Linear free divisors and Frobenius manifolds

Ignacio de Gregorio, David Mond and Christian Sevenheck

Abstract

We study linear functions on fibrations whose central fibre is a linear free divisor. We analyse the Gauß-Manin system associated to these functions, and prove the existence of a primitive and homogenous form. As a consequence, we show that the base space of the semi-universal unfolding of such a function carries a natural Frobenius manifold structure.

Contents

1 Introduction

In this paper we study Frobenius manifolds arising as deformation spaces of linear functions on certain non-isolated singularities, the so-called linear free divisors. It is a nowadays classical result that the semi-universal unfolding space of an isolated hypersurface singularity can be equipped with a Frobenius structure. One of the main motivations to study Frobenius manifolds comes from the fact that they also arise in a very different area: the total cohomology space of a projective manifold carries such a structure, defined by the quantum multiplication. Mirror symmetry postulates an equivalence between these two types of Frobenius structures. In order to carry this program out, one is forced to study not only local singularities (which are in fact never the mirror of a quantum cohomology ring) but polynomial functions on affine manifolds. It has been shown in [DS03] (and later, with a somewhat different strategy in [Dou05]) that given a convenient and non-degenerate Laurent polynomial $\widetilde{f} : (\mathbb{C}^*)^n \to \mathbb{C}$, the base space M of a semi-universal unfolding $\widetilde{F} : (\mathbb{C}^*)^n \times M \to \mathbb{C}$ can be equipped with a (canonical) Frobenius structure. An important example is the function $\widetilde{f} = x_1 + \ldots + x_{n-1} + \frac{t}{x_1 \cdot \ldots \cdot x_{n-1}}$ for some fixed $t \in \mathbb{C}^*$: the Frobenius structure obtained on its unfolding space is known (see [Giv95], [Giv98] and [Bar00]) to be isomorphic to the full quantum cohomology of the projective space $\mathbb{P}^{n-1}$. More generally, one can consider the Laurent polynomial $\widetilde{f} = x_1 + \ldots + x_{n-1} + \frac{t}{x_1^{w_1} \cdot \ldots \cdot x_{n-1}^{w_{n-1}}}$ for some weights $(w_1, \ldots, w_{n-1}) \in \mathbb{N}^{n-1}$, here the Frobenius structure corresponds to the (orbifold) quantum cohomology of the weighted projective space $\mathbb{P}(1, w_1, \ldots, w_{n-1})$ (see [Man08] and [CCLT06]). A detailed analysis on how to construct the Frobenius structure

for these functions is given in [DS04], some of the techniques in this paper are similar to those used here. Notice that the mirror of the ordinary projective space might be interpreted in a slightly different way, namely, as the restriction of the linear polynomial $f = x_1 + \ldots + x_n : \mathbb{C}^n \to \mathbb{C}$ to the non-singular fibre $h(x_1, \ldots, x_n) - t = 0$ of the torus fibration defined by the homogeneous polynomial $h = x_1 \cdot \ldots \cdot x_n$.
In the present work, we construct Frobenius structures on the unfolding spaces of a class of functions generalizing this basic example, namely, we consider homogenous functions h such that its zero fibre is a *linear free divisor*. Linear free divisor have been recently introduced by R.-O. Buchweitz and the second author in [BM06] (see also [GMNS06]), but are closely related to the more classical *prehomogeneous spaces* of T. Kimura and M. Sato ([SK77]). They are defined as free divisors $D = h^{-1}(0)$ in some vector space V whose sheaf of derivations can be generated by vector fields having only linear coefficients. The classical example is of course the normal crossing divisor. Following the analogy with the mirror of $\mathbb{P}^n$, we are interested in characterising when a linear function f has isolated singularities on the Milnor fibre $D_t = h^{-1}(t)$, $t \neq 0$. As it turns out, not all linear free divisors support such functions, but the large class of *reductive* ones do, and for these the set of linear functions having only isolated singularities can be characterised as the complement of the *dual divisor*.
Let us give a short overview on the paper. In section 2 we state and prove some general results on linear free divisors. In particular, we introduce the notion of *special linear free divisors*, and show that reductive ones are always special. This is proved by studying the relative logarithmic de Rham complex (subsection 2.2) which is also important in the later discussion of the Gauß-Manin-system. The cohomology of this complex is computed in reductive case, thanks to a classical theorem of Hochschild and Serre.
Section 3 discusses linear functions f on linear free divisors D, as well as on their Milnor fibres D_t. We show (in an even more general situation where D is not a linear free divisor) that $f_{|D_t}$ is a Morse function if the restriction $f_{|D}$ is right-left stable. This implies in particular that the Frobenius structures associated to the functions $f_{|D_t}$ are all semi-simple. Subsection 3.2 discusses deformation problems associated to the two functions (f, h). In particular, we show that linear forms in the complement of the dual divisor have the necessary finiteness properties. In order to exhibit Frobenius structures, the fibration defined by $f_{|D_t}$ is required to have a good behaviour at infinity, comprised in the notion of *tameness*. In subsection 3.3 it is shown that this property indeed hold for these functions.
In section 4 we study the (algebraic) Gauß-Manin system and the (algebraic) Brieskorn lattice of $f_{|D_t}$. We actually define both as a family over the parameter space of h, and using logarithmic forms along D (more precisely, the relative logarithmic de Rham complex mentioned above) we get very specific extensions of these families over D. The fact that D is a linear free divisor allows us to construct explicitely a basis of this family of Brieskorn lattice, hence showing its freeness. Next we give an explicit solution to the so called Birkhoff problem. Although this solution is not a good basis in the sense of M. Saito [Sai89], that is, it might not compute the spectrum at infinity of $f_{|D_t}$, we give an algorithmic procedure to turn it into one. This allows us in particular to compute the monodromy of $f_{|D_t}$. We finish this section by showing that this solution to the Birkhoff problem is also compatible with a natural pairing defined on the Brieskorn lattice, at least under an additional hypothesis (which is satisfied in many examples) on the spectral numbers.
In section 5 we finally apply all these results to construct Frobenius structures on the unfolding spaces of the functions $f_{|D_t}$ (subsection 5.1) and on $f_{|D}$ (subsection 5.2). Whereas the former exists in all cases, the latter depends on a conjecture concerning a natural pairing on the Gauss-Manin-system. Similarly, assuming this conjecture, we give some partial results concerning *logarithmic Frobenius structures* as defined in [Rei08] in subsection 5.3.
We end the paper with some examples (section 6). On the one hand, they illustrate the different phenomena that can occur, as for instance, the fact that there might not be a *canonical* choice (as in [DS03]) of a primitive form. On the other hand, they support the conjeture concerning the pairing used in the discussion of the limit Frobenius structure.

Acknowledgements. We are grateful to Antoine Douai, Claus Hertling, Dmitry Rumynin, Claude Sabbah and Mathias Schulze for helpful discussions on the subject of this article. We thank the anonymous referee for a careful reading of a first version and a number of very helpful remarks.

2 Reductive and special linear free divisors

2.1 Definition and examples

A hypersurface D in a complex manifold X is a *free divisor* if the $\mathcal{O}_X$-module $\mathrm{Der}(-\log D)$ is locally free. If $X = \mathbb{C}^n$ then D is furthermore a *linear free divisor* if $\mathrm{Der}(-\log D)$ has an $\mathcal{O}_{\mathbb{C}^n}$-basis consisting of weight-zero vector fields – vector fields whose coefficients, with respect to a standard linear coordinate system, are linear functions (see [GMNS06, Section 1]). By Serre's conjecture, if $D \subset \mathbb{C}^n$ is a free divisor then $\mathrm{Der}(-\log D)$ is globally free. If $D \subset \mathbb{C}^n$ is a linear free divisor then the group $G_D := \{A \in \mathrm{Gl}_n(\mathbb{C}) : AD = D\}$ of its linear automorphisms is algebraic of dimension n. We denote by G_D^0 the connected component of G_D containing the identity, and by Sl_D the intersection of G_D^0 with $\mathrm{Sl}_n(\mathbb{C})$. The infinitesimal action of the Lie algebra $\mathfrak{g}_D$ of G_D^0 generates $\mathrm{Der}(-\log D)$ over $\mathcal{O}_{\mathbb{C}^n}$, and it follows that the complement of D is a single G_D^0-orbit ([GMNS06, Section 2]). G_D^0, is a *prehomogeneous vector space* (cf [SK77]). Thus, $\mathbb{C}^n$, with this action of G_D^0, is a *prehomogeneous vector space*(cf [SK77]) i.e. a representation ρ of a group G on a vector space V in which the group has an open orbit. The complement of the open orbit in a prehomogeneous vector space is known as the *discriminant*. The (reduced) discriminant in a prehomogeneous vector space is a linear free divisor if and only if the dimensions of G and V and the degree of the discriminant are all equal.

By Saito's criterion ([Sai80]), the determinant of the matrix of coefficients of a set of generators of $\mathrm{Der}(-\log D)$ is a reduced equation for D, which is therefore homogeneous of degree n. Throughout the paper we will denote the reduced homogeneous equation of the linear free divisor D by h.

If the group G acts on the vector space V, then a rational function $f \in \mathbb{C}(V)$ is a *semi-invariant* (or *relative invariant*) if there is a character $\chi_f : G \to \mathbb{C}^*$ such that for all $g \in G$, $f \circ g = \chi_f(g)f$. In this case χ_f is the *character associated to* f. Sato and Kimura prove ([SK77, §4 Lemma 4]) that semi-invariants with multiplicatively independent associated characters are algebraically independent. If D is a linear free divisor with equation h, then h is a semi-invariant ([SK77, §4]) (for the action of G_D^0). For it is clear that g must leave D invariant, and thus $h \circ g$ is some complex multiple of h. This multiple is easily seen to define a character, which we call χ_h.

Definition 2.1. We call the linear free divisor D *special* if χ_h is equal to the determinant of the representation, and *reductive* if the group G_D^0 is reductive.

We show in 2.9 below that every reductive linear free divisor is special. We do not know if the converse holds. The term "special" is used here because the condition means that the elements of G_D which fix h lie in $\mathrm{Sl}_n(\mathbb{C})$. Not all linear free divisors are special. Consider the example of the group B_k of upper triangular complex matrices acting on the space $V = \mathrm{Sym}_k(\mathbb{C})$ of symmetric $k \times k$ matrices by transpose conjugation,

$$B \cdot S = {}^tBSB \tag{2.1}$$

The discriminant here is a linear free divisor ([GMNS06, Example 5.1]). Its equation is the product of the determinants of the top left-hand $l \times l$ submatrices of the generic $k \times k$ symmetric matrix, for $l = 1, \ldots, k$. It follows that if $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_k) \in B_k$ then

$$h \circ \rho(B) = \lambda_1^{2k} \lambda_2^{2k-2} \cdots \lambda_k^2 \, h,$$

and D is not special. The simplest example is the case $k = 2$, here the divisor has the equation

$$h = x(xz - y^2) \tag{2.2}$$

Irreducible prehomogeneous vector spaces are classified in [SK77]. However, irreducible representations account for very few of the linear free divisors known. For more examples we turn to the representation spaces of quivers:

Proposition 2.2 ([BM06])**.** *(i) Let Q be a quiver without oriented loops and let $\mathbf{d}$ be (a dimension vector which is) a real Schur root of Q. Then the triple $(Gl_{Q,\mathbf{d}}, \rho, \mathrm{Rep}(Q, \mathbf{d}))$ is a prehomogeneous vector space and the complement of the open orbit is a divisor D (the "discriminant" of the representation ρ of the quiver group $Gl_{Q,\mathbf{d}}$ on the representation space $\mathrm{Rep}(Q, \mathbf{d})$)).*

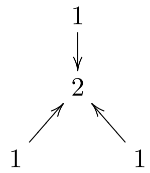
(ii) If in each irreducible component of D there is an open orbit, then D is a linear free divisor.

(iii) If Q is a Dynkin quiver then the condition of (ii) holds for all real Schur roots $\mathbf{d}$.

We note that the normal crossing divisor appears as the discriminant in the representation space $\mathrm{Rep}(Q, \mathbf{1})$ for every quiver Q whose underlying graph is a tree. Here $\mathbf{1}$ is the dimension vector which takes the value 1 at every node.

All of the linear free divisors constructed in Proposition 2.2 are reductive. For if D is the discriminant in $\mathrm{Rep}(Q, \mathbf{d})$ then G_D^0 is the quotient of $\mathrm{Gl}_{Q,\mathbf{d}} = \prod_i \mathrm{Gl}_{d_i}(\mathbb{C})$ by a 1-dimensional central subgroup.

Example 2.3. (i) Consider the quiver of type D_4 with real Schur root

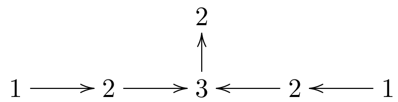


By choosing a basis for each vector space we can identify the representation space $\mathrm{Rep}(Q, \mathbf{d})$ with the space of 2×3 matrices, with each of the three morphisms corresponding to a column. The open orbit in $\mathrm{Rep}(Q, \mathbf{d})$ consists of matrices whose columns are pairwise linearly independent. The discriminant thus has equation

$$h = (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{12}a_{23} - a_{22}a_{13}). \tag{2.3}$$

This example generalises: instead of three arrows converging to the central node, we take n, and set the dimension of the space at the central node to $n-1$. The representation space can now be identified with the space of $(n-1) \times n$ matrices, and the discriminant is once again defined by the vanishing of the product of maximal minors. Again it is a linear free divisor ([GMNS06, Example 5.3]), even though for $n > 3$ the quiver is no longer a Dynkin quiver. We refer to it as the *star quiver*, and denote it by $\star_n$.

(ii) The linear free divisor arising by the construction of Proposition 2.2 from the quiver of type E_6 with real Schur root



has five irreducible components. In the 22-dimensional representation space $\mathrm{Rep}(Q, \mathbf{d})$, we take coordinates $a, b, \ldots, v$. Then

$$h = F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5 \tag{2.4}$$

where four of the components have the equations

$$\boxed{\begin{array}{lcl} F_1 &=& dfpq - cgpq - dfor + cgor + efps - chps + egrs - dhrs - efot + chot - egqt + dhqt \\ F_2 &=& jlpq - impq - jlor + imor + klps - inps + kmrs - jnrs - klot + inot - kmqt + jnqt \\ F_3 &=& -aejl - bhjl + adkl + bgkl + aeim + bhim - ackm - bfkm - adin - bgin + acjn + bfj \\ F_4 &=& egiu - dhiu - efju + chju + dfku - cgku + eglv - dhlv - efmv + chmv + dfnv - cgnv \end{array}}$$

and the fifth has the equation $F_5 = 0$, which is of degree 6, with 48 monomials. This example is discussed in detail in [BM06, Example 7.3]

2.2 The relative logarithmic de Rham complex

Let D be a linear free divisor with equation h. We set $\mathrm{Der}(-\log h) = \{\chi \in \mathrm{Der}(-\log D) : \chi \cdot h = 0\}$. Under the infinitesimal action of G_D^0, the Lie algebra of $\ker(\chi_h)$, which we denote by $\mathfrak{g}_h$, is identified with the weight zero part of $\mathrm{Der}(-\log h)$, which we denote by $\mathrm{Der}(-\log h)_0$. $\mathrm{Der}(-\log h)$ is a summand of $\mathrm{Der}(-\log D)$, as is shown by the equality

$$\xi = \frac{\xi \cdot h}{E \cdot h} E + \left(\xi - \frac{\xi \cdot h}{E \cdot h} E\right)$$

in which E is the Euler vector field and the second summand on the right is easily seen to annihilate h. The quotient complex

$$\Omega^\bullet(\log h) := \frac{\Omega^\bullet(\log D)}{dh/h \wedge \Omega^{\bullet-1}(\log D)} = \frac{\Omega^\bullet(\log D)}{h^*\left(\Omega^1_{\mathbb{C}}(\log\{0\})\right) \wedge \Omega^{\bullet-1}(\log D)}$$

is the "relative logarithmic complex" associated with the function $h : \mathbb{C}^n \to \mathbb{C}$. Each module $\Omega^k(\log h)$ is isomorphic to the submodule

$$\Omega^k(\log h)' := \{\omega \in \Omega^k(\log D) : \iota_E \omega = 0\} \subset \Omega^k(\log D).$$

For he natural map $i : \omega \mapsto \omega + (dh/h)\wedge\Omega^{\bullet-1}(\log D)$ gives an injection $\Omega^k(\log h)' \to \Omega^k(\log h)$, since for $\omega \in \Omega^k(\log h)'$, if $\omega = (dh/h)\wedge\omega_1$ for some ω_1 then

$$0 = \iota_E(\omega) = \iota_E\left(\frac{dh}{h}\wedge\omega_1\right) = n\omega_1 - \frac{dh}{h}\wedge\iota_E(\omega_1)$$

and thus $\omega_1 = (dh/nh)\wedge\iota_E(\omega_1)$ and $\omega = (dh/h)\wedge(dh/nh)\wedge\iota_E(\omega_1) = 0$. Because

$$\frac{1}{n}\iota_E\left(\frac{dh}{h}\wedge\omega\right) \in \Omega^k(\log h)'$$

and

$$\omega - \frac{1}{n}\iota_E\left(\frac{dh}{h}\wedge\omega\right) \in \frac{dh}{h}\wedge\Omega^{k-1}(\log D) \tag{2.5}$$

i is surjective. However, the collection of $\Omega^k(\log h)'$ is not a subcomplex of $\Omega^\bullet(\log D)$: $\iota_E(d\omega)$ may not be zero even when $\iota_E(\omega) = 0$. We define $d' : \Omega^k(\log h)' \to \Omega^{k+1}(\log h)'$ by composing the usual exterior derivative $\Omega^k(\log h)' \to \Omega^{k+1}(\log D)$ with with the projection operator $P : \Omega^\bullet(\log D) \to \Omega^\bullet(\log h)'$ defined by

$$P(\omega) = \frac{1}{n}\iota_E\left(\frac{dh}{h}\wedge\omega\right) = \omega - \frac{1}{n}\frac{dh}{h}\wedge\iota_E(\omega). \tag{2.6}$$

Lemma 2.4. *(i)* $d \circ i = i \circ d'$.

(ii) $(d')^2 = 0$.

(iii) $i : \left(\Omega^\bullet(\log h)', d'\right) \to \left(\Omega^\bullet(\log h), d\right)$ *is an isomorphism of complexes.*

Proof. The first statement is an obvious consequence of the second equality in (2.6). The second follows because $d^2 = 0$ and i is an injection. The third is a consequence of (i) and (ii). □

Lemma 2.5. *The weight zero part of* $\left(\Omega^\bullet(\log h)', d'\right)$ *is a subcomplex of* $\left(\Omega^\bullet(\log D), d\right)$.

Proof. Let $\omega \in \Omega^k(\log h)'$. We have

$$P_{k+1}(d\omega) = \frac{1}{n}\iota_E\left(\frac{dh}{h}\wedge d\omega\right) = d\omega - \frac{1}{n}\frac{dh}{h}\wedge\iota_E(d\omega) = d\omega - \frac{1}{n}\frac{dh}{h}\wedge\big(L_E(\omega) - d\iota_E(\omega)\big)$$

where L_E is the Lie derivative with respect to E. By assumption, $\iota_E(\omega) = 0$, and since $L_E(\sigma) = \mathrm{weight}(\sigma)\sigma$ for any homogeneous form, it follows that if $\mathrm{weight}(\omega) = 0$ then $d'\omega = d\omega$. □

Let

$$\alpha = \iota_E\left(\frac{dx_1\wedge\cdots\wedge dx_n}{h}\right) \tag{2.7}$$

Evidently $\alpha \in \Omega^{n-1}(\log h)'$, and moreover

$$\alpha = n\frac{dx_1\wedge\cdots\wedge dx_n}{dh}.$$

For $\xi \in \mathrm{Der}(-\log h)$, we define the form $\lambda_\xi = \iota_\xi\alpha$. Notice that α generates the rank one $\mathbb{C}[V]$-module $\Omega^{n-1}(\log\, h)$: $\alpha \wedge dh/h$ is a constant multiple of $dx_1 \wedge \ldots dx_n/h$, which is a generator of $\Omega^n(\log\, D)$ (remember that dh/h is the element of $\Omega^1(\log\, D)$ dual to $E \in \mathrm{Der}(-\log\, D)$).

Lemma 2.6. *The linear free divisor* $D \subset \mathbb{C}^n$ *is special if and only if* $d\lambda_\xi = 0$ *for all* $\xi \in \mathrm{Der}(-\log h)_0$.

Proof. Let $\xi \in \mathrm{Der}(-\log h)_0$ and let $\lambda_\xi = \iota_\xi(\alpha) = \iota_\xi\iota_E\mathrm{vol}$. Since α generates $\Omega^{n-1}(\log\, h)$ and λ_ξ has weight zero, $d'\lambda_\xi = c\alpha$ for some scalar c. By the previous lemma, the same is true for $d\lambda_\xi$. Since $dh\wedge\alpha = \mathrm{vol}$, it follows that $dh\wedge d\lambda_\xi = c\mathrm{vol}$. Now $dh\wedge d\lambda_\xi = -d(dh\wedge\lambda_\xi) = d\iota_\xi(\mathrm{vol}) = L_\xi(\mathrm{vol})$. An easy calculation shows that $L_\xi(\mathrm{vol}) = \mathrm{trace}(A)\mathrm{vol}$, where A is the $n \times n$ matrix such that $A \cdot x = \xi(x)$. Hence

$$d\lambda_\xi = 0 \quad \Leftrightarrow \quad \mathrm{trace}(A) = 0.$$

Thus $d\lambda_\xi = 0$ for all $\xi \in \mathrm{Der}(-\log D)$ if and only if $\mathrm{trace}(A) = 0$ for all matrices $A \in \ker d\chi_h$, i.e. if and only if $\ker d\chi_h \subseteq \ker d\det$. Since both kernels have codimension 1, the inclusion holds if and only if equality holds, and this is equivalent to χ_h being a power of det. On the other hand, regarding G_D^0 as a subgroup of $\mathrm{Gl}_n(\mathbb{C})$, both det and χ_h are polynomials of degree n, so they must be equal. □

If D is a linear free divisor with reductive group G_D^0 and reduced homogeneous equation h then by Mather's lemma ([Mat69, lemma 3.1]) the fibre $D_t := h^{-1}(t)$, $t \neq 0$, is a single orbit of the group $\ker(\chi_h)$. It follows that D_t is a finite quotient of $\ker(\chi_h)$ since $\dim(D_t) = \dim(\ker(\chi_h))$ and the action is algebraic. Hence D_t has cohomology isomorphic to $H^*(\ker(\chi_h), \mathbb{C})$. Now $\ker(\chi_h)$ is reductive - its Lie algebra $\mathfrak{g}_h$ has the same semi-simple part as $\mathfrak{g}_D$, and a centre one dimension smaller than that of $\mathfrak{g}_D$. Thus, $\ker(\chi_h)$ has a compact $n-1$-dimensional Lie group K_h as deformation retract. Poincaré duality for K_h implies a duality on the cohomology of $\ker(\chi_h)$, and this duality carries over to $H^*(D_t; \mathbb{C})$. How is this reflected in the cohomology of the complex $\Gamma(V, \Omega^\bullet(\log h))$ of global algebraic logarithmic forms? Notice that evidently $H^0(\Gamma(V, \Omega^\bullet(\log h)) = \mathbb{C}[h]$, since the kernel of d_h consists precisely of functions constant along the fibres of h. It is considerably less obvious that $H^{n-1}(\Gamma(V, \Omega^\bullet(\log h)))$ should be isomorphic to $\mathbb{C}[h]$, for this cohomology group is naturally a quotient, rather than a subspace, of $\mathbb{C}[V]$. We prove it (in Theorem 2.7 below) by showing that thanks to the reductiveness of G_D^0, it follows from a classical theorem of Hochschild and Serre ([HS53, Theorem 10]) on the cohomology of Lie algebras. From Theorem 2.7 we then deduce that every reductive linear free divisor is special.

We write $\Omega_\bullet(\log h)_m$ for the graded part of $\Gamma(V, \Omega^\bullet(\log h))$ of weight m.

