))$, the exceptional set has measure 0.

§ 8. The inhomogeneous equation. Mild solutions.

Suppose that the assumptions of Theorem I are satisfied. The *mild solution* of the inhomogeneous equation (E) is given by

$$u(t) = U(t, 0)\phi + \int_0^t U(t, s)f(s)ds\,. \tag{8.1}$$

For simplicity we abbreviate (8.1) to

$$u = U(\delta \otimes \phi) \oplus Uf = U(\delta \otimes \phi \oplus f)\,. \tag{8.2}$$

In general u need not be a solution of (E) in the strict sense, but we have the following results regarding the map U. Here we use the simplified notation

$$L(X) = L^1(I, X)\,, \qquad C(X) = C(I, X)\,, \quad \text{etc.} \tag{8.3}$$

THEOREM II. *U maps* (a) $\delta\otimes X\oplus L(X)$ *into* $C(X)$, (b) $\delta\otimes Y\oplus L(Y)$ *into* $C(Y)$, *and* (c) $\delta\otimes Y\oplus L(Y)\cap C(X)$ *into* $C(Y)\cap C^1(X)$. *The map is continuous in each case, and we have for* (8.2)

$$\|\|u\|\|_{\infty,X}\leqq\|\|U\|\|_{\infty,X}(\|\phi\|_X+\|\|f\|\|_{1,X}) \qquad \textit{in case (a)}, \tag{8.4a}$$

$$\|\|u\|\|_{\infty,Y}\leqq\|\|U\|\|_{\infty,Y}(\|\phi\|_Y+\|\|f\|\|_{1,Y}) \qquad \textit{in cases (b) and (c)}, \tag{8.4b}$$

$$\|\|du/dt\|\|_{\infty,X}\leqq\|\|f\|\|_{\infty,X}+\|\|A\|\|_{\infty,Y,X}\|\|U\|\|_{\infty,Y}(\|\phi\|_Y+\|\|f\|\|_{1,Y}) \qquad \textit{in case (c)}, \tag{8.4c}$$

(*for notation see* § 4). *In case* (c), *u is a solution of* (E) *in the strict sence.*

PROOF. If $\phi\in X$ and $f\in C(X)$, then it is obvious that $u\in C(X)$ and (8.4a) holds. Since $C(X)$ is dense in $L(X)$, (a) follows with (8.4a). Similarly one proves (b) and (8.4b). To prove (c) we use the result of [1], Theorem 7.1, which shows that if $\phi\in Y$ and $f\in C(Y)$, then $u\in C(Y)\cap C^1(X)$ and solves (E). In this case (8.4c) follows from (E) and (8.4b). Then the desired results follow since $C(Y)$ is dense in $L(Y)\cap C(X)$.

§ 9. Preliminary estimates.

To develop quantitative approximation theory, it is convenient to introduce some primitive constants that describe the behavior of the family $\{A(t)\}$. In particular, we want to estimate $\|\|U\|\|_{\infty,X}$ and $\|\|U\|\|_{\infty,Y}$ in terms of such constants.

The following system of numbers (or any system majorizing these) will be called the *primitive constants for* $\{A(t)\}$:

$$M,\quad \|\|\beta\|\|_1,\quad \|\|S\|\|_{\infty,Y,X},\quad \|\|S^{-1}\|\|_{\infty,X,Y},\quad \|\|B-C\|\|_{1,X}, \tag{9.1}$$

where $\{M, \beta\}$ is a stability index for A. We do not count $\|\|A\|\|_{\infty,Y,X}$ among them, although it was used in (8.4c).

$\|\|U\|\|_{\infty,X}$ and $\|\|U\|\|_{\infty,Y}$ can be easily estimated in terms of the primitive constants. The construction of U given in § 3 gives immediately

$$\|\|U\|\|_{\infty,X}\leqq M\exp\|\|\beta\|\|_1. \tag{9.2}$$

To estimate $\|\|U\|\|_{\infty,Y}$, we note that

$$\|\|U\|\|_{\infty,Y}\leqq\|\|S\|\|_{\infty,Y,X}\|\|S^{-1}\|\|_{\infty,X,Y}\|\|W\|\|_{\infty,X} \tag{9.3}$$

by (5.5). But we have from (5.6)

$$\|\|W\|\|_{\infty,X}\leqq\|\|U\|\|_{\infty,X}\exp(\|\|B-C\|\|_{1,X}\|\|U\|\|_{\infty,X}) \tag{9.4}$$

by using Volterra-type estimates similar to (5.2).

