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LINEAR FDES IN THE FRAME OF GENERALIZED ODES: VARIATION-OF-CONSTANTS FORMULA

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Cordially dedicated to the memory of Štefan Schwabik

Abstract. We present a variation-of-constants formula for functional differential equations of the form

$$\dot{y} = \mathcal{L}(t)y_t + f(y_t, t), \quad y_{t_0} = \varphi,$$

where $\mathcal{L}$ is a bounded linear operator and φ is a regulated function. Unlike the result by G. Shanholt (1972), where the functions involved are continuous, the novelty here is that the application $t \mapsto f(y_t, t)$ is Kurzweil integrable with t in an interval of $\mathbb{R}$, for each regulated function y. This means that $t \mapsto f(y_t, t)$ may admit not only many discontinuities, but it can also be highly oscillating and yet, we are able to obtain a variation-of-constants formula. Our main goal is achieved via theory of generalized ordinary differential equations introduced by J. Kurzweil (1957). As a matter of fact, we establish a variation-of-constants formula for general linear generalized ordinary differential equations in Banach spaces where the functions involved are Kurzweil integrable. We start by establishing a relation between the solutions of the Cauchy problem for a linear generalized ODE of type

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x], \quad x(t_0) = \widetilde{x}$$

and the solutions of the perturbed Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x + F(x,t)], \quad x(t_0) = \widetilde{x}.$$

Then we prove that there exists a one-to-one correspondence between a certain class of linear generalized ODE and the Cauchy problem for a linear functional differential equations of the form

$$\dot{y} = \mathcal{L}(t)y_t, \quad y_{t_0} = \varphi,$$

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where $\mathcal{L}$ is a bounded linear operator and φ is a regulated function. The main result comes as a consequence of such results. Finally, because of the extent of generalized ODEs, we are also able to describe the variation-of-constants formula for both impulsive FDEs and measure neutral FDEs.

Keywords: functional differential equation; variation-of-constants formula

MSC 2010: 34K40, 34K06

1. Introduction

In order to generalize certain classical results on continuous dependence of solutions of ODEs with respect to parameters, Jaroslav Kurzweil introduced in 1957 the concept of generalized ordinary differential equations (we write generalized ODEs, for short). See references [12], [13], [14], [15], [16], [21].

In [4] and [7], it was proved that retarded functional differential equations (we write FDEs, for short) and impulsive FDEs can be regarded as generalized ODEs taking values in Banach spaces. Then several important applications came out. See, for instance, [1], [6], [5].

In the present paper, we establish a variation-of-constants formula for abstract linear generalized ODEs, relating the solutions of the Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x], \quad x(t_0) = \widetilde{x} \tag{1.1}$$

and the solutions of the perturbed problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x + F(x,t)], \quad x(t_0) = \widetilde{x}.$$

Then we establish a relation between a Cauchy problem for linear FDEs of type

$$\begin{cases} \dot{y} = \mathcal{L}(t)y_t, \\ y_{t_0} = \varphi, \end{cases}$$

and linear generalized ODEs of the form (1.1). Such relation leads us to important applications such as a variation-of-constants formula for the perturbed FDEs of the form

$$\begin{cases} \dot{y} = \mathcal{L}(t)y_t + f(y_t,t), \\ y_{t_0} = \varphi, \end{cases}$$

where φ is a regulated function and $t \mapsto f(y_t,t)$ is Kurzweil integrable for every regulated function y. Therefore, our results generalize the variation-of-constants formula for FDEs which appear in [8] and [25].

In order to make our text self-contained, we include the concept and basic facts of the Kurzweil non-absolute integration theory as well as the definition of generalized ODEs and their main features. It is worth mentioning that nonabsolute integrals in the sense of Kurzweil and Henstock have been playing an important role in Quantum Physics and Finance. See, for instance, [2], [18] and [19].

2. Kurzweil integration

A *tagged partition* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection $P = (\tau_i, s_i)$, where $a = s_0 \leqslant s_1 \leqslant \ldots \leqslant s_{|P|} = b$ is a partition of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, |P|$ are tags of the subintervals. Here $|P|$ denotes the number of subintervals $[s_{i-1}, s_i]$ of $[a,b]$.

A *gauge* on a set $E \subset [a,b]$ is any function $\delta\colon E \to (0,\infty)$. Given a gauge δ on $[a,b]$, a tagged partition $P = (\tau_i, s_i)$ is δ*-fine* if for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a,b] \colon |t - \tau_i| < \delta(\tau_i)\}.$$

We sometimes write $P = (\tau_i, [s_{i-1}, s_i])$ instead of $P = (\tau_i, s_i)$ whenever there is need to emphasize the subintervals.

Let X be a Banach space with norm $\|\cdot\|_X$. In the sequel, we will use integration specified by the next definition due to Kurzweil.

Definition 2.1. A function $U(\tau,t)\colon [a,b] \times [a,b] \to X$ is *Kurzweil integrable* if there is a unique element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that for every δ-fine tagged partition $P = (\tau_i, s_i)$ of $[a,b]$ we have

$$\|S(U,P) - I\|_X < \varepsilon,$$

where $S(U,P) = \sum\limits_{i=1}^{|P|} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$. In this case, we write $I = \int_a^b DU(\tau,t)$.

Note that the definition above has sense whenever, for a certain gauge δ on $[a,b]$, one can guarantee the existence of a δ-fine tagged partition of $[a,b]$. Fortunately, Cousin's Lemma (see [9], Theorem 4.1) guarantees this existence.

Lemma 2.2 (Cousin's Lemma)**.** *Given a gauge δ on $[a,b]$, there exists a δ-fine tagged partition of $[a,b]$.*

Remark 2.3. Suppose the integral $\int_a^b DU(\tau,t)$ exists. We define $\int_b^a DU(\tau,t) = -\int_a^b DU(\tau,t)$.

The type of integration we are dealing with (namely the Kurzweil integration) and its properties are described extensively in Chapter I of [21] for the case $X = \mathbb{R}^n$ (see [21], Definition 1.2n). The next results can be found in [21] for the case $X = \mathbb{R}^n$. When X is a general Banach space, analogous results can be obtained. We should mention, in particular, that the Kurzweil integral has the usual properties of linearity and additivity with respect to adjacent intervals. Another important result, which will be used later, concerns the integrability on subintervals. Such result is stated next for the Kurzweil integral (see [21], Theorem 1.10).

Lemma 2.4. *Let $U(\tau,t)\colon [a,b]\times[a,b]\to X$ be integrable over $[a,b]$. Then for each subinterval $[c,d]\subset[a,b]$, the integral $\int_c^d DU(\tau,t)$ exists.*

The next result is known as the Saks-Henstock Lemma for the Kurzweil integral. A proof of it can be found in [21], Lemma 1.13.

Proposition 2.5 (Saks-Henstock Lemma)**.** *Let $U(\tau,t)\colon [a,b]\times[a,b]\to X$. If for every $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that for every δ-fine tagged partition $P=(\tau_i,s_i)$ of $[a,b]$,*

$$\left\|\sum_{i=1}^{|P|}[U(\tau_i,s_i)-U(\tau_i,s_{i-1})]-\int_a^b DU(\tau,t)\right\|_X<\varepsilon,$$

then for $a\leqslant c_1\leqslant \eta_1\leqslant d_1\leqslant c_2\leqslant \eta_2\leqslant d_2\leqslant\ldots\leqslant c_l\leqslant\eta_l\leqslant d_l\leqslant b$ with $\eta_j\in[c_j,d_j]\subset[\eta_j-\delta(\eta_j),\eta_j+\delta(\eta_j)]$, $j=1,2,\ldots,l$,

$$\left\|\sum_{j=1}^{l}[U(\eta_j,d_j)-U(\eta_j,c_j)-\int_{c_j}^{d_j} DU(\tau,t)]\right\|_X<\varepsilon.$$

The following result is an important Hake-type theorem (see [21], Theorem 1.14). It says that the Kurzweil integral is invariant under Cauchy extensions.

Lemma 2.6. *Let a function $U\colon [a,b]\times[a,b]\to X$ be given so that U is integrable over $[a,c]$ for every $c\in[a,b)$, and let the limit*

$$\lim_{c\to b-}\left[\int_a^c DU(\tau,t)-U(b,c)+U(b,b)\right]=I\in X$$

exist. Then the function U is integrable over $[a,b]$ and $\int_a^b DU(\tau,t)=I$. Similarly, if the function U is integrable over $[c,b]$ for every $c\in(a,b]$, and the limit

$$\lim_{c\to a+}\left[\int_c^b DU(\tau,t)+U(a,c)-U(a,a)\right]=I\in X$$

exists, then the function U is integrable over $[a,b]$ and $\int_a^b DU(\tau,t)=I$.

This leads to the following result (see [21], Theorem 1.16).

Lemma 2.7. *Let $U\colon [a,b]\times[a,b]\to X$ be Kurzweil integrable over $[a,b]$ and $c\in[a,b]$. Then U is Kurzweil integrable over $[a,c]$ and*

$$\lim_{s\to c}\biggl[\int_a^s DU(\tau,t)-U(c,s)+U(c,c)\biggr]=\int_a^c DU(\tau,t).$$

Remark 2.8. Lemma 2.7 shows that the function defined by $s\in[a,b]\mapsto \int_a^s DU(\tau,t)\in X$, that is, the *indefinite Kurzweil integral of U*, may not be continuous in general: it is continuous at a point $c\in[a,b]$ if and only if the function $U(c,\cdot)\colon [a,b]\to X$ is continuous at the point c. Notice further, that if $U\colon [a,b]\times[a,b]\to X$ is Kurzweil integrable over $[a,b]$, then by Lemma 2.4, the indefinite integral of the function U is well defined on the whole interval $[a,b]$.

When U from Definition 2.1 is given by $U(\tau,t)=f(\tau)g(t)$ with $f\colon [a,b]\to\mathbb{R}^n$ and $g\colon [a,b]\to\mathbb{R}$, we have

$$\int_a^b DU(\tau,t)=\int_a^b f(s)\,\mathrm{d}g(s)$$

and we refer to the last integral as the Kurzweil-Stieltjes integral. In particular, Lemma 2.7 also says that the Kurzweil-Stieltjes integrals of type $\int_a^b f(s)\,\mathrm{d}g(s)$ encompass their improper integrals.

Now, we present a classic example of a highly oscillating function which is Kurzweil integrable.

Example 2.9. Let $f\colon [0,1]\to\mathbb{R}$ be a function given by

$$f(t)=\begin{cases}2t\sin\dfrac{2}{t^2}-\dfrac{2}{t}\cos\dfrac{2}{t^2}, & t\in(0,1],\\ 0, & t=0.\end{cases}$$

Note that f is a highly oscillating function, it is not absolutely integrable over $[0,1]$ and also it is not Lebesgue integrable over $[0,1]$. Since the improper Riemann integral of f exists, using a Hake-type theorem for the Kurzweil integral, which says that the Kurzweil integral is invariant by Cauchy extensions, the Kurzweil integral of f also exists and it has the same value as the improper Riemann integral of f.