Theorem 2.7. *Let $D \subset \mathbb{C}^n$ be a reductive linear free divisor with homogeneous equation h. There is a natural graded isomorphism*

$$H^*(\Omega^\bullet(\log h)_0) \otimes_{\mathbb{C}} \mathbb{C}[h] \to H^*(\Gamma(V, \Omega^\bullet(\log h))).$$

In particular, $H^*(\Gamma(V, \Omega^\bullet(\log h)))$ is a free $\mathbb{C}[h]$-module.

Proof. The complex $\Omega^\bullet(\log h)_m$ is naturally identified with the complex $\bigwedge^\bullet(\mathfrak{g}_h; \mathrm{Sym}^m(V^\vee))$ whose cohomology is the Lie algebra cohomology of $\mathfrak{g}_h$ with coefficients in the representation $\mathrm{Sym}^m(V^\vee)$, $H^*(\mathfrak{g}_h; \mathrm{Sym}^m(V^\vee))$. For (as vector spaces)

$$\Omega^k(\log h)_m = \Omega^k(\log h)_0 \otimes_{\mathbb{C}} \mathrm{Sym}^m(V^\vee) = \left(\bigwedge^k \mathfrak{g}_h^\vee\right) \otimes_{\mathbb{C}} \mathrm{Sym}^m\left(V^\vee\right) = \bigwedge^k \left(\mathfrak{g}_h^\vee \otimes_{\mathbb{C}} \mathrm{Sym}^m\left(V^\vee\right)\right),$$

and inspection of the formulae for the differentials in the two complexes shows that they are the same under this identification[1]. The representation of $\mathfrak{g}_h$ in $\mathrm{Sym}^k(V^\vee)$ is semi-simple (completely reducible), since $\mathfrak{g}_h$ is a reductive Lie algebra and every finite dimensional complex representation of a reductive Lie algebra is semisimple. By a classical theorem of Hochschild and Serre ([HS53, Theorem 10]), if M is a semi-simple representation of a finite-dimensional complex reductive Lie algebra $\mathfrak{g}$, then

$$H^*(\mathfrak{g}; M) = H^*(\mathfrak{g}; M^0),$$

where M^0 is the submodule of M on which $\mathfrak{g}$ acts trivially. Evidently we have $H^*(\mathfrak{g}; M^0) = H^*(\mathfrak{g}; \mathbb{C}) \otimes_{\mathbb{C}} M^0$. Now

$$\mathrm{Sym}^m(V^\vee)^0 = \begin{cases} \mathbb{C} \cdot h^\ell & \text{if } m = \ell n \\ 0 & \text{otherwise} \end{cases}$$

by the uniqueness, up to scalar multiple, of the semi-invariant with a given character on a prehomogeneous vector space, referred to in the proof of 3.8. It follows that

$$\begin{aligned} H^k(\Gamma(V, \Omega^\bullet(\log h))) &= \bigoplus_m H^k(\Omega^\bullet(\log h)_m) = \bigoplus_m H^k(\mathfrak{g}_h; \mathrm{Sym}^m(V^\vee)) \\ &= \bigoplus_\ell H^*(\mathfrak{g}_h; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C} \cdot h^\ell = H^*(\Omega^\bullet(\log h)_0) \otimes_{\mathbb{C}} \mathbb{C}[h]. \end{aligned}$$

□

Corollary 2.8. *There is a $\mathbb{C}[h]$-perfect pairing*

$$\begin{aligned} H^k((\Gamma(V, \Omega^\bullet(\log h))) \times H^{n-k-1}((\Gamma(V, \Omega^\bullet(\log h))) &\longrightarrow H^{n-1}((\Gamma(V, \Omega^\bullet(\log h))) \simeq \mathbb{C}[h] \\ ([\omega_1], [\omega_2]) &\longmapsto [\omega_1 \wedge \omega_2]. \end{aligned}$$

[1] This identification for the case $m = 0$ was already made in [GMNS06], where it gave a proof of the global logarithmic comparison theorem for reductive linear free divisors.

Proof. The pairing is evidently well defined. Poincaré Duality on the compact deformation retract K_h of $\ker(\chi_h)$ gives rise to a perfect pairing

$$H^k(D_t) \times H^{n-k-1}(D_t) \to H^{n-1}(D_t),$$

Now

$$H^k(D_t) = H^k\big(\Gamma(V, \Omega^\bullet(\log h) \otimes_{\mathbb{C}[h]} \mathbb{C}[h]/(h-t))\big)$$

by the affine de Rham theorem, since $\Omega^k(\log h)/(h-t) = \Omega^k_{D_t}$. In view of 2.7, the perfect pairing on $H^*(D_t)$ lifts to a $\mathbb{C}[h]$-perfect pairing on $H^*\big(\Gamma(V, \Omega^\bullet(\log h))\big)$. □

Corollary 2.9. *A linear free divisor with reductive group is special.*

Proof. By what was said before, $H^{n-1}(\ker(\chi_h), \mathbb{C})$ is isomorphic to $H^{n-1}(\Omega^\bullet(\log\, h)_0)$, so Poincaré duality for $\ker \chi_h$ implies that the class of α in $H^{n-1}(\Omega^\bullet(\log h)_0)$ is non-zero. Recall from the proof of lemma 2.6 that if $\lambda = \iota_\xi \alpha = \iota_\xi \iota_E(\mathrm{vol}/h)$ with $\xi \in \mathrm{Der}(-\log h)_0$, then $d\lambda = c\alpha$ for some $c \in \mathbb{C}$ in $\Omega^\bullet(\log h)_0$. As the class of α is non-zero, this forces $d\lambda$ to be zero. The conclusion follows from 2.6. □

3 Functions on Linear Free Divisors and their Milnor Fibrations

3.1 Right-left stable functions on divisors

Let h and f be homogenous polynomials in n variables, where the degree of h is n. As before, we write $D = h^{-1}(0)$ and $D_t = h^{-1}(t)$ for $t \neq 0$. However, we do not assume in this subsection that D is a free divisor. We call $f_{|D_t}$ a *Morse function* if all its critical points are non-degenerate and all its critical values are distinct.

Lemma 3.1. *$f_{|D_t}$ is a Morse function if and only if $\mathbb{C}[D_t]/J_f$ is generated over $\mathbb{C}$ by the powers of f.*

Proof. Suppose $f_{|D_t}$ is a Morse function, with critical points $p_1, \ldots, p_N$. Since any quotient of $\mathbb{C}[D_t]$ with finite support is a product of its localisations, we have

$$\mathbb{C}[D_t]/J_f \simeq \oplus_{j=1}^N \mathcal{O}_{D_t,p_j}/J_f = \oplus_{j=1}^N \mathbb{C}_{p_j}.$$

The image in $\oplus_{k=j}^N \mathbb{C}_{p_j}$ of f^k is the vector $\big(f(p_1)^k, \ldots, f(p_N)^k\big)$. These vectors, for $0 \leq k \leq N-1$, make up the Vandermonde determinant, which is non-zero because the $f(p_j)$ are pairwise distinct. Hence they span $\oplus_{j=1}^N \mathbb{C}_{p_j}$.

Conversely, if $1, f, \ldots, f^N$ span $\mathbb{C}[D_t]/J_f$ then the powers of f span each local ring $\mathcal{O}_{D_t,p_j}/J_f$. This implies that there is an $\mathscr{R}_e$-versal deformation of the singularity of $f_{|D_t}$ at p_j of the form $F(x,u) = g_u \circ f(x)$. In particular, the critical point of $f_{|D_t}$ at p_j does not split, and so must be non-degenerate. Now choose a minimal R such that $1, f, \ldots, f^{R-1}$ span $\mathbb{C}[D_t]/J_f$. Since all the critical points are non-degenerate, projection of $\mathbb{C}[D_t]/J_f$ to the product of its local rings shows that the matrix $M := [f^{k-1}(p_j)]_{1 \leq k \leq R, 1 \leq j \leq N}$ has rank N. But if $f(p_i) = f(p_j)$ for some $i \neq j$ then M has two equal columns. So the critical values of f must be pairwise distinct. □

If (X, x) is a germ of complex variety, an analytic map-germ $f : (X, x) \to (\mathbb{C}^p, 0)$ is *right-left stable* if every germ of deformation $F : (X \times \mathbb{C}, (x, 0)) \to (\mathbb{C}^p \times \mathbb{C}, (0, 0))$ can be trivialised by suitable parametrised families of bi-analytic diffeomorphisms of source and target. A necessary and sufficient condition for right-left stability is infinitesimal right-left stability: $df(\theta_{X,0}) + f^{-1}(\theta_{\mathbb{C}^p,0}) = \theta(f)$, where $\theta_{X,0}$ is the space of germs of vector fields on X and $\theta(f)$ is the space of infinitesimal deformations of f (freely generated over $\mathcal{O}_{X,0}$ by $\partial/\partial y_1, \ldots, \partial/\partial y_p$, where $y_1, \ldots, y_p$ are coordinates on $\mathbb{C}^p$). When $p = 1$, $\theta(f) \simeq \mathcal{O}_{X,0}$ and $f^{-1}(\theta_{\mathbb{C},0}) \simeq \mathbb{C}\{f\}$. Note also that if $X \subset \mathbb{C}^n$ then $\theta_{X,0}$ is the image of $\mathrm{Der}(-\log X)_0$ under the restriction of $\theta_{\mathbb{C}^n}$ to X.

Proposition 3.2. *If $f_{|D} : D \to \mathbb{C}$ has a right-left stable singularity at 0 then $f_{|D_t}$ is a Morse function, or non-singular.*

Proof. $f_{|D}$ has a stable singularity at 0 if and only if the image in $\mathcal{O}_{D,0}$ of $df(\mathrm{Der}(-\log D)) + \mathbb{C}\{f\}$ is all of $\mathcal{O}_{D,0}$. Write $\mathfrak{m} := \mathfrak{m}_{\mathbb{C}^n,0}$. Since $df(\chi_E) = f$, stability implies

$$df(\mathrm{Der}(-\log h)) + (f) + (h) = \mathfrak{m}. \tag{3.1}$$

As $\deg(h) > 1$, we deduce that

$$\mathfrak{m}/\big(df(\mathrm{Der}(-\log h)) + \mathfrak{m}^2\big) = \langle f \rangle_{\mathbb{C}}.$$

It follows that for all $k \in \mathbb{N}$,

$$\left(\mathfrak{m}^k + df(\mathrm{Der}(-\log h))\right)/\left(df(\mathrm{Der}(-\log h)) + \mathfrak{m}^{k+1}\right) = \langle f^k \rangle_{\mathbb{C}}$$

and thus that

$$df(\mathrm{Der}(-\log h)) + \mathbb{C}\{f\} = \mathcal{O}_{\mathbb{C}^n,0}\,. \tag{3.2}$$

Now (3.1) implies that $V(df(\mathrm{Der}(-\log h)))$ is either a line or a point. Call it L_f. If $L_f \not\subseteq D$, then the sheaf $h_*\left(\mathcal{O}_{\mathbb{C}^n}/df(\mathrm{Der}(-\log h))\right)$ is finite over $\mathcal{O}_{\mathbb{C}}$, and (3.2) shows that its stalk at 0 is generated by $1, f, \ldots, f^R$ for some finite R. Hence these same sections generate $h_*\left(\mathcal{O}_{\mathbb{C}^n}/df(\mathrm{Der}(-\log h))\right)_t$ for t near 0, and therefore for all t, by homogeneity. As $h_*\left(\mathcal{O}_{\mathbb{C}^n}/df(\mathrm{Der}(-\log h))\right)_t = \mathbb{C}[D_t]/J_f$, by 3.1 $f_{|D_t}$ is a Morse function.
On the other hand, if $L_f \subset D$, then $f : D_t \to \mathbb{C}$ is non-singular. □

We do not know of any example where the latter alternative holds.

Proposition 3.3. *If $f : (D,0) \to (\mathbb{C},0)$ is right-left stable then f is linear and $\mathrm{Der}(-\log D)_0$ must contain at least n linearly independent weight zero vector fields. In particular, the only free divisors supporting right-left stable functions are linear free divisors.*

Proof. From equation (3.1) it is obvious that f must be linear, and that $Der(-\log h)$ must contain at least $n-1$ independent weight zero vector fields; these, together with the Euler field, make n in $\mathrm{Der}(-\log D)$. □

We note that the hypothesis of the proposition is fulfilled by a generic linear function on the hypersurface defined by $\sum_j x_j^2 = 0$, which is not a free divisor.

3.2 $\mathscr{R}_D$ and $\mathscr{R}_h$-equivalence of functions on divisors

Let $D \subset \mathbb{C}^n$ be a weighted homogeneous free divisor and let h be its weighted homogeneous equation. We consider functions $f : \mathbb{C}^n \to \mathbb{C}$ and their restrictions to the fibres of h. The natural equivalence relation to impose on functions on D is $\mathscr{R}_D$-equivalence: right-equivalence with respect to the group of bianalytic diffeomorphisms of $\mathbb{C}^n$ which preserve D. However, as we are interested also in the behaviour of f on the other fibres of h we consider also *fibred right-equivalence* with respect to the function $h : (\mathbb{C}^n, 0) \to \mathbb{C}$. That is, right-equivalence under the action of the group $\mathscr{R}_h$ consisting of germs of bianalytic diffeomorphisms $\varphi : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $h \circ \varphi = h$. A standard calculation shows that the tangent spaces to the $\mathscr{R}_D$ and $\mathscr{R}_h$-orbits of f are equal to $df(\mathrm{Der}(-\log D))$ and $df(\mathrm{Der}(-\log h))$ respectively. We define

$$T^1_{\mathscr{R}_D} f := \frac{\mathcal{O}_{\mathbb{C}^n,0}}{df(\mathrm{Der}(-\log D))}$$
$$T^1_{\mathscr{R}_h} f := \frac{\mathcal{O}_{\mathbb{C}^n,0}}{df(\mathrm{Der}(-\log h)) + (h)}$$
$$T^1_{\mathscr{R}_h/\mathbb{C}} f := \frac{\mathcal{O}_{\mathbb{C}^n,0}}{df(\mathrm{Der}(-\log h))}$$

and say that f is $\mathscr{R}_D$-finite or $\mathscr{R}_h$-finite if $\dim_{\mathbb{C}} T^1_{\mathscr{R}_D} f < \infty$ or $\dim_{\mathbb{C}} T^1_{\mathscr{R}_h} f < \infty$ respectively. Note that it is only in the definition of $T^1_{\mathscr{R}_h} f$ that we explicitly restrict to the hypersurface D.
We remark that a closely related notion called ${}_D\mathscr{K}$-equivalence is studied by Damon in [Dam06].

Proposition 3.4. *If the germ $f \in \mathcal{O}_{\mathbb{C}^n,0}$ is $\mathscr{R}_h$-finite then there exist $\varepsilon > 0$ and $\eta > 0$ such that for $t \in \mathbb{C}$ with $|t| < \eta$,*

$$\sum_{x \in D_t \cap B_\varepsilon} \mu(f_{|D_t}; x) = \dim_{\mathbb{C}} T^1_{\mathscr{R}_h} f.$$

If f is weighted homogeneous (with respect to the same weights as h) then ε and η may be taken to be infinite.

Proof. Let $\xi_1, \ldots \xi_{n-1}$ be an $\mathcal{O}_{\mathbb{C}^n,0}$-basis for $\mathrm{Der}(-\log h)$. The $\mathscr{R}_h$-finiteness of f implies that the functions $df(\xi_1), \ldots, df(\xi_{n-1})$ is a regular sequence in $\mathcal{O}_{\mathbb{C}^n,0}$, so that $T^1_{\mathscr{R}_h/\mathbb{C}} f$ is a complete intersection ring, and in particular Cohen-Macaulay, of dimension 1. The condition of $\mathscr{R}_h$-finiteness is equivalent to $T^1_{\mathscr{R}_h/\mathbb{C}} f$ being finite over $\mathcal{O}_{\mathbb{C},0}$. It follows that it is locally free over $\mathcal{O}_{\mathbb{C},0}$. □

Now suppose that $D \subset \mathbb{C}^n = V$ is a linear free divisor. We denote the dual space $\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ by $V^\vee$. The group G_D^0 acts on $V^\vee$ by the contragredient action $\rho^\vee$ in which

$$g \cdot f = f \circ \rho(g)^{-1}.$$

If we write the elements of $V^\vee \simeq \mathbb{C}^n$ as column vectors, then the representation $\rho^\vee$ takes the form $\rho^\vee(g) = {}^t\rho(g)^{-1}$. and the infinitesimal action takes the form $d\rho^\vee(A) = -{}^tA$. Let $A_1, \ldots, A_n$ be a basis for $\mathfrak{g}_D$. Then the vector fields

$$\xi_i(x) = (\partial/\partial x_1, \ldots, \partial/\partial x_n)A_i x, \quad \text{for } i = 1, \ldots, n \tag{3.3}$$

form an $\mathcal{O}_{\mathbb{C}^n}$ basis for $\mathrm{Der}(-\log D)$, and the determinant of the $n \times n$ matrix of their coefficients is a non-zero scalar multiple of h, by Saito's criterion. The vector fields

$$\xi_i(y) = (\partial/\partial y_1, \ldots, \partial/\partial y_n)({}^tA_i)y, \quad \text{for } i = 1, \ldots, n \tag{3.4}$$

generate the infinitesimal action of $\mathfrak{g}_D$ on $V^\vee$. We denote by $h^\vee$ the determinant of the $n \times n$ matrix of their coefficients. Its zero-locus is the complement of the open orbit of G_D^0 on $V^\vee$. In general $\rho^\vee$ and ρ are not equivalent representations. Indeed, it is not always the case that $(G_D^0, \rho^\vee, V^\vee)$ is a prehomogeneous vector space. We describe an example where this occurs in 3.6 below.
Suppose $f \in V^\vee$. Let $L_f = \mathrm{supp}\, T^1_{\mathscr{R}_h/\mathbb{C}} f$. Since $\mathrm{Der}(-\log h)$ is generated by weight zero vector fields, L_f is a linear subspace of V.

Proposition 3.5. *Let $f \in V^\vee$. Then*

(i) L_f is a line transverse to $f^{-1}(0)$ $\Leftrightarrow$ f is $\mathscr{R}_D$-finite $\Leftrightarrow$ the G_D^0-orbit of f in the representation $\rho^\vee$ is open.

(ii) Set $T_pD_t = \{f = 0\}$. Then

$$H(p) \neq 0 \quad \Longrightarrow \quad \mu(f_{|D_t}; p) = 1 \tag{3.5}$$

where H is the Hessian determinant of h.

(iii) If f is $\mathscr{R}_h$-finite then

(a) f is $\mathscr{R}_D$- finite;

(b) the classes of $1, f, \ldots, f^{n-1}$ form a $\mathbb{C}$-basis for $T^1_{\mathscr{R}_h} f$;

(c) on each Milnor fibre $D_t := h^{-1}(t), t \neq 0$, f has n non-degenerate critical points, which form an orbit under the diagonal action of the group of n-th roots of unity on $\mathbb{C}^n$.

Proof. (i) The first equivalence holds simply because

$$df(\mathrm{Der}(-\log D)) = df(\mathrm{Der}(-\log h)) + (df(E)) = df(\mathrm{Der}(-\log h)) + (f).$$

For the second equivalence, observe that the tangent space to the G_D^0-orbit of f is naturally identified with $df(\mathrm{Der}(-\log D)) \subset \mathfrak{m}_{V,0}/\mathfrak{m}_{V,0}^2 = V^\vee$. For given $A \in \mathfrak{g}_D$, we have

$$\left(\frac{d}{dt}\exp(tA) \cdot f\right)_{|t=0}(x) = df\left(\frac{d}{dt}\exp(-tA) \cdot x\right)_{|t=0} = -df(\xi_A) \tag{3.6}$$

where ξ_A is the vector field on V arising from A under the the infinitesimal action of ρ. Because $\mathrm{Der}(-\log D)$ is generated by vector fields of weight zero, $df(\mathrm{Der}(-\log D))$ is generated by linear forms, and so f is $\mathscr{R}_D$-finite if and only if $df(\mathrm{Der}(-\log D)) \supset m_{V,0}$.
(ii) is well-known. To prove it, parametrise D_t around p by $\varphi : (\mathbb{C}^{n-1}, 0) \to (D_t, p)$. Then because f is linear we have

$$\frac{\partial^2(f \circ \varphi)}{\partial u_i \partial u_j} = \sum_s \frac{\partial f}{\partial x_s}\frac{\partial^2 \varphi_s}{\partial u_i \partial u_j}. \tag{3.7}$$

Because $h \circ \varphi$ is constant, we find that

$$0 = \sum_{s,t} \frac{\partial^2 h}{\partial x_s \partial x_t}\frac{\partial \varphi_s}{\partial u_i}\frac{\partial \varphi_t}{\partial u_j} + \sum_s \frac{\partial h}{\partial x_s}\frac{\partial^2 \varphi_s}{\partial u_i \partial u_j}. \tag{3.8}$$

Because $T_pD_t = \{f = 0\}$, d_ph is a scalar multiple of $d_pf = f$. From this, equations (3.7) and (3.8) give an equality (up to non-zero scalar multiple) of $n \times n$ matrices,

$$\left[\frac{\partial^2(f\circ\varphi)}{\partial u_i \partial u_j}\right] = {}^t\left[\frac{\partial \varphi_s}{\partial u_i}\right]\left[\frac{\partial^2 h}{\partial x_s \partial x_t}\circ\varphi\right]\left[\frac{\partial \varphi_t}{\partial u_j}\right] \tag{3.9}$$

It follows that if $H \neq 0$ then the restriction of f to D_t has a non-degenerate critical point at p.

(iii)(a) If f is $\mathscr{R}_h$-finite then L_f must be a line intersecting D only at 0. If $\mathscr{R}_D$ finiteness of f fails, then $L_f \subset \{f = 0\}$, and f is constant along L_f. But at all points $p \in L_f$, $\ker d_pf \subset \ker d_ph$, so h also is constant along L_f.

(iii)(b) As L_f is a line and $\mathcal{O}_V/df(\mathrm{Der}(-\log h)) = \mathcal{O}_{L_f}$, $h_{|L_f}$ is necessarily the n'th power of a generator of $m_{L_f,0}$. It follows that $T^1_{\mathscr{R}_h} f$ is generated by the first n non-negative powers of any linear form whose zero locus is transverse to the line L_f.

(iii)(c) Since f is $\mathscr{R}_D$ finite, L_f is a line transverse to $\{f = 0\}$. The critical points of $f_{|D_t}$ are those points $p \in D_t$ where $T_pD_t = \{f = 0\}$; thus $L_f \pitchfork D_t$ at each critical point. In $\mathcal{O}_{D_t}$, the ideals $df(\mathrm{Der}(-\log h))$ and $J_{f_{|D_t}}$ coincide. Thus the intersection number of L_f with D_t at p, which we already know is equal to 1, is also equal to the Milnor number of $f_{|D_t}$ at p. The fact that there are n critical points, counting multiplicity, is just the fundamental theorem of algebra, applied to the single-variable polynomial $(h-t)_{|L_f}$. The fact that these n points form an orbit under the diagonal action of the group $\mathbb{G}_n$ of n-th roots of unity is a consequence simply of the fact that h is $\mathbb{G}_n$-invariant and L_f is preserved by the action. □

For homogeneous functions f of higher degree, the implications of 3.5(iii) hold only with the additional assumption that f has an isolated singularity at $0 \in \mathbb{C}^n$.

If D is a linear free divisor, there may be no $\mathscr{R}_h$-finite linear forms, or even no $\mathscr{R}_D$ finite linear forms, as the following examples shows.

Example 3.6. Let D be the free divisor in the space V of 2×5 complex matrices defined by the vanishing of the product of the 2×2 minors $m_{12}, m_{13}, m_{23}, m_{34}$ and m_{35}. Then D is a linear free divisor ([GMNS06, Example 5.7(2)]), but $\rho^\vee$ has no open orbit in $V^\vee$: it is easily checked that $h^\vee = 0$. It follows by 3.5 (i)that no linear function $f \in V^\vee$ is $\mathscr{R}_D$-finite, and so by 3.5(iii) that none is R_h-finite.

