Linear Filtering for Bilinear Stochastic Differential Systems With Unknown Inputs

Alfredo Germani, Costanzo Manes, and Pasquale Palumbo

***Abstract*—This note investigates the problem of state estimation for bilinear stochastic multivariable differential systems in presence of an additional disturbance, whose statistics are completely unknown. A linear filter is proposed, based on a suitable decomposition of the state of the bilinear system into two components. The first one is a computable function of the observations while the second component is estimated via a suitable linear filtering algorithm. No *a priori* information on the disturbance is required for the filter implementation. The proposed filter is robust with respect to the unknown input, in that the covariance of the estimation error is not affected by such input. Numerical simulations show the effectiveness of the proposed filter.**

***Index Terms*—Bilinear systems, linear filtering, state estimation, unknown-input systems.**

I. Introduction

In many fields of applications, the mathematical model describing the dynamic relationships among the state variables, the inputs and the measurements is given by the following nonlinear stochastic differential system, described by the Ito equations:

$$\begin{aligned} dx(t) &= A(t)x(t)\,dt + B(t)\,du(t) + \mathcal{B}_1(x(t),\,dW(t)) \\ &\quad t \geq t_0 \\ dy(t) &= C(t)x(t)\,dt + D(t)\,du(t) + \mathcal{B}_2(x(t),\,dW(t)) \\ &\quad x(t_0) = x_0 \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the unknown input, $y(t) \in \mathbb{R}^q$ is the measured output, $W(t) \in \mathbb{R}^b$ is a Wiener process with respect to some increasing family of σ algebras, namely $\{\mathcal{F}_t,\, t \geq t_0\}$, referred to a probability space $(\Omega,\, \mathcal{F},\, P)$; $A(t),\, B(t),\, C(t),\, D(t)$ are matrices of suitable dimensions and $\mathcal{B}_1,\, \mathcal{B}_2$ are bilinear forms (see [1], [3], [4], [20]–[23], and [26] for more details on discrete and continuous-time bilinear systems and filtering problems related to them).

The unknown input $u(t)$ in system (1) may model the presence of an additive noise with no *a priori* statistical informations (deterministic disturbance). The unknown-input can be used also to describe uncertainties in the system equations, for instance derived from linearization errors, or it can be used to model *failure systems.* Among applications, unknown-inputs systems are of great interest in the geophysical and environmental framework, as shown in [19].

This note investigates the problem of estimating the state of a time-varying bilinear stochastic differential system, affected by additive disturbances that involve both the state and measurement equations. No *a priori* knowledge is assumed on the disturbances.

A great deal of literature is available in the field of filtering a discrete-time stochastic linear system with unknown inputs: a first recursive algorithm, consisting of an optimization technique can be found in [19], where Kitanidis developed an unbiased Kalman filter by minimizing the trace of the error covariance matrix. This technique has been recently parameterized [10] [18] to extend previous results. Many contributes treat the loss of information by modeling the unknown-input system as a descriptor system and then applying a previously developed filtering algorithm for this class of systems (see [6]–[8], [11], [17], and [27]). Other contributions (see [5], [12], [13], and [24]) take inspiration from an algorithm, also used for the construction of unknown-input observers, which is able to remove the influence of the disturbance by a clever use of the measurement process. In [15], Hsieh proposes a robust two-stage Kalman filter [14], optimal with respect to the minimum variance, which is shown to be equivalent to the one of Kitanidis [19].

Unfortunately, neither the descriptor system nor the decoupling approach can be directly applied to the continuous-time case, in that they would require the knowledge of the noisy output derivatives. An hypothesis that in the stochastic framework can not be assumed. On the contrary in the deterministic linear continuous-time case, a wide literature is available that treats all the cases of interest (e.g., [9] and the references therein).

This work investigates the problem of defining a robust linear filter for stochastic bilinear differential systems forced by completely unknown inputs. In particular, a suitable class of estimators is introduced and the minimum variance filter in this class is computed.

The sections are structured as follows. In Section II, the class of the system to be filtered is defined. In Section III, the filtering algorithm is proposed. Some simulation results given in Section IV show the effectiveness of the proposed algorithm.