3. Generalized ordinary differential equations

Again, we consider X a Banach space with norm $\|\cdot\|_X$. Let an open set $\Omega \subset X \times \mathbb{R}$ be given and assume that $G\colon \Omega \to X$ is a given X-valued function $G(x,t)$ defined for $(x,t) \in \Omega$.

Definition 3.1. A function $x\colon [\alpha,\beta] \to X$ is called a *solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DG(x,t) \tag{3.1}$$

on the interval $[\alpha,\beta] \subset \mathbb{R}$ if $(x(t),t) \in \Omega$ for all $t \in [\alpha,\beta]$ and if the equality

$$x(v) - x(\gamma) = \int_\gamma^v DG(x(\tau),t)$$

holds for every $\gamma, v \in [\alpha,\beta]$, where the integral is in the sense of Definition 2.1.

Let $-\infty < a < b < \infty$ and set $\Omega = O \times [a,b]$, where $O \subset X$ is an open set (e.g. $O = B_c = \{x \in X\colon \|x\| < c\}$ for some $c > 0$). Next, we define a class of right-hand sides of equation (3.1) for which the local existence and uniqueness of a solution are guaranteed.

Definition 3.2. A function $G\colon \Omega \to X$ belongs to the class $\mathcal{F}(\Omega,h)$ if there exists a nondecreasing function $h\colon [a,b] \to \mathbb{R}$ such that

$$\|G(x,s_2) - G(x,s_1)\| \leqslant |h(s_2) - h(s_1)| \tag{3.2}$$

for all $(x,s_2),(x,s_1) \in \Omega$ and

$$\|G(x,s_2) - G(x,s_1) - G(y,s_2) + G(y,s_1)\| \leqslant \|x-y\||h(s_2) - h(s_1)|$$

for all $(x,s_2),(x,s_1),(y,s_2),(y,s_1) \in \Omega$.

Assume that $G \in \mathcal{F}(\Omega,h)$ for some nondecreasing function $h\colon [a,b] \to \mathbb{R}$. If $[\alpha,\beta] \subset [a,b]$ and $x\colon [\alpha,\beta] \to X$ is a solution of (3.1), then the inequality

$$\|x(s_1) - x(s_2)\| \leqslant |h(s_2) - h(s_1)| \tag{3.3}$$

holds for every $s_1, s_2 \in [\alpha,\beta]$ (see [21], Lemma 3.10).

Let $\operatorname{var}_\alpha^\beta x$ denote the variation of a function $x\colon [\alpha,\beta] \to X$. Then we have the following property of solutions of (3.1), where G belongs to the class $\mathcal{F}(\Omega,h)$.

See [21], Corollary 3.11 for the case $X = \mathbb{R}^n$. The infinite dimensional case follows analogously.

Proposition 3.3. *Assume that* $G \in \mathcal{F}(\Omega, h)$ *for some nondecreasing function* $h\colon [a,b] \to \mathbb{R}$. *If* $[\alpha,\beta] \subset (a,b)$ *and* $x\colon [\alpha,\beta] \to X$ *is a solution of* (3.1), *then* x *is of bounded variation in* $[\alpha,\beta]$ *and* $\operatorname{var}_\alpha^\beta x \leqslant h(\beta) - h(\alpha) < \infty$. *Every point in* $[\alpha,\beta]$ *at which the function* h *is continuous is a continuity point of the solution* $x\colon [\alpha,\beta] \to X$.

Moreover, we have the following result (see [21], Lemma 3.12).

Lemma 3.4. *If* $x\colon [\alpha,\beta] \to X$ *is a solution of* (3.1) *and* $G\colon \Omega \to X$ *satisfies condition* (3.2), *then*

$$x(\sigma+) - x(\sigma) = \lim_{s\to\sigma+} x(s) - x(\sigma) = G(x(\sigma),\sigma+) - G(x(\sigma),\sigma), \quad \sigma \in [\alpha,\beta),$$
$$x(\sigma) - x(\sigma-) = x(\sigma) - \lim_{s\to\sigma-} x(s) = G(x(\sigma),\sigma) - G(x(\sigma),\sigma-), \quad \sigma \in (\alpha,\beta],$$

where

$$G(x,\sigma+) = \lim_{s\to\sigma+} G(x,s), \quad \sigma \in [\alpha,\beta),$$
$$G(x,\sigma-) = \lim_{s\to\sigma-} G(x,s), \quad \sigma \in (\alpha,\beta].$$

Note that all the onesided limits $G(x,\sigma+)$, $G(x,\sigma-)$, $x(\sigma+)$ and $x(\sigma-)$ exist in X, since h is a nondecreasing real function.

We recall that a function $f\colon [a,b] \to X$ is *regulated* if at any point $t \in [a,b]$, it possesses onesided limits, that is, the limits

$$\lim_{s\to t-} f(s) = f(t-) \in X,\ t \in (a,b] \quad \text{and} \quad \lim_{s\to t+} f(s) = f(t+) \in X,\ t \in [a,b)$$

exist. We write $f \in G([a,b],X)$ in this case. Therefore, if $f \in G([a,b],X)$, then for every $\varepsilon > 0$ and $t \in (a,b]$ there exist $\delta > 0$ and $f(t-) \in X$ such that

$$\|f(s) - f(t-)\| < \varepsilon \quad \text{for } t-\delta < s < t,$$

and for every $\varepsilon > 0$ and $t \in [a,b)$ there exist $\delta > 0$ and $f(t+) \in X$ such that

$$\|f(s) - f(t+)\| < \varepsilon \quad \text{for } t < s < t+\delta.$$

If we endow $G([a,b],X)$ with the usual supremum norm $\|f\|_\infty = \sup\limits_{a\leqslant t\leqslant b} \|f(t)\|$, then $(G([a,b],X), \|\cdot\|_\infty)$ becomes a Banach space. It is also useful to know that regulated functions are the uniform limit of step functions. For other properties of this space, the reader may want to consult [10].

Now, we present a result on the existence of the integral involved in the definition of the solution of the generalized equation (3.1). See [1], Lemma 2.24 for a proof when X is of finite dimension. The infinite dimensional case follows analogously.

Lemma 3.5. *Let $G \in \mathcal{F}(\Omega, h)$ and $x\colon [\alpha,\beta] \to X$ be regulated in $[\alpha,\beta] \subset [0,\infty)$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha,\beta]$. Then the Kurzweil integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and the function $s \mapsto \int_\alpha^s DG(x(\tau), t) \in X$ is of bounded variation in $[\alpha,\beta]$ (and therefore also regulated).*

The next result concerns the local existence and uniqueness of a solution of (3.1) (see [4], Theorem 2.15).

Theorem 3.6 (Existence and uniqueness)**.** *Let $G\colon \Omega \to X$ belong to the class $\mathcal{F}(\Omega, h)$, where the function $h\colon [a,b] \to \mathbb{R}$ is left continuous. Then for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$ we have $(\widetilde{x}_+, t_0) \in \Omega$ and there exists $\Delta > 0$ such that on the interval $[t_0, t_0+\Delta]$ there exists a unique solution $x\colon [t_0, t_0+\Delta] \to X$ of the generalized ordinary differential equation* (3.1) *for which $x(t_0) = \widetilde{x}$.*

Remark 3.7. The assumption on the left continuity of the function h in Theorem 3.6 implies that the solutions of (3.1) are also left continuous (see equation (3.3)). Given a solution x of (3.1), the limit $x(\sigma-)$ exists for every σ in the domain of x. This follows again by (3.3) and, by Lemma 3.4, we have the relation $x(\sigma) = x(\sigma-) + G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$, which describes the discontinuity of the given solution.

4. The variation-of-constants formula

Let $[a,b] \subset \mathbb{R}$ be a compact interval, $(\widetilde{x}, t_0) \in X \times [a,b]$, where X is a Banach space and let $L(X)$ be the Banach space of bounded linear operators from X into itself endowed with the usual operator norm in $L(X)$.

Let us assume that $F\colon X \times [a,b] \to X$ is given by $F(x,t) = A(t)x$, where $A\colon [a,b] \to L(X)$ is of bounded variation, that is, $\operatorname{var}_a^b(A) < \infty$. Moreover, we assume that the following conditions are satisfied:

(4.1)
$$\frac{1}{I + [A(t+) - A(t)]} \in L(X) \quad \forall\, t \in [a,b),$$
$$\frac{1}{I - [A(t) - A(t-)]} \in L(X) \quad \forall\, t \in (a,b],$$

where $I \in L(X)$ is the identity operator, $A(t+) = \lim\limits_{s\to t+} A(s)$ and $A(t-) = \lim\limits_{s\to t-} A(s)$.

Consider the following Cauchy problem for a generalized ODE

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t) = D[A(t)x], \\ x(t_0) = \widetilde{x}. \end{cases} \tag{4.2}$$

According to Definition 3.1, a function $x\colon [a,b] \to X$ is a solution of the generalized ODE in (4.2) on the interval $[a,b]$ if

$$x(s_2) = x(s_1) + \int_{s_1}^{s_2} D[A(t)x(\tau)] \quad \forall\, s_1, s_2 \in [a,b]. \tag{4.3}$$

In particular, a function $x\colon [a,b] \to X$ is a solution of the initial value problem (4.2) on $[a,b]$ whenever

$$x(s) = \widetilde{x} + \int_{t_0}^{s} D[A(t)x(\tau)] \quad \forall\, s \in [a,b]. \tag{4.4}$$

Note that the integral term on the right-hand side of the last equality comes from the fact that we are dealing with the Kurzweil-Stieltjes integral as it was described in [22]. Indeed, the Riemann-type sums for $\int_{t_0}^{v} D[A(t)x(\tau)]$ have the form $\sum [A(s_i) - A(s_{i-1})]x(\tau_i)$, which leads us to the more conventional form of notation of the integral, that is, to the form $\int_{t_0}^{v} \mathrm{d}[A(t)]x(t)$. Then (4.3) becomes

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} \mathrm{d}[A(s)]x(s), \quad s_1, s_2 \in [a,b]$$

and, similarly, equation (4.4) can be written as

$$x(s) = \widetilde{x} + \int_{t_0}^{s} \mathrm{d}[A(r)]x(r), \quad s \in [a,b].$$

Let $BV([a,b],X)$ denote the Banach space of functions $x\colon [a,b] \to X$ such that $\operatorname{var}_a^b x < \infty$, endowed with the variation norm $\|x\| = \|x(a)\|_X + \operatorname{var}_a^b x$. By $BV([a,b],L(X))$ we denote the Banach space of operators $F\colon [a,b] \to L(X)$ such that $\operatorname{var}_a^b F < \infty$, endowed with the norm $\|F\| = \|F(a)\|_{L(X)} + \operatorname{var}_a^b F$.

In general, we have the following result for the existence of the Kurzweil-Stieltjes integral in Banach spaces (see [22], Proposition 15).