In Example 3.6, the group G_D^0 is not reductive. Results of Sato and Kimura in [SK77, §4] show that if G_D^0 is reductive then $(G_D^0, \rho^\vee, V^\vee)$ is prehomogeneous, so that almost all $f \in V^\vee$ are $\mathscr{R}_D$-finite, and moreover imply that all f in the open orbit in $V^\vee$ are $\mathscr{R}_h$-finite. We briefly review their results. First, the complement of the open orbit in $V^\vee$ is a divisor whose equation, in suitable coordinates x on V, and dual coordinates y on $V^\vee$, is of the form $h^*(y) = \overline{h(\overline{y})}$. The coordinates in question are chosen as follows: as G_D^0 is reductive, it has a Zariski dense compact subgroup K. In suitable coordinates on $V = \mathbb{C}^n$ the representation ρ places K inside $U(n)$. Call such a coordinate system *unitary.* From this it follows that if f is any rational semi-invariant on V with associated character χ then the function $f^* : V^\vee \to \mathbb{C}$ defined by $f^*(y) = \overline{f(\overline{y})}$ is also a semi-invariant *for the representation of* K with associated character $\bar{\chi}$, which is equal to χ^{-1} since $\chi(K) \subset S^1$ by compactness. Note that f^* cannot be the zero polynomial. As K is Zariski-dense in G_D^0, the rational equality

$$f^*(\rho^\vee(g)y) = \frac{1}{\chi(g)} f^*(y)$$

holds for all $g \in G_D^0$.

Proposition 3.7. *Let $D \subset \mathbb{C}^n$ be a linear free divisor with equation h. If G_D^0 is reductive then*

(i) $(G_D^0, \rho^\vee, V^\vee)$ is a prehomogeneous vector space.

(ii) $D^\vee$, the complement of the open orbit in $V^\vee$, has equation h^*, with respect to dual unitary coordinates on $V^\vee$.

(iii) $D^\vee$ is a linear free divisor.

Proof. As $\mathbb{C}$-basis of the Lie algebra $\mathfrak{g}_D$ of G_D^0 we can take a real basis of the Lie algebra of K. With respect to unitary coordinates, ρ represents K in $U(n)$, so $d\rho(\mathfrak{g}_D) \subset \mathfrak{gl}_n(\mathbb{C})$ has $\mathbb{C}$-basis $A_1, \ldots, A_n$ such that $A_i \in \mathfrak{u}_n$, i.e. ${}^tA_i = -\bar{A}_i$, for $i = 1, \ldots, n$. It follows that the determinant of the matrix of coefficients of the matrix (3.4) above is equal to h^*, and in particular is not zero. This proves (i) and (ii).

That $D^\vee$ is free follows from Saito's criterion ([Sai80]): the n vector fields (3.4) are logarithmic with respect to $D^\vee$, and h^*, the determinant of their matrix of coefficients, is not identically zero, and indeed is a reduced equation for $D^\vee$ because h is reduced. □

We now prove the main result of this section. In order to make the argument clear, we postpone some steps in the proof to a sequence of lemmas, 3.9, 3.10 and 3.11, which we prove immediately afterwards.

Theorem 3.8. *If G^0_D is reductive then $f \in V^\vee$ is $\mathscr{R}_h$-finite if and only if it is $\mathscr{R}_D$-finite. In particular, f is $\mathscr{R}_h$-finite if and only if $f \in V^\vee \backslash D^*$.*

Proof. Let $p \in D_t$ (for $t \neq 0$) and suppose that T_pD_t has equation f, i.e. that $\nabla h(p)$ is a non-zero multiple of f. We claim that f is $\mathscr{R}_h$-finite. For by Lemma (3.10) below, $H(p) \neq 0$, and it follows by 3.5(ii) that the restriction of f to D_t has a non-degenerate critical point at p. The critical locus of $f_{|D_t}$ is precisely $L_f \cap D_t$; so L_f must be a line (recall that it is a linear subspace of V), and must meet D_t transversely at p. By the homogeneity of D, it follows that $L_f \cap D = \{0\}$, so f is $\mathscr{R}_h$-finite. Thus

$$f\ \mathscr{R}_D\text{-finite} \quad \overset{3.5}{\Longrightarrow} \quad f \in V^\vee \smallsetminus D^* \quad \overset{3.11}{\Longrightarrow} \quad f = \nabla h(p) \text{ for some } p \notin D \quad \Longrightarrow \quad f\ \mathscr{R}_h\text{-finite.}$$

We have already proved the opposite implication, in 3.5. □

Lemma 3.9. *Let $D \subset \mathbb{C}^n$ be a linear free divisor with homogeneous equation h, let $h^\vee$ be the determinant of the matrix of coefficients of (3.4), and let H be the Hessian determinant of h. Then*

$$h^\vee\left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right) = (n-1)Hh.$$

Proof. Choose the basis $A_1 = I_n, \ldots A_n$ for $\mathfrak{gl}_D$ so that the asociated vector fields $\xi_2, \ldots, \xi_n$ are in $\mathrm{Der}(-\log h)$. The matrix I gives rise to the Euler vector field E. Write $A_i = [a^k_{ij}]$, with the upper index k referring to columns and the lower index j referring to rows. Let $\alpha_{ji} = \sum_k a^k_{ij} x_k$ denote the coefficient of $\partial/\partial x_j$ in ξ_i for $i = 2, \ldots, n-1$. Then

$$0 = dh(\xi_i) = \sum_j \alpha_{ji} \frac{\partial h}{\partial x_j},$$

so differentiating with respect to x_k,

$$0 = \sum_j \frac{\partial \alpha_{ji}}{\partial x_k}\frac{\partial h}{\partial x_j} + \sum_j \alpha_{ji}\frac{\partial^2 h}{\partial x_k \partial x_j} = \sum_j a^k_{ij}\frac{\partial h}{\partial x_j} + \sum_j \alpha_{ji}\frac{\partial^2 h}{\partial x_k \partial x_j}. \tag{3.10}$$

For the Euler field ξ_1 we have

$$nh = dh(E) = \sum_j \alpha_{j1}\frac{\partial h}{\partial x_j}$$

so

$$n\frac{\partial h}{\partial x_k} = \sum_j a^k_{1j}\frac{\partial h}{\partial x_j} + \sum_j \alpha_{j1}\frac{\partial^2 h}{\partial x_k \partial x_j} = \frac{\partial h}{\partial x_k} + \sum_j \alpha_{j1}\frac{\partial^2 h}{\partial x_k \partial x_j}. \tag{3.11}$$

Putting the n equations (3.10) and (3.11) together in matrix form we get

$$\begin{bmatrix} {}^tE \\ {}^t\xi_1 \\ \cdot \\ {}^t\xi_{n-1} \end{bmatrix} \left[\frac{\partial^2 h}{\partial x_k \partial x_j}\right] = -\begin{bmatrix} (n-1)\nabla h \cdot E \\ \nabla h \cdot A_1 \\ \cdot \\ \nabla h \cdot A_{n-1} \end{bmatrix}.$$

Now take determinants of both sides. The determinant on the right hand side is

$$(n-1)h^\vee\left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right).$$

The determinants of the two matrices on the left are, respectively, h and H. □

Lemma 3.10. ([SK77]) *If D is a reductive linear free divisor, then for all $p \in \mathbb{C}^n$*

$$h(p) \neq 0 \quad \Longrightarrow \quad H(p) \neq 0.$$

Proof. In [SK77, Page 72], Sato and Kimura show that if g is a homogeneous rational semi-invariant of degree r with associated character χ_g then there is a polynomial $b(m)$ of degree r (the b-function of g) such that, with respect to unitary coordinates on $\mathbb{C}^n$,

$$g^*\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)\cdot g^m = b(m)g^{m-1} \tag{3.12}$$

This is proved by showing that the left hand side is a semi-invariant with associated character χ_g^{m-1}, and noting that the semi-invariant corresponding to a given character is unique up to scalar multiple, since the quotient of two semi-invariants with the same character is an absolute invariant, and therefore must be constant (since G_D^0 has a dense orbit). From this it follows ([SK77, page 72]) that

$$g^*\left(\frac{\partial g}{\partial x_1},\dots,\frac{\partial g}{\partial x_n}\right) = b_0 g^{r-1}, \tag{3.13}$$

where b_0 is the (non-zero) leading coefficient of the polynomial $b(m)$, and hence, by 3.9, that

$$(n-1)H = b_0 h^{n-2}. \tag{3.14}$$

□

Proposition 3.11. *If D is a linear free divisor with reductive group G_D^0 and homogeneous equation h with respect to unitary coordinates, then*

(i) the gradient map ∇h maps the fibres $D_t, t \neq 0$ of h diffeomorphically to the fibres of h^*.

(ii) the gradient map ∇h^* maps Milnor fibres of h^* diffeomorphically to Milnor fibres of h.

Proof. The formula (3.13) shows that ∇h maps fibres of h into fibres of h^*. Each fibre of h is a single orbit of the kernel of $\chi_h : G_D^0 \to \mathbb{C}^*$, and each fibre of h^* is a single orbit of the kernel of χ_{h^*}. These two subgroups coincide because $\chi_{h^*} = (\chi_h)^{-1}$. The map is equivariant: $\nabla h(\rho(g)x) = \rho^*(g)^{-1}\nabla h(x)$. It follows that ∇h maps D_t surjectively onto a fibre of h^*. By (3.10), this mapping is a local diffeomorphism. It is easy to check that it is 1-1. Since $(h^*)^* = h$ and dual unitary coordinates are themselves unitary, the same argument, interchanging the roles of h and h^*, gives (ii). □

Question 3.12. If we drop the condition that D be a *linear* free divisor, what condition could replace reductivity to guarantee that for (linear) functions $f \in \mathcal{O}_{\mathbb{C}^n}$, $\mathscr{R}_D$-finiteness implies $\mathscr{R}_h$ finiteness?

Remark 3.13. The following will be used in the proof of lemma 3.19. Let $AT_xD_t := x + T_xD_t$ denote the affine tangent space at x. Lemma 3.11 implies that the affine part $D_t^\vee = \{AT_xD_t : x \in D_t\}$ of the projective dual of D_t is a Milnor fibre of h^*. For AT_xD_t is the set

$$(y_1,\dots,y_n) \in \mathbb{C}^n : d_xh(y_1,\dots,y_n) = d_xh(x_1,\dots,x_n);$$

by homogeneity of h the right hand side is just nt, and thus in dual projective coordinates AT_xD_t is the point $(-nt : \partial h/\partial x_1(x) : \cdots : \partial h/\partial x_n(x))$. In affine coordinates on U_0, this is the point

$$\left(\frac{-1}{nt}\frac{\partial h}{\partial x_1}(x),\cdots,\frac{-1}{nt}\frac{\partial h}{\partial x_n}(x)\right).$$

By (3.13), the function h^* takes the value $b_0t^{n-1}/(nt)^n = b_0/n^nt$ at this point, independent of $x \in D_t$, and so $D_t^\vee \subset (h^*)^{-1}(b_0/n^nt)$. The opposite inclusion holds by openness of the map ∇h, which, in turn, follows from Proposition 3.9.

3.3 Tameness

In this subsection, we study a property of the polynomial functions $f_{|D_t}$ known as tameness. It describes the topological behavior of f at infinity, and is needed in order to use the general results from [Sab06] and [DS03] on the Gauß-Manin system and the construction of Frobenius structures. In fact we discuss two versions, cohomological tameness and M-tameness. Whereas the first will be seen to hold for all $\mathcal{R}_h$-finite linear functions on a linear free divisor D, we show M-tameness only if D is reductive. Cohomological tameness is all that is needed in our later construction of Frobenius manifolds, but we feel that the more evidently geometrical condition of M-tameness is of independent interest.

Definition 3.14 ([Sab06])**.** Let X be an affine algebraic variety and $f : X \to \mathbb{C}$ a regular function. Then f is called cohomologically tame if there is a partial compactification $X \overset{j}{\hookrightarrow} Y$ with Y quasiprojective, and a proper regular function $F : Y \to \mathbb{C}$ extending f, such that for any $c \in \mathbb{C}$, the complex $\varphi_{F-c}(\mathbb{R}j_*\mathbb{Q}_X)$ is supported in a finite number of points, which are contained in X. Here φ is the functor of vanishing cycles of Deligne, see, e.g., [Dim04].

It follows in particular that a cohomologically tame function f has at most isolated critical points.

Proposition 3.15. *Let $D \subset V$ be linear free and $f \in \mathbb{C}[V]_1$ be an $\mathcal{R}_h$-finite linear section. Then the restriction of f to $D_t := h^{-t}(t)$ is cohomologically tame.*

Proof. A similar statement is actually given without proof in [NS99] as an example of a so-called weakly tame function. We consider the standard graph compactification of f: $\overline{\Gamma}(f)$ is the closure of the graph $\Gamma(f) \subset D_t \times \mathbb{C}$ of f in $\overline{D}_t \times \mathbb{C}$ (where $\overline{D}_t$ is the projective closure of D_t in $\mathbb{P}^n$), we identify f with the projection $\Gamma(f) \to \mathbb{C}$, and extend f to the projection $F : \overline{\Gamma(f)} \to \mathbb{C}$. Refine the canonical Whitney stratification of $\overline{D_t}$ by dividing the open stratum, which consists of $D_t \cup (B_h)_{\mathrm{reg}}$, into the two strata D_t and $(B_h)_{\mathrm{reg}}$. Evidently this new stratification $\mathscr{S}$ is still Whitney regular. From $\mathscr{S}$ we obtain a Whitney stratification $\mathscr{S}'$ of $\overline{\Gamma}(f)$, since $\overline{\Gamma}(f)$ is just the transversal intersection of a hyperplane with $\overline{D}_t \times \mathbb{C}$. The isosingular locus of $\overline{D_t}$ through any point $(0 : x_1 : \ldots : x_n) \in B_h$ contains the projectivised isosingular locus of D through $(x_1, \ldots, x_n)$, and so by the $\mathscr{R}_h$-finiteness of f, $\{f = 0\}$ is transverse to the strata of $\mathscr{S}$. This translates into the fact that the restriction of F (i.e., the second projection) to the strata of the stratification $\mathscr{S}'$ (except the stratum over D_t) is regular. It then follows from [Dim04, proposition 4.2.8] that the cohomology sheaves of $\varphi_{F-c}(\mathbb{R}j_*\mathbb{Q}_{D_t})$ are supported in D_t in a finite number of points, namely the critical points of $f_{|D_t}$. Therefore f is cohomologically tame. □

Definition 3.16 ([NS99])**.** Let $X \subset \mathbb{C}^n$ be an affine algebraic variety and $f : X \to \mathbb{C}$ a regular function. Set

$$M_f := \{x \in X : f^{-1}(f(x)) \not\pitchfork S_{\|x\|}\},$$

where $S_{\|x\|}$ is the sphere in $\mathbb{C}^n$ centred at 0 with radius $\|x\|$. We say that $f : X \to \mathbb{C}$ is *M-tame* if there is no sequence $(x^{(k)})$ in M_f such that

(i) $\|x^{(k)}\| \to \infty$ as $k \to \infty$ and (ii) $f(x^{(k)})$ tends to a limit $\ell \in \mathbb{C}$ as $k \to \infty$.

Suppose $x^{(k)}$ is a sequence in M_f satisfying (i) and (ii). After passing to a subsequence, we may suppose also that as $k \to \infty$,

(iii) $(x^{(k)}) \to x^{(0)} \in H_\infty$, where H_∞ is the hyperplane at infinity in $\mathbb{P}^n$,

(iv) $T^{(k)} \to T^{(0)} \in G_{d-1}(\mathbb{P}^n)$ where $T^{(k)}$ denotes the *affine* tangent space $AT_{x^{(k)}} f^{-1}(f(x^{(k)}))$, $d = \dim X$ and $G_{d-1}(\mathbb{P}^n)$ is the Grassmannian of $(d-1)$-planes in $\mathbb{P}^n$.

Let f and h be homogeneous polynomials on $\mathbb{C}^n$ and $X = D_t = h^{-1}(t)$ for some $t \neq 0$. Let

$$B_f = \{(0, x_1, \ldots, x_n) \in \mathbb{P}^n : f(x_1, \ldots, x_n) = 0\} \quad B_h = \{(0, x_1, \ldots, x_n) \in \mathbb{P}^n : h(x_1, \ldots, x_n) = 0\}.$$

Note that B_f and B_h are contained in the projective closure of every affine fibre of f and h respectively. We continue to denote the restriction of f to D_t by f. Let $x^{(k)}$ be a sequence satisfying 3.16(i)–(iv).

Lemma 3.17. $x^{(0)} \in B_f \cap B_h$.

Proof. Evidently $x^{(0)} \in \overline{D}_t \cap H_\infty = B_h$. Let $U_1 = \{(x_0, x_1, \ldots, x_n) \in \mathbb{P}^n : x_1 \neq 0\}$. After permuting the coordinates $x_1, \ldots, x_n$ and passing to a subsequence we may assume that $|x_1^{(k)}| \geq |x_j^{(k)}|$ for $j \geq 1$. It follows that $x^{(0)} \in U_1$. In local coordinates $y_0 = x_0/x_1, y_2 = x_2/x_1, \ldots, y_n = x_n/x_1$ on U_1, B_f is defined by the two equations $y_0 = 0, f(1, y_2, \ldots, y_n) = 0$. Since $f(x_1^{(k)}, \ldots, x_n^{(k)}) \to \ell$ and $x_1^{(k)} \to \infty$, we have $f(1, x_2^{(k)}/x_1^{(k)}, \ldots, x_n^{(k)}/x_1^{(k)}) \to 0$. It follows that $f(x_1^{(0)}, \ldots, x_n^{(0)}) = 0$, and $x^{(0)} \in B_f$. □

Lemma 3.18. *If f is a linear function then $T^{(0)} = B_f$.*

Proof. For all k we have $T^{(k)} \subset AT_{x^{(k)}}S_{\|x^{(k)}\|}$. Let $x^\perp$ denote the Hermitian orthogonal complement of the vector x in $\mathbb{C}^n$. Then $T^{(k)}$ is contained in $(x^{(k)}+{x^{(k)}}^\perp)\cap AT_{x^{(k)}}D_t$, since this is the maximal complex subspace of $AT_{x^{(k)}}D_t \cap AT_{x^{(k)}}S_{\|x^{(k)}\|}$. With respect to dual homogeneous coordinates on $(\mathbb{P}^n)^\vee$, $x^{(k)}+{x^{(k)}}^\perp = (-\|x^{(k)}\|^2 : x_1^{(k)} : \cdots : x_n^{(k)})$. So

$$\lim_{k\to\infty} x^{(k)}+{x^{(k)}}^\perp = \lim_{k\to\infty}(1 : x_1^{(k)}/\|x^{(k)}\|^2 : \cdots : x_n^{(k)}/\|x^{(k)}\|^2) = (1:0: \cdots :0) = H_\infty.$$

It follows that $T^{(0)} \subset H_\infty$. To see that $T^{(0)} \subset B_f$, note that $T^{(k)} \subset f^{-1}(f(x^{(k)}))$. Since $f(x^{(k)})\to\ell$, $\underline{f^{-1}(f(x^{(k)}))}\to f^{-1}(\ell)$ and so in the limit $T^{(0)} \subseteq \overline{f^{-1}(\ell)}$. Since $T^{(0)} \subset H_\infty$, we conclude that $T^{(0)} \subset \overline{f^{-1}(\ell)} \cap H_\infty = B_f$. As $\dim B_f = \dim T^{(0)}$, the two spaces must be equal. □

By passing to a subsequence, we may suppose that $AT_{x^{(k)}}D_t$ tends to a limit L as $k\to\infty$.

Lemma 3.19. *If $D = h^{-1}(0)$ is a reductive linear free divisor then $L \neq H_\infty$.*

Proof. It is only necessary to show that H_∞ does not lie in the projective closure of the dual $D_t^\vee$ of D_t. By Remark 3.13, $D_t^\vee = (h^*)^{-1}(c)$ for some $c \neq 0$. Its projective closure is thus $\{(y_0 : y_1 : \cdots : y_n) \in (\mathbb{P}^n)^\vee : h^*(y_1,\dots,y_n) = cy_0^n\}$, which does not contain $H_\infty = (1:0: \cdots :0)$. □

Let $\{X_\alpha\}_{\alpha\in A}$ be a Whitney stratification of $\overline{D}_t$, with regular stratum D_t, and suppose $x^{(0)} \in X_\alpha$. By Whitney regularity, $L \supset AT_{x^{(0)}}X_\alpha$. Clearly $T^{(0)} \subset L$. As $L \neq H_\infty$ then since $T^{(0)} \subset H_\infty$, for dimensional reasons we must have $T^{(0)} = L \cap H_\infty$. It follows that $T^{(0)} \supset AT_{x^{(0)}}X_\alpha$, and thus, by Lemma 3.18,

$$B_f \supset AT_{x^{(0)}}X_\alpha.$$

We have proved

Proposition 3.20. *If $D = \{h = 0\}$ is a reductive linear free divisor, $D_t = h^{-1}(t)$ for $t \neq 0$, and $f : D_t\to\mathbb{C}$ is not M-tame, then B_f is not transverse to the Whitney stratification $\{X_\alpha\}_{\alpha\in A}$ of $\overline{D}_t$.* □

Now we can prove the main result of this section.

Theorem 3.21. *If D is a reductive linear free divisor with homogeneous equation h, and if the linear function f is $\mathscr{R}_h$ finite, then the restriction of f to D_t is M-tame.*

Proof. $\mathscr{R}_h$-finiteness of f implies that for all $x \in D \cap \{f = 0\} \smallsetminus \{0\}$,

$$T_x\{f = 0\} + \mathrm{Der}(-\log h)(x) = T_x\mathbb{C}^n. \tag{3.15}$$

The strata of the canonical Whitney stratification $\mathscr{S}$ ([Tei82] and [TT83, Corollary 1.3.3]) of D are unions of isosingular loci. So for any $x \in X_\alpha \in \mathscr{S}$, $T_xX_\alpha \supset \mathrm{Der}(-\log D)(x)$. It follows from (3.15) that $\{f = 0\} \pitchfork \mathscr{S}$. Because D is homogeneous, the strata of $\mathscr{S}$ are homogeneous too, and so we may form the projective quotient stratification $\mathbb{P}\mathscr{S}$ of B_h. Transversality of $\{f = 0\}$ to D outside 0 implies that B_f is transverse to $\mathbb{P}\mathscr{S}$. The conclusion follows by Proposition 3.20. □

Remark 3.22. Reductivity is needed in Lemma 3.19 to conclude that $L \neq H_\infty$. Indeed, the example given by Broughton in [Bro88, Example 3.2] of a non-tame function on $\mathbb{C}^2$, $g(x_1, x_2) = x_1(x_1x_2 - 1)$, becomes the defining equation of a non-reductive linear free divisor, $h(x_1, x_2, x_3) = x_1(x_1x_2 - x_3^2)$, on homogenisation (in fact this is the case $n = 2$ of the non-special linear free divisors described after Definition 2.1). The sequence $x^{(k)} = (1/k, k^2, \sqrt{2k})$ lies in D_{-1} and tends to $x^{(0)} = (0:0:1:0)$ in $\mathbb{P}^3$, and $AT_{x^{(k)}}D_{-1}$ has dual projective coordinates $(3:0:1/n^2 : -2n^{-1/2})$ and thus tends to H_∞ as $n\to\infty$.

4 Gauß-Manin systems and Brieskorn lattices

In this section we introduce the family of Gauß-Manin systems and Brieskorn lattices attached to an $\mathcal{R}_h$-finite linear section of the fibration defined by the equation h. All along this section, we suppose that h defines a linear free divisor.

Under this hypothese, we show the freeness of the Brieskorn lattice, and prove that a particular basis can be found yielding a solution of the so called Birkhoff problem. The proof of the freeness relies on two facts,

first, we need that the deformation algebra $T^1_{\mathscr{R}_h/\mathbb{C}}f$ is generated by the powers of f (this would follow only from the hypotheses of lemma 3.1) and second on a division theorem, whose essential ingredient is lemma 4.3 below, which in turn uses the relative logarithmic de Rham complex associated to a LFD which was studied in subsection 2.2. The particular form of the connection that we obtain on the Brieskorn lattice allows us to prove that a solution to the Birkhoff problem always exists. This solution defines an extension to infinity (i.e., a family of trivial algebraic bundles on $\mathbb{P}^1$) of the Brieskorn lattice. However, these solutions miss a crucial property needed in the next section: The extension is not compatible with the canonical V-filtration at infinity, in other words, it is not a V^+-solution in the sense of [DS03, Appendix B]. We provide a very explicit algorithm to compute these V^+-solutions. In particular, this gives the spectral numbers at infinity of the functions $f_{|D_t}$. Using the tameness of the functions $f_{|D_t}$ it is shown in [Sab06] that the Gauß-Manin systems are equipped with a non-degenerate pairing with a specific pole order property on the Brieskorn lattices. A solution to the Birkhoff problem compatible with this pairing is called S-solution in [DS03, Appendix B]. One needs such a solution in order to construct Frobenius structures, see the next section. We prove that our solution is a (V^+, S)-solution under an additional hypothesis, which is nevertheless satisfied for many examples.
Let us start by defining the two basic objects we are interested in this section. We recall that we work in the algebraic category.