II. Bilinear Systems With Unknown Inputs

Let $(\Omega,\, \mathcal{F},\, P)$ be a probability space and $\{\mathcal{F}_t,\, t \geq t_0\}$ be a family of nondecreasing sub-σ algebras of $\mathcal{F}$. As is well known, a bilinear stochastic differential system in the Ito formulation is described by the equations

$$\begin{aligned} dx(t) &= A(t)x(t)\,dt + B(t)\,du(t) + \sum_{k=1}^{b}(N_k(t)x(t) + F_k(t)) \\ &\qquad \cdot dW_k(t) \\ x(t_0) &= x_0 \\ dy(t) &= C(t)x(t)\,dt + D(t)\,du(t) + \sum_{k=1}^{b}(M_k(t)x(t) + G_k(t)) \\ &\qquad \cdot dW_k(t) \end{aligned} \tag{2}$$

with $x(t) \in \mathbb{R}^n$ the state of the system, $u(t) \in \mathbb{R}^p$ an additive unknown input, $y(t) \in \mathbb{R}^q$ the measured output, x_0 a random variable with mean value $m_0 = m_x(t_0) = \mathbb{E}[x_0]$, and covariance matrix $\Psi_0 = \Psi_x(t_0) = \mathrm{Cov}(x_0)$, $W_k(t)$ the kth component of a standard Wiener process $(W(t),\, \mathcal{F}_t)$, $W(t) \in \mathbb{R}^b$, and the matrices $A(t),\, N_k(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times p}$, $C(t),\, M_k(t) \in \mathbb{R}^{q\times n}$, $D(t) \in \mathbb{R}^{q\times p}$, $F_k(t) \in \mathbb{R}^{n\times 1}$, $G_k(t) \in \mathbb{R}^{q\times 1}$. Moreover, it will be assumed that $D(t)$ is full-column rank, $\forall\, t \geq t_0$.

For the sequel, let the matrices $T_1(t), T_2(t)$ defined as

$$T_1(t) = (D^T(t)D(t))^{-1}D^T(t) \in \mathbb{R}^{p\times q} \tag{3}$$

while $T_2(t)$ is chosen in such a way that $\mathcal{R}(T_2^T(t)) = \mathcal{N}(D^T(t))$. In other words $T_2(t) \in \mathbb{R}^{(q-p)\times q}$ is a matrix with $(q-p)$ independent rows that constitute a basis for the left null space of $D(t)$. Matrix $T_2(t)$ allows to define the postprocessed output, useful for the sequel

$$dz(t) = T_2(t)\,dy(t). \tag{4}$$

Manuscript received May 28, 2001; revised April 17, 2002. Recommended by Associate Editor Q. Zhang. This work was supported by the Ministry for Education, University, and Research (MURST) and by the Italian Aerospace Agency (ASI).

A. Germani and C. Manes are with the Dipartimento di Ingegneria Elettrica, Università degli Studi dell'Aquila, 67040 L'Aquila, Italy (e-mail: germani@ing.univaq.it; manes@ing.univaq.it).

P. Palumbo is with the Istituto di Analisi dei Sistemi ed Informatica del CNR (IASI-CNR), 00185 Rome, Italy (e-mail: palumbo@iasi.rm.cnr.it).

Digital Object Identifier 10.1109/TAC.2002.803546.

With respect to the output $z(t)$, the bilinear stochastic system (2) can be represented in the robust form (independent of the unknown input) given by the following lemma.

Lemma 1: The bilinear stochastic differential system described by (2) can be rewritten as

$$\begin{aligned} dx(t) &= \mathcal{A}(t)x(t)\,dt + \mathcal{B}(t)\,dy(t) + \sum_{k=1}^{b}(\mathcal{N}_k(t)x(t) + \mathcal{F}_k(t)) \\ &\quad \cdot dW_k(t) \\ x(t_0) &= x_0 \\ dz(t) &= \mathcal{C}(t)x(t)\,dt + \sum_{k=1}^{b}(\mathcal{M}_k(t)x(t) + \mathcal{G}_k(t))\,dW_k(t) \end{aligned} \tag{5}$$

with

$$\begin{aligned} \mathcal{A}(t) &= A(t) - B(t)T_1(t)C(t) \in \mathbb{R}^{n\times n} \\ \mathcal{B}(t) &= B(t)T_1(t) \in \mathbb{R}^{n\times q} \\ \mathcal{N}_k(t) &= N_k(t) - B(t)T_1(t)M_k(t) \in \mathbb{R}^{n\times n} \\ \mathcal{F}_k(t) &= F_k(t) - B(t)T_1(t)G_k(t) \in \mathbb{R}^{n\times 1} \\ \mathcal{C}(t) &= T_2(t)C(t) \in \mathbb{R}^{(q-p)\times n} \\ \mathcal{M}_k(t) &= T_2(t)M_k(t) \in \mathbb{R}^{(q-p)\times n} \\ \mathcal{G}_k(t) &= T_2(t)G_k(t) \in \mathbb{R}^{(q-p)\times 1}. \end{aligned} \tag{6}$$