Theorem 4.1. *Let X be a Banach space. Suppose $g \in G([a,b],X)$ and $F \in BV([a,b],L(X))$. Then the Kurzweil-Stieltjes integral $\int_a^t \mathrm{d}[F(s)]g(s)$ exists for every $t \in [a,b]$.*

The next result concerns the existence and uniqueness of a solution of the initial value problem (4.2) on the whole interval $[a,b]$. Hence, a consequence of such result is the global existence of a solution. For a proof, see [23], Theorem 2.10 together with the last remark following it.

Theorem 4.2. *Suppose* $A \in BV([a,b], L(X))$. *If* $x\colon [a,b] \to X$ *is a solution of* (4.2) *on* $[a,b]$, *then* $x \in BV([a,b], X)$. *Moreover, if* A *satisfies* (4.1), *then the initial value problem* (4.2) *has a unique solution on* $[a,b]$.

As we look at the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x]$$

in (4.2), we can also consider the equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = D[A(t)\Phi], \tag{4.5}$$

where $\Phi \in L(X)$ and a solution of the second equation on the interval $[a,b]$ is an operator-valued function $\Phi\colon [a,b] \to L(X)$ such that

$$\Phi(s_2) = \Phi(s_1) + \int_{s_1}^{s_2} D[A(t)\Phi(\tau)] = \Phi(s_1) + \int_{s_1}^{s_2} \mathrm{d}[A(s)]\Phi(s) \quad \forall\, s_1, s_2 \in [a,b].$$

Thus, we define an operator $\Phi\colon [a,b] \to L(X)$ by

$$\Phi(t) = \Phi(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]\Phi(s), \quad t \in [a,b]. \tag{4.6}$$

Then Theorem 4.2 implies the operator Φ given by (4.6) is uniquely determined. Furthermore, if (4.6) is satisfied for all $t \in [a,b]$, then $\Phi\colon [a,b] \to L(X)$ is a solution of the generalized equation (4.5).

Now, we define the fundamental operator for linear generalized ODEs (see [21], Theorem 6.13). We reproduce the proof here, applied to the infinite dimensional case.

Theorem 4.3. *Suppose* $A \in BV([a,b], L(X))$ *satisfies* (4.1). *Then there exists a uniquely determined operator* $U\colon [a,b] \times [a,b] \to L(X)$, *called fundamental operator, such that*

$$U(t,s) = I + \int_{s}^{t} \mathrm{d}[A(r)]U(r,s), \quad t, s \in [a,b] \tag{4.7}$$

and for every fixed $s \in [a,b]$, $U(\cdot,s)$ is of bounded variation in $[a,b]$. Moreover, the unique solution $x\colon [a,b] \to X$ of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x]$$

with initial condition $x(s) = \widetilde{x} \in X$, $s \in [a,b]$, is given by the relation

$$x(t) = U(t,s)\widetilde{x}, \quad t \in [a,b]. \tag{4.8}$$

P r o o f. This proof follows as in Theorem 6.13 and 6.14 from [21]. For a given $s \in [a,b]$, U is a solution of

$$\Phi(t) = I + \int_s^t \mathrm{d}[A(r)]\Phi(r)$$

and by Theorem 4.2, this solution exists and is of bounded variation in $[a,b]$ for every fixed $s \in [a,b]$. Then for $s \in [a,b]$ fixed, the function $x\colon [a,b] \to X$ given by (4.8) is of bounded variation. Therefore, for every $t \in [a,b]$, the integral $\int_s^t \mathrm{d}[A(r)]x(r)$ exists (Theorem 4.1) and we have

$$\int_s^t \mathrm{d}[A(r)]x(r) = \int_s^t \mathrm{d}[A(r)]U(r,s)\widetilde{x} = [U(t,s) - I]\widetilde{x} = x(t) - \widetilde{x}$$

for $t \in [a,b]$, and $x(s) = U(s,s)\widetilde{x}$. This means that x is a solution of the initial value problem of the statement and this solution is unique by Theorem 4.2. □

Now, let us mention several properties of the fundamental operator U given by (4.7). The proof follows the steps of Theorem 6.15 in [21]. But will not repeat it here, since it is not so short.

Theorem 4.4. *Suppose $A \in BV([a,b], L(X))$ satisfies (4.1). Then the operator $U\colon [a,b] \times [a,b] \to L(X)$ given by (4.7) satisfies the following properties:*

(i) *$U(t,t) = I$ for all $t \in [a,b]$.*

(ii) *There exists a constant $M > 0$ such that for all $t,s \in [a,b]$ we have*

$$\|U(t,s)\| \leqslant M, \quad \operatorname{var}_a^b U(t,\cdot) \leqslant M \quad \text{and} \quad \operatorname{var}_a^b U(\cdot,s) \leqslant M.$$

(iii) *$U(t,s) = U(t,r)U(r,s)$ for all $t,r,s \in [a,b]$.*

(iv) *There exists $[U(t,s)]^{-1} \in L(X)$ and $[U(t,s)]^{-1} = U(s,t)$ for all $t,s \in [a,b]$.*

(v) *We have for all $t,s \in [a,b]$,*

$$U(t+,s) = [I + \Delta^+ A(t)]U(t,s) \quad \text{and} \quad U(t-,s) = [I - \Delta^- A(t)]U(t,s),$$
$$U(t,s+) = U(t,s)[I + \Delta^+ A(t)]^{-1} \quad \text{and} \quad U(t,s-) = U(t,s)[I - \Delta^- A(t)]^{-1}.$$

The next two lemmas are new and they will be useful for obtaining a Dirichlet-type formula involving regulated functions in Banach spaces.

Lemma 4.5. *Let* $t_0 \in [a,b]$ *and* $U\colon [a,b]\times[a,b] \to L(X)$ *be given by* (4.7) *in Theorem* 4.3. *Then for* $\varphi \in G([a,b],X)$, *the function* $\widehat{\varphi}\colon [a,b] \to X$ *given by* $\widehat{\varphi}(r) = \int_{t_0}^{r} \mathrm{d}_s[U(r,s)]\varphi(s)$ *is regulated.*

P r o o f. Let $\varphi \in G([a,b],X)$ be fixed. Consider $r \in [t_0,t)$ and let r_n be a sequence in $[a,b]$ such that $r_n \searrow r$ (that is, $\{r_m\}$ is a sequence in $[a,b]$ such that $r_n \geqslant r$ for every $n \in \mathbb{N}$ and r_n converges to r). By Theorem 4.4, $\lim\limits_{n\to\infty} U(r_n,s) = U(r+,s)$ exists for all $s \in [a,b]$ and there exists $M > 0$ such that $\operatorname{var}_a^b U(r_n,\cdot) < M$ for all $n \in \mathbb{N}$. Then, by Helly's Choice Theorem (see [10], Theorem I.5.8), $U(r+,\cdot) \in BV([a,b],L(X))$ and thus, the integral $\int_{t_0}^{r} \mathrm{d}_s[U(r+,s)]\varphi(s)$ makes sense.

Define $\widetilde{U}(\sigma,s) = U(\sigma,s)$ for $a \leqslant s \leqslant \sigma \leqslant b$ and $\widetilde{U}(\sigma,s) = I$ for $a \leqslant \sigma < s \leqslant b$. It is difficult to see that U and $\widetilde{U}$ has the same properties. Thus

$$\int_{t_0}^{t} \mathrm{d}_s[\widetilde{U}(r,s)]\,\varphi(s) = \int_{t_0}^{r} \mathrm{d}_s[U(r,s)]\,\varphi(s) + \int_{r}^{t} \mathrm{d}_s[I]\,\varphi(s) = \widehat{\varphi}(r).$$

In order to obtain that $\widehat{\varphi}$ is regulated, let us prove that $\widetilde{U}(r_n,s) \to \widetilde{U}(r+,s)$ uniformly on $[a,b]$. Indeed, by property (ii) in Theorem 4.4, there exists $M > 0$ such that $\|U(r,s)\| \leqslant M$ for all $t,s \in [a,b]$, then

$$\begin{aligned}
\sup_{s\in[a,b]} \|\widetilde{U}(r_n,s) - \widetilde{U}(r+,s)\| &= \sup_{s\in[a,r]} \|U(r_n,s) - U(r+,s)\| \\
&= \sup_{s\in[a,r]} \|U(r_n,r)U(r,s) - [I + \Delta^+ A(r)]U(r,s)\| \\
&= \sup_{s\in[a,r]} \|U(r_n,r) - I - \Delta^+ A(r)\|\|U(r,s)\| \\
&\leqslant M \bigg\| \int_{r}^{r_n} \mathrm{d}[A(\tau)]U(\tau,r) - \Delta^+ A(r) \bigg\| \to 0.
\end{aligned}$$

Thus, by the Uniform Convergence Theorem for the Kurzweil-Stieltjes integral (see [22], Theorem 11), we have

$$\lim_{n\to\infty} \int_{t_0}^{t} \mathrm{d}_s[\widetilde{U}(r_n,s) - \widetilde{U}(r+,s)]\,\varphi(s) = 0$$

and we conclude that the right-hand sided limit $\lim\limits_{s\to r^+} \widehat{\varphi}(s)$ exists for every $r \in [t_0,t)$. Similarly, one can prove that the left-hand sided limit $\lim\limits_{s\to r^-} \widehat{\varphi}(s)$ exists for every $r \in (t_0,t]$. $\square$

Lemma 4.6. *Suppose $K \in BV([a,b],L(X))$ and let $U\colon [a,b]\times[a,b]\to L(X)$ be given by (4.7) in Theorem 4.3 and $t\in[a,b]$. Then the function $\widehat{U}\colon [a,t]\to L(X)$ given by $\widehat{U}(s)=\int_s^t \mathrm{d}[K(r)]U(r,s)$ is of bounded variation on $[a,t]$.*

P r o o f. At first, note that condition (ii) in Theorem 4.4 together with the fact that $K\in BV([a,b],L(X))$, implies that the function $\widehat{U}$ is well defined (Theorem 4.1).

