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LINEAR INDEPENDENCE OF ITERATES OF ENTIRE FUNCTIONS

LUIS BERNAL GONZÁLEZ

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ABSTRACT. We prove the following result: The set $\{h_n : n = 0, 1, \dots\}$ is a linearly independent sequence of entire functions, where $h_0 = 1$, $h_1 = g_1$, $h_2 = g_1 \circ g_2$, $h_3 = g_1 \circ g_2 \circ g_3, \dots$, g_1 is a nonconstant entire function and g_n $(n \geq 2)$ are entire functions which are not polynomials of degree ≤ 1. Our theorem generalizes a previous one about linear independence of iterates.

In [1] it is proved that except for trivial cases a sequence of iterates of entire functions is always linearly independent. As a related result, it is also shown in [1] that the Feigenbaum functional equation:

$$(1)\qquad \begin{aligned} f(f(\lambda x)) + \lambda f(x) = 0 \qquad & (-1 \leq x \leq 1); \\ 0 < \lambda = -f(1) < 1; \qquad & f(0) = 1, \end{aligned}$$

does not have an entire solution.

In this paper we shall generalize the interesting theorems proved in [1]. For the sake of completeness, we state them. In the following we denote by $\mathscr{F}$ the space of all entire functions, and by $\mathbf{C}$ the complex plane.

Theorem A. *Let f be a nonidentically zero entire function that satisfies the functional equation*:

$$f(f(\lambda z)) + \lambda f(z) = 0$$

for all $z \in \mathbf{C}$, where λ is a fixed nonzero complex number, then either $f(z) = -z$ and λ is arbitrary or $f(z)$ is a constant with $\lambda = -1$.

From Theorem A, the nonexistence of entire solutions of (1) is derived.

Theorem B. *Let f be an entire function that is not a polynomial of* degree ≤ 1. *Let $R_f : \mathscr{F} \to \mathscr{F}$ be defined by $R_f(g) = g \circ f$ for all $g \in \mathscr{F}$. Then R_f has no eigenvalue distinct from* $0, 1$ *and any eigenfunction with eigenvalue* 1 *is a constant.*

Theorem C. *Let g be a nonconstant entire function and let f be an entire function which is not a polynomial of* degree ≤ 1. *Denote the successive iterates*

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of f *in the following way:* $f_1 = f$, $f_2 = f \circ f_1, \ldots, f_{n+1} = f \circ f_n$ $(n = 1, 2, \ldots)$, *and* f_0 *will stand for the identity mapping. Then* $\{g \circ f_n : n \geq 0\}$ *is a linearly independent sequence of entire functions.*

The proofs of these results are based upon a classical result of Pólya (see [5; 2, pp. 50–51]) concerning the growth of Nevanlinna's characteristics of composite functions. We prove a more general theorem without using these characteristics.

First, we need several results about the maximum modulus of an entire function. Throughout this paper we use the following notation. If $f, g, h, f_i, g_i, h_i, \ldots$ are entire functions, then $F, G, H, F_i, G_i, H_i, \ldots$ respectively are their maximum modulus functions, i.e., $F(r) = \max\{|f(z)| : |z| = r\}$ $(r > 0)$ and so on. If $f = \sum_{i=1}^{n} \lambda_i f_i$ $(\lambda_i \in \mathbf{C};\ i = 1, 2, \ldots, n)$ and $h = g \circ f$, then, trivially, $F(r) \leq \sum_{i=1}^{n} |\lambda_i| F_i(r)$ and $H(r) \leq G(F(r))$ for all $r > 0$. For functions defined on large enough $r > 0$, we say that $\varphi(r) < \psi(r)$ asymptotically (asymp.) when there is $r_0 > 0$ with $\varphi(r) < \psi(r)$ for every $r > r_0$.

The proof of the following theorem can be found, for instance, in [3, pp. 80–81; 4, pp. 225–227].