Proof: Note first that, by definition, the matrices $T_1(t)$ and $T_2(t)$ are such that $T_1(t)D(t) = I_p$ and $T_2(t)D(t) = O_{(q-p)\times p}$. From these, it follows:

$$\begin{aligned} T_1(t)\,dy(t) &= T_1(t)C(t)x(t)\,dt + du(t) + T_1(t)\sum_{k=1}^{b} \\ &\quad \cdot (M_k(t)x(t) + G_k(t))\,dW_k(t) \end{aligned} \tag{7}$$

$$\begin{aligned} dz(t) &= T_2(t)\,dy(t) \\ &= T_2(t)C(t)x(t)\,dt + T_2(t)\sum_{k=1}^{b} \\ &\quad \cdot (M_k(t)x(t) + G_k(t))\,dW_k(t). \end{aligned} \tag{8}$$

This last equation gives back the new measure equation, while eliminating $du(t)$ from the state equation of (2) by (7), the thesis immediately follows. ■

Remark 2: The structure (5) for the bilinear system with unknown inputs is obtained by suitably exploiting the information brought by the output on the unknown inputs. The "new" measurement process $z(t)$ is, in some sense, what remains after all information available on the unknown input has been exploited, and constitutes the "remaining" part of the output that can be used for filtering. Moreover, as a kind of confirmation, it is worth noting that the new measure equation vanishes if $q = p$, so it follows that the filtering approach presented in this work requires $q > p$. •

III. The Linear Filtering Algorithm

In this section, the new measurement vector $z(t)$ defined in (4) is used to develop a state estimator for system (5). In order to properly take into account the presence of the original output $y(t)$ as a forcing term in the state equation (5), a suitable decomposition of the system is required, as given by the following proposition.

Proposition 3: The system (5) can be written in the split form

$$\begin{aligned} x(t) &= x_d(t) + x_s(t) \\ dz(t) &= dz_s(t) + \mathcal{C}(t)x_d(t)\,dt \end{aligned} \tag{9}$$

where

$$\begin{aligned} dx_d(t) &= \mathcal{A}(t)x_d(t)\,dt + \mathcal{B}(t)\,dy(t) \\ x_d(t_0) &= \mathbb{E}[x_0] \\ dx_s(t) &= \mathcal{A}(t)x_s(t)\,dt + \sum_{k=1}^{b}(\mathcal{N}_k(t)(x_d(t) + x_s(t)) + \mathcal{F}_k(t)) \\ &\quad \cdot dW_k(t) \end{aligned} \tag{10}$$

$$\begin{aligned} x_s(t_0) &= x_0 - \mathbb{E}[x_0] \\ dz_s(t) &= \mathcal{C}(t)x_s(t)\,dt + \sum_{k=1}^{b}(\mathcal{M}_k(t)(x_d(t) + x_s(t)) + \mathcal{G}_k(t)) \\ &\quad \cdot dW_k(t). \end{aligned} \tag{11}$$

Proof: The proof is readily obtained by direct computation. ■

Remark 4: Proposition 3 shows the decomposition of the system state $x(t)$ in two terms: $x_d(t)$ is the totally observed component and $x_s(t)$ is the partially observed zero-mean component endowed with the new measurement process $z_s(t)$ from time t_0 up to time t. •

Remark 5: It must be stressed that the evolution of (10) completely determined by the measurements does not depend on $x_s(t)$ whereas, on the contrary, (11), forced also by the evolution of the totally known $x_d(t)$, admits the representation

$$\begin{aligned} dx_s(t) &= \mathcal{A}(t)x_s(t)\,dt + \sum_{k=1}^{b}\left(\mathcal{N}_k(t)x_s(t) + \tilde{\mathcal{F}}_k(t)\right)dW_k(t) \\ x_s(t_0) &= x_0 - \mathbb{E}[x_0] \\ dz_s(t) &= \mathcal{C}(t)x_s(t)\,dt + \sum_{k=1}^{b}\left(\mathcal{M}_k(t)x_s(t) + \tilde{\mathcal{G}}_k(t)\right)dW_k(t) \end{aligned} \tag{12}$$

where

$$\begin{aligned} \tilde{\mathcal{F}}_k(t) &= \mathcal{F}_k(t) + \mathcal{N}_k(t)x_d(t) \in \mathbb{R}^{n\times 1} \\ \tilde{\mathcal{G}}_k(t) &= \mathcal{G}_k(t) + \mathcal{M}_k(t)x_d(t) \in \mathbb{R}^{(q-p)\times 1}. \end{aligned} \tag{13}$$

•

The aforementioned remarks suggest the following.

Definition 6: A state estimator for the class of bilinear stochastic differential systems with unknown inputs is said to be input insensitive if its structure does not depend explicitly on the unknown input. •

Throughout this note, $x_d^\star(t)$ will denote the process $x_d(t)$ evaluated on the measured output path $y(t)$. Moreover, the superscript * will indicate all the processes that include a dependence on $x_d(t)$ when computed on the detected path $x_d^\star(t)$. For instance, the processes x_s and z_s defined by (12) with substitution of x_d with $x_d^\star$ will be denoted $x_s^\star$ and $z_s^\star$, respectively.