Let $\widetilde{U}(\sigma,s)=U(\sigma,s)$ for $a\leqslant s\leqslant\sigma\leqslant b$ and $\widetilde{U}(\sigma,s)=I$ for $a\leqslant\sigma<s\leqslant b$. It is easy to see that the function $\widetilde{U}$ has the same properties as U does and, for each $\sigma\in[a,b]$, $\operatorname{var}_a^b\widetilde{U}(\sigma,\cdot)=\operatorname{var}_a^\sigma U(\sigma,\cdot)+\operatorname{var}_\sigma^b I=\operatorname{var}_a^\sigma U(\sigma,\cdot)$. Moreover, for every $s\in[a,t]$,

$$\int_a^t \mathrm{d}[K(r)]\widetilde{U}(r,s)=\int_a^s \mathrm{d}[K(r)]I+\int_s^t \mathrm{d}[K(r)]U(r,s)=K(s)-K(a)+\widehat{U}(s).$$

With these equalities in mind, considering a partition $P=(\alpha_j)$ of $[t_0,t]$, $j=1,\dots,|P|$, we have

$$\begin{aligned}\sum_{j=1}^{|P|}\|\widehat{U}(\alpha_j)-\widehat{U}(\alpha_{j-1})\| &\leqslant \sum_{j=1}^{|P|}\bigg\|\int_a^t \mathrm{d}[K(r)](\widetilde{U}(r,\alpha_j)-\widetilde{U}(r,\alpha_{j-1}))\bigg\|+\operatorname{var}_a^t K\\ &\leqslant \operatorname{var}_a^t K\sup_{r\in[a,t]}\sum_{j=1}^{|P|}\|\widetilde{U}(r,\alpha_j)-\widetilde{U}(r,\alpha_{j-1})\|+\operatorname{var}_a^t K\\ &\leqslant \operatorname{var}_a^t K\sup_{r\in[a,t]}\operatorname{var}_a^t\widetilde{U}(r,\cdot)+\operatorname{var}_a^t K\\ &=\operatorname{var}_a^t K\sup_{r\in[a,t]}\operatorname{var}_a^t U(r,\cdot)+\operatorname{var}_a^t K\\ &=\operatorname{var}_a^t K\Big(\sup_{r\in[a,t]}\operatorname{var}_a^t U(r,\cdot)+1\Big).\end{aligned}$$

Then $\widehat{U}\in BV([a,t],L(X))$ and we have finished the proof. □

Remark 4.7. The two lemmas above are similar to Lemma 2.10 and Lemma 2.12 from [27], where they are presented for the finite dimension case though.

Next, we present a new Dirichlet-type formula for regulated functions in Banach spaces. It will be useful to prove our variation-of-constants formula for generalized ODEs. Such result generalizes Lemma 6.16 in [21], which deals with $\mathbb{R}^n$-valued functions of bounded variation. The technique we use here is totally different from that presented in [21]. We first evaluate all the integrals of characteristic functions, then apply a Uniform Convergence Theorem. In [21], the author used a Tonelli-type theorem instead. In [27], a similar result appears (see Theorem 2.13 there), but by the reasons pointed out in the last remark, our result is more general.

Lemma 4.8. *Let $A, K \in BV([a,b], L(X))$ and assume that A satisfies (4.1). Then for $U\colon [a,b]\times[a,b]\to L(X)$ given by (4.7) in Theorem 4.3, and for $t_0, t \in [a,b]$ and $\varphi \in G([a,b], X)$, we have the equality*

$$\int_{t_0}^{t} \mathrm{d}[K(r)] \int_{t_0}^{r} \mathrm{d}_s[U(r,s)]\varphi(s) = \int_{t_0}^{t} \mathrm{d}[K(s)]\varphi(s) + \int_{t_0}^{t} \mathrm{d}_s\Big[\int_{s}^{t} \mathrm{d}_r[K(r)]U(r,s)\Big]\varphi(s). \tag{4.9}$$

P r o o f. Assume that $t_0, t \in [a,b]$ with $t_0 \leqslant t$. Note that by Lemmas 4.5 and 4.6, all the integrals involved in (4.9) are well defined.

Let $(\alpha,\beta) \in [t_0, t]$ and $x_0 \in X$ be fixed. Using Lemmas 12 and 13 from [22], we have

$$\int_{t_0}^{r} \mathrm{d}_s[U(r,s)]\chi_{(\alpha,\beta)}(s)x_o = \begin{cases} 0, & t_0 \leqslant r \leqslant \alpha; \\ x_0 - \lim\limits_{s\to\alpha^+} U(r,s)x_0, & \alpha < r < \beta; \\ \lim\limits_{s\to\beta^-} U(r,s)x_0 - \lim\limits_{s\to\alpha^+} U(r,s)x_0, & \beta \leqslant r \leqslant t. \end{cases}$$

Then

$$\begin{aligned} &\int_{t_0}^{t} \mathrm{d}[K(r)] \int_{t_0}^{r} \mathrm{d}_s[U(r,s)]\chi_{(\alpha,\beta)}(s)x_0 \\ &\quad= \lim_{c\to\alpha^+}\Big(\lim_{s\to\beta^-} K(s)x_0 - K(c)x_0 + \lim_{s\to\beta^-}[K(\beta)-K(s)]U(\beta,s)x_0 \\ &\qquad + \int_{\beta}^{t} \mathrm{d}[K(r)] \lim_{s\to\beta^-} U(r,s)x_0 + \int_{c}^{t} \mathrm{d}[K(r)]\Big(-\lim_{s\to\alpha^+} U(r,s)x_0\Big)\Big) \\ &\quad= \lim_{s\to\beta^-} K(s)x_0 - \lim_{s\to\alpha^+} K(s)x_0 + \lim_{s\to\beta^-}[K(\beta)-K(s)]U(\beta,s)x_0 \\ &\qquad + \int_{\beta}^{t} \mathrm{d}[K(r)] \lim_{s\to\beta^-} U(r,s)x_0 - \lim_{s\to\alpha^+}\int_{s}^{t} \mathrm{d}[K(r)]U(r,s)x_0. \end{aligned} \tag{4.10}$$

On the other hand,

$$\int_{t_0}^{t} \mathrm{d}[K(s)]\chi_{(\alpha,\beta)}(s)x_0 = \lim_{s\to\beta^-} K(s)x_0 - \lim_{s\to\alpha^+} K(s)x_0 \tag{4.11}$$

and, using Theorem 17 in [22], we obtain

$$\begin{aligned}
(4.12)\qquad & \int_{t_0}^{t} \mathrm{d}_s \Big[\int_s^t d_r[K(r)]U(r,s)\Big]\chi_{(\alpha,\beta)}(s)x_0 \\
&= \lim_{s\to\beta^-}\int_s^t \mathrm{d}_r[K(r)]U(r,s)x_0 - \lim_{s\to\alpha^+}\int_s^t \mathrm{d}_r[K(r)]U(r,s)x_0 \\
&= \lim_{s\to\beta^-}[K(\beta)-K(s)]U(\beta,s)x_0 + \int_\beta^t \mathrm{d}_r[K(r)]\lim_{s\to\beta^-}U(r,s)x_0 \\
&\quad - \lim_{s\to\alpha^+}\int_s^t \mathrm{d}_r[K(r)]U(r,s)x_0.
\end{aligned}$$

Then by (4.10), (4.11) and (4.12), equality (4.9) holds for $\varphi(s)=\chi_{(\alpha,\beta)}(s)x_0$.

Now, let $x_1\in X$ and suppose $t_0<\gamma<t$. Using the same ideas as above, we have

$$\int_{t_0}^{r} \mathrm{d}_s[U(r,s)]\chi_{\{\gamma\}}(s)x_1 = \begin{cases} 0, & t_0\leqslant r<\gamma; \\ x_0 - \lim\limits_{s\to\gamma^-}U(\gamma,s)x_0, & r=\gamma; \\ \lim\limits_{s\to\gamma^+}U(r,s)x_0 - \lim\limits_{s\to\gamma^-}U(r,s)x_0, & \gamma<r\leqslant t \end{cases}$$

and

$$\begin{aligned}
(4.13)\qquad & \int_{t_0}^{t}\mathrm{d}[K(r)]\int_{t_0}^{r}\mathrm{d}_s[U(r,s)]\chi_{\{\gamma\}}(s)x_1 \\
&= \int_{t_0}^{\gamma}\mathrm{d}[K(r)]\int_{t_0}^{r}\mathrm{d}_s[U(r,s)]\chi_{\{\gamma\}}(s)x_1 \\
&\quad + \int_{\gamma}^{t}\mathrm{d}[K(r)]\int_{t_0}^{r}\mathrm{d}_s[U(r,s)]\chi_{\{\gamma\}}(s)x_1 \\
&= \lim_{s\to\gamma^-}[K(\gamma)-K(s)](x_1-U(\gamma,s)x_1) - \int_\gamma^t \mathrm{d}[K(r)]\lim_{s\to\gamma^-}U(r,s)x_1 \\
&\quad + \lim_{s\to\gamma^+}\Big(\int_s^t \mathrm{d}[K(r)]\lim_{\sigma\to\gamma^+}U(r,\sigma)x_1 + [K(s)-K(\gamma)]x_1\Big) \\
&= \lim_{s\to\gamma^+}K(s)x_1 - \lim_{s\to\gamma^-}K(s)x_1 - \lim_{s\to\gamma^-}[K(\gamma)-K(s)]U(\gamma,s)x_1 \\
&\quad + \lim_{s\to\gamma^+}\int_s^t \mathrm{d}[K(r)]U(r,s)x_1 - \int_\gamma^t \mathrm{d}[K(r)]\lim_{s\to\gamma^-}U(r,s)x_1.
\end{aligned}$$

For the right-hand side of (4.9) we obtain

$$(4.14)\qquad \int_{t_0}^{t}\mathrm{d}[K(s)]\chi_{\{\gamma\}}(s)x_1 = \lim_{s\to\gamma^+}K(s)x_1 - \lim_{s\to\gamma^-}K(s)x_1$$

and also

$$(4.15)\qquad \int_{t_0}^{t} \mathrm{d}_s\Big[\int_s^t \mathrm{d}[K(r)]U(r,s)\Big]\chi_{\{\gamma\}}(s)x_1$$
$$= \lim_{s\to\gamma^+}\int_s^t \mathrm{d}[K(r)]U(r,s)x_1 - \lim_{s\to\gamma^-}\int_s^t \mathrm{d}[K(r)]U(r,s)x_1$$
$$= \lim_{s\to\gamma^+}\int_s^t \mathrm{d}[K(r)]U(r,s)x_1 - \int_\gamma^t \mathrm{d}[K(r)]\lim_{s\to\gamma^-} U(r,s)x_1$$
$$- \lim_{s\to\gamma^-}[K(\gamma)-K(s)]U(\gamma,s)x_1.$$

Then by (4.13), (4.14) and (4.15), equality (4.9) holds for $\varphi(s)=\chi_{\{\gamma\}}(s)x_1$. Analogously, one can prove equality (4.9) in the case where $\gamma=t_0$ and $\gamma=t$.

Finally, let $\varphi\in G([t_0,t],X)$ and φ_n be a sequence of step functions which is uniformly convergent in $[t_0,t]$ to φ, that is,

$$\lim_{n\to\infty}\sup_{s\in[t_0,t]}\|\varphi_n(s)-\varphi(s)\|_X=0.$$

Since φ_n is a step function, equality (4.9) holds for all $n\in\mathbb{N}$. By the Uniform Convergence Theorem for the Kurzweil-Stieltjes integral (see [22], Theorem 11), (4.9) also holds for $\varphi\in G([t_0,t],X)$. □

Corollary 4.9. *Let $A\in BV([a,b],L(X))$ and assume that A satisfies* (4.1). *Then for $U\colon [a,b]\times[a,b]\to L(X)$ given by* (4.7) *in Theorem* 4.3, *the equality*

$$\int_{t_0}^{t}\mathrm{d}[A(r)]\int_{t_0}^{r}\mathrm{d}_s[U(r,s)]\varphi(s)=\int_{t_0}^{t}\mathrm{d}[A(s)]\varphi(s)+\int_{t_0}^{t}\mathrm{d}_s[U(t,s)]\varphi(s)$$

holds for $t_0,t\in[a,b]$ and $\varphi\in G([a,b],X)$.

