

LINEAR FUNCTIONS ON THE CLASSICAL MATRIX GROUPS

ELIZABETH MECKES

ABSTRACT. Let M be a random matrix in the orthogonal group \mathcal{O}_n , distributed according to Haar measure, and let A be a fixed $n \times n$ matrix over \mathbb{R} such that $\text{Tr}(AA^t) = n$. Then the total variation distance of the random variable $\text{Tr}(AM)$ to a standard normal random variable is bounded by $\frac{2\sqrt{3}}{n-1}$, and this rate is sharp up to the constant. Analogous results are obtained for M a random unitary matrix and A a fixed $n \times n$ matrix over \mathbb{C} . The proofs are applications of a new abstract normal approximation theorem which extends Stein's method of exchangeable pairs to situations in which continuous symmetries are present.

1. INTRODUCTION

Let \mathcal{O}_n denote the group of $n \times n$ orthogonal matrices, and let M be distributed according to Haar measure on \mathcal{O}_n . Let A be a fixed $n \times n$ matrix over \mathbb{R} , subject to the condition that $\text{Tr}(AA^t) = n$, and let $W = \text{Tr}(AM)$. D'Aristotile, Diaconis, and Newman showed in [4] that

$$\sup_{\substack{A: \text{Tr}(AA^t)=n \\ -\infty < x < \infty}} |\mathbb{P}(W \leq x) - \Phi(x)| \rightarrow 0$$

as $n \rightarrow \infty$. Their argument uses classical methods involving sub-subsequences and tightness, and cannot be improved to yield a theorem for finite n . Theorem 4 below gives an explicit rate of convergence of the law of W to the standard normal distribution in the total variation metric on probability measures, specifically,

$$(1) \quad d_{TV}(\mathbf{L}_W, \mathfrak{N}(0, 1)) \leq \frac{2\sqrt{3}}{n-1}$$

for all $n \geq 2$.

The history of this problem begins with the following theorem, first given rigorous proof by Borel in [2]: let X be a random vector on the unit sphere S^{n-1} , and let X_1 be the first coordinate of X . Then $\mathbb{P}(\sqrt{n}X_1 \leq t) \rightarrow \Phi(t)$ as $n \rightarrow \infty$, where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{x^2}{2}} dx$. Since the first column of a Haar distributed orthogonal matrix is uniformly distributed on the unit sphere, Borel's theorem follows from Theorem 4 by taking $A = \sqrt{n} \oplus \mathbf{0}$, where $\mathbf{0}$ is the $n-1 \times n-1$ matrix with all zeros. Borel's theorem was generalized in one direction by Diaconis and Freedman [8], who proved the convergence of the first k coordinates of $\sqrt{n}X$ to independent standard

Received by the editors September 22, 2006.

2000 *Mathematics Subject Classification*. Primary 60F05; Secondary 60B15, 60B10.

Key words and phrases. Stein's method, normal approximation, random matrices.

This research was supported in part by the ARCS Foundation.

©2008 American Mathematical Society
Reverts to public domain 28 years from publication

normal random variables in total variation distance for $k = o(n)$; [8] also contains a detailed history of this problem. This line of research was further developed in [7], where a total variation bound was given between an $r \times r$ block of a random orthogonal matrix and an $r \times r$ matrix of independent standard Gaussians, for $r = O(n^{1/3})$. This was later improved by Jiang (see [12]) to $r = O(n^{1/2})$, which he proved was sharp. In the same paper, Jiang also showed that given a sequence of Haar distributed random matrices $\{M_n\}$, there is a sequence of Gaussian matrices $\{Y_n\}$ with Y_j defined on the same probability space as M_j such that if

$$\epsilon_n = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m_n}} |\sqrt{n}M_{ij} - Y_{ij}|$$

with $m_n = O(\frac{n}{\log n})$, then $\epsilon_n \rightarrow 0$ in probability as $n \rightarrow \infty$. Thus an $n \times m_n$ block of a Haar distributed matrix can be approximated by a Gaussian matrix ‘in probability’. Theorem 4 gives another sense in which a random orthogonal matrix is close to a matrix of independent normals by giving a uniform bound of distance to normal over all linear combinations of entries of M .

Another special case of Theorem 4 is $A = I$, so that $W = \text{Tr}(M)$. Diaconis and Mallows (see [5]) first proved that $\text{Tr}(M)$ is approximately normal; Stein [15] and Johansson [13] later independently obtained fast rates of convergence to normal of $\text{Tr}(M^k)$ for fixed k , with Johansson’s rates an improvement on Stein’s. In studying eigenvalues of random orthogonal matrices, Diaconis and Shahshahani [9] extended this to show that the joint limiting distribution of $\text{Tr}(M), \text{Tr}(M^2), \dots, \text{Tr}(M^k)$ converges to that of independent normal variables as $n \rightarrow \infty$, for k fixed.