According to its definition (10), the best estimate of $x_d(t)$, given the observations, is indeed $x_d(t)$ itself. Therefore, taking into account the decomposition (9), a possible estimator of $x(t)$ can be obtained by adding to $x_d(t)$ an estimate of its not $\mathcal{F}_t^Y$-measurable part, $x_s(t)$. A way to define such estimate is through the best linear estimate of the process $x_s^\star(t)$ given the observation $z_s^\star$. This approach is summarized in the following definition.

Definition 7: The class $\mathcal{P}$ of estimators is defined as the set of all input-insensitive state estimators $\tilde{x}(t)$ such that

$$\tilde{x}(t) = x_d(t) + \tilde{x}_s^\star(t) \tag{14}$$

where $\tilde{x}_s^\star(t)$ is any estimate of the process $x_s^\star(t)$ among all the $\mathcal{F}_t^{Z_s^\star}$-measurable functions. A $\mathcal{P}$-estimator is any state estimator in the class $\mathcal{P}$. •

As is well known, the optimal choice for $\tilde{x}_s^\star(t)$ is given by $\mathbb{E}[x_s^\star(t)|\mathcal{F}_t^{Z_s^\star}]$, whose computation in general can not be obtained through algorithms of finite-dimension. Nevertheless, from an applicative point of view, it is useful to look for finite-dimensional approximations of the optimal filter, that is estimates described by stochastic differential equations of the form

$$\begin{aligned} d\xi(t) &= f(\xi(t))\,dt + g(\xi(t))\,dz_s^\star(t) \\ \tilde{x}_s^\star(t) &= h(\xi(t)) \end{aligned} \tag{15}$$

where $\{\xi(t),\, t \geq t_0\}$ is a process taking its values on a finite-dimensional space. The finite-dimensional system (15) is an optimal linear filter for the random process $x_s^\star(t)$ if

$$\tilde{x}_s^\star(t) = \Pi[x_s^\star(t)|L_t(z_s^\star)] \tag{16}$$

where $\Pi[\cdot|L_t(z_s^\star)]$ denotes the projection onto the space $L_t(z_s^\star)$ linearly spanned by the family of random variables $\{z_s^\star(\tau),\, t_0 \leq \tau \leq t\}$. It follows that the state estimator

$$x_d(t) + \Pi[x_s^\star(t)|L_t(z_s^\star)] \tag{17}$$

is a $\mathcal{P}$ estimator, and in the following it will be denoted as the *optimal $\mathcal{P}$-linear estimator* for (2).

Remark 8: It can be readily proved that the optimal $\mathcal{P}$-linear estimator (17) is unbiased. Actually, any $\mathcal{P}$-estimator of the type (14) with $\tilde{x}_s^\star(t) \in L_t(z_s^\star)$ is unbiased. •

The computation of $\Pi[x_s^\star(t)|L_t(z_s^\star)]$ for the bilinear system (12) forced by $x_d^\star$ can be done following the approach in [4].

Theorem 9: Let $\Psi_{x_s^\star}(t) = \mathrm{Cov}(x_s^\star(t))$ be the covariance matrix of $x_s^\star(t)$, whose evolution is given by the following equations:

$$\begin{aligned} \dot{\Psi}_{x_s^\star}(t) &= \mathcal{A}(t)\Psi_{x_s^\star}(t) + \Psi_{x_s^\star}(t)\mathcal{A}^T(t) \\ &\quad + \sum_{k=1}^{b}\left(\mathcal{N}_k(t)\Psi_{x_s^\star}(t)\mathcal{N}_k^T(t) + \tilde{\mathcal{F}}_k^\star(t)\tilde{\mathcal{F}}_k^{\star T}(t)\right) \\ \Psi_{x_s^\star}(t_0) &= \Psi_0 \end{aligned} \tag{18}$$

where $\tilde{\mathcal{F}}_k^\star(t) = \mathcal{F}_k(t) + \mathcal{N}_k(t)x_d^\star(t)$ and $\tilde{\mathcal{G}}_k^\star(t) = \mathcal{G}_k(t) + \mathcal{M}_k(t)x_d^\star(t)$. Moreover, let $R^\star(t)$ be the square matrix defined as follows:

$$R^\star(t) = \sum_{k=1}^{b}\left(\mathcal{M}_k(t)\Psi_{x_s^\star}(t)\mathcal{M}_k^T(t) + \tilde{\mathcal{G}}_k^\star(t)\tilde{\mathcal{G}}_k^{\star T}(t)\right). \tag{19}$$