P r o o f. Let us consider $K=A$ in Lemma 4.8. Then

$$\int_{t_0}^{t}\mathrm{d}[A(r)]\int_{t_0}^{r}\mathrm{d}_s[U(r,s)]\varphi(s)=\int_{t_0}^{t}\mathrm{d}[A(s)]\varphi(s)+\int_{t_0}^{t}\mathrm{d}_s\Big[\int_s^t\mathrm{d}_r[A(r)]U(r,s)\Big]\varphi(s)$$

and by (4.7),

$$\int_{t_0}^{t}\mathrm{d}[A(r)]\int_{t_0}^{r}\mathrm{d}_s[U(r,s)]\varphi(s)=\int_{t_0}^{t}\mathrm{d}[A(s)]\varphi(s)+\int_{t_0}^{t}\mathrm{d}_s[U(t,s)]\varphi(s)$$

for $t_0,t\in[a,b]$ and $\varphi\in G([a,b],X)$. □

The proof of the next theorem follows the main ideas of [21], Theorem 6.17 (see also [24]). However, while in [21] the Cauchy problem concerns a linear generalized ODE perturbed by a function of t only, here we consider a perturbation which also depends on the state x.

Theorem 4.10. *Let $A \in BV([a,b], L(X))$ and $F\colon X \times [a,b] \to L(X)$ be given and let $[\alpha,\beta] \subseteq [a,b]$. Then $x\colon [\alpha,\beta] \to X$ is a solution of*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x + F(x,t)], \\ x(t_0) = \widetilde{x} \end{cases} \tag{4.16}$$

on $[\alpha,\beta]$ if and only if it is a solution of the integral equation

$$\begin{aligned} x(t) = U(t,t_0)\widetilde{x} + \int_{t_0}^t DF(x(\tau),s) \\ - \int_{t_0}^t \mathrm{d}_\sigma[U(t,\sigma)] \int_{t_0}^\sigma DF(x(\tau),s), \quad t \in [\alpha,\beta] \end{aligned} \tag{4.17}$$

on $[\alpha,\beta]$, where $U\colon [a,b] \times [a,b] \to L(X)$ is given by (4.7).

P r o o f. Let $t_0 \in [a,b]$ and $\varphi(\sigma) = \int_{t_0}^\sigma DF(x(\tau),s)$ for $\sigma \in [a,b]$. Since $\varphi \in BV([a,b],X)$ (by Lemma 3.5), Corollary 4.9 implies that if $x\colon [\alpha,\beta] \to X$ is a solution of (4.17), then

$$\int_{t_0}^t \mathrm{d}[A(r)]x(r) = U(t,t_0)\widetilde{x} - \widetilde{x} - \int_{t_0}^t \mathrm{d}_\sigma[U(t,\sigma)]\varphi(\sigma) = x(t) - \widetilde{x} - \int_{t_0}^t DF(x(\tau),s),$$

that is, x is a solution of (4.16). On the other hand, if $x\colon [\alpha,\beta] \to X$ is a solution of (4.16) and we define

$$y(t) = U(t,t_0)\widetilde{x} + \int_{t_0}^t DF(x(\tau),s) - \int_{t_0}^t \mathrm{d}_\sigma[U(t,\sigma)] \int_{t_0}^\sigma DF(x(\tau),s), \quad t \in [\alpha,\beta],$$

then by Corollary 4.9, y satisfies

$$\int_{t_0}^t \mathrm{d}[A(r)]y(r) = U(t,t_0)\widetilde{x} - \widetilde{x} - \int_{t_0}^t \mathrm{d}_\sigma[U(t,\sigma)]\varphi(\sigma) = y(t) - \widetilde{x} - \int_{t_0}^t DF(x(\tau),s).$$

Thus,

$$x(t) - y(t) = \int_{t_0}^t \mathrm{d}[A(r)](x(r) - y(r)), \quad t \in [\alpha,\beta]$$

and since the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\tau} = D[A(t)z], \\ z(t_0) = 0 \end{cases}$$

is the function $z(t) = 0$ for all $t \in [\alpha, \beta]$, we have $x(t) = y(t)$ for all $t \in [\alpha, \beta]$. Then we conclude that x is a solution of (4.17). □

5. Linear FDEs regarded as linear generalized ODEs

We recall that $G([a,b], \mathbb{R}^n)$ denotes the Banach space, endowed with the usual supremum norm, of regulated functions from the compact interval $[a,b] \subset \mathbb{R}$ to $\mathbb{R}^n$. Denote by $|\cdot|$ any norm in $\mathbb{R}^n$.

Let $r, \sigma > 0$ and $t_0 \in \mathbb{R}$. Given a function $y\colon \mathbb{R} \to \mathbb{R}^n$, let $y_t\colon [-r, 0] \to \mathbb{R}^n$ be given by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$, $t \in \mathbb{R}$. It is clear that for any function $y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ we have $y_t \in G([-r, 0], \mathbb{R}^n)$ for all $t \in [t_0, t_0 + \sigma]$.

Now, let us consider the following initial value problem for a linear functional differential equation

$$\begin{cases} \dot{y} = \mathcal{L}(t) y_t, \\ y_{t_0} = \varphi, \end{cases} \tag{5.1}$$

where $\varphi \in G([-r, 0], \mathbb{R}^n)$ and $\mathcal{L}(t)\colon G([-r,0], \mathbb{R}^n) \to \mathbb{R}^n$ is bounded and linear for every $t \in [t_0, t_0 + \sigma]$.

We will show that the linear functional differential equation in (5.1) (we write linear FDE, for short) can be transformed into a linear generalized ordinary differential equation of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x], \\ x(t_0) = \widetilde{x}, \end{cases} \tag{5.2}$$

whose solution x takes values in a subspace of $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$. For this purpose, we introduce two conditions that the function $\mathcal{L}$ must fulfill:

(A) For every $y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, the application $t \mapsto \mathcal{L}(t) y_t$ is Kurzweil integrable over $[t_0, t_0 + \sigma]$.

(B) There exists a Kurzweil integrable function $M\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ such that

$$\left| \int_{s_1}^{s_2} \mathcal{L}(s) y_s \,\mathrm{d}s \right| \leqslant \int_{s_1}^{s_2} M(s) \|y_s\| \,\mathrm{d}s$$

for every $s_1, s_2 \in [t_0, t_0 + \sigma]$ and $y \in G([t_0, t_0 + \sigma], \mathbb{R}^n)$.

Remark 5.1. Condition (B) holds in the following particular situation. Let $\eta\colon \mathbb{R}\times\mathbb{R}\to L(\mathbb{R}^n)$ be a function such that $\eta(t,\cdot)$ is left continuous on $(-r,0)$ and of bounded variation on $[-r,0]$ for a fixed t. Suppose $M(t)=\operatorname{var}_{-r}^{0}\eta(t,\cdot)$. Then M is clearly nonnegative. Is is also Lebesgue integrable over $[t_0,t_0+\sigma]$ (and hence Kurzweil integrable). Define

$$\mathcal{L}(t)\psi=\int_{-r}^{0}\mathrm{d}_\theta[\eta(t,\theta)]\psi(\theta). \tag{5.3}$$

By Theorem 4.1, the Kurzweil-Stieltjes integral on the right-hand side of (5.3) exists for each $t\in[t_0,t_0+\sigma]$. Note that for $t\in[t_0,t_0+\sigma]$ and $\psi\in G([-r,0],\mathbb{R}^n)$ we have

$$|\mathcal{L}(t)\psi|=\left|\int_{-r}^{0}\mathrm{d}_\theta[\eta(t,\theta)]\psi(\theta)\right|\leqslant\operatorname{var}_{-r}^{0}\eta(t,\cdot)\|\psi\|=M(t)\|\psi\|$$

and this, together with (A), implies (B).

Similarly as in [4], for $y\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ and $t\in[t_0,t_0+\sigma]$, define

$$F(y,t)(\theta)=\begin{cases}0, & t_0-r\leqslant\theta\leqslant t_0,\\ \displaystyle\int_{t_0}^{\theta}\mathcal{L}(s)y_s\,\mathrm{d}s, & t_0\leqslant\theta\leqslant t\leqslant t_0+\sigma,\\ \displaystyle\int_{t_0}^{t}\mathcal{L}(s)y_s\,\mathrm{d}s, & t_0\leqslant t\leqslant\theta\leqslant t_0+\sigma.\end{cases} \tag{5.4}$$

For each $y\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ and $t\in[t_0,t_0+\sigma]$, one can notice, looking at (5.4), that $F(y,t)$ defines a continuous function on $[t_0-r,t_0+\sigma]$, that is, $F(y,t)\in C([t_0-r,t_0+\sigma],\mathbb{R}^n)$. This means that for any fixed $t\in[t_0,t_0+\sigma]$, by the relations in (5.4), an operator acting on $G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ is defined. Formally, $F(\cdot,t)\colon G([t_0-r,t_0+\sigma],\mathbb{R}^n)\to G([t_0-r,t_0+\sigma],\mathbb{R}^n)$.

The second easy observation, coming from (5.4), is the fact that $F(y,t)$ is linear in the first variable, that is, $F(\alpha y_1+\beta y_2,t)=\alpha F(y_1,t)+\beta F(y_2,t)$, for every $t\in[t_0,t_0+\sigma]$, $\alpha,\beta\in\mathbb{R}$ and $y_1,y_2\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$. This follows from the linearity of both the integral and the operator $\mathcal{L}(t)$ involved in (5.4).

Let us rewrite F in the more conventional form $F(y,t)=A(t)y$, that is,

$$[A(t)y](\theta)=\begin{cases}0, & t_0-r\leqslant\theta\leqslant t_0,\\ \displaystyle\int_{t_0}^{\theta}\mathcal{L}(s)y_s\,\mathrm{d}s, & t_0\leqslant\theta\leqslant t\leqslant t_0+\sigma,\\ \displaystyle\int_{t_0}^{t}\mathcal{L}(s)y_s\,\mathrm{d}s, & t_0\leqslant t\leqslant\theta\leqslant t_0+\sigma.\end{cases} \tag{5.5}$$

Using this convention, we have

$$A(t)\colon\ G([t_0-r,t_0+\sigma],\mathbb{R}^n)\to G([t_0-r,t_0+\sigma],\mathbb{R}^n)$$

for every fixed $t\in[t_0,t_0+\sigma]$.