Definition 4.1. Let D be a linear free divisor with defining equation $h \in \mathbb{C}[V]_n$ and $f \in \mathbb{C}[V]_1$ linear and $\mathcal{R}_h$-finite. Let

$$\mathbf{G} := \frac{\Omega^{n-1}(\log\, h)[\tau, \tau^{-1}]}{(d - \tau df\wedge)(\Omega^{n-2}(\log\, h)[\tau, \tau^{-1}])}$$

be the family of *algebraic* Gauß-Manin systems of (f, h) and

$$G := \text{Image of } \Omega^{n-1}(\log h)[\tau^{-1}] \text{ in } \mathbf{G}$$

be the family of *algebraic* Brieskorn lattices of (f, h).

Lemma 4.2. $\mathbf{G}$ *is a free* $\mathbb{C}[t, \tau, \tau^{-1}]$*-module of rank* n*, and* G *is free over* $\mathbb{C}[t, \tau^{-1}]$ *and is a lattice inside* $\mathbf{G}$*, i.e.,* $\mathbf{G} = G \otimes_{\mathbb{C}[t,\tau^{-1}]} \mathbb{C}[t, \tau, \tau^{-1}]$*. A* $\mathbb{C}[t, \tau, \tau^{-1}]$*-basis of* $\mathbf{G}$ *(resp. a* $\mathbb{C}[t, \tau^{-1}]$*-basis of* G*) is given by* $(f^i\alpha)_{i\in\{0,\dots,n-1\}}$*, where* $\alpha := n \cdot vol/dh = \iota_E(vol/h)$*.*

Proof. As it is clear that $\mathbf{G} = G \otimes \mathbb{C}[t, \tau, \tau^{-1}]$, we only have to show that the family $(f^i\alpha)_{i\in\{0,\dots,n-1\}}$ freely generates G. This is done along the line of [dG07, proposition 8]. Remember from the discussion in subsection 2.2 that $\Omega^{n-1}(\log h)$ is $\mathbb{C}[V]$-free of rank one, generated by the form α. The next fact needed is that

$$G/\tau^{-1}G \cong \frac{\Omega^{n-1}(\log h)}{df \wedge \Omega^{n-2}(\log h)} \cong \left(h_*\mathcal{T}^1_{\mathscr{R}_h/\mathbb{C}}f\right)\alpha = \left(\frac{\mathbb{C}[V]}{\xi_1(f), \dots, \xi_{n-1}(f)}\right)\alpha$$

which is a graded free $\mathbb{C}[t]$-module of rank $n = \deg(h)$ by proposition 3.4 and proposition 3.5. Let $1, f, f^2, \dots, f^{n-1}$ be the homogeneous $\mathbb{C}[t]$-basis of $h_*\mathcal{T}^1_{\mathscr{R}_h/\mathbb{C}}f$ constructed in proposition 3.5, and $\omega = g\alpha$ be a representative for a section $[\omega]$ of G, where $g \in \mathbb{C}[V]_l$ is a homogeneous polynomial of degree l. Then g can be written as $g(x) = \widetilde{g}(h) \cdot f^i + \eta(f)$ where $\widetilde{g} \in \mathbb{C}[t]_{\lfloor l/n \rfloor}$, $i = l \bmod n$ and $\eta \in \mathrm{Der}(-\log h)$. Using the linear basis $\xi_1, \dots, \xi_{n-1}$ of $\mathrm{Der}(-\log h)$, we find homogeneous functions $k_j \in \mathbb{C}[V]_{l-1}$, $j = 1, \dots, n-1$ such that

$$\omega = \widetilde{g}(h)f^i\alpha + \sum_{j=1}^{n-1} k_j\xi_j(f)\alpha$$

It follows from the next lemma that in $\mathbf{G}$ we have

$$[\omega] = \widetilde{g}(h)f^i\alpha + \tau^{-1}\sum_{j=1}^{n-1} \left(\xi_j(k_j) + \mathrm{trace}(\xi_j) \cdot k_j\right)\alpha$$

As $\deg(\xi_j(k_j) + \mathrm{trace}(\xi_j) \cdot k_j) = \deg(g) - 1$, we see by iterating the argument (i.e., applying it to all the classes $[(\xi_j(k_j) + \mathrm{trace}(\xi_j) \cdot k_j)\alpha] \in G$) that $(f^i\alpha)_{i=0,\dots,n-1}$ gives a system of generators for G over $\mathbb{C}[t, \tau^{-1}]$. To show that they freely generate, let us consider a relation

$$\sum_{i=0}^{n-1} a_i(t, \tau^{-1})f^i\alpha = (d - \tau df\wedge)\sum_{i=-l}^{L} \tau^i\omega_i,\ \omega_i \in \Omega^{n-2}(\log h)$$

Rewriting the left-hand side as a polynomial in τ^{-1} with coefficients in $\Omega^{n-1}(\log h)$, the above equation becomes

$$\sum_{i=-m}^{M} \tau^i \eta_i = (d - \tau df\wedge) \sum_{i=-l}^{L} \tau^i \omega_i \tag{4.1}$$

where we have written $\eta_i = \sum_{j=0}^{n-1} b_{ij}(t)(f^j\alpha)$ and set $\eta_i = 0, i > 0$ for convenience. The above implies $\eta_{L+1} = df \wedge \omega_L$, so that if $L > -1$, we must have $df \wedge \omega_L = 0$, This in turn implies $\omega_L = df \wedge \omega'_L$ for some $\omega'_L \in \Omega^{n-2}(\log h)$ and hence $\eta_L = d\omega_L + df \wedge \omega_{L-1} = df \wedge (d\omega'_L - \omega_{L-1})$. By induction we obtain $\eta_0 \in df \wedge \Omega^{n-2}(\log h)$ and since $f^i\alpha, i = 0, \ldots, n-1$ form a $\mathbb{C}[t]$-basis of $\Omega^{n-1}(\log h)/df \wedge \Omega^{n-2}(\log h)$, it follows that $b_{0j} = 0, j = 0, \ldots, n-1$. Notice that the same conclusion is obtained if $L \leq -1$. In both cases, succesive multiplication by τ in both sides of (4.1) yields $\eta_{-i} \in df \wedge \Omega^{n-2}(\log h)$.

□

Lemma 4.3. *For any ξ in $\mathrm{Der}(-\log h)_0$ and $g \in \mathbb{C}[V]$, the following relation holds in $\mathbf{G}$*

$$\tau g\xi(f)\alpha = (\xi(g) + g \cdot \mathrm{trace}(\xi))\, \alpha$$

Proof. We have that

$$\begin{aligned}\tau g\xi(f)\alpha &= \tau g i_\xi(df)\alpha = \tau g(i_\xi(df \wedge \alpha) + df \wedge i_\xi\alpha) = \tau g df \wedge i_\xi\alpha \\ &= d(g i_\xi\alpha) = dg \wedge i_\xi\alpha + g d i_\xi\alpha = i_\xi(dg \wedge \alpha) + dg \wedge i_\xi\alpha + g d i_\xi\alpha \\ &= i_\xi(dg)\alpha + g d i_\xi\alpha = \xi(g)\alpha + g d i_\xi\alpha \\ &= (\xi(g) + g \cdot \mathrm{trace}(\xi))\, \alpha.\end{aligned}$$

In this computation, we have twice used the fact that for any function $r \in \mathbb{C}[V]$, the class $i_\xi(dr \wedge \alpha)$ is zero in $\Omega^{n-1}(\log h)$. This holds because for $\xi \in \mathrm{Der}(-\log h)$ and for $r \in \mathbb{C}[V]$ the operations i_ξ and $dr\wedge$ are well defined on $\Omega^\bullet(\log h)$ and moreover, $\Omega^n(\log h) = 0$, so that already $dr \wedge \alpha = 0 \in \Omega^\bullet(\log h)$. □

We denote by $T := \mathrm{Spec}\,\mathbb{C}[t]$ the base of the family defined by h. Then $\mathbf{G}$ corresponds to a rational vector bundle of rank n over $\mathbb{P}^1 \times T$, with poles along $\{0, \infty\} \times T$. Here we consider the two standard charts of $\mathbb{P}^1$ where τ is a coordinate centered at infinity. G defines an extension over $\{0\} \times T$, i.e., an algebraic bundle over $\mathbb{C} \times T$ of the same rank as $\mathbf{G}$.
We define a (relative) connection operator on $\mathbf{G}$ by

$$\nabla_{\partial_\tau}\left(\sum_{i=i_0}^{i_1} \omega_i \tau^i\right) := \sum_{i=i_0-1}^{i_1} \left((i+1)\omega_{i+1} - f \cdot \omega_i\right) \tau^i$$

where $\omega_{i_1+1} := 0, \omega_{i_0-1} := 0$. Then it is easy to check that this gives a well defined operator on the quotient $\mathbf{G}$, and that it satisfies the Leibniz rule, so that we obtain a relative connection

$$\nabla : \mathbf{G} \longrightarrow \mathbf{G} \otimes \Omega^1_{\mathbb{C}\times T/T}(*\{0\} \times T).$$

As multiplication with f leaves invariant the module $\Omega^{n-1}(\log h)$, we see that the operator ∇_{∂_τ} sends G to itself, in other words, G is stable under $\nabla_{\partial_\tau} = -\tau^{-2}\nabla_{\partial_{\tau^{-1}}} = -\theta^2\nabla_{\partial_\theta}$, where we write $\theta := \tau^{-1}$. This shows that the relative connection ∇ has a pole of order at most two on G along $\{0\} \times T$ (i.e., along $\tau = \infty$).
Consider, for any $t \in T$, the restrictions $\mathbf{G}_t := \mathbf{G}/\mathfrak{m}_t\mathbf{G}$ and $G_t := G/\mathfrak{m}_t G$. Then $\mathbf{G}_t$ is a free $\mathbb{C}[\tau, \tau^{-1}]$-module and G_t is a $\mathbb{C}[\tau^{-1}]$-lattice in it. It follows from the definition that this is exactly the (localized partial Fourier-Laplace transformation of the) Gauß-Manin system resp. the Brieskorn lattice of the function $f : D_t \to \mathbb{C}$, as studied in [Sab06]. We will make use of the results of loc.cit. applied to $f_{|D_t}$ in the sequel. Let us remark that the freeness of the individual Brieskorn lattices G_t (and consequently also of the Gauß-Manin systems $\mathbf{G}_t$) follows from the fact that $f_{|D_t}$ is cohomological tame ([Sab06, theorem 10.1]). In our situation we have the stronger statement of lemma 4.2, which gives the $\mathbb{C}[\tau^{-1}, t]$-freeness of the whole module G.
Our next aim is to consider the so-called Birkhoff problem, that is, to find a basis $\underline{\omega}^{(1)}$ of G such that the connection take the particularly simple form

$$\partial_\tau(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left(\Omega_0 + \tau^{-1} A_\infty\right),$$

(from now on, we write ∂_τ instead of ∇_{∂_τ} for short) where we require additionally that A_∞ is diagonal. We start with the basis $\underline{\omega}^{(0)}$, defined by

$$\omega_i^{(0)} := (-f)^{i-1} \cdot \alpha \quad \forall i \in \{1, \ldots, n\}. \tag{4.2}$$

Then we have $\partial_\tau(\omega_i^{(0)}) = \omega_{i+1}^{(0)}$ for all $i \in \{1, \ldots, n-1\}$ and

$$\partial_\tau(\omega_n^{(0)}) = (-f)^n \alpha.$$

As $\deg((-f)^n) = n$, $(-f)^n$ is a non-zero multiple of h in the Jacobian algebra $\mathbb{C}[V]/(df(Der(-\log h)))$, so that we have an expression $(-f)^n = c_0 \cdot h + \sum_{j=1}^{n-1} d_j^{(1)} \xi_j(f)$, where $c_0 \in \mathbb{C}^*$, $d_j^{(1)} \in \mathbb{C}[V]_{n-1}$. This gives by using lemma 4.3 again that

$$\partial_\tau(\omega_n^{(0)}) = (-f)^n \alpha = \left(c_0 t + \sum_{j=1}^{n-1} d_j^{(1)} \xi_j(f) \right) \alpha = \left(c_0 t + \tau^{-1} \sum_{j=1}^{n-1} \Big(\xi_j(d_j^{(1)}) + \mathrm{trace}(\xi_j) \Big) \right) \alpha.$$

As $\deg \Big(\xi_j(d_j^{(1)}) + \mathrm{trace}(\xi_j) \Big) = n-1$, there exist $c_1 \in \mathbb{C}$ and $d_r^{(2)} \in \mathbb{C}[V]_{n-2}$ such that

$$\begin{aligned}\left(\sum_{j=1}^{n-1} \Big(\xi_j(d_j^{(1)}) + \mathrm{trace}(\xi_j) \Big) \right) \alpha &= \left(c_1(-f)^{n-1} + \sum_{r=1}^{n-1} d_r^{(2)} \xi_r(f) \right) \alpha \\ &= \left(c_1(-f)^{n-1} + \tau^{-1} \sum_{r=1}^{n-1} \Big(\xi_r(d_r^{(2)}) + \mathrm{trace}(\xi_r) d_r^{(2)} \Big) \right) \alpha,\end{aligned}$$

and $\deg \Big(\xi_r(d_r^{(2)}) + \mathrm{trace}(\xi_r) d_r^{(2)} \Big) = n-2$. We see by iteration that the connection operator ∂_τ takes the following form with respect to $\underline{\omega}^{(0)}$:

$$\partial_\tau(\underline{\omega}^{(0)}) = \underline{\omega}^{(0)} \cdot \begin{pmatrix} 0 & 0 & \ldots & 0 & c_0 t + c_n \tau^{-n} \\ 1 & 0 & \ldots & 0 & c_{n-1}\tau^{-n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 & c_2 \tau^{-2} \\ 0 & 0 & \ldots & 1 & c_1 \tau^{-1} \end{pmatrix} =: \underline{\omega}^{(0)} \cdot \Omega =: \underline{\omega}^{(0)} \cdot \left(\sum_{k=0}^{n} \Omega_k \tau^{-k} \right). \tag{4.3}$$

Notice that if D is special then $c_n = 0$, i.e., $\Omega_n = 0$.

The matrix Ω_0 has a very particular form, due to the fact that the jacobian algebra $h_* \mathcal{T}^1_{\mathcal{R}_h/\mathbb{C}} f$ is generated by the powers of f. Notice also that the restriction $(\Omega_0)_{|t=0}$ is nilpotent, with a single Jordan block with eigenvalue zero. This reflects the fact that (G_0, ∇) is *regular singular* at $\tau = \infty$, which is not the case for any $t \neq 0$. Remember that although D is singular itself, so that it is not quite true that there is only one critical value of f on D, we have that f is regular on $D \backslash \{0\}$ in the stratified sense (see the proof of proposition 3.15). The particular form of the matrix Ω_0 is the key ingredient to solve the Birkhoff problem, which can actually be done by a triangular change of basis.

Lemma 4.4. *There exists a base change*

$$\omega_j^{(1)} := \omega_j^{(0)} + \sum_{i=1}^{j-1} b_i^j \tau^{-i} \omega_{j-i}^{(0)}, \tag{4.4}$$

such that the matrix of the connection with respect to $\underline{\omega}^{(1)}$ *is given by*

$$\Omega_0 + \tau^{-1} A_\infty,$$

where $A_\infty^{(1)}$ *is diagonal. Moreover, if* D *is special, then* b_i^j *can be chosen such that* $b_i^{i+1} = 0$ *for* $i = 1, \ldots, n-1$.

Proof. Let us regard b_i^j as unknown constants to be determined and then let

$$B := (b_{j-k}^j \tau^{k-j})_{kj} =: \sum_{i=0}^{n-1} B_i \tau^{-i} = \mathrm{Id} + \sum_{i=1}^{n-1} B_i \tau^{-i}.$$

Here $b^i_j = 0$ for $j < 0$. Notice that B_i is a matrix whose only non-zero entries are in the position $(j, j + i)$ for $j = 1, \dots, n - i$.
The matrix of the action of ∂_τ changes according to the formula:

$$X := B^{-1} \cdot \frac{dB}{d\tau} + B^{-1}\Omega B =: \sum_{i=0}^{n} X_i \tau^{-i}. \tag{4.5}$$

Multiplying by B both sides of the above equation we find

$$BX = \sum_{i=0}^{n} \left(\sum_{j=0}^{i} B_j X_{i-j} \right) \tau^{-i} = \sum_{i=0}^{n} (-(i-1)B_{i-1} + \Omega_0 B_i + \Omega_i)\tau^{-i}. \tag{4.6}$$

where $B_{-1} := 0$. Let $N = (n_{ij})$ be the matrix with $n_{ij} = 1$ if $j = i - 1$ or 0 otherwise. Hence $\Omega_0 = N + C_0$ where C_0 is the matrix whose only non-zero entry is $c_0 t$ in the right top corner. It follows that $X_0 = \Omega_0$ and that

$$X_i = -\sum_{j=1}^{i-1} B_j X_{i-j} - [B_i, N] - (i-1)B_{i-1} + \Omega_i. \tag{4.7}$$

We are looking for a solution to the system $X_1 = A^{(1)}_\infty$, $X_i = 0$, $i = 2, \dots, n$, where $A^{(1)}_\infty$ is diagonal with entries yet to be determined. In view of the above, this system is equivalent to:

$$\begin{gathered} X_1 = -[B_1, N] + \Omega_1 = A^{(1)}_\infty, \\ [B_{i+1}, N] = -B_i X_1 - iB_i + \Omega_{i+1}, \ i = 1, \dots, n-1. \end{gathered} \tag{4.8}$$

We are going to show that this system of polynomial equations in the variables b^j_i can always be reduced to a triangular system in b^j_1, so that there exists a solution. In particular, this determines the entries of the diagonal matrix $-[B_1, N] + \Omega_1$, i.e., the matrix $A^{(1)}_\infty$ we are looking for.
A direct calculation shows that if we substitute the first equation of (4.8) into the right hand side of the second one, we obtain $[B_{i+1}, N] = B_i([B_1, N] - \Omega_1 + i\mathrm{Id}) + \Omega_{i+1} =: P^i$, where the only non-zero coefficients of the matrix P^i are $P^i_{j,i+j}$, namely:

$$\begin{aligned} P^i_{j,i+j} &= b^{i+j}_i (b^{i+j+1}_1 - b^{i+j}_1 + i), j = 1, \dots, n-i-1, \\ P^i_{n-i,n} &= b^n_i(-b^n_1 - c_1 + i) + c_{i+1}. \end{aligned} \tag{4.9}$$

A matrix B_{i+1} satisfying $[B_{i+1}, N] = P^i$ exists if and only if $Q^i := \sum_{j=1}^{n-i} P^i_{j,i+j} = 0$, and if this is case, the solution is given by setting:

$$b^{i+k+1}_{i+1} = -\sum_{j=k+1}^{n-i} P^i_{j,j+i}, \ k = 1, \dots, n-i-1. \tag{4.10}$$

For $i = 1$ and $j = 1, \dots, n-1$, we have from (4.9)

$$P^1_{j,j+1} = -(b^{j+1}_1)^2 + \text{lower degree terms in } b^{j+1}_1 \text{ with coefficients in } b^k_1, k > j+1$$

so that substituting (4.10) in (4.9) for $i = 2$

$$P^2_{j,2+j} = -(b^{j+2}_1)^3 + \text{lower degree terms in } b^{j+2}_1 \text{ with coefficients in } b^1_k, \ k > j+2$$

By induction we see that after substitution we have

$$P^i_{j,i+j} = -(b^{i+j}_1)^{i+1} + \text{lower degree terms in } b^{i+j}_1 \text{ with coefficients in } b^k_1, \ k > i+j$$

from which it follows that

$$Q^i = -(b^{i+1}_1)^{i+1} + \text{lower degree terms in } b^{i+1}_1 \text{ with coefficients in } b^k_1, \ k > i+1$$

The system $Q^i = 0, i = 1, \dots, n-1$ is triangular (e.g., $Q^{n-1} \in \mathbb{C}[b^n_1]$) and thus has a solution.
In the case where D is special, the vanishing of Ω_n can be used to set $b^{i+1}_i = 0$ from the start. The above proof then works verbatim. □

Notice that we can assume by a change of coordinates on T that the non-zero constant c_0 is actually normalized to 1. We will make this assumption from now on.
In the next section, we are interested in constructing Frobenius structures associated to the tame functions $f_{|D_t}$ and to study their limit behaviour when t goes to zero. For that purpose, it is desirable to complete the relative connection ∇ from above to an absolute one, which will acquire an additional pole at $t = 0$. This can be done in general, however, in order to give an explicit expression for this connection, we will need the special form it takes in the basis $\underline{\omega}^{(1)}$ as well as theorem 2.7. For this reason, we restrict to the reductive case here. It is however true that the definition given below defines an integrable connection on $\mathbf{G}$ in all cases, more precisely, it defines the (partial Fourier-Laplace transformation of the) Gauß-Manin connection for the complete intersection given by the two functions (f, h). We will not discuss this in detail here.
The completion of the relative connection ∇ on $\mathbf{G}$ refered to above is given by the formula

$$\nabla_{\partial_t}(\omega) := \frac{1}{n \cdot t} \left(L_E(\omega) - \tau L_E(f) \cdot \omega\right), \tag{4.11}$$

for any $[\omega] \in \Omega^{n-1}(\log h)$ and extending τ-linearly. One checks that

$$(t\nabla_{\partial_t}) \left((d - \tau df\wedge)(\Omega^{n-2}(\log\, h)[\tau, \tau^{-1}])\right) \subset (d - \tau df\wedge)(\Omega^{n-2}(\log\, h)[\tau, \tau^{-1}]),$$

so that we obtain operator

$$\nabla : \mathbf{G} \longrightarrow \mathbf{G} \otimes \tau\Omega^1_{\mathbb{C}\times T}(\log \mathcal{D}), \tag{4.12}$$

where $\mathcal{D}$ is the divisor $(\{0\} \times T) \cup (\mathbb{C} \times \{0\}) \subset \mathbb{C} \times T$.

Proposition 4.5. *Let D be reductive. Then:*

(i) The elements of the basis $\underline{\omega}^{(1)}$ constructed above can be represented by differential forms $\omega_i^{(1)} = [g_i\alpha]$ with g_i homogeneous of degree $i = 0, \ldots, n-1$, i.e., by elements outside of $\tau^{-1}\Omega^{n-1}(\log h)[\tau^{-1}]$.

(ii) The connection operator defined above is flat outside $\theta = 0, t = 0$. We denote by $\mathbf{G}^\nabla$ the corresponding local system and by $\mathbf{G}^\infty$ its space of multivalued flat sections.

(iii) Consider the Gauß-Manin system, localized at $t = 0$, i.e.

$$\mathbf{G}[t^{-1}] := \mathbf{G} \otimes_{\mathbb{C}[\tau,\tau^{-1},t]} \mathbb{C}[\tau, \tau^{-1}, t, t^{-1}] \cong \frac{\Omega^{n-1}_{V/T}(*D)[\tau, \tau^{-1}]}{(d - \tau df\wedge)\Omega^{n-2}_{V/T}(*D)[\tau, \tau^{-1}]}.$$

and similarly, the localized Brieskorn lattice

$$G[t^{-1}] := G \otimes_{\mathbb{C}[\tau^{-1},t]} \mathbb{C}[\tau^{-1}, t, t^{-1}] \cong \frac{\Omega^{n-1}_{V/T}(*D)[\tau^{-1}]}{(\tau^{-1}d - df\wedge)\Omega^{n-2}_{V/T}(*D)[\tau^{-1}]} \subset \mathbf{G}[t^{-1}].$$

Then $\underline{\omega}^{(1)}$ provides a solution to the Birkhoff problem for $(G[t^{-1}], \nabla)$ "in a family", i.e., an extension to a trivial algebraic bundle $\widehat{G}[t^{-1}] \subset \widetilde{i}_* G[t^{-1}]$ (here $\widetilde{i} : \mathbb{C} \times (T\backslash\{0\}) \hookrightarrow \mathbb{P}^1 \times (T\backslash\{0\})$) on $\mathbb{P}^1 \times (T\backslash\{0\})$, on which the connection has a logarithmic pole along $\{\infty\} \times (T\backslash\{0\})$ (and, as before, a pole of type one along $\{0\} \times (T\backslash\{0\})$).