Then, the optimal linear estimate of the state $x_s^\star(t)$ of (12), is given by

$$\begin{aligned} d\tilde{x}_s^\star(t) &= \mathcal{A}(t)\tilde{x}_s^\star(t)\,dt + \left(\sum_{k=1}^{b}\tilde{\mathcal{F}}_k^\star(t)\tilde{\mathcal{G}}_k^{\star T}(t) + P^\star(t)\mathcal{C}^T(t)\right) \\ &\quad \cdot R^\star(t)^{-1}(dz_s^\star(t) - \mathcal{C}(t)\tilde{x}_s^\star(t)\,dt) \\ \tilde{x}_s^\star(t_0) &= 0 \end{aligned} \tag{20}$$

where $P^\star(t) = \mathrm{Cov}(x_s^\star(t) - \tilde{x}_s^\star(t))$ is the error covariance matrix, given by the following equations:

$$\begin{aligned} \dot{P}^\star(t) &= \mathcal{A}(t)P^\star(t) + P^\star(t)\mathcal{A}^T(t) + R^\star(t) \\ &\quad - \left(\sum_{k=1}^{b}\tilde{\mathcal{F}}_k^\star(t)\tilde{\mathcal{G}}_k^{\star T}(t) + P^\star(t)\mathcal{C}^T(t)\right) \\ &\quad \cdot R^\star(t)^{-1}\left(\sum_{k=1}^{b}\tilde{\mathcal{F}}_k^\star(t)\tilde{\mathcal{G}}_k^{\star T}(t) + P^\star(t)\mathcal{C}^T(t)\right)^T \\ P^\star(t_0) &= \Psi_0 \end{aligned} \tag{21}$$

provided that $R^\star(t)$ is positive–definite $\forall t \geq t_0$.

Proof: The proof is just a straightforward extension to the time varying case of the proof of [4, Th. 4.4]. ■

Remark 10: Note that a sufficient condition for the positive definiteness of matrix $R^\star(t)$ defined in (19) is the nonsingularity of

$$\sum_{k=1}^{b}\tilde{\mathcal{G}}_k^\star(t)\tilde{\mathcal{G}}_k^{\star T}(t). \tag{22}$$

•

Now, the optimal $\mathcal{P}$-linear state estimator for system (2) can be presented.

Theorem 11: Under the same hypotheses of Theorem 9, according to (17), the optimal $\mathcal{P}$-linear estimate of the bilinear stochastic differential system (2) is given by the following algorithm:

$$\begin{aligned} d\tilde{x}(t) &= \mathcal{A}(t)\tilde{x}(t)\,dt + \mathcal{B}(t)\,dy(t) \\ &\quad + \left(\sum_{k=1}^{b}\tilde{\mathcal{F}}_k^\star(t)\tilde{G}_k^{\star T}(t) + P^\star(t)C^T(t)\right) \\ &\quad \cdot \mathcal{R}^\star(t)(dy^\star(t) - C(t)\tilde{x}(t)\,dt) \\ \tilde{x}(t_0) &= \mathbb{E}[x_0] \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{P}^\star(t) &= \mathcal{A}(t)P^\star(t) + P^\star(t)\mathcal{A}^T(t) + R^\star(t) \\ &\quad - \left(\sum_{k=1}^{b}\tilde{\mathcal{F}}_k^\star(t)\tilde{G}_k^{\star T}(t) + P^\star(t)C^T(t)\right) \\ &\quad \cdot \mathcal{R}^\star(t)\left(\sum_{k=1}^{b}\tilde{\mathcal{F}}_k^\star(t)\tilde{G}_k^{\star T}(t) + P^\star(t)C^T(t)\right)^T \\ P^\star(t_0) &= \Psi_0. \end{aligned} \tag{24}$$

where

$$\begin{aligned} \tilde{G}_k^\star(t) &= G_k(t) + M_k(t)x_d^\star(t) \\ \mathcal{R}^\star(t) &= T_2^T(t)R^{\star -1}(t)T_2(t) \end{aligned} \tag{25}$$

and $R^\star(t)$ is given by (19), defined in Theorem 9.