For $t\in[t_0,t_0+\sigma]$ and $y\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$, consider the norm

$$\|A(t)y\|=\sup_{\theta\in[t_0-r,t_0+\sigma]}|[A(t)y](\theta)|.$$

Using the definition (5.5) and conditions (A) and (B) we deduce that the estimate

$$|[A(t)y](\theta)|\leqslant\left|\int_{t_0}^{t}\mathcal{L}(s)y_s\,\mathrm{d}s\right|\leqslant\int_{t_0}^{t}M(s)\,\mathrm{d}s\,\|y\|$$

holds for all $t\in[t_0,t_0+\sigma]$, $\theta\in[t_0-r,t_0+\sigma]$ and $y\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$. Therefore

$$\|A(t)y\|\leqslant\int_{t_0}^{t}M(s)\,\mathrm{d}s\,\|y\|$$

for each $t\in[t_0,t_0+\sigma]$ and $y\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$. This means that $A(t)$ is for each $t\in[t_0,t_0+\sigma]$ a linear bounded operator on $G(t\in[t_0-r,t_0+\sigma],\mathbb{R}^n)$, i.e. for each $t\in[t_0,t_0+\sigma]$, $A(t)\in L(G(t\in[t_0-r,t_0+\sigma],\mathbb{R}^n))$.

Furthermore, for $y\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ and $t_0\leqslant s_1\leqslant s_2\leqslant t_0+\sigma$, by the definition in (5.5) and by conditions (A) and (B), we have

$$\begin{aligned}\|[A(s_2)-A(s_1)]y\|&=\sup_{\theta\in[t_0-r,t_0+\sigma]}|[A(s_2)y](\theta)-[A(s_1)y](\theta)|\\&=\sup_{\theta\in[s_1,s_2]}|[A(s_2)y](\theta)-[A(s_1)y](\theta)|\\&=\sup_{\theta\in[s_1,s_2]}\left|\int_{s_1}^{\theta}\mathcal{L}(s)y_s\,\mathrm{d}s\right|\leqslant\int_{s_1}^{s_2}M(s)\,\mathrm{d}s\,\|y\|.\end{aligned}$$

Thus,

$$\|A(s_2)-A(s_1)\|_{L(G([t_0-r,t_0+\sigma],\mathbb{R}^n))}\leqslant\int_{s_1}^{s_2}M(s)\,\mathrm{d}s \tag{5.6}$$

for any $s_1,s_2\in[t_0,t_0+\sigma]$, $s_1\leqslant s_2$. This implies that $A\colon\ [t_0,t_0+\sigma]\to L(G([t_0-r,t_0+\sigma],\mathbb{R}^n))$ is of bounded variation on $[t_0-r,t_0+\sigma]$ and $\mathrm{var}_{t_0-r}^{t_0+\sigma}(A)\leqslant\int_{t_0-r}^{t_0+\sigma}M(s)\,\mathrm{d}s$. Actually, (5.6) implies that $A\colon\ [t_0,t_0+\sigma]\to L(G([t_0-r,t_0+\sigma],\mathbb{R}^n))$ is continuos on $[t_0,t_0+\sigma]$.

The construction of the operator $A(t)$ described in (5.5) is based on the idea coming from [20] and [11] and from the results presented in [5] and [4] which relate initial value problems for FDEs, in particular, linear systems of type (5.1), with a certain type of Kurzweil's generalized ODEs in Banach spaces. We now describe this connection in more detail.

Let $\varphi \in G([-r,0],\mathbb{R}^n)$ be given. Define

$$x(t_0)(\theta) = \widetilde{x}(\theta) = \begin{cases} \varphi(\theta - t_0), & t_0 - r \leqslant \theta \leqslant t_0, \\ \varphi(0) = x(t_0)(t_0), & t_0 \leqslant \theta \leqslant t_0 + \sigma. \end{cases} \tag{5.7}$$

It is easy to see that $\widetilde{x} \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and this function is constructed from the initial condition of problem (5.1).

The generalized ODE related to problem (5.1) is linear and has the form of (5.2) with phase space $X = G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$. The integral form of problem (5.2) is

$$x(t) = \widetilde{x} + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s), \quad t \in [t_0 - r, t_0 + \sigma],$$

and the integral is in the sense of Kurzweil-Stieltjes.

By (5.6), the operator A in (5.2) satisfies the assumptions in Theorem 4.2 and we have global existence and uniqueness of a solution of the initial value problem (5.2) (see Theorem 4.2).

Before presenting a correspondence between linear functional differential equations of type (5.1) and a certain class of linear generalized ordinary differential equations of type (5.2), we mention an important result borrowed from [4] (see Lemma 3.3 there).

Lemma 5.2. *Let $x\colon [t_0, t_0+\sigma] \to G([t_0-r, t_0+\sigma], \mathbb{R}^n)$ be a solution of the linear generalized ODE* (5.2) *on the interval $[t_0, t_0+\sigma]$. If $v \in [t_0, t_0+\sigma]$, then*

$$x(v)(\theta) = x(v)(v), \quad \theta \geqslant v,\, \theta \in [t_0 - r, t_0 + \sigma],$$
$$x(v)(\theta) = x(\theta)(\theta), \quad v \geqslant \theta,\, \theta \in [t_0 - r, t_0 + \sigma].$$

The following result relates a solution of the linear FDE (5.1) and a solution of the linear generalized ODE (5.2). The proof can be found in [17], Theorems 4.4 and 4.5.

Theorem 5.3. *Let $x(t_0) = \widetilde{x}$ be given by* (5.7).

(a) *If $y\colon [t_0-r, t_0+\sigma] \to \mathbb{R}^n$ is a solution of* (5.1) *on $[t_0, t_0+\sigma]$, then $x\colon [t_0, t_0+\sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ given by*

$$x(t)(\theta) = \begin{cases} y(\theta), & \theta \in [t_0 - r, t], \\ y(t), & \theta \in [t, t_0 + \sigma] \end{cases} \tag{5.8}$$

for $t \in [t_0, t_0+\sigma]$, is a solution of (5.2) *on $[t_0 - r, t_0 + \sigma]$.*

(b) *If* $x\colon [t_0, t_0+\sigma] \to G([t_0-r, t_0+\sigma], \mathbb{R}^n)$ *is a solution of* (5.2) *on* $[t_0, t_0+\sigma]$, *then* $y\colon [t_0-r, t_0+\sigma] \to \mathbb{R}^n$ *given by*

$$(5.9)\qquad y(\theta) = \begin{cases} x(t_0)(\theta), & t_0 - r \leqslant \theta \leqslant t_0, \\ x(\theta)(\theta), & t_0 \leqslant \theta \leqslant t_0 + \sigma \end{cases}$$

is a solution of (5.1) *on* $[t_0-r, t_0+\sigma]$ *and* $y(\theta) = x(t_0+\sigma)(\theta)$ *for all* $\theta \in [t_0-r, t_0+\sigma]$.

Remark 5.4. In [4], FDEs (not necessarily linear) and generalized ODEs are related (see Theorems 3.4 and 3.5 there). Clearly, then linear FDEs can be regarded as generalized ODEs. But Theorem 5.3 says that, as a matter of fact, linear FDEs can be embeded in a class of linear generalized ODEs. Conversely, as a particular case of Theorem 3.5 in [4], a solution of a linear generalized ODE can be described as a solution of a general FDE. Here, though, we prove that, in fact, a solution of a linear generalized ODE can be viewed as a solution of a linear FDE.

The last result, namely Theorem 5.3, produce a one-to-one correspondence between the solution of the linear FDE (5.1) and the solution of the linear generalized ODE (5.2). Thus, for results involving the solutions, it is possible to transfer the properties of a solution of a functional differential equation to a solution of the corresponding generalized ODE and vice-versa. In this paper, we use the variation-of-constants formula we have established for generalized ODEs in Section 4 (see Theorem 4.10) to obtain a variation-of-constants formula for perturbed linear FDEs.

6. A nonlinear variation-of-constants formula for FDEs

Consider the perturbed linear FDE

$$(6.1)\qquad \begin{cases} \dot{y} = \mathcal{L}(t)y_t + f(y_t, t), \\ y_{t_0} = \varphi, \end{cases}$$

where $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma] \to \mathbb{R}^n$ and $\mathcal{L}\colon [t_0,t_0+\sigma] \to L(G([-r,0],\mathbb{R}^n),\mathbb{R}^n)$ satisfy conditions (A) and (B) and $\varphi \in G([-r,0],\mathbb{R}^n)$.

As in [4], we define for $y \in G([-r,0],\mathbb{R}^n)$ and $t \in [t_0-r, t_0+\sigma)$,

$$(6.2)\qquad F(y,t)(\theta) = \begin{cases} 0, & t_0 - r \leqslant \theta \leqslant t_0, \\ \displaystyle\int_{t_0}^{\theta} f(y_s, s)\,\mathrm{d}s, & t_0 \leqslant \theta \leqslant t, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)\,\mathrm{d}s, & t \leqslant \theta \leqslant t_0 + \sigma. \end{cases}$$

By Theorems 3.4 and 3.5 from [4] and, in particular, considering the linear case and Theorem 5.3 in the present paper, it is clear that there is a one-to-one correspondence between equation (6.1) and the following perturbed generalized ODE:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x + F(x,t)], \\ x(t_0) = \widetilde{x}, \end{cases} \tag{6.3}$$

where $A\colon [t_0 - r, t_0 + \sigma] \to L(G([t_0 - r, t_0 + \sigma], \mathbb{R}^n))$ is given by (5.5) and $\widetilde{x}$ is given by (5.7).

Before establishing a variation-of-constants formula for the perturbed linear FDE (6.1), we need some auxiliary results which we present next. The first result, namely Lemma 6.1, relates the integrals of the nonlinear perturbations of the Cauchy problems (6.1) and (6.3).

Lemma 6.1. *Let y and x be the solutions of the perturbed Cauchy problems* (6.1) *and* (6.3)*, respectively, corresponding to one another. Then $\int_{t_0}^{t} DF(x(\tau), s)(\theta) = F(y,t)(\theta)$ whenever $t_0 \leqslant t \leqslant t_0 + \sigma$.*

P r o o f. By the definition of F in (6.2), it is easy to see that $\int_{t_0}^{t} DF(x(\tau), s)(\theta) = 0$ for $\theta \in [t_0 - r, t_0]$.

Let $\varepsilon > 0$ be given. Since $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a regulated function, there exists a sequence in $[t_0, t]$, $t_0 < t_1 < \ldots < t_m = t$ such that $|y(\varrho) - y(s)| < \varepsilon$ whenever $t_{k-1} < \varrho, s < t_k$, $k = 1, \ldots, m$ (see [10], Theorem 3.1).

Consider a gauge δ on $[t, t_0]$ such that for $\tau \in [t, t_0]$,

$$\delta(\tau) < \min\Big\{\frac{t_k - t_{k-1}}{2},\ k = 1, \ldots, m\Big\}$$

and

$$\delta(\tau) < \min\{|\tau - t_k|, |\tau - t_{k-1}|,\ \tau \in [t_{k-1}, t_k],\ k = 1, \ldots, m\}.$$

This choice of δ ensures us that if (τ_k, s_k) is a δ-fine partition, then each subinterval $[s_{k-1}, s_k]$ contains at most one of the points $t_0, \ldots, t_m$ and, in this case, t_k is the corresponding tag of that subinterval.

