PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 9, September 1996

LINEAR INDEPENDENCE OF TIME-FREQUENCY TRANSLATES

CHRISTOPHER HEIL, JAYAKUMAR RAMANATHAN, AND PANKAJ TOPIWALA

(Communicated by Palle E. T. Jorgensen)

Abstract. The refinement equation $\varphi(t) = \sum_{k=N_1}^{N_2} c_k\, \varphi(2t-k)$ plays a key role in wavelet theory and in subdivision schemes in approximation theory. Viewed as an expression of linear dependence among the time-scale translates $|a|^{1/2}\varphi(at-b)$ of $\varphi \in L^2(\mathbf{R})$, it is natural to ask if there exist similar dependencies among the time-frequency translates $e^{2\pi ibt} f(t+a)$ of $f \in L^2(\mathbf{R})$. In other words, what is the effect of replacing the group representation of $L^2(\mathbf{R})$ induced by the affine group with the corresponding representation induced by the Heisenberg group? This paper proves that there are no nonzero solutions to lattice-type generalizations of the refinement equation to the Heisenberg group. Moreover, it is proved that for each arbitrary finite collection $\{(a_k, b_k)\}_{k=1}^N$, the set of all functions $f \in L^2(\mathbf{R})$ such that $\{e^{2\pi ib_k t} f(t+a_k)\}_{k=1}^N$ is independent is an open, dense subset of $L^2(\mathbf{R})$. It is conjectured that this set is all of $L^2(\mathbf{R}) \setminus \{0\}$.

1. Introduction

Gabor analysis and wavelet analysis are two important classes of techniques with applications in mathematics, physics, and engineering. Each can be described in terms of the action of a particular group on a function space, such as $L^2(\mathbf{R}) = \{f : \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty\}$. This point of view has been beautifully advanced in the atomic decomposition theory of Feichtinger and Gröchenig [FG1], [FG2], which applies to general representations acting on Banach spaces. A survey specifically of Gabor and wavelet analysis on $L^2(\mathbf{R})$ from the group viewpoint can be found in the research/tutorial article [HW].

The group for Gabor analysis is the Heisenberg group $\mathbf{H} = \mathbf{T} \times \mathbf{R} \times \mathbf{R}$, where $\mathbf{T}$ is the unit circle in the complex plane and $\mathbf{R}$ is the real line. The group operation is induced by the *Schroedinger representation* ρ of $\mathbf{H}$ on $L^2(\mathbf{R})$ [F], which is defined

Received by the editors March 13, 1995.

1991 *Mathematics Subject Classification.* Primary 42C99, 46B15; Secondary 46C15.

Key words and phrases. Affine group, frames, Gabor analysis, Heisenberg group, linear independence, phase space, refinement equations, Schroedinger representation, time-frequency, wavelet analysis.

The first author was partially supported by National Science Foundation Grant DMS-9401340 and by the MITRE Sponsored Research Program. The second author was partially supported by National Science Foundation Grant DMS-9401859 and by a Faculty Research and Creative Project Fellowship from Eastern Michigan University. The third author was supported by the MITRE Sponsored Research Program.

©1996 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

by

$$\rho(z,a,b)f(t) \;=\; z\,e^{\pi iab}\,e^{2\pi ibt}\,f(t+a) \qquad \text{for } (z,a,b)\in \mathbf{H} \text{ and } f\in L^2(\mathbf{R}).$$

In this paper the toral component of $\mathbf{H}$ is inconsequential, as multiplication by the scalar z and phase factor $e^{\pi iab}$ can always be absorbed into other scalar multiplications that will appear. We therefore write simply $\rho(a,b)f(t) = e^{2\pi ibt}\,f(t+a)$. We call $\rho(a,b)f$ a *time-frequency translate* of f, and refer to $\mathbf{R}^2 = \mathbf{R}\times\mathbf{R}$ as *phase space*.