Proof: Taking into account the decomposition (9) in Proposition 3 and (17) of the optimal $\mathcal{P}$-linear estimate, (23) for $d\tilde{x}(t)$ is obtained by adding $dx_d(t)$ from (10) and $d\tilde{x}_s^\star(t)$ from (20). ■

Remark 12: Note that if $B = O_{n\times p}$ and $D = O_{q\times p}$ in system (2), the problem reduces to the filtering of a stochastic bilinear system without unknown inputs. The linear and polynomial optimal solutions of the filtering problem for this class of systems are reported in [4]. On the other hand, when N_k, M_k, F_k, G_k are zero matrices the system reduces to a deterministic linear one. Some solutions of the state-observation problem for this class of systems can be found in [9] and the references therein. •

Remark 13: The estimator presented in Theorem 11 can be regarded as the optimal linear filter when x_d is not a random variable, but is the output of (10) driven by the "detected path" $y(t)$. Therefore, the presented filter in some sense is designed considering the detected path $y(t)$ as a deterministic input to (2), once that is rewritten in the split form (9). In this framework the filter in some sense is designed for the "open-loop" system and then it is used in "closed-loop," because $y(t)$ is in fact the output of system (9). Note that the optimal filter for an open-loop system, when used for state estimation of the closed-loop system, gives back the optimal filter for the closed-loop system, and therefore is efficient. This result, recently presented in [2], strongly suggests the conjecture that the proposed algorithm is indeed efficient in the class of $\mathcal{P}$-linear estimators. The formal proof of this assertion would require the extension of the results given in [2] to suboptimal

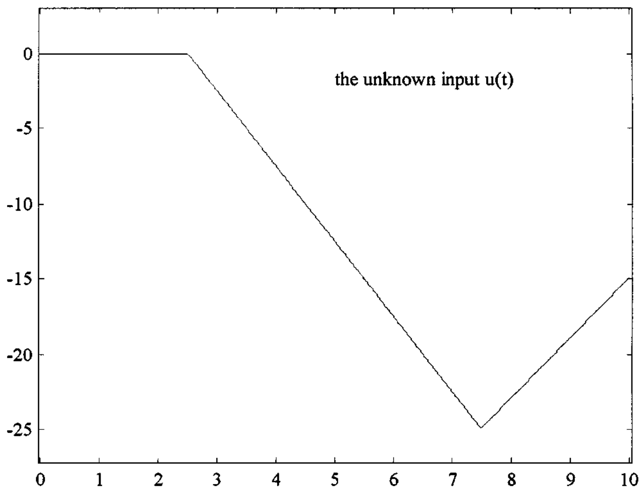

Fig. 1. The unknown input.

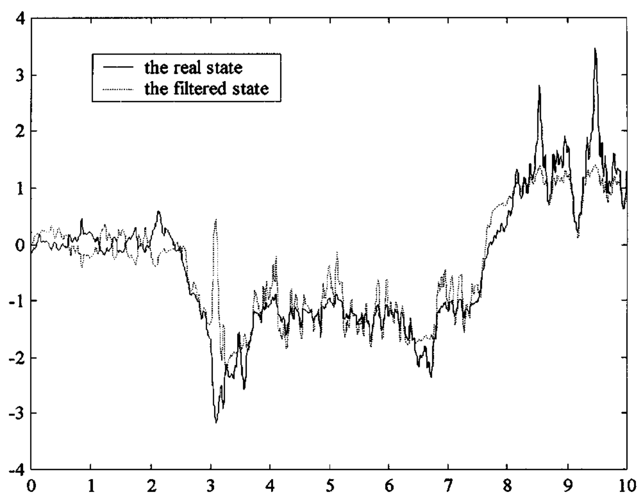

Fig. 2. The true and estimated states: first component.

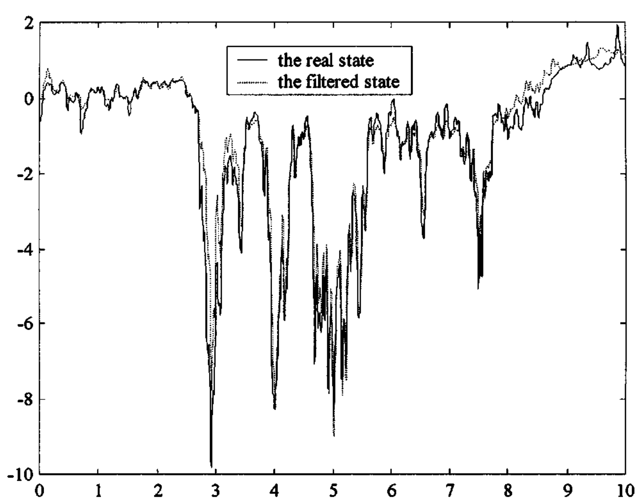

Fig. 3. The true and estimated states: second component.

filtering. This is indeed not an easy task, and is the object of a current research work.