The other source of motivation for theorems like Theorem 4 is Hoeffding’s combinatorial central limit theorem [11], which can be stated as follows. Let $A = (a_{ij})$ be a fixed $n \times n$ matrix over \mathbb{R} , normalized to have row and column sums equal to zero and $\frac{1}{n-1} \sum_{i,j} a_{ij}^2 = 1$. Let π be a random permutation in S_n , and let $W(\pi) = \sum_i a_{i\pi(i)}$. Then, under certain conditions on A , W is approximately normal. Later, Bolthausen [1] proved an explicit rate of convergence via Stein’s method. Note that if

$$M_{ij} = \begin{cases} 1, & \pi(j) = i, \\ 0, & \text{otherwise,} \end{cases}$$

then $W = \text{Tr}(AM)$, and so Hoeffding’s theorem is really a theorem about the distribution of linear functions on the set of permutation matrices.

The unitary group is another source of many important applications; see, e.g. [6]. In Section 4, the random variable $\text{Tr}(AM)$ for A a fixed matrix over \mathbb{C} and M a random unitary matrix distributed according to Haar measure on \mathcal{U}_n is considered. The main theorem of the section, Theorem 6, gives a bound on the total variation distance of $\text{Re}[\text{Tr}(AM)]$ to standard normal analogous to that of Theorem 4; this can be viewed as a theorem about real-linear functions on \mathcal{U}_n . Corollary 7 shows that in the limit, the complex random variable $\text{Tr}(AM)$ is close to standard complex normal. The methods used here cannot be used directly to prove the convergence of $\text{Tr}(AM)$ to the standard complex normal; they work for approximation of real-valued random variables only. A version of the present methods in a multivariate context is forthcoming in [3], which includes a rate of convergence for Corollary 7.

Notation and conventions. The total variation distance $d_{TV}(\mu, \nu)$ between the measures μ and ν on \mathbb{R} is defined by

$$d_{TV}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)|,$$

where the supremum is over measurable sets A . This is equivalent to

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sup_f \left| \int f(t) d\mu(t) - \int f(t) d\nu(t) \right|,$$

where the supremum is taken over continuous functions which are bounded by 1 and vanish at infinity; this is the definition used in what follows. The total variation distance between two random variables X and Y is defined to be the total variation distance between their distributions:

$$d_{TV}(X, Y) = \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| = \frac{1}{2} \sup_f |\mathbb{E}f(X) - \mathbb{E}f(Y)|.$$

We will use $\mathfrak{N}(\mu, \sigma^2)$ to denote the normal distribution on \mathbb{R} with mean μ and variance σ^2 .

2. AN ABSTRACT NORMAL APPROXIMATION THEOREM

In this section, a general approach for normal approximation to random variables with continuous symmetries is developed. The ideas which give rise to Theorem 1 below first appeared in Stein [15], where fast rates of convergence to Gaussian (as $n \rightarrow \infty$) were obtained for $\text{Tr}(M^k)$, with $k \in \mathbb{N}$ fixed and M a random $n \times n$ orthogonal matrix.

Theorem 1. *Suppose that (W, W_ϵ) is a family of exchangeable pairs defined on a common probability space with $\mathbb{E}W = 0$ and $\mathbb{E}W^2 = \sigma^2$. Suppose that there are functions α and β with*

$$\mathbb{E}|\alpha(\sigma^{-1}W)| < \infty, \quad \mathbb{E}|\beta(\sigma^{-1}W)| < \infty,$$

a random variable E , and a constant λ such that

(i)

$$\frac{1}{\epsilon^2} \mathbb{E} [W_\epsilon - W | W] = -\lambda W + o(1)\alpha(W),$$

(ii)

$$\frac{1}{\epsilon^2} \mathbb{E} [(W_\epsilon - W)^2 | W] = 2\lambda\sigma^2 + E\sigma^2 + o(1)\beta(W),$$

(iii)

$$\frac{1}{\epsilon^2} \mathbb{E} |W_\epsilon - W|^3 = o(1),$$

where $o(1)$ refers to the limit as $\epsilon \rightarrow 0$, with the implied constants deterministic.

Then

$$d_{TV}(W, Z) \leq \frac{1}{\lambda} \mathbb{E} |E|,$$

where $Z \sim \mathfrak{N}(0, \sigma^2)$.

Remarks. (i) A straightforward computation using condition (i) shows that the random variable E of condition (ii) has $\mathbb{E}E = 0$.

- (ii) The factor of $\frac{1}{\epsilon^2}$ in each of the three expressions above could be replaced by a general function $f(\epsilon)$. In practice, W_ϵ is typically constructed such that $W_\epsilon - W = O(\epsilon)$. This makes it clear that $f(\epsilon) = \frac{1}{\epsilon^2}$ is the suitable choice for condition (ii). It is less clear that $f(\epsilon) = \frac{1}{\epsilon^2}$ is the suitable choice for condition (i). In the applications given here, while $W_\epsilon - W = O(\epsilon)$, symmetry conditions imply that

$$\mathbb{E} [W_\epsilon - W | W] = O(\epsilon^2).$$

Before beginning the proof, some background on Stein’s method is helpful. The following lemma is key.