§ 10. Perturbation theory.

Suppose we have another equation of the same type

(E′) $$du'/dt+A'(t)u'=f'(t), \qquad 0\leqq t\leqq T,\ u'(0)=\phi',$$

in the same Banach space X. For $\{A'(t)\}$ we make the same basic assumptions (i′), (ii‴), (iii), and use the obvious notation such as $S'(t)$, $B'(t)$, etc. The space $Y\subset X$ is assumed to be common to the two systems.

Then there exists the evolution operator U' for the new system, and we construct the associated kernel W' as above. The mild solution of (E′) is given by $u'=U'(\delta\otimes\phi'\oplus f')$.

The purpose of this section is to estimate $u'-u$ uniformly in X- and Y-norms, where $u=U(\delta\otimes\phi\oplus f)$ is the solution of (E). The following are our main results. For notation see §§ 4 and 8.

THEOREM III. *Let $\phi\in Y$, $f\in L(Y)$, $\phi'\in X$, and $f'\in L(X)$. Then*

(10.1) $$|||u'-u|||_{\infty,X}\leqq K'[\|\phi'-\phi\|_X+|||f'-f|||_{1,X}+|||(A'-A)u|||_{1,X}],$$

where K' is a constant depending only on the primitive constants for A', and $A'-A$ is regarded as a multiplication operator on $C(Y)$ to $L(X)$; note that $u\in C(Y)$ by Theorem II, (b).

THEOREM IV. *Let ϕ, $\phi'\in Y$ and f, $f'\in L(Y)$. Then*

(10.2) $$\begin{aligned}|||u'-u|||_{\infty,Y}\leqq\ & K'(\|\phi'-\phi\|_Y+|||f'-f|||_{1,Y})\\ &+K'(\|[S'(0)-S(0)]\phi\|_X+|||(S'-S)f|||_{1,X})+K'|||(S'-S)u|||_{\infty,X}\\ &+K'|||[(B'-C')-(B-C)]Su|||_{1,X}+K'|||(U'-U)[\delta\otimes\psi\oplus g]|||_{\infty,X},\end{aligned}$$

where K' is a constant depending only on the primitive constants for A', and

(10.3) $$\psi=S(0)\phi\in X, \qquad g=Sf+(C-B)Su\in L(X).$$

PROOF OF THEOREM III. We start from the well-known identity

(10.4) $$U'(t,r)y-U(t,r)y=-\int_r^t U'(t,s)[A'(s)-A(s)]U(s,r)y\,ds,$$

where $y\in Y$. As usual (10.4) may be deduced by differentiating and integrating $U'(t,s)U(s,r)y$ in s (see [1]). Setting $r=0$ and $y=\phi$ gives $(U'-U)(\delta\otimes\phi)=-U'(A'-A)U(\delta\otimes\phi)$. Setting $y=f(r)$ and integrating in r gives $(U'-U)f=-U'(A'-A)Uf$. Hence

(10.5) $$(U'-U)(\delta\otimes\phi\oplus f)=-U'(A'-A)u.$$

On the other hand we have

(10.6) $$u'-u=U'[\delta\otimes(\phi'-\phi)\oplus(f'-f)]+(U'-U)(\delta\otimes\phi\oplus f).$$

(10.1) follows from (10.5) and (10.6); we can take $K'=|||U'|||_{\infty,X}$.

PROOF OF THEOREM IV. Since

$$|||u'-u|||_{\infty,Y}\leqq|||S'^{-1}|||_{\infty,X,Y}|||S'(u'-u)|||_{\infty,X}, \tag{10.7}$$

it suffices to estimate the right member of (10.7). We compute $S'(u'-u)$ as follows. Using (5.5) in the form $SU=WS$, we obtain

$$\begin{aligned} S'(u'-u) &= (S-S')u+S'u'-Su \\ &= (S-S')u+W'S'[(\delta\otimes\phi'\oplus f')-(\delta\otimes\phi\oplus f)] \\ &\quad +W'(S'-S)(\delta\otimes\phi\oplus f)+(W'-W)S(\delta\otimes\phi\oplus f). \end{aligned} \tag{10.8}$$