By the continuity of the Kurzweil indefinite integral (see Remark 2.8), we can assume that

$$\int_{t_k}^{t_k + \delta(t_k)} M(s) \|y_s - x(t_k)_s\| \, \mathrm{d}s < \frac{\varepsilon}{2m + 1}, \qquad k = 1, \ldots, m.$$

Let $P = (\tau_k, s_k)$ be a δ-fine partition of $[t_0, t]$ such that

$$\Big\| \int_{t_0}^{t} DF(x(\tau), s) - \sum_{k=1}^{|P|} [F(x(\tau_k), s_k) - F(x(\tau_{k-1}), s_{k-1})] \Big\| < \varepsilon.$$

By (6.2), for every $k = 1, \dots, |P|$ we have

$$[F(x(\tau_k), s_k) - F(x(\tau_{k-1}), s_{k-1})](\theta) = \begin{cases} 0, & \theta \in [t_0 - r, s_{k-1}], \\ \displaystyle\int_{s_{k-1}}^{\theta} f(x(\tau_k)_s, s)\,\mathrm{d}s, & \theta \in [s_{k-1}, s_k], \\ \displaystyle\int_{s_{k-1}}^{s_k} f(x(\tau_k)_s, s)\,\mathrm{d}s, & \theta \in [s_k, t_0 + \sigma]. \end{cases}$$

Suppose $\theta \in [t_0, t]$. Then $\theta \in [s_{j-1}, s_j]$ for some $j = 1, \dots, |P|$ and

$$\begin{aligned} &\left|\sum_{k=1}^{|P|}[F(x(\tau_k), s_k) - F(x(\tau_{k-1}), s_{k-1})](\theta) - \int_{t_0}^{\theta} f(y_u, u)\,\mathrm{d}u\right| \\ &\quad= \Bigg|\sum_{k=1}^{j-1}[F(x(\tau_k), s_k) - F(x(\tau_{k-1}), s_{k-1})](\theta) - \int_{s_{k-1}}^{s_k} f(y_u, u)\,\mathrm{d}u \\ &\qquad + [F(x(\tau_j), s_j) - F(x(\tau_{j-1}), s_{j-1})](\theta) - \int_{s_{j-1}}^{\theta} f(y_u, u)\,\mathrm{d}u \\ &\qquad + \sum_{k=j}^{|P|}[F(x(\tau_k), s_k) - F(x(\tau_{k-1}), s_{k-1})](\theta)\Bigg| \\ &\quad= \left|\sum_{k=1}^{j-1}\int_{s_{k-1}}^{s_k}[f(x(\tau_k)_u, u) - f(y_u, u)]\,\mathrm{d}u + \int_{s_{j-1}}^{\theta}[f(x(\tau_j)_u, u) - f(y_u, u)]\,\mathrm{d}u\right|. \end{aligned}$$

Note that if $u \leqslant \tau_k$, then $x(\tau_k)_u = y_u$ and consequently,

$$\int_{s_{k-1}}^{s_k}[f(x(\tau_k)_u, u) - f(y_u, u)]\,\mathrm{d}u = \int_{\tau_k}^{s_k}[f(x(\tau_k)_u, u) - f(y_u, u)]\,\mathrm{d}u$$

for every $k = 1, \dots, j-1$ and

$$\int_{s_{j-1}}^{\theta}[f(x(\tau_j)_u, u) - f(y_u, u)]\,\mathrm{d}u = \begin{cases} 0, & \theta \in [s_{j-1}, \tau_j], \\ \displaystyle\int_{\tau_j}^{s_j}[f(x(\tau_j)_u, u) - f(y_u, u)]\,\mathrm{d}u, & \theta \in [\tau_j, s_j]. \end{cases}$$

By condition (B),

$$\left|\int_{\tau_k}^{s_k}[f(x(\tau_k)_u, u) - f(y_u, u)]\,\mathrm{d}u\right| \leqslant \int_{\tau_k}^{s_k} M(u)\|x(\tau_k)_u - y_u\|\,\mathrm{d}u$$

for every $k = 1, \dots, j-1$ and

$$\left|\int_{\tau_j}^{\theta}[f(x(\tau_j)_u, u) - f(y_u, u)]\,\mathrm{d}u\right| \leqslant \int_{\tau_j}^{\theta} M(u)\|x(\tau_k)_u - y_u\|\,\mathrm{d}u.$$

If $u \in [\tau_k, s_k]$, then

$$x(\tau_k)_u(\theta) = x(\tau_k)(u+\theta) = \begin{cases} y(u+\theta) = y_u(\theta), & u+\theta \leqslant \tau_k, \\ y(\tau_k), & \tau_k \leqslant u+\theta \end{cases}$$

for every $k = 1, \ldots, j-1$. Therefore, by the definition of the gauge δ, we have $\|x(\tau_k)_u - y_u\| = \sup_{\varrho \in [\tau_k, u_k]} |y(\varrho) - y(\tau_k)| \leqslant \varepsilon$ and if the intersection of $[s_{k-1}, s_k]$ and $\{t_0, \ldots, t_m\}$ is empty, then

$$\int_{\tau_k}^{s_k} M(u)\|x(\tau_k)_u - y_u\| \, \mathrm{d}u \leqslant \varepsilon \int_{\tau_k}^{s_k} M(u) \, \mathrm{d}u$$

for every $k = 1, \ldots, j-1$.

Analogously, we have

$$\int_{\tau_j}^{\theta} M(u)\|x(\tau_j)_u - y_u\| \, \mathrm{d}u \leqslant \varepsilon \int_{\tau_j}^{\theta} M(u) \, \mathrm{d}u.$$

If the intersection of $[s_{k-1}, s_k]$ and $\{t_0, \ldots, t_m\}$ is not empty, then $\tau_k = t_j$ for some $j \in \{1, \ldots, m\}$ and in this case,

$$\int_{\tau_k}^{s_k} M(s)\|y_s - x(\tau_k)_s\| \, \mathrm{d}s \leqslant \int_{t_j}^{t_j+\delta(t_j)} M(s)\|y_s - x(t_j)_s\| \, \mathrm{d}s < \frac{\varepsilon}{2m+1}.$$

Thus,

$$\left| \sum_{k=1}^{|P|} [F(x(\tau_k), s_k) - F(x(\tau_{k-1}), s_{k-1})](\theta) - \int_{t_0}^{\theta} f(y_u, u) \, \mathrm{d}u \right| \leqslant \varepsilon \int_{t_0}^{\theta} M(u) \, \mathrm{d}u + \varepsilon.$$

Finally, since $\varepsilon > 0$ is arbitrary, $\int_{t_0}^{t} DF(x(\tau), s)(\theta) = \int_{t_0}^{\theta} f(y_u, u) \, \mathrm{d}u$ for $\theta \in [t_0, t]$. Following the same steps as above, one can prove the case where $\theta \in [t, t_0 + \sigma]$. $\square$

In order to present a variation-of-constants formula for perturbed linear FDEs, let us define a solution operator for linear FDEs.

Definition 6.2. Let $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ be the solution of the linear FDE

$$\dot{y} = \mathcal{L}(t) y_t \tag{6.4}$$

with initial condition $y_{t_0} = \varphi \in G([-r, 0], \mathbb{R}^n)$. For $t, s \in [t_0, t_0 + \sigma]$, $t \geqslant s$, the operator $T(t, s)\colon G([-r, 0], \mathbb{R}^n) \to G([-r, 0], \mathbb{R}^n)$ defined by

$$T(t, s) y_s = y_t, \quad t, s \in [t_0, t_0 + \sigma], \ t \geqslant s,$$

will be called *solution operator* of the linear FDE (6.4).

From the existence and uniqueness of a solution of the linear FDE (5.1), the operator $T(t,s)$ satisfies:

(i) $T(t,t)=I$, $t\in[t_0,t_0+\sigma]$ (identity property),

(ii) $T(t,w)T(w,s)=T(t,s)$, $t,w,s\in[t_0,t_0+\sigma]$ and $t\geqslant w\geqslant s$ (semigroup property).

The above definition, restricted to continuous functions, coincides with the definition of solution operator presented in [8] (see Section 6.2 there).

Remark 6.3. Definition 6.2 can be generalized in the following way. Let $g\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ be given and for $s\in[t_0,t_0+\sigma]$ let $y\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ be the solution of the linear FDE (6.4) with initial condition $y_s=g_s$. Then for $t\in[t_0,t_0+\sigma]$, $t\geqslant s$, we define $T(t,s)\colon G([t_0-r,t_0+\sigma],\mathbb{R}^n)\to G([-r,0],\mathbb{R}^n)$ by $T(t,s)g=y_t$.

Lemma 6.4. *Let y and x be the corresponding solutions of the perturbed Cauchy problems* (6.1) *and* (6.3), *respectively. Let $T(t,s)$ be the solution operator of the linear* FDE (6.4) *and $U(t,s)$ be the fundamental operator of the corresponding linear generalized* ODE (5.2). *Then for $t_0\leqslant s\leqslant t\leqslant t_0+\sigma$ and $t_0\leqslant w\leqslant t_0+\sigma$ we have*

$$U(t,s)\Big(\int_{t_0}^{w} DF(x(\tau),u)\Big)(t)=T(t,s)(h(w)_s)(0),$$

where

$$h(w)(\theta)=\begin{cases}0, & t_0-r\leqslant\theta\leqslant t_0,\\ \displaystyle\int_{t_0}^{\theta} f(y_u,u)\,\mathrm{d}u, & t_0\leqslant\theta\leqslant w,\\ \displaystyle\int_{t_0}^{w} f(y_u,u)\,\mathrm{d}u, & w\leqslant\theta\leqslant t_0+\sigma.\end{cases}\tag{6.5}$$

P r o o f. By Lemma 6.1, $\int_{t_0}^{w} DF(x(\tau),u)=h(w)$ for $w\in[t_0,t_0+\sigma]$. Also, Definition 6.2 implies that $T(t,s)(h(w)_s)(0)$ describes the solution of the linear FDE

$$\begin{cases}\dot y=\mathcal{L}(t)y_t,\\ y_s=h(w)_s\end{cases}$$

and, by Theorem 4.3, $U(t,s)h(w)$ describes the solution of the corresponding linear generalized ODE

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}\tau}=D[A(t)x],\\ x(s)=h(w),\end{cases}$$

where A is given by (5.5). Then, Theorem 5.3 implies

$$U(t,s)\Big(\int_{t_0}^{w} DF(x(\tau),u)\Big)(t) = T(t,s)(h(w)_s)(0)$$

for all $t_0 \leqslant s \leqslant t \leqslant t_0 + \sigma$ and $t_0 \leqslant w \leqslant t_0 + \sigma$. □

The next result will be useful to get a variation-of-constants formula for the perturbed linear FDE (6.1).