(iv) Let γ resp. γ' be a small counterclockwise loop around the divisor $\{0\} \times T$ resp. $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times T$. Let M resp. M' denote the mondromy endomorphisms on $\mathbf{G}^\infty$ corresponding to γ resp. γ'. Then

$$M^{-1} = (M')^n.$$

(v) Let $u : \mathbb{C}^2 \to \mathbb{C} \times T$, $(\theta, s) \mapsto (\theta, s^n)$. Consider the pullback $u^*(G, \nabla)$ and denote by $(\widetilde{G}, \nabla)$ the restriction to $\mathbb{C} \times \mathbb{C}^*$ of the analytic bundle corresponding to $u^*(G, \nabla)$. Then $\widetilde{G}$ underlies a Sabbah-orbit *of TERP-structures, as defined in [HS07, definition 4.1].*

Proof. (i) It follows from theorem 2.7 that for $g \in \mathbb{C}[V]_i$ with $1 < i < n$, the $n-1$-form $g\alpha$ is exact in the complex $\Omega^\bullet(\log h)$. Therefore in $\mathbf{G}$ we have $\tau^{-1}g\alpha = \tau^{-1}d\omega' = df \wedge \omega' = g'\alpha$ for some $\omega' \in \Omega^{n-2}(\log h)$ and $g' \in \mathbb{C}[V]$. Note that necessarily $g' \in \mathbb{C}[V]_{i+1}$. Moreover, in the above constructed base change matrix we had $B_{1l} = \delta_{1l}$ (as D is reductive hence special), which implies that for all $i > 0$, $\omega_i^{(1)}$ is represented by an element in $f\mathbb{C}[V]\alpha[\tau^{-1}]$, i.e, by a sum of terms of the form $\tau^{-k}g\alpha$ with $g \in \mathbb{C}[V]_{\geq 1}$. This proves that we can successively erase all negative powers of τ, i.e., represent all $\omega_i^{(1)}$, $i > 0$ by pure forms (i.e., without τ^{-1}'s), and $\omega_0^{(1)} = \omega_0^{(0)} = \alpha$ is pure anyhow.

(ii) From (i) and the definition of ∇_{∂_t} in (4.11) we obtain

$$\nabla(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left[(\Omega_0 + \tau^{-1} A_\infty^{(1)}) d\tau + (D + \tau\Omega_0 + A_\infty^{(1)}) \frac{dt}{nt} \right]$$

where $D = \mathrm{diag}(0, \ldots, n-1)$. The flatness conditions of an arbitrary connection of the form

$$\nabla(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left[(\tau A + B) \frac{d\tau}{\tau} + (\tau A' + B') \frac{dt}{t} \right]$$

with $A, A' \in M(n \times n, \mathbb{C}[t])$ and $B, B' \in M(n \times n, \mathbb{C})$ is given by the following system of equations:

$$[A, A'] = 0 \quad ; \quad [B, B'] = 0 \quad ; \quad (t\partial_t)A - A' = [A, B'] - [A', B]$$

One checks that for $A = \Omega_0$, $A' = \frac{1}{n}\Omega_0$, $B = A_\infty^{(1)}$ and $B' = \frac{1}{n}(A_\infty^{(1)} + D)$, these equations are satisfied.

(iii) The extension defined by $\underline{\omega}^{(1)}$, i.e., $\widehat{G}[t^{-1}] := \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1 \times T}[t^{-1}] \omega_i^{(1)}$ provides the solution in a family to the Birkhoff problem, i.e., we have $\nabla_X \widehat{G}[t^{-1}] \subset \widehat{G}[t^{-1}]$ for any $X \in \mathrm{Der}(-\log(\{\infty\} \times (T \backslash \{0\})))$.

(iv) If we restrict $(\mathbf{G}, \nabla)$ to the curve $C := \left\{ (\tau, t) \in (\mathbb{C}^*)^2 \,|\, \tau^n t = 1 \right\}$ we obtain

$$\nabla_{|C} = -D \frac{d\tau}{\tau}.$$

As the diagonal entries of D are integers, the monodromy of $(\mathbf{G}, \nabla)_{|C}$ is trivial which implies the result (notice that the composition of γ_1 and γ_2 is homotopic to a loop around the origin in C).

(v) That the restriction to $\mathbb{C} \times (T \backslash \{0\})$ of the (analytic bundle corresponding to) G underlies a variation of pure polarized TERP-structures is a general fact, due to the tameness of the functions $f_{|D_t}$ (see [Sab06] and [Sab08], [HS07, theorem 11.1]). Using the connection matrix from (ii), it is an easy calculation that $\nabla_{s\partial_s - \tau\partial_\tau}(\underline{\widetilde{\omega}}^{(1)}) = 0$, where $\underline{\widetilde{\omega}}^{(1)} := u^* \underline{\omega}^{(1)} \cdot s^{-D}$ so that $(\widetilde{G}, \nabla)$ satisfies condition 2.(a) in [HS07, definition 4.1].

□

For the purpose of the next section, we need to find a much more special solution to the Birkhoff problem, which is called V^+-solution in [DS03]. It takes into account the Kashiwara-Malgrange filtration of $\mathbf{G}$ at infinity (i.e., at $\tau = 0$). We briefly recall the notations and explain how to construct the V^+-solution starting from our basis $\underline{\omega}^{(1)}$.

Fix $t \in T$ and consider, as before, the restrictions $\mathbf{G}_t$ resp. G_t of the family of Gauß-Manin-systems resp. Brieskorn lattices $\mathbf{G}$ resp. G. As already pointed out, for $t \neq 0$, these are the Gauß-Manin-system resp. the Brieskorn lattice of the tame of the function $f_{|D_t}$. $\mathbf{G}_t$ is known to be a holonomic left $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, with singularities at $\tau = 0$ and $\tau = \infty$ only. The one at infinity, i.e. $\tau = 0$ is regular singular, but not necessarily the one at zero (at $\tau = \infty$). Similarly to the notation used above, we have the the local system $\mathbf{G}_t^\nabla$ and its space of multivalued global flat sections $\mathbf{G}_t^\infty$. Recall that for any $t \neq 0$, the monodromy of $\mathbf{G}_t^\nabla$ is quasi-unipotent, so any logarithm of any of its eigenvalues is a rational number. As we will see in section 6, the same is true in all examples for $t = 0$, but this is not proved for the moment. Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{Q}$, depending on whether $t = 0$ or $t \neq 0$. In the former case, we chose the lexicographic ordering on $\mathbb{C}$ which extends the usual ordering of $\mathbb{R}$. Recall that there is a unique increasing exhaustive filtration $V_\bullet \mathbf{G}_t$ indexed by $\mathbb{K}$, called the Kashiwara-Malgrange or canonical V-filtration on $\mathbf{G}_t$ with the properties

(i) It is a good filtration with respect to the V-filtration $V_\bullet \mathbb{C}[\tau]\langle\partial_\tau\rangle$ of the Weyl-algebra, i.e. it satisfies $V_k \mathbb{C}[\tau]\langle\partial_\tau\rangle V_l \mathbf{G}_t \subset V_{k+l} \mathbf{G}_t$ and this is an equality for any $k \leq 0$, $l \leq -l_0$, $k \geq 0$, $l \geq l_0$ for some sufficiently large positive integer l_0.

(ii) For any $\alpha \in \mathbb{K}$, the operator $\tau\partial_\tau + \alpha$ is nilpotent on the quotient $gr_\alpha^V \mathbf{G}_t$

We have an induced V-filtration on the Brieskorn lattice G_t, and we denote by

$$\mathrm{Sp}(G_t, \nabla) := \sum_{\alpha \in \mathbb{K}} \dim_{\mathbb{C}} \left(\frac{V_\alpha \cap G_t}{V_{<\alpha} \cap G_t + \tau^{-1} G_t \cap V_\alpha} \right) \alpha \in \mathbb{Z}[\mathbb{K}]$$

the spectrum of G_t at infinity (for $t \neq 0$ it is also called the spectrum at infinity associated to $f_{|D_t}$). We also write it as an ordered tuple of (possibly repeated) rational numbers $\alpha_1 \leq \ldots \leq \alpha_n$. We recall the following notions from [DS03, appendix B].

Lemma and Definition 4.6. *(i) The following conditions are equivalent.*

(a) *There is a solution to the Birkhoff problem, i.e, a basis* $\underline{\omega}$ *of* G_t *with* $\partial_\tau(\underline{\omega}) = \underline{\omega}(\Omega_0 + \tau^{-1}A_\infty)$ *(where* A_∞ *is not necessarily semi-simple).*

(b) *There is a* $\mathbb{C}[\tau]$*-lattice* G'_t *of* $\mathbf{G}_t$ *which is stable under* $\tau\partial_\tau$*, and such that* $G_t = (G_t \cap G'_t) \oplus \tau^{-1}G_t$*.*

(c) *There is an extension to a free* $\mathcal{O}_{\mathbb{P}^1}$*-module* $\widehat{G}_t \subset \widetilde{i}_* G_t$ *(where* $\widetilde{i} : \mathbb{C} \hookrightarrow \mathbb{P}^1$*) with the property that* $(\tau\nabla_\tau)\widehat{G}_t \subset \widehat{G}_t$*.*

(ii) A solution to the Birkhoff problem G'_t *is called a* V*-solution iff*

$$G_t \cap V_\alpha \mathbf{G}_t = (G_t \cap G'_t \cap V_\alpha \mathbf{G}_t) \oplus (\tau^{-1}G_t \cap V_\alpha \mathbf{G}_t).$$

(iii) G'_t *is called a* V^+*-solution if moreover we have*

$$(\tau\partial_\tau + \alpha)(G_t \cap G'_t \cap V_\alpha \mathbf{G}_t) \subset (G_t \cap G'_t \cap V_{<\alpha}\mathbf{G}_t) \oplus \tau(G_t \cap G'_t \cap V_{\alpha+1}\mathbf{G}_t).$$

In this case, a basis as in (i) (a) can be chosen such that the matrix A_∞ *is diagonal, and the diagonal entries are the spectral numbers of* (G_t, ∇) *at infinity.*

(iv) Suppose that we are moreover given a non-degenerate flat Hermitian pairing on $\mathbf{G}_t$ *which has weight* $n-1$ *on* G_t*, more precisely (see [DS03, secton 1.f.] or [DS04, section 4]) a morphism* $S : \mathbf{G}_t \otimes_{\mathbb{C}[\tau,\tau^{-1}]} \overline{\mathbf{G}}_t \to \mathbb{C}[\tau, \tau^{-1}]$ *(where* $\overline{\mathbf{G}}_t$ *denotes the module* $\mathbf{G}_t$ *on which* τ *acts as* $-\tau$*) with the following properties*

(a) $\tau\partial_\tau S(a, \overline{b}) = S(\tau\partial_\tau a, \overline{b}) + S(a, \tau\partial_\tau \overline{b})$*,*

(b) $S : V_0 \otimes \overline{V}_{<1} \to \mathbb{C}[\tau]$*,*

(c) $S(G_t, \overline{G}_t) \subset \tau^{-n+1}\mathbb{C}[\tau^{-1}]$*, and the induced symmetric pairing* $G_t/\tau^{-1}G_t \otimes G_t/\tau^{-1}G_t \to \tau^{-n+1}\mathbb{C}$ *is non-degenerate.*

In particular, the spectral numbers then obey the symmetry $\alpha_1 + \alpha_{n+1-i} = n-1$*. A* V^+*-solution* G'_t *is called* (V^+, S)*-solution if* $S(G_t \cap G'_t, \overline{G}_t \cap \overline{G}'_t) \subset \mathbb{C}\tau^{-n+1}$*.*

We will see in the sequel (theorem 4.13) that under a technical hypothesis (which is however satisfied in many examples) we are able to construct directly a (V^+, S)-solution. Without this hypothesis, we can for the moment only construct a V^+-solution. In order to obtain Frobenius structures in all cases, we need the following general result, which we quote from [Sab06] and [DS03].

Theorem 4.7. *Let* Y *be a smooth affine complex algebraic variety and* $f : Y \to \mathbb{C}$ *be a cohomologically tame function. Then the Gauß-Manin system of* f *is equipped a pairing* S *as above, and there is a canonical* (V^+, S)*-solution to the Birkhoff problem for the Brieskorn lattice of* f*, defined by a (canonical choice of an) opposite filtration to the Hodge filtration of the mixed Hodge structure associated to* f*.*

The key tool to compute the spectrum and to obtain such a V^+-solution to the Birkhoff problem is the following result.

Proposition 4.8. *Let* $t \in T$ *be arbitrary,* $G_t \subset \mathbf{G}_t$ *as before and consider any solution to the Birkhoff problem for* (G_t, ∇)*, given by a basis* $\underline{\omega}$ *of* G_t *such that* $\partial_\tau(\underline{\omega}) = \underline{\omega}(\Omega_0 + \tau^{-1}A_\infty)$ *with* Ω_0 *as above and such that* $A_\infty = \mathrm{diag}(-\nu_1, \ldots, -\nu_n)$ *is diagonal. Suppose moreover that* $\nu_i - \nu_{i-1} \le 1$ *for all* $i \in \{1, \ldots, i\}$ *and additionally that* $\nu_1 - \nu_n \le 1$ *if* $t \ne 0$*.*
Then $\underline{\omega}$ *is a* V^+*-solution to the Birkhoff problem and the numbers* $(-\nu_i)_{i=1,\ldots,n}$ *give the spectrum* $\mathrm{Sp}(G_t, \nabla)$ *of* G_t *at infinity.*

Proof. The basic idea is similar to [DS04] and [dG07], namely, that the spectrum of A_∞ can be used to *define* a filtration which turns out to coincide with the V-filtration using that the latter is unique with the above properties. More precisely, we define a $\mathbb{K}$-grading on $\mathbf{G}_t$ by $\deg(\tau^k\omega_i) := \nu_i - k$ and consider the associated increasing filtration $\widetilde{V}_\bullet \mathbf{G}_t$ given by

$$\widetilde{V}_\alpha \mathbf{G}_t := \left\{ \sum_{i=1}^n c_i \tau^{k_i}\omega_i \in \mathbf{G}_t \,|\, \max_i(\nu_i - k_i) \le \alpha \right\}$$

$$\widetilde{V}_{<\alpha} \mathbf{G}_t := \left\{ \sum_{i=1}^n c_i \tau^{k_i}\omega_i \in \mathbf{G}_t \,|\, \max_i(\nu_i - k_i) < \alpha \right\}.$$

By definition $\partial_\tau \widetilde{V}_\bullet \mathbf{G}_t \subset \widetilde{V}_{\bullet+1}\mathbf{G}_t$ and $\tau\widetilde{V}_\bullet \mathbf{G}_t \subset \widetilde{V}_{\bullet-1}\mathbf{G}_t$ and moreover, τ is obviously bijective on $\mathbf{G}$. Thus to verify that $V_\bullet \mathbf{G}_t = \widetilde{V}_\bullet \mathbf{G}_t$, we only have to show that $\tau\partial_\tau + \alpha$ is nilpotent on $gr_\alpha^{\widetilde{V}}\mathbf{G}_t$. This will prove both statements of the proposition: the conditions in definition 4.6 for $\underline{\omega}$ to be a V^+-solution are trivially satisfied if we replace V by $\widetilde{V}$. The nilpotency of $\tau\partial_\tau + \alpha \in End_\mathbb{C}(gr_\alpha^{\widetilde{V}}\mathbf{G}_t)$ follows from the assumption $\nu_i - \nu_{i-1} \leq 1$: First define a block decomposition of the ordered tuple $(1,\ldots,n)$ by putting $(1,\ldots,n) = (I_1,\ldots,I_s)$, where $I_r = (i_r, i_r+1,\ldots,i_r+l_r = i_{r+1}-1)$ such that $\nu_{i_r+l+1} - \nu_{i_r+l} = 1$ for all $l \in \{0,\ldots,l_r-1\}$ and $\nu_{i_r} - \nu_{i_r-1} < 1$, $\nu_{i_{r+1}} - \nu_{i_r+l_r} < 1$. Then in $\mathbf{G}_t$ we have $(\tau\partial_\tau + (\nu_i - k_i))(\tau^{k_i}\omega_i) = \tau^{k_i+1}\omega_{i+1}$ for $i \in \{1,\ldots,n-1\}$ and $(\tau\partial_\tau + (\nu_n - k_n))(\tau^{k_n}\omega_n) = t\tau^{k_n+1}\omega_1$, so that $(\tau\partial_\tau + (\nu_i - k_i))^{i_{r+1}-i}(\tau^{k_i}\omega_i) = 0$ in $gr_{\nu_i-k_i}^{\widetilde{V}}\mathbf{G}_t$ for all $i \in I_r$ (here we put $i_{s+1} := n+1$, note also that if $t \neq 0$ we suppose that $\nu_1 - \nu_n \leq 1$). $\square$

As a by-product, a solution with the above properties also makes it possible to compute the monodromy of $\mathbf{G}_t$. Consider the local system $\mathbf{G}_t^\nabla$ and the space $\mathbf{G}_t^\infty$ of its multivalued flat sections. There is a natural isomorphism $\oplus_{\alpha\in(0,1]} gr_\alpha^V \mathbf{G}_t \xrightarrow{\psi} \mathbf{G}_t^\infty$. The monodromy $M \in Aut(\mathbf{G}_t^\infty)$, which corresponds to a counter-clockwise loop around $\tau = \infty$, decomposes as $M = M_s \cdot M_u$ into semi-simple and unipotent part, and we write $N := \log(M_u)$ for the nilpotent part of M. N corresponds under the isomorphism ψ, up to a constant factor, to the operator $\oplus_{\alpha\in(0,1]}(\tau\partial_\tau + \alpha) \in \oplus_{\alpha\in(0,1]} End_\mathbb{C}(gr_\alpha^V \mathbf{G})$. This gives the following result, notice that a similar statement and proof are given in [DS04, end of section 3].

Corollary 4.9. *Consider the basis of $\mathbf{G}_t^\infty$ induced from a basis $\underline{\omega}$ as above, i.e.,*

$$\mathbf{G}_t^\infty = \oplus_{i=1}^n \mathbb{C}\psi^{-1}([\tau^{l_i}\omega_i]),$$

where $l_i = \lfloor \nu_i \rfloor + 1$. Then $M_s\psi^{-1}[\tau^{l_i}\omega_i] = e^{-2\pi i\nu_i} \cdot \psi^{-1}[\tau^{l_i}\omega_i]$ and

$$N(\psi^{-1}[\tau^{l_i}\omega_i]) = \begin{cases} 2\pi i\psi^{-1}[\tau^{l_i}\omega_{i+1}] & \text{if } \nu_{i+1} - \nu_i = 1 \\ 0 & \text{else,} \end{cases}$$

where $\omega_{n+1} = \omega_1$ if $t \neq 0$ and $\omega_{n+1} = 0$ if $t = 0$. Thus the Jordan blocks of N are exactly the blocks appearing above in the decomposition of the tuple $(1,\ldots,n)$.

We can now use proposition 4.8 to compute a V^+-solution and the spectrum of G_t. We give an explicit algorithm, which we split into two parts for the sake of clarity. Once again it should be emphasized that the special form of the matrix Ω_0 is the main ingredient for this algorithm.

Algorithm 1. Given $\underline{\omega}^{(1)}$ from lemma 4.4, i.e., $\partial_\tau(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)}(\Omega_0 + \tau^{-1}A_\infty)$ and $A_\infty^{(1)} = \mathrm{diag}(-\nu_1^{(1)},\ldots,-\nu_n^{(1)})$, whenever there is $i \in \{2,\ldots,n\}$ with $\nu_i^{(1)} - \nu_{i-1}^{(1)} > 1$, put

$$\begin{array}{rcl} \widetilde{\omega}_i^{(1)} & := & \omega_i^{(1)} + \tau^{-1}(\nu_i^{(1)} - \nu_{i-1}^{(1)} - 1)\omega_{i-1}^{(1)} \\ \widetilde{\omega}_j^{(1)} & := & \omega_j^{(1)} \quad \forall j \neq i \end{array} \tag{4.13}$$

so that $\partial_\tau(\underline{\widetilde{\omega}}^{(1)}) = \underline{\widetilde{\omega}}^{(1)}(\Omega_0 + \tau^{-1}\widetilde{A}_\infty^{(1)})$ and $\widetilde{A}_\infty^{(1)} = \mathrm{diag}(-\widetilde{\nu}_1^{(1)},\ldots,-\widetilde{\nu}_n^{(1)})$, where $\widetilde{\nu}_i^{(1)} = \nu_{i-1}^{(1)} + 1$, $\widetilde{\nu}_{i-1}^{(1)} = \nu_i^{(1)} - 1$ and $\widetilde{\nu}_j^{(1)} = \nu_j$ for any $j \notin \{i, i-1\}$. Restart algorithm 1 with input $\underline{\widetilde{\omega}}^{(1)}$.

Now we have

Lemma 4.10. *Given any basis $\underline{\omega}^{(1)}$ of G_t as above, algorithm 1 terminates. Its output $\underline{\omega}^{(2)}$ is a V^+-solution for G_t if $t = 0$.*

Proof. The first statement is a simple analysis on the action of the algorithm on the array $(\nu_1^{(1)},\ldots,\nu_n^{(1)})$, namely, if $(\nu_1^{(1)},\ldots,\nu_k^{(1)})$ is ordered (i.e., $\nu_i^{(1)} - \nu_{i-1}^{(1)} \leq 1$ for all $i \in \{2,\ldots k\}$), then after a finite number of steps the array $(\widetilde{\nu}_1^{(1)},\ldots,\widetilde{\nu}_{k+1}^{(1)})$ will be ordered. This shows that the algorithm will eventually terminate. Its output is then a V^+-solution for G_t if $t = 0$ by proposition 4.8. $\square$

If we want to compute the spectrum and a V^+-solution of G_t for $t \neq 0$, we also have to make sure that $\nu_1 - \nu_n \leq 1$. This is done by the following procedure.

Algorithm 2. Run algorithm 1 on the input $\underline{\omega}^{(1)}$ with output $\underline{\omega}^{(2)}$ where $A_\infty^{(2)} = (-\nu_1^{(2)},\ldots,-\nu_n^{(2)})$. As long as $\nu_1^{(2)} - \nu_n^{(2)} > 1$, put

$$\begin{array}{rcl} \widetilde{\omega}_1^{(2)} & := & t\omega_1^{(2)} + \tau^{-1}(\nu_1^{(2)} - \nu_n^{(2)} - 1)\omega_n^{(2)} \\ \widetilde{\omega}_i^{(2)} & := & t\omega_i^{(2)} \quad \forall i \neq 1. \end{array} \tag{4.14}$$

so that $\partial_\tau(\underline{\widetilde{\omega}}^{(2)}) = \underline{\widetilde{\omega}}^{(2)}(\Omega_0+\tau^{-1}\widetilde{A}^{(2)}_\infty)$ where $\widetilde{A}^{(2)}_\infty = \mathrm{diag}(-\widetilde{\nu}^{(2)}_1,\ldots,-\widetilde{\nu}^{(2)}_n)$, where $\widetilde{\nu}^{(2)}_1 = \nu^{(2)}_n+1$, $\widetilde{\nu}^{(2)}_n = \nu^{(2)}_1-1$ and $\widetilde{\nu}^{(2)}_i = \nu^{(2)}_i$ for any $i \notin \{1,n\}$. Run algorithm 2 again on input $\underline{\widetilde{\omega}}^{(2)}$.