Taking the $|||\ |||_{\infty,X}$-norm of the first three terms in the last member leads to the first three terms of (10.2); note, for example, that

$$|||W'S'[\cdots]|||_{\infty,X}\leqq|||W'|||_{\infty,X}|||S'[\cdots]|||_{1,X}\leqq|||W'|||_{\infty,X}\|S'\|_{\infty,Y,X}|||\cdots|||_{1,Y}.$$

To estimate the last term of (10.8), we use the identity

$$W'-W=(1+U'D')^{-1}[(U'-U)-(U'-U)DW-U'(D'-D)W], \tag{10.9}$$

which is an immediate analog of (6.4) due to (5.6). The contribution of the last term in [ ] of (10.9) to $(W'-W)S(\delta\otimes\phi\oplus f)$ is equal to $-(1+U'D')^{-1}U'(D'-D)Su$, since $WS(\delta\otimes\phi\oplus f)=SU(\delta\otimes\phi\oplus f)=Su$. Since $(1+U'D')^{-1}U'$ can be estimated in terms of the primitive constants for A', it leads to the fourth term on the right of (10.2). Similarly, the remaining terms contribute $K'|||(U'-U)(1-DW)S(\delta\otimes\phi\oplus f)|||_{\infty,X}$, which is easily seen to equal the last term of (10.2) (again use $WS=SU$).

REMARK. We do not deduce explicit estimates for $(d/dt)(u'-u)$ here, but it can be done easily by using (E), (E′), (10.1), and some additional assumptions on A'.

§ 11. Convergence theorems.

In this section we prove some convergence theorems for the solutions of evolution equations.

Suppose we have a sequence of equations in X:

$$du^n/dt+A^n(t)u^n=f^n(t),\qquad 0\leqq t\leqq T,\quad u^n(0)=\phi^n, \tag{E^n}$$

$n=1, 2, \cdots$. Suppose the A^n satisfy conditions (i′), (ii‴), (iii) *uniformly* in n, by which we mean that the primitive constants for them can be chosen independent of n. We use the obvious notation like S^n, B^n, C^n, M^n, β^n, etc. Again the space Y is assumed to be common to all A^n.

The associated evolution operators U^n then exist, and we can compute the mild solutions $u^n=U^n(\delta\otimes\phi^n\oplus f^n)$ of (E^n). We ask when u^n converges to the solution $u=U(\delta\otimes\phi\oplus f)$ of (E).

THEOREM V. *In addition to the above assumptions, suppose that*

$$A^n(t)\rightarrow A(t) \text{ strongly in } B(Y, X) \text{ for a.e. } t\in I. \tag{11.1}$$

$$\int_E \|A^n(t)\|_{Y,X}dt\rightarrow 0 \text{ as } |E|\rightarrow 0, \text{ uniformly in } n. \tag{11.2}$$

Then we have

$$U^n(t, s)\rightarrow U(t, s) \text{ strongly in } B(X), \text{ uniformly in } t, s\in\Delta. \tag{11.3}$$

THEOREM Va. *If in addition $\phi^n\rightarrow\phi$ in X and $f^n\rightarrow f$ in $L(X)$, then $u^n\rightarrow u$ in $C(X)$.*

THEOREM VI. *In addition to the assumptions of Theorem V, suppose that*

$$B^n(t)-C^n(t)\rightarrow B(t)-C(t) \text{ strongly in } B(X) \text{ for a.e. } t\in I. \tag{11.4}$$

$$\int_E^* \|B^n(t)-C^n(t)\|_X dt\rightarrow 0 \text{ as } |E|\rightarrow 0, \text{ uniformly in } n. \tag{11.5}$$

$$S^n(t)\rightarrow S(t) \text{ strongly in } B(Y, X), \text{ uniformly in } t\in I. \tag{11.6}$$

Then we have

$$U^n(t, s)\rightarrow U(t, s) \text{ strongly in } B(Y), \text{ uniformly in } t, s\in\Delta. \tag{11.7}$$

THEOREM VIa. *If in addition $\phi^n\rightarrow\phi$ in Y and $f^n\rightarrow f$ in $L(Y)$, then $u^n\rightarrow u$ in $C(Y)$.*

REMARK. If we assume (11.4) and (11.5) for B^n and C^n separately, (11.6) is automatically satisfied if $S^n(0)\rightarrow S(0)$ strongly in $B(Y, X)$. This is due to the fact that S^n satisfies the differential equation $dS^n/dt=C^n(t)S^n$ and similarly for S.