In Gabor analysis, a function $g \in L^2(\mathbf{R})$ is analyzed in terms of the collection of inner products $\{\langle g, \rho(a,b)f\rangle : (a,b)\in\Lambda\}$, where $f\in L^2(\mathbf{R})$ and $\Lambda\subset\mathbf{R}^2$ are fixed and are selected according to some constraints. These inner products $\langle g,\rho(a,b)f\rangle$ are termed *Gabor coefficients*. Retaining all possible Gabor coefficients (i.e., $\Lambda = \mathbf{R}^2$) permits a stable integral reconstruction of g from those coefficients for any f. The mapping $g\mapsto\{\langle g,\rho(a,b)f\rangle : (a,b)\in\mathbf{R}^2\}$ is called the *continuous Gabor transform* of g by f. Less redundant transforms can be obtained by selecting a discrete subset Λ of $\mathbf{R}^2$. In order that g be completely determined by the Gabor coefficients corresponding only to $(a,b)\in\Lambda$, the collection

$$S(f,\Lambda) \;=\; \{\rho(a,b)f : (a,b)\in\Lambda\}$$

of time-frequency translates of f along Λ must be complete in $L^2(\mathbf{R})$. In order to allow stable reconstruction of g from these Gabor coefficients, $S(f,\Lambda)$ must form a frame for $L^2(\mathbf{R})$. (Frames are generalizations of Riesz bases, allowing stable basis-like representations of elements but without the requirement that these representations be unique.)

Extensive effort has been put into analyzing the properties of $S(f,\Lambda)$. Feichtinger–Gröchenig theory ensures that $S(f,\Lambda)$ will be a frame for $L^2(\mathbf{R})$ for many f as long as Λ is dense enough. This result is not dependent on the representation or the function space, although the required density does depend on these factors. Aside from density, no structural properties of Λ are required.

In the opposite direction, it is known that $S(f,\Lambda)$ cannot be complete if the density of Λ is too low. Specifically, for the lattice $\Lambda = a\mathbf{Z}\times b\mathbf{Z}$, Daubechies [D1] and Rieffel [R] proved that $S(f,\Lambda)$ is incomplete if the density $1/(ab)$ is strictly less than one. Both proofs rely heavily on the algebraic structure of Λ. Rieffel's result is a corollary of powerful C^*-algebra theorems. Daubechies' result, which applies only to rational values of ab, relies on the *Zak transform*, a unitary mapping of $L^2(\mathbf{R})$ onto $L^2([0,1)^2)$ with numerous applications in Gabor analysis. Daubechies' monograph [D2] contains a wealth of expository information relating to the analysis of $S(f,\Lambda)$ with lattice Λ. Recently, Ramanathan and Steger [RS] proved for arbitrary irregular Λ that $S(f,\Lambda)$ cannot be a frame for any f if Λ has density strictly less than one. The proof requires only simple dimension-counting techniques. Moreover, if $S(f,\Lambda)$ is a frame and Λ has density greater than one, then $S(f,\Lambda)$ is overcomplete, hence not a basis for $L^2(\mathbf{R})$. Thus $S(f,\Lambda)$ can form a Riesz basis for $L^2(\mathbf{R})$ only if Λ has density exactly one. For the case of lattice $\Lambda = a\mathbf{Z}\times b\mathbf{Z}$ with density $1/(ab) = 1$, the *Balian–Low Theorem* places severe restrictions on those $f\in L^2(\mathbf{R})$ such that $S(f,\Lambda)$ can form a Riesz basis. Specifically, f must "maximize uncertainty," meaning that $\left(\int |t\,f(t)|^2\,dt\right)\left(\int |\gamma\,\hat{f}(\gamma)|^2\,d\gamma\right) = +\infty$. In particular, either f is not smooth or it has very slow decay at infinity. Proofs of the Balian–Low Theorem are exposited in [BHW] and [D2].

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Note added in proof. Walnut and Heil recently observed that the work of Landau [L] on sparse sets of exponentials on unions of intervals implies the existence of *complete* Gabor systems $S(f,\Lambda)$, where Λ is *not* a lattice and has density strictly less than one [BHW].