Lemma 4.11. *Let $t \neq 0$, given any solution $\underline{\omega}^{(1)}$ to the Birkhoff problem for G_t, such that $\partial_\tau(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)}(\Omega_0+\tau^{-1}A^{(1)}_\infty)$ with Ω_0 as above and $A^{(1)}_\infty$ diagonal, then algorithm 2 with input $\underline{\omega}^{(1)}$ terminates and yields a basis $\underline{\omega}^{(3)}$ with $\partial_\tau(\underline{\omega}^{(3)}) = \underline{\omega}^{(3)}(\Omega_0+\tau^{-1}A^{(3)}_\infty)$, where $A^{(3)}_\infty = (-\nu^{(3)}_1,\ldots,-\nu^{(3)}_n)$ with $\nu^{(3)}_{i+1}-\nu^{(3)}_i \leq 1$ for $i \in \{1,\ldots n\}$ (here $\nu^{(3)}_{n+1} := \nu^{(3)}_1$).*

Proof. We only have to prove that algorithm 2 terminates. This is easily be done by showing that in each step, the number $\widetilde{\nu}^{(2)}_1 - \widetilde{\nu}^{(2)}_n$ does not increase, that it strictly decreases after a finite number of steps, and that the possible values for this number are contained in the set $\{a-b \,|\, a,b \in \{\nu_1,\ldots,\nu_n\}\}+\mathbb{Z}$ (which have no accumulation points), so that after a finite number of steps we necessarily have $\widetilde{\nu}^{(2)}_1 - \widetilde{\nu}^{(2)}_n \leq 1$. □

Note that for any fixed $t \neq 0$, algorithm 2 produces a base change of G_t, but this does not lift to a base change of G itself, i.e., $G^{(3)} := \oplus_{i=1}^n \mathbb{C}[\tau^{-1},t]\omega^{(3)}_i$ is a proper free submodule of G which coincides with G only after localization off $t=0$. In other words, it is a $\mathbb{C}[t]$-lattice of $G[t^{-1}]$ which is in general different from G. Summarizing the above calculations, we have shown the following.

Corollary 4.12. *(i) Let $D \subset V$ be a linear free divisor with defining equation $h \in \mathbb{C}[V]_n$, seen as a morphism $h: V \to T$. Let $f \in \mathbb{C}[V]_1$ be linear and $\mathcal{R}_h$-finite. Then for any $t \in T$, there is a V^+-solution of the Birkhoff problem for (G_t,∇), defined by bases $\underline{\omega}^{(2)}$ if $t=0$ resp. $\underline{\omega}^{(3)}$ if $t \neq 0$ as constructed above. If $\nu^{(2)}_1-\nu^{(2)}_n \leq 1$ then $\underline{\omega}^{(3)} = \underline{\omega}^{(2)}$. Moreover, we have that $\underline{\omega}^{(2)}_i - (-f)^{i-1}\alpha$ and $\omega^{(3)}_i - (-f)^{i-1}\alpha$ lie in $\tau^{-1}G_t$ for all $i \in \{1,\ldots,n\}$.*

(ii) Let D be reductive. Then the integrable connection ∇ on $\mathbf{G}[t^{-1}]$ defined by formula (4.11) takes the following form in the basis $\underline{\omega}^{(3)}$:

$$\nabla(\underline{\omega}^{(3)}) = \underline{\omega}^{(3)} \cdot \left[(\Omega_0+\tau^{-1}A^{(3)}_\infty)d\tau + (\widetilde{D}+\tau\Omega_0+A^{(3)}_\infty)\frac{dt}{nt}\right]$$

where $\widetilde{D} := D+k\cdot n\cdot Id$, here k is the number of times the (meromorphic) base change (4.14) in algorithm 2 is performed.

Hence, in the reductive case, $\underline{\omega}^{(3)}$ gives a V^+-solution to the Birkhoff problem for $(G[t^{-1}],\nabla)$.

Proof. Starting with the basis $\omega^{(0)}_i = (-f)^{i-1}\alpha$ of G_t, we construct $\underline{\omega}^{(2)}$ resp. $\underline{\omega}^{(3)}$ using lemma 4.4, proposition 4.8 and lemma 4.10 resp. lemma 4.11. In both cases, the base change matrix $P \in \mathrm{Gl}(n,\mathbb{C}[\tau^{-1}])$ defined by $\underline{\omega}^{(2)} = \underline{\omega}^{(0)}\cdot P$ resp. $\underline{\omega}^{(3)} = \underline{\omega}^{(0)}\cdot P$ has the property that $P - \mathrm{Id} \in \tau^{-1}\mathrm{Gl}(n,\mathbb{C}[\tau^{-1}])$ which shows the second statement of the first part. As to the second part, one checks that the base change steps (4.13) performed in algorithm 1 have the effect that $n\cdot t\partial_t(\underline{\widetilde{\omega}}^{(1)}) = \underline{\widetilde{\omega}}^{(1)}(\tau\Omega_0+D+\widetilde{A}^{(1)}_\infty)$, whereas step (4.14) in algorithm 2 gives $n\cdot t\partial_t(\underline{\widetilde{\omega}}^{(2)}) = \underline{\widetilde{\omega}}^{(2)}(\tau\Omega_0+D+n\cdot Id+\widetilde{A}^{(2)}_\infty)$. □

As already indicated above, we can show that the solution obtained behaves well with respect to the pairing S, provided that a technical hypothesis holds true. More precisely, we have the following statement.

Theorem 4.13. *Let $t \neq 0$. Suppose that the minimal spectral number of the tame function $f_{|D_t}$ is of multiplicity one, i.e., there is a unique $i \in \{1,\ldots,n\}$ such that $-\nu^{(3)}_i = \min_{j\in\{1,\ldots,n\}}(-\nu^{(3)}_j)$. Then $\underline{\omega}^{(3)}$ is a (V^+,S)-solution of the Birkhoff problem for (G_t,∇), i.e., $S(G_t\cap G'_t, \overline{G}_t\cap\overline{G}'_t) \subset \mathbb{C}\tau^{-w}$, where $G'_t := \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1\setminus\{0\}\times\{t\}}\,\omega^{(3)}_i$.*

Proof. The proof is essentially a refined version of the proof of the similar statement [DS04, lemma 4.1]. Denote by $\alpha_1,\ldots,\alpha_n$ a non-decreasing sequence of rational numbers such that we have an equality of sets $\{-\nu^{(3)}_1,\ldots,-\nu^{(3)}_n\} = \{\alpha_1,\ldots,\alpha_n\}$. Then, as was stated in lemma 4.6 (iv), we have $\alpha_i+\alpha_{n+1-i} = n-1$ for all $k \in \{1,\ldots,n\}$.

Let i be the index of the smallest spectral number $-\nu^{(3)}_i$. The symmetry $\alpha_k+\alpha_{n+1-k} = n-1$ implies that there is a unique $j \in \{1,\ldots,n\}$ such that $\nu^{(3)}_i+\nu^{(3)}_j = n-1$, or, equivalently, that $-\nu^{(3)}_j = \max_{l\in\{1,\ldots,n\}}(-\nu^{(3)}_l)$. Then, as in the proof of loc.cit., we have that for all $l \in \{1,\ldots,n\}$

$$S(\omega^{(3)}_i,\overline{\omega}^{(3)}_l) = \begin{cases} 0 & \text{if } \; l \neq j \\ c\cdot\tau^{-n+1}, c\in\mathbb{C} & \text{if } \; l = j \end{cases}$$

This follows from the compatibility of S with the V-filtration and the pole order property of S on the Brieskorn lattice G_t (i.e., properties (iv) (b) and (c) in definition 4.6). Suppose without loss of generality that $i < j$, if $i = j$, i.e., if there is only one spectral number, then the result is clear. Now the proof of the theorem follows from the next lemma. □

Lemma 4.14. *Let i and j as above. Then*

(i) For any $k \in \{i, \dots, j\}$, we have

$$S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) = \begin{cases} 0 & \text{for all} \quad l \neq i+j-k \\ S(\omega_i^{(3)}, \overline{\omega}_j^{(3)}) \textbf{ and } \nu_k^{(3)} + \nu_l^{(3)} = -n+1 & \text{for} \quad l = i+j-k \end{cases}$$

(ii) For any $k \in \{1, \dots, n\} \backslash \{i, \dots, j\}$, we have that

$$S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) = \begin{cases} 0 & \text{for all} \quad l \neq i+j-k \\ c_{kl} \cdot S(\omega_i^{(3)}, \overline{\omega}_j^{(3)}) \textbf{ and } \nu_k^{(3)} + \nu_l^{(3)} = -n+1 & \text{for} \quad l = i+j-k \end{cases}$$

where $c_{kl} \in \mathbb{C}$.

Proof. (i) We will prove the statement by induction over k. It is obviously true for $k = i$ by the hypothesis above. Hence we suppose that there is $r \in \{i, \dots, j\}$ such that statement in (i) is true for all k with $i \le k < r \le j$. The following identity is a direct consequence of property (iv) (a) in definition (4.6).

$$(\tau\partial_\tau + (n-1))S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) = S(\tau\partial_\tau\omega_k^{(3)}, \overline{\omega}_l^{(3)}) + S(\omega_k^{(3)}, \tau\partial_\tau\overline{\omega}_l^{(3)}) + (n-1)S(\omega_k^{(3)}, \overline{\omega}_l^{(3)})$$

$$= S(\tau\partial_\tau\omega_k^{(3)}, \overline{\omega}_l^{(3)}) + S(\omega_k^{(3)}, \overline{\tau\partial_\tau\omega_l}^{(3)}) + (n-1)S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) =$$

$$S(\tau\omega_{k+1}^{(3)} - \nu_k^{(3)}\omega_k^{(3)}, \overline{\omega}_l^{(3)}) + S(\omega_k^{(3)}, \overline{\tau\omega_{l+1}^{(3)} - \nu_l^{(3)}\omega_k^{(3)}}) + (n-1)S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) =$$

$$(n-1-\nu_k^{(3)} - \nu_l^{(3)})S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) + \tau\left(S(\omega_{k+1}^{(3)}, \overline{\omega}_l^{(3)}) - S(\omega_k^{(3)}, \overline{\omega}_{l+1}^{(3)})\right).$$

By induction hypothesis, we have that $(\tau\partial_\tau + (n-1))S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) = 0$ for all $l \in \{1, \dots, n\}$.

Now we distinguish several cases depending on the value of l: If $l \notin \{i+j-k, i+j-k-1\}$, then by the induction hypothesis, both $S(\omega_k^{(3)}, \overline{\omega}_l^{(3)})$ and $S(\omega_k^{(3)}, \overline{\omega}_{l+1}^{(3)})$ are zero. Hence it follows that $S(\omega_{k+1}^{(3)}, \overline{\omega}_l^{(3)}) = 0$ in this case.

If $l = i+j-k$ then again by the induction hypothesis we know that $(n-1) - \nu_k^{(3)} - \nu_l^{(3)} = 0$ and that moreover $S(\omega_k^{(3)}, \overline{\omega}_{l+1}^{(3)}) = 0$. Thus we have $S(\omega_{k+1}^{(3)}, \overline{\omega}_{i+j-k}^{(3)}) = 0$.

Finally, if $l = i+j-k-1$, then $S(\omega_k^{(3)}, \overline{\omega}_l^{(3)}) = 0$, and so $S(\omega_{k+1}^{(3)}, \overline{\omega}_l^{(3)}) = S(\omega_k^{(3)}, \overline{\omega}_{l+1}^{(3)})$ in other words: $S(\omega_{k+1}^{(3)}, \overline{\omega}_{i+j-(k+1)}^{(3)}) = S(\omega_k^{(3)}, \overline{\omega}_{i+j-k}^{(3)})$. In conclusion, we obtain that

$$S(\omega_{k+1}^{(3)}, \overline{\omega}_l^{(3)}) = \begin{cases} 0 & \text{if} \quad l \neq i+j-(k+1) \\ S(\omega_k^{(3)}, \overline{\omega}_{i+j-k}^{(3)}) & \text{if} \quad l = i+j-(k+1). \end{cases}$$

In order to make the induction work, it remains to show that $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} = n-1$. It is obvious that $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} \ge n-1$ for otherwise we would have $S(\omega_{k+1}^{(3)}, \overline{\omega}_{i+j-(k+1)}^{(3)}) = 0$. (Remember that it follows from the flatness of S, i.e. from condition (iv) (a) in 4.6, that $S : V_\alpha \otimes \overline{V}_{<1-\alpha+m} \to \tau^{-m}\mathbb{C}[\tau]$ for any $\alpha \in \mathbb{Q}, m \in \mathbb{Z}$, so that $S(\omega_{k+1}^{(3)}, \overline{\omega}_{i+j-(k+1)}^{(3)}) \in \tau^{-n+2}\mathbb{C}[\tau]$ if $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} < n-1$, which is impossible since $S : G_t \otimes_{\mathbb{C}[\tau^{-1}]} \overline{G}_t \to \tau^{-n+1}\mathbb{C}[\tau^{-1}]$). Thus the only case to exclude is $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} > n-1$.

First notice that it follows from property (iv) (c) of definition 4.6 that S induces an isomorphism

$$\overline{\tau^{n-1}G_t} \cong G_t^* := \mathit{Hom}_{\mathbb{C}[\tau^{-1}]}(G_t, \mathbb{C}[\tau^{-1}]).$$

On the other hand, we deduce from [Sab06, remark 3.6] that for any $\alpha \in \{-\nu_1^{(3)}, \dots, -\nu_n^{(3)}\}$,

$$gr_\alpha^{V^*}(G_t^*/\tau^{-1}G_t^*) \cong gr_{-\alpha}^V(G_t/\tau^{-1}G_t),$$

where V^* denotes the canonical V-filtration on the dual module $(G_t, \nabla)^*$. In conclusion, S induces a non-degenerate pairing

$$S : gr^V_\alpha(G_t/\tau^{-1}G_t) \otimes gr^V_{n-1-\alpha}(G_t/\tau^{-1}G_t) \to \tau^{-n+1}\mathbb{C}$$

which yields a non-degenerate pairing on the sum $gr^V_\bullet(G_t/\tau^{-1}G_t) := \oplus_{\alpha\in\mathbb{Q}} gr^V_\alpha(G_t/\tau^{-1}G_t)$. However, we know that $\underline{\omega}^{(3)}$ induces a basis of $gr^V_\bullet(G_t/\tau^{-1}G_t)$, compatible with the above decomposition. This, together with the fact that $S(\omega^{(3)}_{k+1}, \overline{\omega}^{(3)}_l) \in \tau^{-n+1}\mathbb{C}\delta_{i+j,k+1+l}$, yields that $-\nu^{(3)}_{k+1} - \nu^{(3)}_{i+j-(k+1)} = n-1$, as required.

(ii) For this second statement, we consider the constant (in τ^{-1}) base change given by $\omega'^{(3)}_{k+1} := \omega^{(3)}_{j+k}$ for all $k \in \{0, \ldots, n-j\}$ and $\omega'^{(3)}_{k+1+n-j} := t\omega^{(3)}_k$ for all $k \in \{1, \ldots, j-1\}$. Then we have

$$\partial_\tau(\underline{\omega}'^{(3)}) = \underline{\omega}'^{(3)} \cdot (\Omega_0 + \tau^{-1}(A^{(3)}_\infty)'),$$

where $(A^{(3)}_\infty)' = \mathrm{diag}(-\nu^{(3)}_j, -\nu^{(3)}_{j+1}, \ldots, -\nu^{(3)}_n, -\nu^{(3)}_1, \ldots, -\nu^{(3)}_{j-1})$. Now the proof of (i) works verbatim for the basis $\underline{\omega}'^{(3)}$, with the index i from above replaced by 1 and the index j from above replaced by $n-j+i+2$. Notice that then the spectral number corresponding to 1 is the biggest one and the one corresponding to $n-j+i+2$ is the smallest one, but this does not affect the proof. Depending on the value of the indices k and l, we have that $c_{kl}(t)$ is either t^{-1}, 1 or t.

□

5 Frobenius structures

5.1 Frobenius structures for linear functions on Milnor fibres

In this subsection, we derive one of the main results of this paper: the existence of a Frobenius structure on the unfolding space of the function $f_{|D_t}, t \neq 0$. Depending on whether we restrict to the class of examples satisfying the hypotheses of theorem 4.13, the Frobenius structure can be derived directly from the (V^+, S)-solution $\underline{\omega}^{(3)}$ of the Birkhoff problem constructed in the last section, or otherwise is obtained by appealing to theorem 4.7. We refer to [Her02] or [Sab02] for the definition of a Frobenius manifold. It is well known that a Frobenius structure on a complex manifold M is equivalent to the following set of data.

(i) a holomorphic vector bundle E on $\mathbb{P}^1 \times M$ such that $\mathrm{rank}(E) = \dim(M)$, which is fibrewise trivial, i.e. $\mathcal{E} = p^*p_*\mathcal{E}$, (where $p : \mathbb{P}^1 \times M \to M$ is the projection) equipped with an integrable connection with a logarithmic pole along $\{\infty\} \times M$ and a pole of type one along $\{0\} \times M$,

(ii) an integer w,

(iii) A non-degenerate, $(-1)^w$-symmetric pairing $S : \mathcal{E} \otimes j^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1\times M}(-w, w)$ (here $j(\tau, u) = (-\tau, u)$, with, as before, τ a coordinate on $\mathbb{P}^1$ centered at infinity and u a coordinate on M and we write $\mathcal{O}_{\mathbb{P}^1\times M}(a, b)$ for the sheaf of meromorphic functions on $\mathbb{P}^1 \times M$ with a pole of order a along $\{0\} \times M$ and order b along $\{\infty\} \times M$) the restriction of which to $\mathbb{C}^* \times M$ is flat,

(iv) A global section $\xi \in H^0(\mathbb{P}^1 \times M, \mathcal{E})$, which is flat with respect to the residue connection $\nabla^{res} : \mathcal{E}/\tau\mathcal{E} \to \mathcal{E}/\tau\mathcal{E} \otimes \Omega^1_M$ with the following two properties

  (a) The morphism

$$\begin{aligned} \Phi_\xi : \mathcal{T}_M &\longrightarrow \mathcal{E}/\tau^{-1}\mathcal{E} \cong p_*\mathcal{E} \\ X &\longmapsto -[\tau^{-1}\nabla_X](\xi) \end{aligned}$$

  is an isomorphism of vector bundles (a section ξ with this property is called primitive),

  (b) ξ is an eigenvector of the residue endomorphism $[\tau\nabla_\tau] \in \mathcal{E}nd_{\mathcal{O}_M}(p_*\mathcal{E}) \cong \mathcal{E}nd_{\mathcal{O}_M}(\mathcal{E}/\tau\mathcal{E})$ (a section with this property is called homogeneous).

In many application one is only interested in constructing a Frobenius structure on a germ at a given point, in that case M is a sufficiently small representative of such a germ.
We now come back to our situation of a $\mathcal{R}_h$-finite linear section f on the Milnor fibration $h : V \to T$. In this subsection, we are interested to construct Frobenius structures on the (germ of a) semi-universal unfolding of the function $f_{|D_t}$, $t \neq 0$. It is well known that in contrast to the local case, such an unfolding does not have obvious universality properties. One defines, according to [DS03, 2.a.], a deformation

$$F = f + \sum_{i=1}^{n} u_i g_i : B_t \times M \to D$$

of the restriction $f_{|B_t}$ to some intersection $D_t \cap B_\epsilon$ such that the critical locus C of F is finite over M via the projection $q : B_t \times M \twoheadrightarrow M$ to be a semi-universal unfolding if the Kodaira-Spencer map $\mathcal{T}_M \to q_*\mathcal{O}_C$, $X \mapsto [X(F)]$ is an isomorphism.
From proposition 3.4 we know that any basis $g_1, \ldots, g_n$ of $T^1_{\mathscr{R}_h} f$ gives a representative

$$F = f + \sum_{i=1}^{n} u_i g_i : B_t \times M \to D$$

of this unfolding, where M is a sufficiently small neighborhood of the origin in $\mathbb{C}^n$, with coordinates $u_1, \ldots, u_n$.
In order to exhibit Frobenius structures via the approach sketched in the beginning of this section, one has to find a (V^+, S)-solution to the Birkhoff problem for G_t. If the minimal spectral number of (G_t, ∇) has multiplicity one, then, according to corollary 4.12 and theorem 4.13, the basis $\underline{\omega}^{(3)}$ yields such a solution, which we denote by $\widehat{G}_t$ (which is, if D is reductive, the restriction of $\widehat{G}$ from corollary 4.12 (ii) to $\mathbb{P}^1 \times \{t\}$). Otherwise, we consider the canonical solution from theorem 4.7, which is denoted by $\widehat{G}_t^{can}$. The bundle called $\mathcal{E}$ in the beginning of this subsection is then obtained by *unfolding* the solution $\widehat{G}_t$ resp. $\widehat{G}_t^{can}$. We will not describe $\mathcal{E}$ explicitly, but use a standard result due to Dubrovin which gives directly the corresponding Frobenius structure provided that one can construct a homogenous and primitive form for $\widehat{G}_t$ resp. $\widehat{G}_t^{can}$, i.e., a section called ξ above at the point t.
We can now state and prove the main result of this section.

Theorem 5.1. *Let $f \in \mathbb{C}[V]_1$ be an $\mathcal{R}_h$-finite linear function. Write M_t for the parameter space of a semi-universal unfolding $F : B_t \times M_t \to D$ of $f_{|B_t}$, $t \neq 0$ as described above. Let $\alpha_{\min} = \alpha_1$ be the minimal spectral number of (G_t, ∇).*

(i) Suppose that $\alpha_{\min}$ has multiplicity one, i.e., $\alpha_2 > \alpha_1$. Then any of the sections $t^{-k}\omega_i^{(3)} \in H^0(\mathbb{P}^1, \widehat{G}_t)$ is primitive and homogenous. Any choice of such a section yields a Frobenius structure on M_t, which we denote by $(M_t, \circ, g, e, E)^{(i)}$.

(ii) Without any assumption on the multiplicity of $\alpha_{\min}$, we have that any of the sections $\omega_i^{(3)}$ which lies in $H^0(\mathbb{P}^1, \widehat{G}_t^{can}) \cap V_\alpha$, where $\alpha \in [\alpha_{min}, \alpha_{min} + 1)$ is primitive and homogenous and hence yields a Frobenius structure on M_t, denoted by $(M_t, \circ, g, e, E)^{(i),can}$.

Remark: In (i), we can also use the sections $\omega_i^{(3)}$ or any other non-zero multiple of them to construct Frobenius structures. However, for technical reasons, it is preferable to work with the rescaled sections $t^{-k}\omega_i^{(3)}$ (see proposition 5.4).

Proof. In both cases, we use the universal semi-simple Frobenius structure defined by a finite set of given initial data as constructed by Dubrovin ([Dub96], see also [Sab02, théorème VII.4.2]). The initial set of data we need to construct is

(i) an n-dimensional complex vector space W,

(ii) a symmetric, bilinear, non-degenerate pairing $g : W \otimes_{\mathbb{C}} W \to \mathbb{C}$,

(iii) two endomorphisms $B_0, B_\infty \in \mathrm{End}_{\mathbb{C}}(W)$ such that B_0 is semi-simple with distinct eigenvalues and g-selfadjoint and such that $B_\infty + B_\infty^* = (n-1)\mathrm{Id}$, where B_∞^* is the g-adjoint of B_∞.

(iv) an eigenvector $\xi \in W$ for B_∞, which is a cyclic generator of W with respect to B_0.