PROOF. For $y\in Y$ we see from (10.4) that

$$\|(U^n-U)(t, r)y\|_X\leqq K^n\int_I\|(A^n-A)(s)U(s, r)y\|_X ds, \tag{11.8}$$

where we again suppose that U has been trivially extended to $I\times I$. Since $K^n=|||U^n|||_{\infty,X}$ is uniformly bounded, the right member of (11.8) tends to 0 uniformly in t, r; the proof is similar to the one for (6.7) and depends on the Vitali convergence theorem based on (11.2). Since Y is dense in X, Theorem V follows.

Then Theorem Va follows from (10.6).

To prove Theorem VI, it suffices to show that $W^n(t, s)\rightarrow W(t, s)$ strongly in $B(X)$, uniformly in t, s. This can be done using (10.9) and again imitating the proof of Lemma 4.

Then Theorem VIa follows from (10.8) and (10.7).

§ 12. Yosida approximation.

As a simple application of the convergence theorems, we shall prove the convergence of the *Yosida approximation.*

Suppose $A=\{A(t)\}$ satisfies the assumptions of Theorem I. For simplicity we assume in addition that $\beta(t)=\beta$ is constant (so that A is stable) and that (B) holds for *every* $t\in I$ with $\|B(t)\|_X\leqq b=\text{const}$. We may further assume $\beta=0$ without loss of generality.

Under these conditions we can define the Yosida approximation (E^n) to (E), $n=1,2,\cdots$, with

$$A^n(t)=A(t)[1+n^{-1}A(t)]^{-1}\in B(X), \qquad t\in I. \tag{12.1}$$

Then it is not difficult to prove the following results.

1° A^n is stable with stability constants $\{M,0\}$.

2° $S(t)A^n(t)S(t)^{-1}=A^n(t)+B^n(t)$, with

$$\|B^n(t)\|_X\leqq M^2b(1-n^{-1}Mb)^{-1}, \qquad n>Mb.$$

3° $A^n\colon I\to B(Y,X)$ is norm-continuous.

4° For each $t\in I$, we have $A^n(t)\to A(t)$, $n\to\infty$, strongly in $B(Y,X)$, with

$$\|A^n(t)\|_{Y,X}\leqq\|A(t)\|_{Y,X}(1-n^{-1}Mb)^{-1}, \qquad n>Mb.$$

5° For each $t\in I$, we have $B^n(t)\to B(t)$, $n\to\infty$, strongly in $B(X)$.

It follows from 1° to 3° that A^n satisfies the assumptions of Theorem I *uniformly in n,* with S independent of n, so that the associated evolution operator U^n exists. Then 2°, 4°, and 5° show that Theorems V and VI are applicable. Thus we conclude that $U^n(t,s)\to U(t,s)$ holds strongly in $B(X)$ as well as $B(Y)$, uniformly in t, $s\in\Delta$.

REMARK. The proof given above depends on the existence of U, which has been proved by other means. It is an open question whether or not U can be *constructed* directly as the limit of the U^n. Since $A^n(t)\in B(X)$ is strongly continuous in t (as is easily proved), U^n can be constructed by an elementary method. Thus a direct construction of U via U^n is highly desirable. It seems that this is possible if X and Y are reflexive (cf. Yosida [7] and Hackman [5]); we plan to study this problem in another publication.

§ 13. Concluding remark.

It seems that the estimate (10.2) is the most useful one in this paper; it contains more information than the convergence theorem VIa, for example. (10.2) is useful even for some nonlinear evolution equation (E) in which $A(t)$ and $f(t)$ depend on the unknown $u(t)$ itself. Suppose we have another such equation (E′) and want to estimate $u'-u$ by using (10.2). Here $|||f'-f|||_{1,Y}$ on the right-hand side may be estimated in terms of the unknown quantity $|||u'-u|||_{\infty,Y}$. If the latter appears with a sufficiently small coefficient (which may occur if T is sufficiently small), it can be absorbed into the left-hand side. The same remark applies to other terms on the right that involve the unknowns u and u'. Applications of (10.2) to nonlinear evolution equations will be discussed elsewhere.

Appendix.

Here we prove some lemmas regarding stongly measurable vector- and operator-valued functions that are used in the text. In particular, we consider approximation of such functions by Riemann step functions. In what follows all functions are defined on a finite closed interval I of real numbers.