Wavelet analysis is parallel in certain respects to Gabor analysis, with the affine group $\mathbf{A} = (\mathbf{R} \setminus \{0\}) \times \mathbf{R}$ substituting for the Heisenberg group. *Time-scale* translates replace time-frequency translates: we let σ be the representation of $\mathbf{A}$ on $L^2(\mathbf{R})$ defined by $\sigma(a,b)\psi(t) = |a|^{1/2}\,\psi(at-b)$ for $(a,b) \in \mathbf{A}$ and $\psi \in L^2(\mathbf{R})$. Again, we may define a continuous transform, or seek frames or bases of the form

$$T(\psi,\Lambda) \;=\; \{\sigma(a,b)\psi : (a,b) \in \Lambda\}.$$

Feichtinger–Gröchenig theory again ensures the existence of frames for Λ which are "dense enough." However, because the affine group differs considerably in structure from the Heisenberg group, there are several striking differences in the obtainable properties of $S(f,\Lambda)$ versus $T(\psi,\Lambda)$. Perhaps the most striking is that $T(\psi,\Lambda)$ can form an orthonormal basis with smooth, well-localized ψ. The smoothness can be C^k if ψ is compactly supported, or C^∞ (with exponential decay) if ψ is infinitely supported. The typical choice for Λ in these constructions is the "regular" discrete subset $\Lambda = \{(a^n, mb) : m,n \in \mathbf{Z}\}$, most often with $a=2$ and $b=1$. This Λ is not a subgroup of $\mathbf{A}$—in fact, $\mathbf{A}$ contains no discrete subgroups. Daubechies' monograph [D2] is again an excellent reference for details of these constructions.

This paper analyzes another fundamental property of $S(f,\Lambda)$ in terms of f and Λ. That property is *finite linear independence*. An immediate motivation arises from the fact that any practical implementation of frames must be finite. And since any finite set of independent vectors is a Riesz basis for its linear span, the first basic question is whether $S(f,\Lambda)$ is independent when Λ is finite. Our results here on finite independence of $S(f,\Lambda)$ have inspired new results from Christensen on finite implementations of frames [C].

A second motivation follows immediately upon comparison with the independence properties of time-scale translates. The key step in the construction of compactly supported wavelets ψ such that $T(\psi,\Lambda)$ forms an orthonormal basis for $L^2(\mathbf{R})$ is the solution of a *refinement equation*

$$\varphi(t) \;=\; \sum_{k=N_1}^{N_2} c_k\,\varphi(2t-k) \tag{1}$$

to find a *scaling function* φ. This refinement equation is also the starting point for the generation of subdivision schemes in approximation theory [CDM]. Note that it is an expression of linear *dependence* among the time-scale translates of φ: $\sigma(1,0)\varphi = \sum_{k=N_1}^{N_2} c_k\,\sigma(2,k)\varphi$. Thus $T(\varphi,\Lambda)$ is dependent when $\Lambda = \{(1,0)\} \cup \{(2,k)\}_{k=N_1}^{N_2}$, a set of regularly spaced points with one additional point. A natural question is whether there are analogous dependencies among time-*frequency* translates: if we take Λ to be any set of regularly spaced points in $\mathbf{R}^2$ along with one additional point, i.e., $\Lambda = \{(p,q)\} \cup \{(ak+c, bk+d)\}_{k=N_1}^{N_2}$, will $S(f,\Lambda)$ be dependent for some $f \in L^2(\mathbf{R})$? In other words, are there solutions to the "Heisenberg refinement equation"

$$e^{2\pi ipt}\,f(t+q) \;=\; \sum_{k=N_1}^{N_2} c_k\,e^{2\pi i(bk+d)t}\,f(t+ak+c)? \tag{2}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

We prove in Section 3 that the answer to this question is no (and therefore that there are no "Heisenberg multiresolution analyses" for $L^2(\mathbf{R})$). In particular, this implies that any collection of three or fewer time-frequency translates of any nonzero f must be independent.

The assumption made for this result, that the points of Λ are regularly spaced and collinear with at most one non-collinear point, is crucial to our method of proof, which is based on the Ergodic Theorem. The question for arbitrary finite Λ is open. We prove in Section 4 that $S(f,\Lambda)$ is independent for several special classes of f and Λ. In particular, we prove that for each fixed finite Λ, the set of functions $f \in L^2(\mathbf{R})$ such that $S(f,\Lambda)$ is independent is an open, dense subset of $L^2(\mathbf{R})$. Thus dependency of $S(f,\Lambda)$ occurs at most rarely, at least in the Baire category sense: the set of f whose translates are dependent is a nowhere dense subset of $L^2(\mathbf{R})$. We conjecture that this set actually consists of the zero function alone.