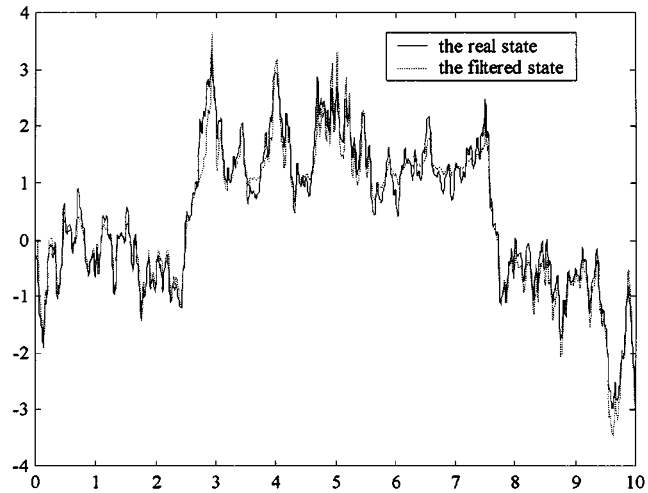

Fig. 4. The true and estimated states: third component.

IV. Numerical Simulations

This section presents simulation results on a bilinear system whose state and measurement equations are forced by a standard Wiener process. A scalar unknown input is also considered on the system. The matrices in (2), in which $p = b = 1$, $n = 3$, $q = 2$, are the following:

$$A = \begin{bmatrix} -4 & 0.1 & 0 \\ 0 & -3 & -1 \\ 0.4 & 0 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0.5 \\ -1.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0.5 & -2 \\ 1 & 1 & -0.2 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{27}$$

$$N = \begin{bmatrix} -1 & 0 & 0.1 \\ 0 & -2 & 0 \\ 0 & 0.4 & 0.5 \end{bmatrix} \quad F = \begin{bmatrix} -1 \\ 1.2 \\ -2 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & -0.5 & -1 \\ 0 & -2 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}. \tag{28}$$

The simulation results here reported are obtained with the unknown input plotted in Fig. 1. Figs. 2–4 present the comparison between the real and the estimated state for each state component. The good behavior of the optimal $\mathcal{P}$-linear filter can be appreciated.

References

[1] C. Bruni, G. Di Pillo, and G. Koch, "Bilinear systems: An appealing class of 'nearly linear' systems in theory and applications," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 334–348, Aug. 1974.

[2] F. Carravetta, A. Germani, R. Liptser, and C. Manes, "Filtering of nonlinear stochastic feedback systems," *SIAM J. Control Optim.*, vol. 40, no. 5, pp. 1576–1584, 2002.

[3] F. Carravetta, A. Germani, and M. Raimondi, "Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1106–1126, Aug. 1997.

[4] F. Carravetta, A. Germani, and M. K. Shuakayev, "A new suboptimal approach to the filtering problem for bilinear stochastic differential systems," *SIAM J. Control Optim.*, vol. 34, no. 4, pp. 1171–1203, 2000.

[5] J. Chen and R. J. Patton, "Optimal filtering and robust fault diagnosis of stochastic systems with unknown disturbances," in *Proc. IEE Control Theory Applications*, vol. 143, 1996, pp. 31–36.

[6] M. Darouach and M. Zasadzinski, "State estimation for a class of singular systems," *Int. J. Syst. Sci.*, vol. 23, no. 4, pp. 517–530, 1992.

[7] M. Darouach, M. Zasadzinski, and D. Mehdi, "State estimation of stochastic singular linear systems," *Int. J. Syst. Sci.*, vol. 2, no. 2, pp. 345–354, 1993.

[8] M. Darouach, M. Zasadzinski, A. Basson Onana, and S. Nowakowski, "Kalman filtering with unknown inputs via optimal state estimation of singular systems," *Int. J. Syst. Sci.*, vol. 26, no. 10, pp. 2015–2028, 1995.

[9] M. Darouach, M. Zasadzinski, and M. Hayar, "Reduced-order observer design for descriptor systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1068–1072, July 1996.

[10] M. Darouach and M. Zasadzinski, "Unbiased minimum variance estimation for systems with unknown exogenous inputs," *Automatica*, vol. 33, no. 4, pp. 717–719, 1997.

[11] A. Germani, C. Manes, and P. Palumbo, "Polynomial filtering for stochastic non-Gaussian descriptor systems," IASI-CNR Research, Rep. no. 526, 2000.

[12] M. Hou and P. C. Muller, "Disturbance decoupled observer design: A unified viewpoint," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1338–1341, June 1994.

[13] M. Hou and R. J. Patton, "Optimal filtering for systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 445–449, Mar. 1998.

[14] C. S. Hsieh and F. C. Chen, "Optimal solution of the two-stage Kalman estimator," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 194–199, Jan. 1999.

[15] C. S. Hsieh, "Robust two-stage Kalman filters for systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 2374–2378, Dec. 2000.

[16] R. E. Kalman, "A new approach to linear filtering and prediction problems," *ASME J. Basic Eng.*, ser. D, vol. 82, 1960.