Lemma 2 (Stein). *Let $Z \sim \mathfrak{N}(0, 1)$. Then*

- (i) For all $f \in C^1_o(\mathbb{R})$,

$$\mathbb{E} [f'(Z) - Zf(Z)] = 0.$$

- (ii) If Y is a random variable such that

$$\mathbb{E} [f'(Y) - Yf(Y)] = 0$$

for all $f \in C^1_b(\mathbb{R})$, then $L(Y) = L(Z)$; i.e., Y is also distributed as a standard Gaussian random variable.

- (iii) For $g : \mathbb{R} \rightarrow \mathbb{R}$ with $\mathbb{E}g(Z) < \infty$ given, the function

$$(2) \quad U_o g(t) = e^{t^2/2} \int_{-\infty}^t [g(x) - \mathbb{E}g(Z)] e^{-x^2/2} dx$$

is a solution to the differential equation

$$f'(x) - xf(x) = g(x) - \mathbb{E}g(Z).$$

The lemma says that the standard Gaussian distribution γ on \mathbb{R} is the unique distribution with the property that $\int_{\mathbb{R}} (f'(x) - xf(x)) d\gamma(x)$ is always zero. The idea of Stein’s method is that if W is a random variable such that $\mathbb{E} [f'(W) - Wf(W)]$ is always *small*, then the distribution of W is close to γ . There are several approaches to bounding this quantity; the approach taken here is modelled on the method of exchangeable pairs (see [14]). In any of the approaches, the following bounds on U_o are useful.

Lemma 3 (Stein). *Let U_o be the operator defined in equation (2). Then*

- (i) $\|U_o g\|_\infty \leq \sqrt{\frac{\pi}{2}} \|g - \mathbb{E}g(Z)\|_\infty \leq \sqrt{2\pi} \|g\|_\infty$.
- (ii) $\|(U_o g)'\|_\infty \leq 2 \|g - \mathbb{E}g(Z)\|_\infty \leq 4 \|g\|_\infty$.
- (iii) $\|(U_o g)''\|_\infty \leq 2 \|g'\|_\infty$.

With this background, the proof of Theorem 1 is straightforward.

Proof of Theorem 1. By considering $\sigma^{-1}W$ instead of W , we may without loss assume that $\sigma = 1$. For $g \in C^\infty_o(\mathbb{R})$ fixed, let f be the solution given in equation (2) to the differential equation

$$f'(x) - xf(x) = g(x) - \mathbb{E}g(Z).$$

Fix ϵ . By the exchangeability of (W, W_ϵ) ,

$$(3) \quad \begin{aligned} 0 &= \mathbb{E} [(W_\epsilon - W)(f(W_\epsilon) + f(W))] \\ &= \mathbb{E} [(W_\epsilon - W)(f(W_\epsilon) - f(W)) + 2(W_\epsilon - W)f(W)] \\ &= \mathbb{E} [\mathbb{E} [(W_\epsilon - W)^2 | W] f'(W) + 2\mathbb{E} [(W_\epsilon - W) | W] f(W) + R], \end{aligned}$$

where R is the error in the derivative approximation. By Taylor’s theorem and Lemma 3,

$$|R| \leq \frac{\|f''\|_\infty}{2} |W_\epsilon - W|^3 \leq \|g'\|_\infty |W_\epsilon - W|^3,$$

and so by condition (iii),

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E}|R| = 0.$$

Dividing both sides of (3) by $2\lambda\epsilon^2$ and taking the limit as $\epsilon \rightarrow 0$, conditions (i) and (ii) yield

$$0 = \mathbb{E} \left[f'(W) - Wf(W) + \frac{E}{2\lambda} f'(W) \right] = \mathbb{E} \left[g(W) - g(Z) + \frac{E}{2\lambda} f'(W) \right].$$

Rearranging and applying the bound on $\|f'\|$ from Lemma 3 yields

$$|\mathbb{E}g(W) - \mathbb{E}g(Z)| \leq \frac{2\|g\|_\infty}{\lambda} \mathbb{E}|E|.$$

Since $C_\infty^\infty(\mathbb{R})$ is dense (with respect to the supremum norm) in the class of bounded continuous functions vanishing at infinity, this completes the proof. \square

3. THE ORTHOGONAL GROUP

This section is mainly devoted to the proof of the following theorem.