In both cases of the theorem, the vector space W will be identified with $G_t/\tau^{-1}G_t$. Dubrovin's theorem yields a germ of a universal Frobenius structure on a certain n-dimensional manifold such that its first structure connection restricts to the data $(W, B_0, B_\infty, g, \xi)$ over the origin. The universality property then induces a Frobenius structure on the germ $(M_t, 0)$, as the tangent space of the latter at the origin is canonically identified with $T^1_{\mathscr{R}_h} f \cong T^1_{\mathscr{R}_h} f \cdot \alpha \cong G_t/\tau^{-1}G_t$.
Let us show how to construct the initial data needed in case (i) and (ii) of the theorem:

(i) We put $W := H^0(\mathbb{P}^1, \widehat{G}_t)$, $g := \tau^{-n+1}S$ (notice that this is possible due to theorem 4.13), $B_0 := [\nabla_\tau] \in \mathrm{End}_{\mathbb{C}}(G_t/\tau^{-1}G_t) \cong \mathrm{End}_{\mathbb{C}}(W)$ and $B_\infty := [\tau\nabla_\tau] \in \mathrm{End}_{\mathbb{C}}(\widehat{G}_t/\tau\widehat{G}_t) \cong \mathrm{End}_{\mathbb{C}}(W)$. In order to verify the conditions from above on these initial data, consider the basis $t^{-k}\underline{\omega}^{(3)}$ of W. Then B_0 is given by the matrix Ω_0, which is obviously semi-simple with distinct eigenvalues (these are the critical values of $f_{|D_t}$). Its self-adjointsness follows from the flatness of S. The endomorphism B_∞ corresponds to the matrix $\widetilde{A}_\infty$, so that the symmetry of the spectrum as well as the proof of lemma 4.14 show that $B_\infty + B_\infty^* = (n-1)\mathrm{Id}$. Finally, it follows from corollary 4.12 that for all $i \in \{1, \dots, n\}$, the class of $t^{-k}\omega_i^{(3)}$ in $G_t/\tau^{-1}G_t$ is equal to the class of $f^{i-1}\alpha$. By definition, $B_0 = [\nabla_\tau]$ is the multiplication by $-f$ on $W \cong G_t/\tau^{-1}G_t$, hence, any of the classes of the sections $t^{-k}\omega_i^{(3)}$ is a cyclic generator of W with respect to $[\nabla_\tau]$. It is homogenous, i.e., an eigenvector of B_∞ by construction. This proves the theorem in case (i).

(ii) We use the canonical solution to the Birkhoff problem described in theorem 4.7. Namely, we put $W := H^0(\mathbb{P}^1, \widehat{G}_t^{can})$ and again $g := \tau^{-n+1}S$, $B_0 := [\nabla_\tau] \in \mathrm{End}_{\mathbb{C}}(G_t/\tau^{-1}G_t) \cong \mathrm{End}_{\mathbb{C}}(W)$ and $B_\infty := [\tau\nabla_\tau] \in \mathrm{End}_{\mathbb{C}}(\widehat{G}_t^{can}/\tau\widehat{G}_t^{can}) \cong \mathrm{End}_{\mathbb{C}}(W)$. The eigenvalues of the endomorphism B_∞ are always the critical values of $f_{|D_t}$ so as in (i) it follows that B_∞ is semi-simple with distinct eigenvalues. It is g-self-adjoint by the same argument as in (i). B_∞ is also semi-simple, as $\widehat{G}_t^{can}$ is a V^+-solution. The property $B_\infty + B_\infty^* = (n-1)\mathrm{Id}$ follows then as in (i) by the fact that it is also a (V^+, S)-solution (more precisely, by choosing a basis $\underline{w}$ of W such that B_∞ is again given by the matrix $\widetilde{A}_\infty$ and such that $g(w_i, w_j) = 1$ if $-\widetilde{\nu}_i - \widetilde{\nu}_j = n-1$ and $g(w_i, w_j) = 0$ otherwise). In order to obtain a primitive form, we use the fact (see [DS03, appendix B.b.]) that the space $H^0(\mathbb{P}^1, G_t) \cap V_\alpha$ for $\alpha \in [\alpha_{min}, \alpha_{min}+1)$ is independent of the choice of the V^+ solution $\widehat{G}_t$ of the Birkhoff problem for (G_t, ∇). Hence the same argument as in (i) shows that any section $\omega_i^{(3)}$ which is an element in $H^0(\mathbb{P}^1, \widehat{G}_t^{can}) \cap V_\alpha$ is primitive and homogenous.

□

The previous theorem yields for fixed i Frobenius structures $M_t^{(i)}$ resp. $M_t^{(i),can}$ for any t. One might ask whether they are related in some way. It turns out that for a specific choice of the index i they are (at least in the reductive case), namely, one of them can be seen as analytic continuation of the other. The proof relies on the fact that it is possible to construct a Frobenius structure from the bundle G simultanously for all values of t at least on a small disc outside of $t = 0$. This is done using a generalization of Dubrovins theorem, due to Hertling and Manin [HM04, theorem 4.5]. In loc.cit., Frobenius manifolds are constructed from so-called "trTLEP-structures". The following result shows how they arise in our situation.

Lemma 5.2. *Suppose that D is reductive. Fix $t \in T$ and suppose that the minimal spectral number α_{min} of (G_t, ∇) has multiplicity one, so that theorem 4.13 applies. Let Δ_t be a sufficiently small disc centered at t. Denote by $\widehat{H}^{(t)}$ the restriction to Δ_t of the analytic bundle corresponding to $\widehat{G}$. Then $\widehat{H}^{(t)}$ underlies a trTLEP-structure on Δ_t, and any of the sections $t^{-k}\omega_i^{(3)}$ satisfy the conditions (IC), (GC) and (EC) of [HM04, theorem 4.5]. Hence, the construction in loc.cit. yields a universal Frobenius structure on a germ $(\widetilde{M}^{(i)}, t) := (\Delta_t \times \mathbb{C}^{n-1}, (t, 0))$.*

Proof. That $\widehat{H}^{(t)}$ underlies a trTLEP-structure is a consequence of corollary 4.12 (i) and theorem 4.13. We have already seen that the sections $t^{-k}\omega_i^{(3)}$ are homogenous and primitive, i.e., satisfy conditions (EC) and (GC) of loc.cit. It follows from the connection form computed in corollary 4.12 (ii), that they also satisfy condition (IC). Thus the theorem of Hertling and Manin gives a universal Frobenius structure on $\widetilde{M}^{(i)}$ such that its first structure connection restricts to $\widehat{H}^{(t)}$ on Δ_t. □

In order to apply this lemma we need to find a homogenous and primitive section of $\widehat{H}^{(t)}$ which is also ∇_t-flat. This is done in the following lemma.

Lemma 5.3. *Let D be reductive. Consider the V^+-solution to the Birkhoff-problem for (G_0, ∇) resp. (G_t, ∇) given by $\underline{\omega}^{(2)}$ resp. $\underline{\omega}^{(3)}$. Then there is an index $j \in \{1, \ldots, n\}$ such that $\deg(\omega_j^{(2)}) = -\nu_i^{(2)}$ and an index $i \in \{1, \ldots, n\}$ such that $\deg(\omega_i^{(3)}) = -\nu_i^{(3)} + k \cdot n$. In particular, $\nu_j^{(2)}, \nu_i^{(3)} \in \mathbb{N}$. Moreover, $\nabla_t^{res}(t^{-k}\omega_i^{(3)}) = 0$, where $\nabla_t^{res} : \widehat{G}/\tau\widehat{G} \to \widehat{G}/\tau\widehat{G}$ is the residue connection.*

Proof. By construction we have $\omega_1^{(1)} = \omega_1^{(0)} = \alpha$, so in particular $\deg(\omega_1^{(1)}) = 0$. We also have $\nu^{(1)} = 0$. Now it suffices to remark that in algorithm 1 (formula (4.13)), whenever we have an index $j \in \{1, \ldots, n\}$ with $\deg(\omega_j^{(1)}) = -\nu_j^{(1)}$, then either $\deg(\widetilde{\omega}_j^{(1)}) = -\widetilde{\nu}_j^{(1)}$ (this happens if the index i in formula (4.13) is different from $j+1$) or $\deg(\widetilde{\omega}_{j+1}^{(1)}) = -\widetilde{\nu}_{j+1}^{(1)}$ (if $i = j+1$). It follows that we always conserve some index k with $\deg(\widetilde{\omega}_k^{(1)}) = -\widetilde{\nu}_k^{(1)}$. A similar argument works for algorithm 2, which gives the second statement of the first part. The residue connection is given by the matrix $\frac{1}{nt}\left(\widetilde{D} + A_\infty^{(3)}\right)$ in the basis $\underline{\omega}^{(3)}$ of $\widehat{G}/\tau\widehat{G}$ (see corollary 4.12 (ii)). This yields the ∇^{res}-flatness of $t^{-k}\omega_i^{(3)}$. □

Finally, the comparision result can be stated as follows.

Proposition 5.4. *Let i be the index from the previous lemma such that $\nabla^{res}(t^{-k}\omega_i^{(3)}) = 0$. Then for any $t' \in \Delta_t$, the germs of Frobenius structures $(\widetilde{M}^{(i)}, t')$ (from lemma 5.2) and $(M^{(i)}, t')$ (from theorem 5.1) are isomorphic.*

Proof. We argue as in [Dou08, proposition 5.5.2]: The trTLEP-structure $\widehat{H}^{(t)}$ is a deformation (in the sense of [HM04, definition 2.3]) of the fibre $\widehat{G}/t'\widehat{G}$, hence contained in the universal deformation of the latter. Thus the (germs at t' of the) universal deformations of $\widehat{H}^{(t)}$ and $\widehat{G}/t'\widehat{G}$ are isomorphic. This gives the result as the homogenous and primitive section $t^{-k}\omega_i^{(3)}$ of $\widehat{H}^{(t)}$ that we choose in order to apply lemma 5.2 is ∇^{res}-flat. □

5.2 Limit Frobenius structures

In the last subsection, we constructed Frobenius structures on the unfolding spaces M_t. It is a natural question to know whether one can attach a "limit" Frobenius structure to the restriction of f on D. In order to carry this out, one is faced with the difficulty that the pairing S from theorem 4.7 is not, a priori, defined on $\mathbf{G}_0$. Hence a more precise control on this pairing on $\mathbf{G}[t^{-1}]$ is needed in order to make a statement on the limit. The following conjecture provides exactly this additional information.

Conjecture 5.5. The pairing S from theorem 4.7 is defined on $\mathbf{G}[t^{-1}]$ and meromorphic at $t = 0$, i.e., induces a pairing $S : \mathbf{G}[t^{-1}] \otimes \overline{\mathbf{G}}[t^{-1}] \to \mathbb{C}[\tau, \tau^{-1}, t, t^{-1}]$. Moreover, consider the natural grading of $\mathbf{G}$ resp. on $\mathbf{G}[t^{-1}]$ induced from the grading of $\Omega^{n-1}(\log h)$ by putting $\deg(\tau) = -1$ and $\deg(t) = n$. Then

(i) S is homogenous, i.e., it sends $(\mathbf{G}[t^{-1}])_k \otimes (\overline{\mathbf{G}[t^{-1}]})_l$ into $\mathbb{C}[\tau, \tau^{-1}, t, t^{-1}]_{k+l}$ (remember that the ring $\mathbb{C}[\tau, \tau^{-1}, t, t^{-1}]$ is graded by $\deg(\tau) = -1$, $\deg(t) = n$).

(ii) S sends G into $\tau^{-n+1}\mathbb{C}[\tau^{-1}, t]$.

Some evidence supporting the first part of this conjecture comes from the computation of the examples in section 6. Namely, it appears that in all cases, there is an extra symmetry satisfied by the spectral numbers, i.e., we have $\nu_k^{(3)} + \nu_{n+1-k}^{(3)} = -n+1$, and not only $\alpha_k + \alpha_{n+1-k} = n-1$ for all $k \in \{1, \ldots, n\}$ (Remember that $\alpha_1, \ldots, \alpha_n$ was the ordered sequence of spectral numbers). Moreover, the eigenvalues of the residue of $t\partial_t$ on $(G/tG)_{|\tau \neq 0}$ are constant in τ and symmetric around zero, which indicates that S extends without poles and as a non-degenerate pairing to G. In particular, one obtains a pairing on G_0, which would explain the symmetry $\nu_k^{(2)} + \nu_{n+1-k}^{(2)} = -n+1$ observed in the examples (notice that even the symmetry of the spectral numbers at $t = 0$, written as an order sequence, is not a priori clear).
The following corollary draws some consequences of the above conjecture.

Corollary 5.6. *Suppose that conjecture 5.5 holds true and that the minimal spectral number α_{min} of (G_t, ∇), $t \neq 0$ has multiplicity one so that theorem 4.13 holds. Then*

(i) The pairing S is expressed in the basis $\underline{\omega}^{(3)}$ as

$$S(\omega_i^{(3)}, \overline{\omega}_j^{(3)}) = \begin{cases} c \cdot t^{2k} \cdot \tau^{-n+1} & \text{if} \quad i + j = n+1 \\ 0 & \text{else} \end{cases}$$

for some constant $c \in \mathbb{C}$, where, as before, $k \in \mathbb{N}$ counts the number of meromorphic base changes in algorithm 2. Moreover, we have $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n+1$ for all $i \in \{1, \ldots, n\}$.

(ii) The pairing S is expressed in the basis $\underline{\omega}^{(2)}$ as

$$S(\omega_i^{(2)}, \overline{\omega}_j^{(2)}) = \begin{cases} c \cdot \tau^{-n+1} & \text{if} \quad i+j = n+1 \\ 0 & \text{else} \end{cases}$$

for the same constant $c \in \mathbb{C}$ as in (i).

(iii) S extends to a non-degenerate paring on G, i.e., it induces a pairing $S : G_0 \otimes_{\mathbb{C}[\tau^{-1}]} \overline{G}_0 \to \tau^{-n+1}\mathbb{C}[\tau^{-1}]$ with all the properties of definition 4.6 (iv). Moreover, $\underline{\omega}^{(2)}$ defines a (V^+, S)-solution for the Birkhoff problem for (G_0, ∇) with respect to S.

Proof. (i) Following the construction of the bases $\underline{\omega}^{(1)}$, $\underline{\omega}^{(2)}$ and $\underline{\omega}^{(3)}$, starting from the basis $\underline{\omega}^{(0)}$ (via lemma 4.4 and algorithms 1 and 2), it is easily seen that $\deg(\omega_i^{(1)}) = \deg(\omega_i^{(2)}) = i-1$ and that $\deg(\omega_i^{(3)}) = k \cdot n + i - 1$. The (conjectured) homogeneity of S yields that $S(\omega_i^{(2)}, \overline{\omega}_j^{(2)}) = i+j-2$ and $\deg(S(\omega_i^{(3)}, \overline{\omega}_j^{(3)}) = 2kn + i + j - 2$.

The proof of lemma 4.14 shows that $\tau^{n-1} S(\omega_i^{(3)}, \overline{\omega}_j^{(3)})$ is either zero or constant in τ, hence, by part (ii) of conjecture 5.5, $S(\omega_i^{(3)}, \overline{\omega}_j^{(3)}) = c(t) \cdot \tau^{-n+1}$, with $c(t) \in \mathbb{C}[t]$, which is actually homogenous by part (i) of conjecture 5.5. Now since $i + j - 2 < 2(n-1)$, $\deg(c(t) \cdot \tau^{-n+1}) = 2kn + (i+j-2)$ is only possible if $i + j = n+1$, and then $c(t) = c \cdot t^{2k}$, in particular, the numbers c_{kl} in lemma 4.14 (ii) are always equal to one, and we have $\nu_i^{(3)} + \nu_j^{(3)} = -n+1$.

(ii) Using (i), one has to analyse the behaviour of S under the base changes inverse to 4.13 (algorithm 1) and 4.14 (algorithm 2). Suppose that $\underline{\omega}$ is a basis of $\mathbf{G}[t^{-1}]$ with $\deg(\omega_i) = l \cdot n + i - 1$, $l \in \{0, \ldots, k\}$ and such that $S(\omega_i, \overline{\omega}_j) = c \cdot t^{2l} \cdot \tau^{-n+1} \cdot \delta_{i+j,n+1}$, then if we define for any $i \in \{1, \ldots, n\}$ a new basis $\underline{\omega}'$ by

$$\begin{array}{rcl} \omega'_i & := & \omega_i - \tau^{-1} \cdot \nu \cdot \omega_{i-1}, \\ \omega'_j & := & \omega_j \quad \forall j \neq i, \end{array} \tag{5.1}$$

where $\nu \in \mathbb{C}$ is any constant, we see that we still have $S(\omega'_i, \overline{\omega}'_j) = c \cdot t^{2l} \cdot \tau^{-n+1} \cdot \delta_{i+j,n+1}$. Notice that if $j = i+1$ and $i + j = n+1$, then in order to show $S(\omega'_i, \overline{\omega}'_i) = 0$, one uses that if $i + (i-1) = n+1$, then $S(\omega_i, \overline{\omega}_{i-1}) = (-1)^{n-1}\overline{S(\omega_{i-1}, \overline{\omega}_i)} = S(\omega_{i-1}, \overline{\omega}_i)$ since $S(\omega_{i-1}, \overline{\omega}_i)$ is homogenous in τ^{-1} of degree $-n+1$.

Similarly, if we put, for any constant $\nu \in \mathbb{C}$,

$$\begin{array}{rcl} \omega''_1 & := & t^{-2}\omega_1 - t^{-1}\tau^{-1} \cdot \nu \cdot \omega_n, \\ \omega''_i & := & t^{-1}\omega_i \quad \forall i \neq 1, \end{array} \tag{5.2}$$

then we have $S(\omega''_i, \overline{\omega}''_j) = c \cdot t^{2(l-1)} \cdot \tau^{-n+1} \cdot \delta_{i+j,n+1}$.

(iii) This follows from (ii) and the fact that $\underline{\omega}^{(2)}$ is a V^+-solution for (G_0, ∇).

$\square$

As a consequence, we show that under the hypothesis of conjecture 5.5, we obtain indeed a limit Frobenius structure.

Theorem 5.7. *Suppose that conjecture 5.5 holds true and that the minimal spectral number α_{min} of (G_t, ∇) for $t \neq 0$ has multiplicity one, so that theorem 4.13 applies. Then the (germ at the origin of the) $\mathcal{R}_h$-deformation space of f, which we call M_0, carries a Frobenius structure, which is constant, i.e., given by a potential of degree at most three (or, expressed otherwise, such that the structure constants c_{ij}^k defined by $\partial_{t_i} \circ \partial_{t_j} = \sum_k c_{ij}^k \partial_{t_k}$ are constant in the flat coordinates $t_1, \ldots, t_n$). It is the limit of the Frobenius structures $M_t^{(1)}$ for $t \to 0$ if $\underline{\omega}^{(3)} = \underline{\omega}^{(2)}$.*

Proof. Remember that $(M_0, 0)$ is a smooth germ of dimension n, with tangent space given by $T^1_{\mathcal{R}_h} f \cong G_0/\tau^{-1}G_0$ (notice that the deformation functor in question is evidently unobstructed). As usual, a $\mathcal{R}_h$-semi-universal unfolding of f is given as

$$F = f + \sum_{i=1}^{n} u_i g_i : V \times M_0 \longrightarrow \mathbb{C},$$

where $u_1, \ldots, u_n$ are coordinates on M_0 and $g_1, \ldots, g_n$ is a basis of $T^1_{\mathcal{R}_h} f$.
In order to endow M_0 with a Frobenius structure, we will use a similar strategy as in subsection 5.1, namely, we construct a germ of an n-dimensional Frobenius manifold which induces a Frobenius structure on M_0 by a universality property. However, we have to use, as in lemma 5.2, the generalization due to Hertling and Manin instead of Dubrovins classical result as the Frobenius manifold we obtain will not be semi-simple. We will use a particular case of this result, namely, [HM04, remark 4.6]. Thus we have to construct a *Frobenius type structure* on a point, and a section satisfying the conditions called (GC) and (EC) in loc.cit. This is nothing but a tuple $(W, g, B_0, B_\infty, \xi)$ as in the proof of theorem 5.1, except that we do not require the endomorphism B_0 to be semi-simple, but to be *regular*, i.e., its characteristic and minimal polynomial must coincide. Consider the (V^+, S)-solution defined by $\widehat{G}_0 := \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1 \times \{0\}} \, \omega_i^{(2)}$, and put, as before, $W := H^0(\mathbb{P}^1, \widehat{G}_0)$, $g := \tau^{-n+1} S$, $B_0 := [\nabla_\tau]$ and $B_\infty := [\tau \nabla_\tau]$. Considering the matrices $(\Omega_0)_{|t=0}$ resp. $A_\infty^{(2)}$ of B_0 resp. B_∞ with respect to the basis $\underline{\omega}^{(2)}$ of W, we see immediately that $g(B_0-,-) = g(-,B_0-)$, $g(B_\infty-,-) = g(-,(n-1)\mathrm{Id} - B_\infty-)$ and that B_0 is regular since $(\Omega_0)_{|t=0}$ is nilpotent with a single Jordan block. Notice that the assumption that conjecture 5.5 holds is used through corollary 5.6 (ii), (iii). The section $\xi := \omega_1^{(2)}$ is obviously homogenous and primitive, i.e., sastisfies (EC) and (GC). Notice that it is, up to constant multiplication, the only primitive section, contrary to the case $t \neq 0$, where we could chose any of the sections $\omega_i^{(3)}$, $i \in \{1, \ldots, n\}$. We have thus verified all conditions of the theorem of Hertling and Manin, and obtain, as indicated above, a Frobenius structure on M_0, which might be considered as a limit $\lim_{t \to 0} M_t^{(1)}$ if the two modules G and $G^{(3)}$ coincide, i.e., if $\omega^{(2)} = \omega^{(3)}$ (equivalently, if $k = 0$).
It remains to show that it is given by potential of degree at most three. The argument is exactly the same as in [Dou08, lemma 6.4.1 and corollary 6.4.2.] so that we omit the details here. The main point is that if one uses the unfolding theorem [HM04, theorem 4.5], then it is easily seen that the matrices of the Higgs field in the natural basis given by the flat extension of $\underline{\omega}^{(2)}$ are polynomials in the matrix $(\Omega_0)_{|t=0}$, but their first columns are fixed, by the fact that $\omega_1^{(2)}$ is the primitive form, i.e., corresponds to the unit field on M_0. In particular, they are constant. □

5.3 Logarithmic Frobenius structures

The pole order property of the connection ∇ on G (see formula 4.12) suggests that the family of germs of Frobenius manifolds M_t studied above can be put together in a single Frobenius manifold with a *logarithmic* degeneration behavior at the divisor $t = 0$. We show that this is actually the case for the normal crossing divisor, the same result has been obtained from a slightly different viewpoint in [Dou08]. In the general case, we observe a phenomenon which also occurs in loc.cit.: one obtains a Frobenius manifold where the multiplication is defined on the logarithmic tangent bundle, but the metric might be degenerate on it (see cit.loc., section 7.1.).
We recall the following definition from [Rei08], which we extend to the more general situation studied here.

Definition 5.8. (i) Let M be a complex manifold and $\Sigma \subset M$ be a normal crossing divisor. Suppose that $(M\backslash\Sigma, \circ, g, E, e)$ is a Frobenius manifold. One says that it has a logarithmic pole along Σ if $\circ \in \Omega^1(\log \Sigma)^{\otimes 2} \otimes \mathrm{Der}(-\log \Sigma)$, $g \in \Omega^1(\log \Sigma)^{\otimes 2}$ and g is non-degenerate as a pairing on $\mathrm{Der}(-\log \Sigma)$.

(ii) If, in the previous definition, we relax the condition of g being non-degenerate on $\mathrm{Der}(-\log \Sigma)$, then we say that (M, Σ) is a weak logarithmic Frobenius manifold.

In [Rei08], logarithmic Frobenius manifolds are constructed using a generalisation of the main theorem of [HM04]. More precisely, universal deformations of so called "logΣ-trTLEP-structures" (see [Rei08, definition 1.8.]) are constructed. In our situation, the base of such an object is the space T, and the divisor $\Sigma := \{0\} \subset T$. In order to adapt the construction to the more general situation that we discuss here, we define a weak logΣ-trTLEP-structures to be such a vector bundle on $\mathbb{P}^1 \times T$ with connection and pairing, where the latter is supposed to be non-degenerate only on $\mathbb{P}^1 \times T\backslash\Sigma$. The result can then be stated as follows.