Theorem D. *Assume that* $f, g \in \mathscr{F}$, $h = g \circ f$, *and* $f(0) = 0$. *Then, there exists* $c \in (0, 1)$ *independent from* f *and* g *such that*

$$G(cF(r/2)) \leq H(r) \quad \textit{for all} \quad r > 0.$$

Next, we state two previous lemmas.

Lemma 1. *Suppose that* $f \in \mathscr{F}$, f *is not constant,* $\alpha > 1$ *and* $0 < \beta < \alpha$. *Then* $\beta F(r) < F(\alpha r)$ *asymp.*

Proof. We obtain $|g(z)| \leq |z| G(R)/R$ if $|z| \leq R$ by applying Schwarz's lemma to $g(z) = f(z) - f(0)$. If $R = \alpha r$, then $F(r) - |f(0)| \leq G(r) \leq (r/R)(F(R) + |f(0)|) = (F(\alpha r) + |f(0)|)/\alpha$. Choose $\varepsilon = (\alpha - \beta)/(2\alpha + 2\beta)$. Then there is $r_0 > 0$ with $|f(0)| < \varepsilon F(r_0)$. Thus $F(\alpha r) \geq ((1-\varepsilon)\alpha/(1+\varepsilon)) F(r) > \beta F(r)$ for all $r > r_0$. □

Lemma 2. *Assume that* p *is a nonnegative integer,* $p \geq 2$, $g_i \in \mathscr{F}$ $(i = 1, 2, \ldots, p)$, g_i *is a nonconstant function* $(i = 2, 3, \ldots, p)$ *and* $h = g_1 \circ g_2 \circ \cdots \circ g_p$. *Then, there exists* $d \in (0, 1)$ *such that*

$$H(r) \geq (G_1 \circ G_2 \circ \cdots \circ G_p)(dr) \quad \text{asymp.}$$

Proof. Let us prove the lemma for $p = 2$ and put $g(z) = g_1(z + g_2(0))$ and $f(z) = g_2(z) - g_2(0)$. Evidently, $h = g \circ f$, $f(0) = 0$, $F(r) \geq G_2(r) - |g_2(0)|$ $(r > 0)$, $G(s) \geq G_1(s - |g_2(0)|)$ $(s > |g_2(0)|)$ and $G_2(r) > (2 + (2/c))|g_2(0)|$ asymp., where c is the constant in Theorem D. Then by this theorem we can conclude that

$$G(cF(r/2)) \geq G_1(cF(r/2) - |g_2(0)|) \geq G_1(cG_2(r/2) - (c+1)|g_2(0)|).$$

Hence

$$H(r) \geq G_1((c/2)G_2(r/2)) \quad \text{asymp.} \tag{2}$$

Let $d = c/6$. By applying Lemma 1 to $\alpha = 3/c$, $\beta = 2/c$ we obtain $G_2(r/2) = G_2(\alpha dr) \geq \beta G_2(dr)$ asymp. This inequality and (2) complete the case $p = 2$.

Let us go on by induction. Assume that there is $d_1 \in (0, 1)$ such that

$$Q(r) \geq (G_1 \circ G_2 \circ \cdots \circ G_{p-1})(d_1 r) \quad \text{asymp.}, \tag{3}$$

where $q(z) = g_1 \circ g_2 \circ \cdots \circ g_{p-1}$. We have $h = q \circ g_p$, so we can apply the case $p = 2$. There exists $d_2 \in (0, 1)$ such that

$$H(r) \geq Q(G_p(d_2 r)) \quad \text{asymp.} \tag{4}$$

Again apply Lemma 1 with $\alpha = 2/d_1$, $\beta = 1/d_1$ and set $d = d_1 d_2/2$. Then

$$d_1 G_p(d_2 r) = (1/\beta) G_p(\alpha dr) \geq G_p(dr) \quad \text{asymp.} \tag{5}$$