Theorem 4. *Let A be a fixed $n \times n$ matrix over \mathbb{R} such that $\text{Tr}(AA^t) = n$, $M \in \mathcal{O}_n$ distributed according to Haar measure, and $W = \text{Tr}(AM)$. Let Z be a standard normal random variable. Then for $n > 1$,*

$$(4) \quad d(W, Z)_{TV} \leq \frac{2\sqrt{3}}{n-1}.$$

The bound in Theorem 4 is sharp up to the constant; consider the matrix $A = \sqrt{n} \oplus \mathbf{0}$ where $\mathbf{0}$ is the $(n-1) \times (n-1)$ matrix with all zeros. For this A , Theorem 4 reproves the following theorem, proved in [8] with slightly worse constant.

Theorem 5. *Let $\mathbf{x} \in \sqrt{n}S^{n-1}$ be uniformly distributed, and let Z be a standard normal random variable. Then*

$$d_{TV}(x_1, Z) \leq \frac{2\sqrt{3}}{n-1}.$$

It is shown in [8] that the order of this error term is correct.

Proof of Theorem 4. First note that one can assume without loss of generality that A is diagonal: let $A = UDV$ be the singular value decomposition of A . Then $W = \text{Tr}(UDVM) = \text{Tr}(DVMU)$, and the distribution of VMU is the same as the distribution of M by the translation invariance of Haar measure.

Now define the pair (W, W_ϵ) for each ϵ as follows. Choose $H = (h_{ij}) \in \mathcal{O}(n)$ according to Haar measure, independent of M , and let $M_\epsilon = HA_\epsilon H^t M$, where

$$A_\epsilon = \begin{bmatrix} \sqrt{1-\epsilon^2} & \epsilon & & & \\ -\epsilon & \sqrt{1-\epsilon^2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \end{bmatrix},$$

thus M_ϵ can be thought of as a small random rotation of M . Let $W_\epsilon = W(M_\epsilon)$; (W, W_ϵ) is an exchangeable pair by construction.

It is convenient to rewrite M_ϵ as follows. Let I_2 be the 2×2 identity matrix, let K be the $n \times 2$ matrix consisting of the first two columns of H , and let

$$C_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} M_\epsilon &= M + K [(\sqrt{1 - \epsilon^2} - 1)I_2 + \epsilon C_2] K^t M \\ &= M + K \left[\left(-\frac{\epsilon^2}{2} + O(\epsilon^4) \right) I_2 + \epsilon C_2 \right] K^t M, \end{aligned}$$

and so

$$(5) \quad W_\epsilon - W = \epsilon \left[\left(-\frac{\epsilon}{2} + O(\epsilon^3) \right) \text{Tr}(AKK^tM) + \text{Tr}(AKC_2K^tM) \right].$$

Now, the distribution of H is unchanged by multiplying a fixed row or column by -1 and H is orthogonal, thus $\mathbb{E}h_{ij}h_{k\ell} = \frac{1}{n}\delta_{ik}\delta_{j\ell}$. This implies that

$$\mathbb{E}[KK^t] = \frac{2}{n}I_n$$

and

$$\mathbb{E}[KC_2K^t] = 0;$$

combining this with (5) yields:

$$\begin{aligned} &\frac{n}{\epsilon^2} \mathbb{E}[(W_\epsilon - W)|W] \\ &= -\frac{n}{2} \mathbb{E}[\mathbb{E}[\text{Tr}(AKK^tM)|M] |W] + \frac{n}{\epsilon} \mathbb{E}[\mathbb{E}[\text{Tr}(AKC_2K^tM)|M] |W] + O(\epsilon) \\ &= -\mathbb{E}[\mathbb{E}[\text{Tr}(AM)|M] |W] + O(\epsilon) \\ &= -W + O(\epsilon), \end{aligned}$$

where the independence of M and H has been used to get the third line, and the implied constants in the $O(\epsilon)$ here and in what follows may depend on n . Condition (i) of Theorem 1 is thus satisfied with $\lambda = \frac{1}{n}$.

Recall now that A is assumed to be diagonal. The second condition of Theorem 1 can also be verified using the expression in (5) as follows:

$$\begin{aligned} (6) \quad &\frac{n}{2\epsilon^2} \mathbb{E}[(W_\epsilon - W)^2|W] \\ &= \frac{n}{2} \mathbb{E}[\mathbb{E}[(\text{Tr}(AKC_2K^tM))^2|M] |W] + O(\epsilon) \\ &= \frac{n}{2} \mathbb{E} \left[\sum_{\substack{i,j \\ i' \neq i \\ j' \neq j}} m_{i'i} m_{j'j} a_{ii} a_{jj} \mathbb{E}[(h_{i1}h_{i'2} - h_{i2}h_{i'1})(h_{j1}h_{j'2} - h_{j2}h_{j'1})|M] \middle| W \right] \\ &\quad + O(\epsilon), \end{aligned}$$

where the conditions on i' and j' are justified as the expression inside the expectation is identically zero when either $i = i'$ or $j = j'$.