Lemma 6.5. *Let y and x be the corresponding solutions of the perturbed linear Cauchy problems* (6.1) *and* (6.3)*, respectively. Let $T(t,s)$ be the solution operator of the linear* FDE (6.4) *and $U(t,s)$ be the fundamental operator of the corresponding linear generalized* ODE (5.2). *Then for $t_0 \leqslant t \leqslant t_0 + \sigma$ we have*

$$\int_{t_0}^{t} \mathrm{d}_s[U(t,s)]\Big(\int_{t_0}^{s} DF(x(\tau),u)\Big)(t) = \int_{t_0}^{t} \mathrm{d}_s[\widetilde{T}(t,s)]h(s)(0),$$

where h is given by (6.5) *and $\widetilde{T}(t,s)y = T(t,s)y_s$ for $y \in G([t_0 - r, t_0 + \sigma, \mathbb{R}^n])$.*

Proof. Let $\varepsilon > 0$ be given. Then by the definition of the Kurzweil integral, there exists a gauge δ on $[t_0,t]$ such that for every δ-fine tagged partition P we have

$$\Big\| \sum_{i=1}^{|P|} [U(t,s_i) - U(t,s_{i-1})] \int_{t_0}^{\tau_i} DF(x(\tau),u) - \int_{t_0}^{t} \mathrm{d}_s[U(t,s)] \int_{t_0}^{s} DF(x(\tau),u) \Big\| < \varepsilon.$$

By Lemma 6.4, we have

$$\begin{aligned}\sum_{i=1}^{|P|} &[U(t,s_i) - U(t,s_{i-1})]\Big(\int_{t_0}^{s_i - 1} DF(x(\tau),u)\Big)(t) \\ &= \sum_{i=1}^{|P|} [T(t,s_i)(h(\tau_i)_{s_i}) - T(t,s_{i-1})(h(\tau_i)_{s_{i-1}})](0) \\ &= \sum_{i=1}^{|P|} [\widetilde{T}(t,s_i) - \widetilde{T}(t,s_{i-1})]h(\tau_i)(0)\end{aligned}$$

and this implies

$$\int_{t_0}^{t} \mathrm{d}_s[U(t,s)]\Big(\int_{t_0}^{s} DF(x(\tau),u)\Big)(t) = \int_{t_0}^{t} \mathrm{d}_s[\widetilde{T}(t,s)]h(s)(0),$$

which completes the proof. □

The next result follows from our main result and the correspondence of differential equations. It gives a variation-of-contansts formula for perturbed linear FDE (6.1).

Theorem 6.6. *Let y be the solution of the perturbed linear* FDE (6.1) *and* $T(t,s)$ *be the solution operator of the linear* FDE (5.1). *Then for $t_0 \leqslant t \leqslant t_0+\sigma$ we have*

$$y(t) = T(t,t_0)\varphi(0) + \int_{t_0}^{t} f(y_u,u)\,\mathrm{d}u - \int_{t_0}^{t} \mathrm{d}_s[T(t,s)]h(s)(0),$$

where h is given by (6.5).

Proof. Given $t\in[t_0,t_0+\sigma]$, Theorems 3.4 and 3.5 in [4] imply $y(t)=x(t)(t)$ and by Theorem 4.10 we obtain

$$x(t)(t) = U(t,t_0)\widetilde{x}(t) + \int_{t_0}^{t} DF(x(\tau),s)(t) - \int_{t_0}^{t} \mathrm{d}_s[U(t,s)]\Big(\int_{t_0}^{s} DF(x(\tau),u)\Big)(t),$$

where x is the solution of the perturbed problem (6.3) and $U(t,s)$ is the fundamental operator of the linear generalized ODE (5.2).

Note that by Theorem 5.3, $U(t,t_0)\widetilde{x}(t) = T(t,t_0)\varphi(0)$. By Lemmas 6.1 and 6.5,

$$\int_{t_0}^{t} DF(x(\tau),s)(t) = \int_{t_0}^{t} f(y_u,u)\,\mathrm{d}u$$

and

$$\int_{t_0}^{t} \mathrm{d}_s[U(t,s)]\Big(\int_{t_0}^{s} DF(x(\tau),u)\Big)(t) = \int_{t_0}^{t} \mathrm{d}_s[\widetilde{T}(t,s)]h(s)(0).$$

Hence

$$y(t) = T(t,t_0)\varphi(0) + \int_{t_0}^{t} f(y_u,u)\,\mathrm{d}u - \int_{t_0}^{t} \mathrm{d}_s[\widetilde{T}(t,s)]h(s)(0)$$

and the proof is complete. □

7. Variation-of-constants formula for impulsive FDEs and measure neutral FDEs

We turn our attention to impulsive FDEs. At first, consider the Cauchy problem

$$\begin{cases} \dot{y} = \mathcal{L}(t)y_t + f(y_t,t), & t \neq t_i, \\ \Delta y(t_i) = I_k(y(t_k)), & i = 1,2,\ldots,m, \\ y_{t_0} = \varphi, \end{cases} \tag{7.1}$$

where the assumptions of equation (6.1) are fulfilled, $t_0 < t_1 < \ldots < t_k < \ldots < t_m \leqslant t_0 + \sigma$ are pre-assigned moments of impulse, $y \mapsto I_k(y)$ maps $\mathbb{R}^n$ into itself and $\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k)$ for $k = 1, 2, \ldots, m$, that is, we suppose y is left continuous at $t = t_k$ and the lateral limit $y(t_k+)$ exists for $k = 1, 2, \ldots, m$.

Concerning the impulse functions $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$, we assume:

(A′) There is a constant $K_1 > 0$ such that $|I_k(x)| \leqslant K_1$ for $k = 1, \ldots, m$ and $x \in \mathbb{R}^n$.

(B′) There is a constant $K_2 > 0$ such that $|I_k(x) - I_k(y)| \leqslant K_2|x - y|$ for $k = 1, \ldots, m$ and $x, y \in \mathbb{R}^n$.

For $y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and $t \in [t_0, t_0 + \sigma]$, let $H(y, t) = F(y, t) + J(y, t)$, where F is given as in (6.2) and, as in [4], $J\colon G([t_0, t_0 + \sigma], \mathbb{R}^n) \times [t_0 - r, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is given by $J(y, t)(\theta) = \sum\limits_{k=1}^{m} H_{t_k}(t) H_{t_k}(\theta) I_k(y(t_k))$ for $\theta \in [t_0 - r, t_0 + \sigma]$ with H_{t_k} denoting the left continuous Heaviside function concentrated at t_k. Then the Cauchy problem (7.1) corresponds to the perturbed generalized ODE

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)x + H(x, t)] = D[A(t)x + F(x, t) + J(x, t)], \\ x(t_0) = \widetilde{x}, \end{cases}$$

and by Theorem 4.10 and the correspondence of equations (see [4], Theorems 3.4 and 3.5), the solution y of (7.1) can be described explicitly as

$$\begin{aligned} y(t) &= x(t)(t) \\ &= U(t, t_0)\widetilde{x}(t) + \int_{t_0}^{t} DH(x(\tau), s)(t) - \int_{t_0}^{t} \mathrm{d}_s[U(t, s)]\Big(\int_{t_0}^{s} DH(x(\tau), u)\Big)(t) \\ &= U(t, t_0)\widetilde{x}(t) + \int_{t_0}^{t} DF(x(\tau), s)(t) - \int_{t_0}^{t} \mathrm{d}_s[U(t, s)]\Big(\int_{t_0}^{s} DF(x(\tau), u)\Big)(t) \\ &\quad + \int_{t_0}^{t} DJ(x(\tau), s)(t) - \int_{t_0}^{t} \mathrm{d}_s[U(t, s)]\Big(\int_{t_0}^{s} DJ(x(\tau), u)\Big)(t), \end{aligned}$$

where, following the ideas in Theorems 3.4 and 3.5 from [4], we obtain

$$\int_{t_0}^{t} DJ(x(\tau), s)(\theta) = \sum_{k=1}^{m} H_{t_k}(t) H_{t_k}(\theta) I_k(y(t_k))$$

for every $t, \theta \in [t_0, t_0 + \sigma]$, and using the same arguments as in Theorem 6.6, we have

$$\begin{aligned} y(t) &= T(t, t_0)\varphi(0) + \int_{t_0}^{t} f(y_u, u)\,\mathrm{d}u - \int_{t_0}^{t} \mathrm{d}_s[\widetilde{T}(t, s)]h(s)(0) \\ &\quad + \sum_{k=1}^{m} I_k(y(t_k))H_{t_k}(t) - \int_{t_0}^{t} \mathrm{d}_s[\widetilde{T}(t, s)]\Big(\sum_{k=1}^{m} I_k(y(t_k))H_{t_k}(s)\Big)(0). \end{aligned}$$

We can also obtain a variation-of-constants formula for a more general class of functions. Consider the equation of the form

$$D[N(y_t,t)] = f(y_t,t)Dg, \tag{7.2}$$

where $D[N(y_t,t)]$ and $Dg(t)$ are the distributional derivatives of $N(y_t,t)$ and $g(t)$, respectively, in the sense of Schwartz (see the references [3], [26]). We call equation (7.2) a *measure neutral functional differential equation* or simply *measure NFDE*. All results presented in Section 5 for linear FDEs such as the correspondence results (Theorem 5.3) have their analogous for measure NFDEs (see [17]). We assume that there exists a matrix $\mu\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}$ such that $\mu(t,\theta)=0$, $\theta\geqslant 0$, and $\mu(t,\theta)=\mu(t,-r)$, $\theta\leqslant -r$, and also that $\mu(t,\cdot)$ is left-continuous in $(-r,0)$, of bounded variation on $[-r,0]$, and $\operatorname{var}_{[s,0]}\mu(t,\cdot)$ goes to zero as $s\to 0$ for each fixed $t\in\mathbb{R}$, such that the operator N is given by

$$N(\varphi,t) = \varphi(0) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(t,\theta)]\varphi(\theta), \tag{7.3}$$

where $\varphi\in G([-r,0],\mathbb{R}^n)$. Thus, by (7.3) the equation

$$D[N(y_t,t)] = \mathcal{L}(t)y_t Dg, \tag{7.4}$$

which we call linear measure NFDE, with the initial condition $y_{t_0}=\varphi\in G([-r,0],\mathbb{R}^n)$ can be rewritten in the form

$$y(t) = \varphi(0) + \int_{t_0}^{t} \mathcal{L}(s)y_s\,\mathrm{d}g(s) + \int_{-r}^{0}\mathrm{d}_\theta[\mu(t,\theta)]y(t+\theta) - \int_{-r}^{0}\mathrm{d}_\theta[\mu(t_0,\theta)]\varphi(t_0+\theta). \tag{7.5}$$

If we define a solution operator $T(t,s)$ for the linear measure NFDE (7.4) such as in Definition 6.2, and use again the same arguments as in Theorem 6.6, the solution of the perturbed equation

$$\begin{cases} D[N(y_t,t)] = [\mathcal{L}(t)y_t + f(y_t,t)]Dg, \\ y_{t_0} = \varphi \end{cases}$$

can be rewritten in the form

$$y(t) = T(t,t_0)\varphi(0) + \int_{t_0}^{t} f(y_u,u)\,\mathrm{d}g(u) - \int_{t_0}^{t}\mathrm{d}_s[\widetilde{T}(t,s)]h(s)(0).$$

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