[17] J. Y. Keller, L. Summer, and M. Darouach, "Extension of Friedland's bias filtering technique to discrete-time systems with unknown inputs," *Int. J. Syst. Sci.*, vol. 27, no. 12, pp. 1219–1229, 1996.

[18] J. Y. Keller, M. Darouach, and L. Caramelle, "Kalman filter with unknown inputs and robust two stage filter," *Int. J. Syst. Sci.*, vol. 29, no. 1, pp. 1–47, 1998.

[19] P. K. Kitanidis, "Unbiased minimum-variance linear state estimation," *Automatica*, vol. 23, pp. 775–778, 1987.

[20] W. J. Kolodziej and R. R. Mohler, "An overview of bilinear system theory and applications," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-10, pp. 683–688, Oct. 1980.

[21] W. J. Kolodziej and R. R. Mohler, "Analysis of a new nonlinear filter and tracking methodology," *IEEE Trans. Inform. Theory*, vol. 30, pp. 677–681, Apr. 1984.

[22] ——, "State estimation and control of conditionally linear systems," *SIAM J. Control Optim.*, vol. 24, pp. 497–508, 1986.

[23] W. J. Kolodziej and R. R. Mohler, "On structurally decomposing processes," *IEEE Trans. Circuits Syst.*, vol. 35, pp. 1417–1424, Nov. 1988.

[24] J. E. Kurek, "The state vector reconstruction for linear systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 1120–1122, Dec. 1983.

[25] R. S. Lipster and A. N. Shiryayev, *Statistics of Random Process*. New York: Springer-Verlag, 1978, vol. 2.

[26] R. R. Mohler, *Bilinear Control Processes*. New York: Academic, 1970.

[27] R. Nikoukhah, B. C. Levy, and A. S. Willsky, "Kalman filtering and Riccati equations for descriptor systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1325–1342, Sept. 1992.

[28] R. Nikoukhah, S. L. Campbell, and F. Delebecque, "Kalman filtering for general discrete-time linear systems," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 1829–1839, Oct. 1999.

[29] S. S. Rappaport and L. Kurz, "An optimal nonlinear detector for digital data transmission through non-Gaussian channels," *IEEE Trans. Commun. Tech.*, vol. CT-14, pp. 266–274, Mar. 1966.

Observer Linearization by Output-Dependent Time-Scale Transformations

M. Guay

***Abstract*—In this note, we study the problem of observer linearization for single-output dynamical systems in the presence of output-dependent time-scaling changes. An alternative algorithm for the solution of the observer linearization problem is introduced. The algorithm employs an exterior calculus approach that provides a simple procedure for the solution of the observer linearization problem by means of an output dependent time-scale transformation.**

***Index Terms*—Linearization, observers, time scaling.**

I. Introduction

The problem of the equivalence of a nonlinear observable systems to observers with linear error dynamics has been studied extensively in the literature. The single-output problem was first proposed and solved in [2] and [6]. The multi-output problem was later considered in [7], [13], [14] and, more recently, in [5], [15], and [11]. In this note, we propose and solve the problem of observer linearization for single-output dynamical systems by means of an output-dependent time-scale transformation and a state-space diffeomorphism. An exterior calculus approach is used to develop an alternative computational procedure for observer linearization of locally observable nonlinear systems. The approach is closely related to the orbital feedback linearization problem solved in [4] and [9].

The note is organized as follows. Some preliminary information is provided in Section II. In Section III, the linearization procedure is presented and the solution of the observer linearization problem by means of output dependent time-scale transformations is given. A three-dimensional example is presented in Section IV to demonstrate the application of the technique. Brief conclusions are presented in Section V.

II. Preliminaries

The class of systems of interest in this note is that of observable single-output nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $f(x)$ is a smooth vector field defined in a subset of $\mathbb{R}^n$, $h(x)$ is a smooth function of $x \in M \subset \mathbb{R}^n$. The measurable output is given by $y \in \mathbb{R}$.

The observer linearization problem treated in this note can be stated as follows.

Problem Statement 2.1: Observer Linearization by Output-Dependent Time-Scaling Transformation: Find a smooth time-scale transformation $\gamma(h) > 0$ with

$$\frac{dt}{d\tau} = \gamma(h) \tag{2}$$

Manuscript received January 10, 2001; revised July 12, 2001, January 10, 2002, and May16, 2002. Recommended by Associate Editor M. E. Valcher. This work was supported by the Natural Sciences and Engineering Council of Canada and the Canadian Foundation for Innovation.

The author is with the Department of Chemical Engineering, Queen's University, K7L 3N6 Kingston, ON, Canada (e-mail: guaym@chee.queensu.ca).

Digital Object Identifier 10.1109/TAC.2002.803547.