Standard techniques are available for computing the mixed moments of entries of H ; see e.g. [10], section 4.2. Using these techniques and the independence of M and H gives that for $i' \neq i$ and $j' \neq j$,

$$\begin{aligned}
 & \mathbb{E} [(h_{i1}h_{i'2} - h_{i2}h_{i'1})(h_{j1}h_{j'2} - h_{j2}h_{j'1})|M] \\
 (7) \quad &= \mathbb{E} [(h_{i1}h_{i'2} - h_{i2}h_{i'1})(h_{j1}h_{j'2} - h_{j2}h_{j'1})] \\
 &= \frac{2}{n(n-1)} [\delta_{ij}\delta_{i'j'} - \delta_{ij'}\delta_{ji'}];
 \end{aligned}$$

putting this into (6) yields

$$\begin{aligned}
 & \frac{n}{2\epsilon^2} \mathbb{E} [(W_\epsilon - W)^2|W] \\
 &= \frac{1}{n-1} \sum_{i,j} \sum_{\substack{i' \neq i \\ j' \neq j}} m_{i'i} m_{j'j} a_{ii} a_{jj} [\delta_{i'j'} \delta_{ij} - \delta_{ij'} \delta_{ji'}] + O(\epsilon) \\
 &= \frac{1}{n-1} \left[\sum_i a_{ii}^2 [(M^t M)_{ii} - m_{ii}^2] - \sum_{i,i' \neq i} (MA)_{ii'} (MA)_{i'i} \right] + O(\epsilon) \\
 &= \frac{1}{n-1} \left[n - \sum_i a_{ii}^2 m_{ii}^2 - \left[\text{Tr}((MA)^2) - \sum_i a_{ii}^2 m_{ii}^2 \right] \right] + O(\epsilon) \\
 &= 1 + \frac{1}{n-1} [1 - \text{Tr}((AM)^2)] + O(\epsilon),
 \end{aligned}$$

thus

$$(8) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E} [(W_\epsilon - W)^2|W] = \frac{2}{n} + \frac{2}{n(n-1)} [1 - \text{Tr}((AM)^2)],$$

and so

$$(9) \quad E = \frac{2}{n(n-1)} [1 - \text{Tr}((AM)^2)].$$

Finally, (5) gives immediately that

$$\mathbb{E} [|W_\epsilon - W|^3|W] = O(\epsilon^3).$$

It remains to bound $n\mathbb{E}|E|$. Let \sum' stand for summing over distinct indices.

$$\begin{aligned}
 (10) \quad \mathbb{E} [\text{Tr}((AM)^2)] &= \mathbb{E} \left[\sum_{i,j} a_{ii} a_{jj} m_{ij} m_{ji} \right] \\
 &= \frac{1}{n} \sum_i a_{ii}^2 = 1,
 \end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}[(\operatorname{Tr}((AM)^2))^2] \\
&= \mathbb{E}\left[\left(\sum_{i,j} a_{ii}a_{jj}m_{ij}m_{ji}\right)\left(\sum_{k,l} a_{kk}a_{ll}m_{kl}m_{lk}\right)\right] \\
&= \sum_{i,j,k,l} a_{ii}a_{jj}a_{kk}a_{ll} \left[\frac{n+1}{(n-1)n(n+2)} [\delta_{ij}\delta_{kl}(1-\delta_{ik}) + \delta_{ik}\delta_{jl}(1-\delta_{ij}) \right. \\
&\quad \left. + \delta_{il}\delta_{jk}(1-\delta_{ij})] + \frac{3}{n(n+2)} \mathbb{I}(i=j=k=l) \right] \\
&= \frac{n+1}{n(n-1)(n+2)} \left(\sum'_{i,k} a_{ii}^2 a_{kk}^2 + \sum'_{i,j} a_{ii}^2 a_{jj}^2 + \sum'_{i,j} a_{ii}^2 a_{jj}^2 \right) \\
&\quad + \frac{3}{n(n+2)} \sum_i a_{ii}^4.
\end{aligned}$$

Now,

$$\sum'_{i,j} a_{ii}^2 a_{jj}^2 = \sum_i a_{ii}^2 (n - a_{ii}^2) = n^2 - \sum_i a_{ii}^4.$$

Applying the above gives

$$\begin{aligned}
(11) \quad \mathbb{E}[(\operatorname{Tr}((AM)^2))^2] &= \frac{3(n+1)n^2}{(n-1)n(n+2)} - \frac{3(n+1)}{(n-1)n(n+2)} \sum_i a_{ii}^4 + \frac{3}{n(n+2)} \sum_i a_{ii}^4 \\
&\leq 3 + \frac{6}{(n-1)(n+2)}.
\end{aligned}$$

Putting these estimates into Theorem 1 gives:

$$(12) \quad d_{TV}(W, Z) \leq \frac{2\sqrt{2 + \frac{6}{(n-1)(n+2)}}}{(n-1)}.$$

Noting that $\frac{6}{(n-1)(n+2)} \leq 1$ for $n \geq 3$ and that the bound in Theorem 4 is trivially true for $n = 2$ completes the proof. \square