Conjecture. *If $f \in L^2(\mathbf{R})$ is nonzero, then $S(f,\Lambda)$ is linearly independent for every finite $\Lambda \subset \mathbf{R}^2$.*

We note that our results and this Conjecture generalize easily to higher dimensions. Most of the results also apply to $L^p(\mathbf{R})$ for $1 \le p < \infty$. The analogue of the Conjecture for $L^\infty(\mathbf{R})$ is clearly false. An interesting counterexample for this case is any trigonometric polynomial $f(t) = \sum_{k=1}^N c_k\, e^{2\pi i b_k t}$—since $f(t+a)\, f(t) - f(t)\, f(t+a) = 0$ for each a, we have $S(f,\Lambda)$ dependent with $\Lambda = \{(0,b_k)\} \cup \{(a,b_k)\}$.

In this paper we focus on the Schroedinger representation induced by the Heisenberg group, with motivation provided by comparison with the representation induced by the affine group. It would be of great interest to understand the problem of independence more generally, i.e., to tie the existence of dependency relations among group translates of a function to some specific structures in the group or its representation.

2. Operations on phase space

Independence of $S(f,\Lambda)$ is preserved by area-preserving affine transformations of phase space, also called *metaplectic transforms*. In particular, if $\alpha \in \mathbf{R}^2$ is fixed and M is a 2×2 matrix with determinant one, then there are unitary operators U, V on $L^2(\mathbf{R})$ such that $S(f,\Lambda) = V(S(Uf, M\Lambda + \alpha))$. Therefore $S(f,\Lambda)$ is independent if and only if $S(Uf, M\Lambda+\alpha)$ is independent. This is most easily seen by writing M as a composition of shears and axis rescalings, and using the following facts.

a. *Translations*. The time and frequency translation operators $T_r f(t) = f(t+r)$ and $M_r f(t) = e^{2\pi i r t}\, f(t)$ translate phase space parallel to the axes: $\rho(a,b) = e^{\pi i r b}\, \rho(a-r,b)\, T_r$ and $\rho(a,b) = e^{-\pi i r a}\, \rho(a,b-r)\, M_r$.

b. *Rescalings*. The dilation operator $D_r f(t) = |r|^{1/2}\, f(rt)$ rescales the axes: $\rho(a,b) = D_{1/r}\, \rho(a/r, br)\, D_r$.

c. *Shears*. Modulation by a linear-FM chirp shears phase space along the frequency axis: if $S_r f(t) = e^{\pi i r t^2}\, f(t)$, then $\rho(a,b) = S_{-r}\, \rho(a, b-ar)\, S_r$. Similarly, convolution with a linear-FM chirp shears along the time axis: if $U_r f = (e^{\pi i r t^2})^\vee * f = (S_r \hat{f})^\vee$, then $\rho(a,b) = U_{-r}\, \rho(-a-br, -b)\, U_r$.

Metaplectic transforms have recently been applied to time-frequency analysis in novel ways by Mann, Baraniuk, Haykin, and Jones [MBHJ]. In addition, Kaiser [K] has shown how rotations of phase space can be useful in constructing collections

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$S(f,\Lambda)$ which form frames for $L^2(\mathbf{R})$ when $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$ is a lattice in $\mathbf{R}^2$. The Fourier transform rotates phase space by $\pi/2$: $\rho(a,b) = \mathcal{F}^{-1}\,\rho(-b,a)\,\mathcal{F}$, where $\mathcal{F}g(\gamma) = \hat{g}(\gamma) = \int g(t)\,e^{-2\pi i\gamma t}\,dt$. Arbitrary rotations can be achieved using the Hermite operator, or harmonic oscillator Hamiltonian, $H = t^2 - \frac{1}{4\pi^2}\frac{d^2}{dt^2}$. The unitary operator $R_\theta = e^{-2\pi i\theta H}$ rotates phase space by θ:

$$\rho(a,b) = R_{-\theta}\,\rho(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)\,R_\theta.$$

We use metaplectic transforms to simplify several of the proofs. For example, here are two simple results establishing that $S(f,\Lambda)$ is always independent for some specific Λ.