Theorem 5.9. *Let D be reductive, $i \in \{1, \ldots, n\}$ be the index from lemma 5.3 such that $\deg(t^{-k}\omega_i) = -\nu_i^{(3)}$ and suppose that the minimal spectral number α_{min} of (G_t, ∇) has multiplicity one (so that theorem 4.7 applies).*

Then the (analytic bundle corresponding to the) module

$$\begin{aligned} \widehat{G}' &:= \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}^1\times T}\,\omega_j^{(4)} \quad \text{where} \\ \omega_j^{(4)} &:= t^{-k}\omega_j^{(3)} \quad \forall j\in\{i,\ldots,n\} \\ \omega_j^{(4)} &:= t^{-k+1}\omega_j^{(3)} \quad \forall j\in\{1,\ldots,i-1\} \end{aligned}$$

underlies a weak $\log\Sigma$*-trTLEP-structure, and a* $\log\Sigma$*-trTLEP-structure if conjecture 5.5 holds true and if* $i=1$*. The form* $t^{-k}\omega^{(3)}$ *is homogenous and primitive and yields a weak logarithmic Frobenius manifold. It yields a logarithmic Frobenius manifold if conjecture 5.5 holds true and if* $i=1$*, e.g., in the case of a linear section* f *of the normal crossing divisor.*

Proof. It is clear by definition that $(\widehat{G}',\nabla,S)$ is a weak $\log\Sigma$-trTLEP-structure (of weight $n-1$). It is easy to see that the connection takes the form

$$\nabla(\underline{\omega}^{(4)}) = \underline{\omega}^{(4)}\cdot\left[(\Omega_0\tau + A_\infty^{(4)})\frac{d\tau}{\tau} + (\Omega_0\tau + \widetilde{A}_\infty^{(4)})\frac{dt}{nt}\right],$$

where

$$\begin{aligned} A_\infty^{(4)} &= \mathrm{diag}(-\nu_i^{(3)},\ldots,-\nu_n^{(3)},-\nu_1^{(3)},\ldots,-\nu_{i-1}^{(3)}) \\ \widetilde{A}_\infty^{(4)} &= A_\infty^{(4)} + \mathrm{diag}\left(\deg(\omega_i^{(4)}),\ldots,\deg(\omega_n^{(4)}),\deg(\omega_1^{(4)}),\ldots,\deg(\omega_{i-1}^{(4)})\right). \end{aligned}$$

In particular, $\omega_1^{(4)}$ is ∇^{res}-flat, $[\nabla_\tau]$-homogenous and a cyclic generator of $H^0(\mathbb{P}^1\times\{0\},\widehat{G}'/t\widehat{G}')$ with respect to $[\nabla_\tau]$ and $[\tau^{-1}\nabla_{t\partial_t}]$ (even with respect to $[\nabla_\tau]$ alone). Moreover, $[\tau^{-1}\nabla_{t\partial_t}(\omega_1^{(4)})]$ is non-zero in $H^0(\mathbb{P}^1\times\{0\},\widehat{G}'/t\widehat{G}')$, so that $\omega_1^{(4)}$ satisfies the conditions (EC), (GC) and (IC) of [Rei08, theorem 1.12], except that the form S might be degenerate on $\widehat{G}'_{|t=0}$ (correspondingly, the metric g on $K:=\widehat{G}'/\tau\widehat{G}'$ from loc.cit. might be degenerate on $K_{|t=0}$). Hence theorem 1.12 of loc.cit yields a weak logarithmic Frobenius structure.
Now assume conjecture 5.5 and suppose that $i=1$. Then $\underline{\omega}^{(4)} = t^{-k}\underline{\omega}^{(3)}$, and we get that S is non-degenerate on $\widehat{G}'$ by corollary 5.6. In particular, $(\widehat{G}',\nabla,S)$ underlies a $\log\Sigma$-trTLEP-structure in this case. This yields a logarithmic Frobenius structure by applying [Rei08, theorem 1.12]. That the pairing is non-degenerate and that $i=1$ holds for the normal crossing divisor case follows, e.g., from the computations in [DS04] (which, as already pointed out above, have been taken up in [Dou08] to give the same result as here). □

Let us remark that one might consider the result for the normal crossing divisor as being "well-known" by the mirror principle: as already stated in the introduction, the Frobenius structure for fixed t is known to be isomorphic to the quantum cohomology ring of the ordinary projective space. But in fact we have more: the parameter t corresponds exactly to the parameter in the small quantum cohomology ring (note that the convention for the name of the coordinate on the parameter space differs from the usual one in quantum cohomology, our t is usually called q and defined as $q=e^t$, where this t corresponds to a basis vector in the second cohomology of the underlying variety, e.g., $\mathbb{P}^{n-1}$). Using this interpretation, the logarithmic structure as defined above is the same as the one obtained in [Rei08, subsection 2.1.2.].

6 Examples

We have computed the spectrum and monodromy for some of the discriminants in quiver representation spaces described in [BM06]. In some cases, we have implemented the methods explained in the previous sections in Singular ([GPS05]). For the infinite families given in the Table 1 below, we have solved the Birkhoff problem by essentially building the seminvariants h_i, where $h=h_1\cdots h_k$ is the equation of D, by successive multiplication by $(-f)$.
We will present two types of examples. On the one hand, we will explain in detail some specific examples, namely, the normal crossing divisor, the star quiver with three exterior vertices (denoted by $\star_n$ in example 2.3 (i)), and the non-reductive example discussed after definition 2.1 for $k=2$. We also give the spectral numbers for the linear free divisor associated to the E_6 quiver (see example 2.3 (ii)), but we do not write down the corresponding good basis, which is quite complicated (remember that already the equation 2.4 was not completely given).

On the other hand, we are able to determine the spectrum for (G_t, ∇) $(t \neq 0)$ and (G_0, ∇) for the whole D_n- and $\star_n$-series by a combinatorical procedure. The details are rather involved, therefore, we present the results, but refer to the forthcoming paper [dGS09] for full details and proofs. It should be noticed that except in the case of the normal crossing divisor and in very small dimensions for other examples, it is very hard to write down explicitly elements for the good bases $\underline{\omega}^{(2)}$ and $\underline{\omega}^{(3)}$ as already the equation for the divisor becomes quickly quite involved.

Let us start with the three explicit examples mentioned above. Notice that is all examples discussed below the linear function is the sum of the coordinates, which is, in each case, easily seen to lie in the open orbit of the dual action.

The case of the normal crossing divisor: As noticed in the first section, this is the discriminant in the representation space of any quiver with a tree as underlying (oriented) graph. In particular, it is the discriminant of the Dynkin A_{n+1}-quiver. Chosing coordinates $x_0, \ldots, x_n$ on V, we have $h = x_0 \cdot \ldots \cdot x_n$. A direct calculation (i.e., without using lemma 4.4 and algorithm 1) shows that $\underline{\omega}^{(1)} = \underline{\omega}^{(2)} = \underline{\omega}^{(3)} = \left(\prod_{j=0}^{i-1} x_j \cdot \alpha\right)_{i=1,\ldots,n}$. This is consistent with the basis found in [DS04, proposition 3.2]. In particular, we have $A^{(2)} = A^{(3)} = -\mathrm{diag}(0, \ldots, n-1)$, so the spectral numbers of (G_t, ∇) for $t \neq 0$ and (G_0, ∇) are $(0, \ldots, n-1)$. We also see that $(nt\partial_t)\underline{\omega}^{(2)} = \underline{\omega}^{(2)} \cdot \tau\Omega_0$, which is a well known result from the calculation of the quantum cohomology of $\mathbb{P}^{n-1}$ (see the last remark in subsection 5.3).

The case $\star_3$ (see example 2.3 (i)): Here the bases of the Brieskorn lattice G produced by lemma 4.4 is

$$\begin{array}{ccccccccccc} \omega_1^{(1)} & = & \alpha & ; & \omega_2^{(1)} & = & ??? \cdot \alpha & ; & \omega_3^{(1)} & = & ??? \cdot \alpha \\ \omega_4^{(1)} & = & ??? \cdot \alpha & ; & \omega_5^{(1)} & = & ??? \cdot \alpha & ; & \omega_6^{(1)} & = & ??? \cdot \alpha \end{array} \tag{6.1}$$

and we have $A_\infty^{(1)} = \mathrm{diag}(-0, -3, -2, -3, -4, -3)$. Algorithm 1 yields that $\omega_2^{(2)} = \omega_2^{(1)} + 2\tau^{-1}\omega_1^{(1)}$ and $\omega_i^{(2)} = \omega_i^{(1)}$ for all $i \neq 2$, and we obtain $A_\infty^{(2)} = \mathrm{diag}(-2, -1, -2, -3, -4, -3)$. As $\nu_1^{(2)} - \nu_6^{(2)} = -1 \leq 1$, we have $\underline{\omega}^{(2)} = \underline{\omega}^{(3)}$, hence $G^{(3)} = G$ and $(2, 1, 2, 3, 4, 3)$ is the spectrum for (G_t, ∇), $t \neq 0$ as well as for (G_0, ∇). We see that the minimal spectral number is unique, therefore, $\underline{\omega}^{(2)}$ yields a (V^+, S)-solution for any t. Moreover, we have $(nt\partial_t)\underline{\omega}^{(2)} = \underline{\omega}^{(2)} \cdot [\tau\Omega_0 + \mathrm{diag}(-2, 0, 0, 0, 0, 2)]$, so that in this case the ∇^{res}-flat section $\omega_i^{(3)}$ from lemma 5.3 is $\omega_2^{(2)}$, which is an eigenvector of $A_\infty^{(2)}$ with respect to the minimal spectral number.

The case E_6 (see example 2.3 (ii)): The spectrum of both (G_t, ∇), $t \neq 0$ and (G_0, ∇) is

$$\left(\frac{44}{5}, \frac{25}{3}, \frac{28}{3}, \frac{31}{3}, \frac{34}{3}, \frac{47}{5}, 6, \ldots, 15, \frac{58}{5}, \frac{29}{3}, \frac{32}{3}, \frac{35}{3}, \frac{38}{3}, \frac{61}{5}\right)$$

Again we have a unique minimal spectral, hence, theorem 4.13 applies. The symmetry $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n + 1$ holds. Moreover, we obtain the following eigenvalues for the residue of $t\partial_t$ on $G_{|\mathbb{C}^* \times T}$ at $t = 0$:

$$\frac{1}{12} \cdot \left(-\frac{44}{5}, \left(-\frac{22}{3}\right)^4, -\frac{22}{5}, 0^{10}, \frac{22}{5}, \left(\frac{22}{3}\right)^4, \frac{44}{5}\right)$$

which are (again) symmetric around zero (hence supporting conjecture 5.5 (ii)).

A non-reductive example in dimension 3 (see (2.2)**)**: The linear free divisor in $\mathbb{C}^3$ with equation $h = x(xz - y^2)$ is not special and therefore not reductive. The basis $\underline{\omega}^{(1)}$ is given as

$$\omega_1^{(1)} = \alpha \quad ; \quad \omega_2^{(1)} = ??? \cdot \alpha \quad ; \quad \omega_3^{(1)} = ??? \cdot \alpha \tag{6.2}$$

and we have $A_\infty^{(1)} = \mathrm{diag}(0, ?, ?)$. Algorithm 1 yields

$$\omega_1^{(2)} = ??? \cdot \alpha \quad ; \quad \omega_2^{(2)} = ??? \cdot \alpha \quad ; \quad \omega_3^{(2)} = ??? \cdot \alpha \tag{6.3}$$

and $A_\infty^{(1)} = \mathrm{diag}(-\frac{3}{4}, -1, -\frac{5}{4})$. Again, as $\nu_1^2 - \nu_3^2 = -\frac{1}{2} \leq 1$, we obtain $\underline{\omega}^{(2)} = \underline{\omega}^{(3)}$, $G = G^{(3)}$, and $(\frac{3}{4}, 1, \frac{5}{4})$ is the spectrum of both (G_t, ∇), $t \neq 0$ and (G_0, ∇). We can also compute the spectral numbers for the case $k = 3, 4$

and 5 (these are again the same for (G_t, ∇), $t \neq 0$ and (G_0, ∇), namely:

size of matrices	$\dim(V)$	**Spectrum of** (G_t, ∇)
$k=3$	$n=6$	$(2, \frac{5}{2}, 2, 3, \frac{5}{2}, 3)$
$k=4$	$n=10$	$(\frac{15}{4}, \frac{13}{3}, \frac{9}{2}, \frac{17}{4}, 4, 5, \frac{19}{4}, \frac{9}{2}, \frac{14}{3}, \frac{21}{4})$
$k=5$	$n=15$	$(6, \frac{53}{8}, 7, \frac{27}{4}, 7, \frac{55}{8}, 6, 7, 8, \frac{57}{8}, 7, \frac{29}{4}, 7, \frac{59}{8}, 8)$

The case $k = 5$ is an example where the minimal spectral number is not unique, hence, theorem 4.13 does not apply. However, we observe that the "extra symmetry" $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n-1$ from corollary 5.6 still holds, which supports conjecture 5.5. One might speculate that although the eigenspace of the smallest spectral number is two-dimensional (generated by $\omega_1^{(3)}$ and $\omega_7^{(3)}$), we still have $\tau^{n-1} S(\omega_1^{(3)}, \omega_j^{(3)}) \in \mathbb{C}\delta_{j,15}$ (resp. $\tau^{n-1} S(\omega_7^{(3)}, \omega_j^{(3)}) \in \mathbb{C}\delta_{j,9}$) which would imply that the conclusions of theorem 4.13 still hold.

Now we turn to the series D_m resp. $\star_m$. The results are given in table 1 below. We write $(p_1, q_1), \ldots, (p_k, q_k)$ to indicate that the output of algorithm 1 resp. algorithm 2 is a basis $\underline{\omega}^{(2)}$ resp. $\underline{\omega}^{(3)}$ which decomposes into k blocks as in the proof of proposition 4.8, where in each block (p_i, q_i) the eigenvalues of the residue endomorphism $\tau\partial_\tau$ along $\tau = 0$ are $-p_i, -p_i - 1, \ldots, -p_i - q_i + 1$. In particular, this gives the monodromy of (G_t, ∇) according to corollary 4.9. We observe that in all cases the symmetries $\nu_i^{(2)} + \nu_{n+1-i}^{(2)} = -n+1$ and $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n+1$ hold, and that the eigenvalues of the residue of $t\partial_t$ on G_0 are symmetric around zero.

	D_m	$\star_{m:=2k+1}$	$\star_{m:=2k}$
$\dim(D)=n-1$	$4m-11$	m^2-m-1	m^2-m-1
$Sp(G_0,\nabla)$	$\left(\frac{4m-10}{3}, m-3\right)$, $(m-3, 2m-4)$, $(\frac{5m-11}{3}, m-3)$	???	???
$Sp(G_t,\nabla)$ $t\neq 0$	$\left(\frac{4m-10}{3}, m-3\right)$, $(m-3, 2m-4)$, $\left(\frac{5m-11}{3}, m-3\right)$	$\overbrace{(2k^2, m-2), (2k^2-1, m-2), \ldots, \left(2k^2-k+1, m-2\right)}^{k}$, $\left(2k^2-k, 2m-2\right)$, $\overbrace{(2k^2+k, m-2), (2k^2+k-1, m-2), \ldots, \left(2k^2+1, m-2\right)}^{k}$	$(mk-k, k-1), \overbrace{(mk-m, m-2), (mk-m-1, m-2), \ldots, (mk-3k+2, m-2)}^{k-1}$, $\left(2k^2-3k+1, 2m-2\right)$, $\overbrace{(mk-k, m-2), (mk-k-1, m-2), \ldots, (mk-m+2, m-2)}^{k-1}, (mk-m+1, k-1)$
$\mathrm{Res}[t\partial_t]$ on $(G_0/tG_0)_{\|\tau\neq\infty}$	$\left(-\frac{4m-10}{3}, m-3\right)$, $(0, 2m-4)$, $\left(\frac{4m-10}{3}, m-3\right)$	???	???

Table 1: Spectra of f on the Milnor resp. zero fibre for fibrations defined by some quiver discriminants.

Remark 6.1. (i) We see that the jumping phenomenon (i.e., the fact that the spectrum of (G_t, ∇), $t \neq 0$ and (G_0, ∇) are different) occurs in our examples only for the star quiver for $n \geq 5$. However, there are probably many more examples where this happens, if the divisor D has sufficiently high degree.

(ii) Each Dynkin diagram supports many different quivers, distinguished by their edge orientations. Nevertheless, each of these quivers has the same set of roots. For quivers of type A_n and D_n, the discriminants in the corresponding representation spaces are also the same, up to isomorphism. However, for the quivers of type E_6, there are three non-isomorphic linear free divisors associated to the highest root (the dimension vector shown). Their generic hyperplane sections all have the same spectrum and monodromy.

(iii) For the case of the star quiver with $n = 2k$, the last and first blocks actually form a single block. We have split them into two to respect the order given by the weight of the corresponding elements in the Gauß-Manin system.

(iv) In all examples presented above the ∇^{res}-flat basis element $t^{-k}\omega_i^{(3)}$ from lemma 5.3 was an eigenvector of $A_\infty^{(3)}$ for the smallest spectral number. An example where the latter does not hold is provided by the *bracelet*, the discriminant in the space of binary cubics (the last example in 4.4 of [GMNS06]). The spectrum of the generic hyperplane section is $(\frac{2}{3}, 1, 2, \frac{7}{3})$, and hence the minimal spectral number is not an integer. However, the ∇^{res}-flat section is an eigenvector for a spectral number within the intervall $[\alpha_{min}, \alpha_{min}+1)$, which is thus still a primitive and homogenous section for the canonical solution $\widehat{G}_t^{can}$ used in theorem 5.1 (ii).

Let us finish the paper by a few remarks on open questions and problems related to the results obtained.
In [DS04], where similar questions for certain Laurent polynomials are studied, it is shown that the (V^+, S)-solution constructed coincides in fact with the canonical solution as described in theorem 4.7 (see of loc.cit.). A natural question is to ask whether the same holds true in our situation.
A second problem is to come to a more conceptual understanding of the degeneration behavior of the various Frobenius structures M_t as discussed in theorem 5.9 in those cases (i.e., all examples except the normal crossing case), where only a weak logarithmic Frobenius manifold is found. As already pointed out, a rather similar phenomenon occurs in [Dou08].
Another very interesting point is to better understand the relation of the Frobenius structure constructed to the called tt^*-geometry (also known as variation of TERP- resp. integrable twistor structures, see, e.g., [Her03]). We know from corollary 4.5 (v) that the families studied here are examples of *Sabbah orbits*. The degeneration behavior of such variations of integrable twistor structures has been studied in [HS07] using methods from [Moc07]. However, the extensions over the boundary point $0 \in T$ used in loc.cit. is in general different from the lattices G resp. $G^{(3)}$ considered here, as the eigenvalues of the residue $[t\partial_t]$ computed above does not always lie in a half-open interval of length one (i.e., $G_{|\mathbb{C}^*\times T}$ is not always a Deligne extension of $G_{|\mathbb{C}^*\times(T\setminus\{0\})}$). One might want to better understand what kind of information is exactly contained in the extension G. Again, a similar problem is studied to some extend for Laurent polynomials in [Dou08].
Finally, as we already remarked, the connection ∂_τ is regular singular at $\tau = \infty$ on $\mathbf{G}_0$ but irregular for $t \neq 0$. Irregular connections are characterized by a subtle set of topological data, the so-called Stokes matrices. It might be interesting to calculate these matrices for the examples we studied, extending the calculations done in [Guz99] for the A_n-case.

References

[Bar00] Serguei Barannikov, *Semi-infinite Hodge structures and mirror symmetry for projective spaces*, Preprint math.AG/0010157, 2000.

[BM06] Ragnar-Olaf Buchweitz and David Mond, *Linear free divisors and quiver representations*, Singularities and computer algebra, London Math. Soc. Lecture Note Ser., vol. 324, Cambridge Univ. Press, Cambridge, 2006, pp. 41–77.

[Bro88] S. A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math. **92** (1988), no. 2, 217–241.

[CCLT06] Tom Coates, Alessio Corti, Yuan-Pin Lee, and Hsian-Hua Tseng, *The Quantum Orbifold Cohomology of weighted projective spaces*, Preprint math.AG/0608481, to appear in "Acta Mathematica", 2006.

[Dam06] James Damon, *On the legacy of free divisors. III. Functions and divisors on complete intersections*, Q. J. Math. **57** (2006), no. 1, 49–79.

[dG07] Ignacio de Gregorio, *Some examples of non-massive Frobenius manifolds in singularity theory*, J. Geom. Phys. **57** (2007), no. 9, 1829–1841.

[dGS09] Ignacio de Gregorio and Christian Sevenheck, *Good bases for some linear free divisors associated to quiver representations*, work in progress, 2009.

[Dim04] Alexandru Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004.

[Dou05] Antoine Douai, *Construction de variétés de Frobenius via les polynômes de Laurent: une autre approche*, Singularités, Inst. Élie Cartan, vol. 18, Univ. Nancy, Nancy, 2005, pp. 105–123.

[Dou08] ———, *Examples of limits of Frobenius (type) structures: The singularity case*, Preprint math.AG/0806.2011, 2008.

[DS03] Antoine Douai and Claude Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. I*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 4, 1055–1116.

[DS04] ———, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 1–18.

[Dub96] Boris Dubrovin, *Geometry of 2D topological field theories.*, Integrable systems and quantum groups. Lectures given at the 1st session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 14-22, 1993. (M. Francaviglia and S. Greco, eds.), Lecture Notes in Mathematics, vol. 1620, Springer-Verlag, 1996, p. 488.

[Giv95] Alexander Givental, *Homological geometry and mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 472–480.

[Giv98] ———, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175.

[GMNS06] Michel Granger, David Mond, Alicia Nieto, and Mathias Schulze, *Linear free divisors and the global logarithmic comparison theorem.*, Preprint math.AG/0607045, to appear in "Ann. Inst. Fourier (Grenoble)", 2006.

[GPS05] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular *3.0*, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2005, `http://www.singular.uni-kl.de`.

[Guz99] Davide Guzzetti, *Stokes matrices and monodromy of the quantum cohomology of projective spaces*, Comm. Math. Phys. **207** (1999), no. 2, 341–383.

[Her02] Claus Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, vol. 151, Cambridge University Press, Cambridge, 2002.

[Her03] ———, *tt^*-geometry, Frobenius manifolds, their connections, and the construction for singularities*, J. Reine Angew. Math. **555** (2003), 77–161.

[HM04] Claus Hertling and Yuri Manin, *Unfoldings of meromorphic connections and a construction of Frobenius manifolds*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 113–144.

[HS53] G. Hochschild and J.-P. Serre, *Cohomology of Lie algebras*, Ann. of Math. (2) **57** (1953), 591–603.

[HS07] Claus Hertling and Christian Sevenheck, *Nilpotent orbits of a generalization of Hodge structures*, J. Reine Angew. Math. **609** (2007), 23–80.

[Man08] Etienne Mann, *Orbifold quantum cohomology of weighted projective spaces*, J. Algebraic Geom. **17** (2008), no. 1, 137–166.

[Mat69] John N. Mather, *Stability of C^∞ mappings. IV. Classification of stable germs by R-algebras*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 37, 223–248.

[Moc07] Takuro Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor $\mathcal{D}$-modules, Part 1*, Mem. Amer. Math. Soc. **185** (2007), no. 869, xi+324.

[NS99] Andras Némethi and Claude Sabbah, *Semicontinuity of the spectrum at infinity*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 25–35.

[Rei08] Thomas Reichelt, *A construction of Frobenius manifolds with logarithmic poles and applications*, Preprint math.AG/0808.2698, to appear in "Communications in Mathematical Physics", 2008.

[Sab02] Claude Sabbah, *Déformations isomonodromiques et variétés de Frobenius*, Savoirs Actuels, EDP Sciences, Les Ulis, 2002, Mathématiques.

[Sab06] ———, *Hypergeometric periods for a tame polynomial*, Port. Math. (N.S.) **63** (2006), no. 2, 173–226, written in 1998.

[Sab08] ———, *Fourier-Laplace transform of a variation of polarized complex Hodge structure*, J. Reine Angew. Math. **621** (2008), 123–158.

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291.

[Sai89] Morihiko Saito, *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 1, 27–72.

[SK77] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

[Tei82] Bernard Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney*, Algebraic geometry (La Rábida, 1981) (José Manuel Aroca, Ragnar Buchweitz, Marc Giusti, and Michel Merle, eds.), Lecture Notes in Math., vol. 961, Springer, Berlin, 1982, pp. 314–491.

[TT83] Lê Dũng Tráng and B. Teissier, *Cycles evanescents, sections planes et conditions de Whitney. II*, Singularities, Part 2 (Arcata, Calif., 1981) (Peter Orlik, ed.), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 65–103.

Ignacio de Gregorio `degregorio@gmail.com`
Mathematics Institute, University of Warwick, Coventry CV4 7AL, England.

David Mond `D.M.Q.Mond@warwick.ac.uk`
Mathematics Institute, University of Warwick, Coventry CV4 7AL, England.

Christian Sevenheck `Christian.Sevenheck@uni-mannheim.de`
Lehrstuhl VI für Mathematik, Universität Mannheim, A6 5, 68131 Mannheim, Germany.