4. THE UNITARY GROUP

Now let $M \in \mathcal{U}_n$ be distributed according to Haar measure, let A be an $n \times n$ matrix over \mathbb{C} , and let $W = \operatorname{Tr}(AM)$. In [4] it was shown that if $M = \Gamma + i\Lambda$ and A and B are fixed real diagonal matrices with $\operatorname{Tr}(AA^t) = \operatorname{Tr}(BB^t) = n$, then $\operatorname{Tr}(A\Gamma) + i\operatorname{Tr}(B\Lambda)$ converges in distribution to a standard complex normal random variable. This implies in particular that $\operatorname{Re}(W)$ converges in distribution to $\mathfrak{N}(0, \frac{1}{2})$. The main theorem of this section gives a rate of this convergence in total variation distance.

A more natural question might be the convergence of W to a standard complex random variable. As this is a multivariate problem, Theorem 1 cannot be applied. A multivariate version of Theorem 1 is forthcoming in [3], which also includes a rate of convergence of W to a standard complex Gaussian random variable.

Theorem 6. *With $M, A,$ and W as above, let W_θ be the inner product of W with the unit vector making angle θ with the real axis. Then*

$$(13) \quad d_{TV} \left(W_\theta, \mathfrak{N} \left(0, \frac{1}{2} \right) \right) \leq \frac{c}{n}$$

for a constant c which is independent of θ .

The constant c is asymptotically equal to $2\sqrt{2}$; for $n \geq 8$ it can be taken to be 4.

Proof. To prove the theorem, first note that it suffices to consider the case $\theta = 0$, that is, to prove that

$$d_{TV} \left(\operatorname{Re}(W), \mathfrak{N} \left(0, \frac{1}{2} \right) \right) \leq \frac{c}{n}.$$

The theorem then follows as stated since the distribution of W is invariant under multiplication by any complex number of unit modulus. Also, A can again be assumed diagonal with positive real entries by the singular value decomposition.

The proof is almost identical to the orthogonal case. Let $H \in \mathcal{U}_n$ be a random unitary matrix, independent of M , and let $M_\epsilon = HA_\epsilon H^* M$, where A_ϵ is as in the orthogonal case.

Let I_2 be the 2×2 identity matrix, let K be the $n \times 2$ matrix consisting of the first two columns of H , and let C_2 be as before. Then

$$(14) \quad \begin{aligned} W_\epsilon - W &= \operatorname{Tr} \left(\left(-\frac{\epsilon^2}{2} + O(\epsilon^4) \right) AKK^*M + \epsilon AKC_2K^*M \right) \\ &= \epsilon \left[\left(-\frac{\epsilon}{2} + O(\epsilon^3) \right) \operatorname{Tr}(AKK^*M) + \operatorname{Tr}(AKC_2K^*M) \right]. \end{aligned}$$

Let $W^r = \operatorname{Re}(W)$ and $W_\epsilon^r = \operatorname{Re}(W_\epsilon)$. As in the orthogonal case, to verify the conditions of Theorem 1, various mixed moments of the entries of H are needed. The relevant unitary integrals can also be found in [10], section 4.2. They imply in particular that

$$(15) \quad \mathbb{E} [(KK^*)_{ij}] = \frac{2}{n} \delta_{ij},$$

$$(16) \quad \mathbb{E} [(KC_2K^*)_{ij}] = 0,$$

thus

$$(17) \quad \lim_{\epsilon \rightarrow 0} \frac{n}{\epsilon^2} \mathbb{E} [W_\epsilon^r - W^r | W] = -W^r;$$

condition (i) is satisfied with $\lambda = \frac{1}{n}$. Also by (14),

$$(18) \quad \begin{aligned} &\lim_{\epsilon \rightarrow 0} \frac{n}{2\epsilon^2} \mathbb{E} [(W_\epsilon^r - W^r)^2 | W] \\ &= \lim_{\epsilon \rightarrow 0} \frac{n}{2} \mathbb{E} \left[(\operatorname{Re}(\operatorname{Tr}(AKC_2K^*M)))^2 | W \right] \\ &= \frac{n}{4} \operatorname{Re} \mathbb{E} \left[\sum_{i,j,k,l} a_{ii} m_{ji} a_{kk} m_{lk} (h_{i1} \bar{h}_{j2} - h_{i2} \bar{h}_{j1})(h_{k1} \bar{h}_{l2} - h_{k2} \bar{h}_{l1}) \right. \\ &\quad \left. + a_{ii} m_{ji} a_{kk} \bar{m}_{lk} (h_{i1} \bar{h}_{j2} - h_{i2} \bar{h}_{j1})(\bar{h}_{k1} h_{l2} - \bar{h}_{k2} h_{l1}) \middle| W \right]. \end{aligned}$$