Let S be any set. A function $g: I\to S$ is a *step function* if there is a finite partition $\{I_k\}_{k=1,\cdots,m}$ of I into subintervals such that g takes a constant value c_k on I_k for each k. We shall express this by

(A1) $$g=\{I_k, c_k; k=1, \cdots, m\} \quad \text{or simply} \quad g=\{I_k, c_k\}.$$

Given a function $f: I\to S$, the step function (A1) will be called a *Riemann step function associated with* f (or simply *for* f) if $c_k=f(t_k)$ for some $t_k\in I_k$, $k=1, \cdots, m$.

LEMMA A1. *Let X be a Banach space, and let $f: I\to X$ a.e. be strongly measurable. Then there exists a sequence of Riemann step functions $f_n=\{I_{nk}, f(t_{nk})\}$, $n=1, 2, \cdots$, for f such that $\sup_k|I_{nk}|\to 0$ and $f_n\to f$ pointwise a.e. If f is strongly integrable, then we can achieve that $f_n\to f$ also in $L^1(I, X)$.*

PROOF. By definition f can be approximated pointwise a.e. by a sequence of simple functions. Hence it is easy to see that there is a sequence of step functions $g_n=\{I_{nk}, x_{nk}\}$ satisfying the requirements of the lemma except that g_n may not be Riemann for f. We shall modify g_n to obtain Riemann step functions f_n.

Fix n for the moment. For each k, choose $t_{nk}\in I_{nk}$ such that

$$\|x_{nk}-f(t_{nk})\|\leqq \inf_{s\in I_{nk}}\|x_{nk}-f(s)\|+n^{-1}.$$

Then $f_n=\{I_{nk}, f(t_{nk})\}$ is a Riemann step function for f.

We shall show that $\{f_n\}$ satisfies the requirements. For any $t\in I$ where $f(t)$ is defined and any n, there is I_{nk} such that $t\in I_{nk}$. Then

$$\begin{aligned}\|f(t)-f_n(t)\| &\leqq \|f(t)-x_{nk}\|+\|x_{nk}-f(t_{nk})\| \\ &\leqq \|f(t)-x_{nk}\|+\inf_{s\in I_{nk}}\|x_{nk}-f(s)\|+n^{-1} \\ &\leqq 2\|f(t)-x_{nk}\|+n^{-1}=2\|f(t)-g_n(t)\|+n^{-1}\to 0 \text{ a.e.}\end{aligned}$$

If f is integrable, we have also $g_n \to f$ in L^1-norm. Hence $f_n \to f$ in L^1-norm too.

LEMMA A2. *Let X, Y be Banach spaces, with X separable, and let $B: I \to B(X, Y)$ a.e. be strongly measurable. Then there exists a sequence of Riemann step functions $B_n = \{I_{nk}, B(t_{nk})\}$, $n = 1, 2, \cdots$, such that $\sup_k |I_{nk}| \to 0$, $B_n(t) \to B(t)$ strongly and $\|B_n(t)\| \to \|B(t)\|$ for a.e. t. If $\|B(t)\|$ is integrable in t, we can also achieve that $\|B_n(\cdot)\| \to \|B(\cdot)\|$ in $L^1(I)$ and $B_n(\cdot)x \to B(\cdot)x$ in $L^1(I, Y)$ for each $x \in X$.*

PROOF. The assumption means that $B(t)x$ is strongly measurable in t for each $x \in X$. Since X is separable, it follows that $b(t) = \|B(t)\|$ is measurable in t. Let $\{x_j\}$ be a fundamental subset of X; we may assume that $\Sigma_j \|x_j\| \leqq 1$. Let $Y' = l(Y)$ be the l-space of Y-valued sequences regarded as a real Banach space, and set

$$f(t) = b(t) \oplus \{B(t)x_j\} \in R \oplus Y' . \tag{A2}$$

Since b is measurable and each $B(\cdot)x_j$ is strongly measurable, it is easy to see that $f: I \to R \oplus Y'$ is strongly measurable.

According to Lemma A1, f can be approximated pointwise a.e. by a sequence of Riemann step functions $f_n = \{I_{nk}, f(t_{nk})\}$. If we define the corresponding step functions $b_n = \{I_{nk}, b(t_{nk})\}$, $B_n = \{I_{nk}, B(t_{nk})\}$ for b, B respectively, we have

$$f_n(t) = b_n(t) \oplus \{B_n(t)x_j\} , \qquad b_n(t) = \|B_n(t)\| .$$

Since $f_n \to f$ a.e., we have $b_n(t) \to b(t)$ and $B_n(t)x_j \to B(t)x_j$ for all j for a.e. t. Since $\|B_n(t)\| = b_n(t)$ is bounded in n for such t and since $\{x_j\}$ is fundamental in X, it follows that $B_n(t) \to B(t)$ strongly for such t.