Equations (3)–(5) complete the proof. □

Theorem. *Suppose that* g_1 *is a nonconstant entire function and* g_n $(n \geq 2)$ *are entire functions which are not polynomials of* degree ≤ 1. *Define* $h_0 = 1$, $h_1 = g_1$, $h_n = h_{n-1} \circ g_n$ $(n \geq 2)$. *Then* $\{h_n : n = 0, 1, 2, \dots\}$ *is a linearly independent sequence of entire functions.*

Proof. By contradiction, let p be an integer ≥ 2 and let $\sum_{i=0}^{p} \lambda_i h_i(z) = 0$ for all $z \in \mathbf{C}$, with $\lambda_i \in \mathbf{C}$ $(i = 0, 1, \dots, p)$ and $\lambda_p \neq 0$ (cases $p = 0$ and $p = 1$ are trivial). Then $h_p = \sum_{i=0}^{p-1} \mu_i h_i$, where $\mu_i = -\lambda_i/\lambda_p$ $(i = 0, 1, \dots, p-1)$. We have $H_p(r) < \mu \cdot \sum_{i=0}^{p-1} H_i(r)$ $(r > 0)$, with μ a fixed positive real number greater than $\max\{|\mu_i| : i = 0, 1, \dots, p-1\}$. By applying Lemma 2, we have $(G_1 \circ G_2 \circ \cdots \circ G_p)(dr) < \mu + \mu \sum_{i=1}^{p-1} (G_1 \circ G_2 \circ \cdots \circ G_i)(r)$ asymp. Define φ by

$$\varphi(r) = (G_1 \circ G_2 \circ \cdots \circ G_{p-1})(r) \qquad (r > 0).$$

Evidently, $1 < \varphi(r)$ asymp. and $(G_1 \circ G_2 \circ \cdots \circ G_i)(r) \leq \varphi(r)$ asymp. $(i = 1, 2, \dots, p-1)$, because $G_j(r) > r$ asymp. for all $j \geq 2$ and each G_j $(j \geq 1)$ is strictly increasing, by hypothesis. Hence,

$$\varphi(G_p(dr)) < p\mu\varphi(r) \quad \text{asymp.} \tag{6}$$

Since g_p is not a polynomial of degree ≤ 1, the inequality

$$G_p(r) > (p/d)(1 + \mu) r \quad \text{asymp.}$$

holds. Then we apply Lemma 1 $p - 1$ times to obtain $\varphi(G_p(dr)) > \varphi((p + p\mu)r) > (G_1 \circ G_2 \circ \cdots \circ G_{p-2})((p - 1 + p\mu) G_{p-1}(r)) > (G_1 \circ \cdots \circ G_{p-3})((p-2+p\mu)(G_{p-2} \circ G_{p-1})(r)) > \cdots > (1+p\mu)(G_1 \circ \cdots \circ G_{p-1})(r) > p\mu\varphi(r)$ asymp., that is a contradiction with (6). □

Finally, we derive Theorems A, B, and C as a corollary.

Proof of Theorem A. If f is a polynomial of degree ≤ 1, then the statement is obvious. Now, let f be an entire function that is not a polynomial of degree ≤ 1. Then f does not satisfy $f(f(\lambda z)) + \lambda f(z) = 0 (z \in \mathbb{C})$ for any nonzero

$\lambda \in \mathbb{C}$. To see this, we only have to apply our theorem to $g_1 = f$, $g_n = f(\lambda z)(n \geq 2)$.

Proof of Theorem B. Let g be an eigenfunction for R_f. Applying our theorem to $g_1 = g$, $g_n = f$ $(n \geq 2)$, we deduce that g is a constant $(g(z) = \mu \neq 0)$. Consequently, there is $\lambda \in \mathbf{C}$ such that $\mu - \lambda\mu = 0$. Thus, in fact, $\lambda = 1$ is the unique eigenvalue of R_f and the eigenfunctions are precisely the set of all nonzero constant entire functions.

Proof of Theorem C. Again apply our theorem to $g_1 = g$, $g_n = f$ $(n \geq 2)$.

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Departamento de Análisis Matemático, Facultad de Matemáticas, Avenida Reina Mercedes, 41080-Sevilla, Spain