Using the formulae from [10], it is straightforward to show that

$$(19) \quad \begin{aligned} & \mathbb{E}[(h_{i1}\bar{h}_{j2} - h_{i2}\bar{h}_{j1})(h_{k1}\bar{h}_{l2} - h_{k2}\bar{h}_{l1})] \\ &= -\frac{2\delta_{il}\delta_{jk}(1 - \delta_{ij})}{(n-1)(n+1)} + \frac{2\delta_{ij}\delta_{kl}(1 - \delta_{ik})}{(n-1)n(n+1)} - \frac{2\mathbb{I}(i=j=k=l)}{n(n+1)} \end{aligned}$$

and

$$(20) \quad \begin{aligned} & \mathbb{E}[(h_{i1}\bar{h}_{j2}h_{i2}\bar{h}_{j1})(\bar{h}_{k1}h_{l2} - \bar{h}_{k2}h_{l1})] \\ &= \frac{2(\delta_{ik}\delta_{jl}(1 - \delta_{ij}))}{(n-1)(n+1)} - \frac{2(\delta_{ij}\delta_{kl}(1 - \delta_{ik}))}{n(n-1)(n+1)} + \frac{2\mathbb{I}(i=j=k=l)}{n(n+1)}. \end{aligned}$$

Recall that $\sum'_{i,j}$ stands for summing over all pairs (i, j) where i and j are distinct.

Putting (19) and (20) into (18) and using the independence of M and H gives:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{n}{2\epsilon^2} \mathbb{E}[(W_\epsilon^r - W^r)^2 | W] \\ &= \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[\sum_{i,j,k,\ell} a_{ii}m_{ji}a_{kk}m_{\ell k} (-\delta_{i\ell}\delta_{jk}(1 - \delta_{ij}) \right. \\ & \quad \left. + \frac{1}{n}\delta_{ij}\delta_{k\ell}(1 - \delta_{ik}) - \left(\frac{n-1}{n}\right)\mathbb{I}(i=j=k=\ell) \right. \\ & \quad \left. + \sum_{i,j,k,\ell} a_{ii}m_{ji}a_{kk}m_{\ell k} (\delta_{ik}\delta_{j\ell}(1 - \delta_{ij}) \right. \\ & \quad \left. - \frac{1}{n}\delta_{ij}\delta_{k\ell}(1 - \delta_{ik}) + \left(\frac{n-1}{n}\right)\mathbb{I}(i=j=k=\ell) \right) \Big| W \Big] \\ &= \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\sum'_{i,j} a_{ii}a_{jj}m_{ij}m_{ji} + \frac{1}{n} \sum'_{i,k} a_{ii}a_{kk}m_{ii}m_{kk} \right. \\ & \quad \left. - \left(\frac{n-1}{n}\right) \sum_i a_{ii}^2 m_{ii}^2 + \sum'_{i,j} a_{ii}^2 |m_{ji}|^2 \right. \\ & \quad \left. - \frac{1}{n} \sum'_{i,k} a_{ii}a_{kk}m_{ii}\bar{m}_{kk} + \frac{n-1}{n} \sum_i a_{ii}^2 |m_{ii}|^2 \Big| W \right] \\ &= \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\left(\operatorname{Tr}((AM)^2) - \sum_i (AM)_{ii}^2 \right) \right. \\ & \quad \left. + \frac{1}{n} \left(W^2 - \sum_i (AM)_{ii}^2 \right) - \left(\frac{n-1}{n}\right) \sum_i (AM)_{ii}^2 \right. \\ & \quad \left. + \sum_i a_{ii}^2 (1 - |m_{ii}|^2) \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{n} \left(|W|^2 - \sum_i a_{ii}^2 |m_{ii}|^2 \right) \\
 & + \frac{n-1}{n} \sum_i a_{ii}^2 |m_{ii}|^2 \left| W \right] \\
 = & \frac{1}{2} + \frac{1}{2(n-1)(n+1)} \\
 & + \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\operatorname{Tr}((AM)^2) + \frac{W^2 - |W|^2}{n} \left| W \right. \right].
 \end{aligned}$$

Condition (2) of Theorem 1 is thus satisfied with

$$(21) \quad nE = \frac{1}{2(n-1)(n+1)} + \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\operatorname{Tr}((AM)^2) + \frac{W^2 - |W|^2}{n} \left| W \right. \right].$$

It remains to estimate $n\mathbb{E}|E|$. First,

$$\begin{aligned}
 \mathbb{E} |\operatorname{Tr}((AM)^2)| &= \mathbb{E} \sqrt{\sum_{i,j,k,l} a_{ii} a_{jj} m_{ij} m_{ji} a_{kk} a_{ll} \overline{m_{kl}} \overline{m_{lk}}} \\
 &\leq \sqrt{\sum_{i,j,k,l} a_{ii} a_{jj} a_{kk} a_{ll} \mathbb{E} [m_{ij} m_{ji} \overline{m_{kl}} \overline{m_{lk}}]} \\
 &= \sqrt{\frac{2n^2}{(n-1)(n+1)} - \frac{2}{(n-1)n(n+1)} \left(\sum_i a_{ii}^4 \right)} \\
 &\leq \sqrt{2 + \frac{1}{n^2 - 1}},
 \end{aligned}$$

using the formulae of [10] to evaluate the integrals.