If $b(t) = \|B(t)\|$ is integrable in t, the same is true of f because

$$\|f(t)\| = b(t) + \sum_j \|B(t)x_j\| \leqq b(t) + \|B(t)\| = 2b(t) .$$

By Lemma A1, we may thus assume that $f_n \to f$ in L^1-norm too. Then $b_n \to b$ and $B_n(\cdot)x_j \to B(\cdot)x_j$ also in L^1-norm. Since $\|B_n(\cdot)\| = b_n$ is bounded in n in L^1-norm, it follows that $B_n(\cdot)x \to B(\cdot)x$ in L^1-norm for each x.

REMARK. Lemmas A1 and A2 can be extended to *simultaneous approximation* by Riemann step functions of several vector- and operator-valued functions. By "simultaneous" we mean that the I_{nk} and t_{nk} are common to all approximating step functions. The proof is immediate by applying Lemma A2 to appropriate direct sums.

LEMMA A3. *Let X, Y be Banach spaces, with X separable, and let $B: I \to B(X, Y)$ be strongly measurable. Then*

(1) *There is a separable subspace Y_0 of Y such that $B(t)X \subset Y_0$ for a.e. t.*

(2) *For each* $\varepsilon>0$, *there is a closed subset* E *of* I *such that* $|I-E|<\varepsilon$ *and* B *is strongly continuous on* E.

PROOF. We use the notation of the proof of Lemma A2. Since each B_n is finitely-valued, the union of the $B_n(t)X$ for all t and all n spans a separable subspace Y_0 of Y. Since $B_n(t)x\rightarrow B(t)x$ for a.e. t for all $x\in X$, we have $B(t)X\subset Y_0$ for a.e. t. This proves (1).

For each j, $B_n(\cdot)x_j$ is piecewise continuous and converges as $n\rightarrow\infty$ a.e. to $B(\cdot)x_j$. Hence it is easy to see, using Egoroff's theorem, that there is a closed set $E_1\subset I$ with $|I-E_1|<\varepsilon/2$ such that for each fixed j, the $B_n(\cdot)x_j$ are continuous and uniformly convergent on E_1. Their limit $B(\cdot)x_j$ is continuous on E_1. On the other hand, there is a closed set $E_2\subset I$ with $|I-E_2|<\varepsilon/2$ such that b is bounded on E_2. Since the $B(\cdot)x_j$ are continuous and $\|B(\cdot)\|=b$ is bounded on $E=E_1\cap E_2$, it follows that $B(\cdot)x$ is continuous on E for each $x\in X$. Since $|I-E|<\varepsilon$, we have proved (2).

LEMMA A4. *Let* $A: I\rightarrow B(X, Y)$ *and* $B: I\rightarrow B(Y, Z)$ *be strongly measurable, where* X, Y, *and* Z *are arbitrary Banach spaces. Then* $BA: I\rightarrow B(X, Z)$ *is strongly measurable.*

PROOF. Let $x\in X$. We have to show that $B(\cdot)A(\cdot)x: I\rightarrow Z$ is strongly measurable. Since $f=A(\cdot)x: I\rightarrow Y$ is strongly measurable, there is by Lemma A1 a sequence $\{f_n\}$ of Riemann step functions for f such that $f_n(t)\rightarrow f(t)$ for $t\in I-N_1$, where N_1 is a null set. Let Y_0 be the subspace of Y spanned by all the values of the f_n; Y_0 is separable since each f_n is finitely-valued. Then $f(t)\in Y_0$ for $t\in I-N_1$.

According to Lemma A2, there is a sequence of Riemann step functions $B_n: I\rightarrow B(Y_0, Z)$ for $B|Y_0$ such that $B_n(t)\rightarrow B(t)$ strongly in $B(Y_0, Z)$ for $t\in I-N_2$, where N_2 is a null set. Since $f_n(I)\subset Y_0$, $B_n(\cdot)f_n(\cdot): I\rightarrow Z$ is a well-defined step function and $B_n(t)f_n(t)\rightarrow B(t)f(t)$ for $I-N_1-N_2$. Hence $B(\cdot)f(\cdot)$ is strongly measurable, as required.

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