Next,

$$\begin{aligned}
 \mathbb{E}|W|^2 &= \mathbb{E} \left[\sum_{i,j} a_{ii} a_{jj} m_{ii} \overline{m_{jj}} \right] \\
 &= \frac{1}{n} \sum_i a_{ii}^2 \\
 &= 1.
 \end{aligned}$$

Putting these estimates into (21) proves the theorem. □

Theorem 6 yields the following bivariate corollary, which can also be seen as a corollary of the main unitary lemma of [4].

Corollary 7. *For each n , let M_n be a random $n \times n$ unitary matrix, A_n an $n \times n$ matrix over \mathbb{C} with $\operatorname{Tr}(A_n A_n^*) = n$, and $W_n = \operatorname{Tr}(A_n M_n)$. Then the distribution of W_n converges to the standard complex normal distribution in the weak-star topology.*

Proof. The result follows immediately from Theorem 6 by considering the characteristic function of W_n . □

ACKNOWLEDGEMENTS

The author would like to thank Persi Diaconis for sharing his many insights. The results of this paper are part of a Ph.D. thesis written under his supervision at Stanford University.

REFERENCES

1. E. Bolthausen, *An estimate of the remainder in a combinatorial central limit theorem*, Z. Wahrsch. Verw. Gebiete **66** (1984), no. 3, 379–386. MR751577 (85j:60032)
2. E. Borel, *Sur les principes de la theorie cinétique des gaz*, Annales de l'école normale sup. **23** (1906), 9–32. MR1509063
3. S. Chatterjee and E. Meckes, *Multivariate central limit theorems via the method of exchangeable pairs*, 2005, Preprint.
4. A. D'Aristotile, P. Diaconis, and C. Newman, *Brownian motion and the classical groups*, Probability, Statistics and Their Applications: Papers in Honor of Rabi Bhattacharya, IMS Lecture Notes Monogr. Ser., vol. 41, Inst. Math. Statist., Beachwood, OH, 2003, pp. 97–116. MR1999417
5. P. Diaconis, *Application of the method of moments in probability and statistics*, Moments in Mathematics (San Antonio, Tex., 1987), Proc. Sympos. Appl. Math., vol. 37, Amer. Math. Soc., Providence, RI, 1987, pp. 125–142. MR921087 (89m:60006)
6. ———, *Patterns in eigenvalues: the 70th Josiah Willard Gibbs lecture*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), no. 2, 155–178 (electronic). MR1962294 (2004d:15017)
7. P. Diaconis, M. Eaton, and S. Lauritzen, *Finite de Finetti theorems in linear models and multivariate analysis*, Scand. J. Statist. **19** (1992), no. 4, 289–315. MR1211786 (94g:60065)
8. P. Diaconis and D. Freedman, *A dozen de Finetti-style results in search of a theory*, Ann. Inst. H. Poincaré Probab. Statist. **23** (1987), no. 2, suppl., 397–423. MR898502 (88f:60072)
9. P. Diaconis and M. Shahshahani, *On the eigenvalues of random matrices*, J. Appl. Probab. **31A** (1994), 49–62, Studies in applied probability. MR1274717 (95m:60011)
10. F. Hiai and D. Petz, *The semicircle law, free random variables and entropy*, Mathematical Surveys and Monographs, vol. 77, American Mathematical Society, Providence, RI, 2000. MR1746976 (2001j:46099)
11. W. Hoeffding, *A combinatorial central limit theorem*, Ann. Math. Statistics **22** (1951), 558–566. MR0044058 (13:363b)
12. T. Jiang, *How many entries of a typical orthogonal matrix can be approximated by independent normals?*, Ann. Probab. **34** (2006), no. 4, 1497–1529. MR2257653 (2007m:60011)
13. K. Johansson, *On random matrices from the compact classical groups*, Ann. of Math. (2) **145** (1997), no. 3, 519–545. MR1454702 (98e:60016)
14. C. Stein, *Approximate computation of expectations*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, 7, Institute of Mathematical Statistics, Hayward, CA, 1986. MR882007 (88j:60055)
15. ———, *The accuracy of the normal approximation to the distribution of the traces of powers of random orthogonal matrices*, Technical Report No. 470, Stanford University Department of Statistics, 1995.

DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106

E-mail address: ese3@cwru.edu

Current address: American Institute of Mathematics, 360 Portage Avenue, Palo Alto, California 94306