Proposition 1. *Assume $f \in L^2(\mathbf{R})$ is nonzero. If the elements of Λ are collinear, then $S(f,\Lambda)$ is linearly independent.*

Proof. By applying a metaplectic transform, we may assume that $\Lambda = \{(0,b_k)\}_{k=1}^N$ with distinct b_k. However, $\sum_{k=1}^N c_k\,\rho(0,b_k)f(t) = \left(\sum_{k=1}^N c_k\,e^{2\pi i b_k t}\right) f(t)$, and trigonometric polynomials are nonzero almost everywhere. □

A *lattice* in $\mathbf{R}^2$ is any rigid translation of a discrete subgroup of $\mathbf{R}^2$ generated by two independent vectors in $\mathbf{R}^2$. It is a *unit lattice* if a fundamental tile has area one.

Proposition 2. *Fix any nonzero $f \in L^2(\mathbf{R})$. If Λ is a finite subset of a unit lattice in $\mathbf{R}^2$, then $S(f,\Lambda)$ is linearly independent.*

Proof. By applying a metaplectic transform, we may assume that Λ is a subset of $\mathbf{Z}^2$. Let Z be the *Zak transform* $Zf(t,\omega) = \sum_{k\in\mathbf{Z}} f(t+k)\,e^{-2\pi i k\omega}$. This is a unitary mapping of $L^2(\mathbf{R})$ onto $L^2([0,1)^2)$. Properties of the Zak transform are surveyed in [J] and [HW, Section 4.3]. We need only the fact that $Z(\rho(m,n)f)(t,\omega) = e^{2\pi i n t}\,e^{2\pi i m\omega}\,Zf(t,\omega)$, so that $Z\left(\sum_{(m,n)\in\Lambda} c_{mn}\,\rho(m,n)f\right)(t,\omega) = E(t,\omega)\,Zf(t,\omega)$, with E a two-dimensional trigonometric polynomial, which must be nonzero almost everywhere. □

3. Heisenberg refinement equations

Proposition 1 states that $S(f,\Lambda)$ is always independent if the elements of Λ are collinear. The addition of one non-collinear point to Λ greatly complicates the question of independence. In this section we prove that $S(f,\Lambda)$ remains independent with one non-collinear point, as long as the collinear points are regularly spaced. In particular, there are no solutions to the Heisenberg refinement equation (2).

Theorem 1. *Assume $f \in L^2(\mathbf{R})$ is nonzero. If Λ consists of a finite collection of regularly spaced collinear points in phase space together with one additional point, i.e., if*

$$\Lambda \;=\; \{(p,q)\} \;\cup\; \{(ak+c, bk+d)\}_{k=N_1}^{N_2},$$

then $S(f,\Lambda)$ is linearly independent. In particular, this is the case for any Λ consisting of three or fewer points.

Proof. By applying a metaplectic transform, we may assume that Λ is a finite subset of $\{(a,0)\} \cup \{(0,k)\}_{k\in\mathbf{Z}}$, with $a \geq 0$. If $a = 0$ then the points of Λ are collinear, and the proof follows by Proposition 1. Assume therefore that $a > 0$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Suppose that $S(f,\Lambda)$ is dependent. Then there would exist scalars c, c_k so that

$$c\, f(t+a) \;=\; \sum_{(0,k)\in\Lambda} c_k\, e^{2\pi i k t}\, f(t) \quad \text{a.e.} \tag{3}$$

In light of Proposition 1 we cannot have $c = 0$, so we may normalize to $c = 1$. By replacing f by a translate $f(t+r)$ and changing the values of the c_k in (3) correspondingly, we may assume that (3) holds and that

$$A \;=\; \operatorname{supp}(f) \cap [0, \min(a,1)]$$

has positive measure. Let $M(t)$ be the 1-periodic trigonometric polynomial $M(t) = \sum_{(0,k)\in\Lambda} c_k\, e^{2\pi i k t}$, so that

$$f(t+a) \;=\; M(t)\, f(t) \quad \text{a.e.} \tag{4}$$

Iterating (4), we have

$$f(t+an) \;=\; f(t) \prod_{j=0}^{n-1} M(t+ja) \;=\; f(t)\, P_n(t), \qquad n \geq 0. \tag{5}$$

Replacing t by $t-a$ in (4), we have $f(t) = M(t-a)\, f(t-a)$. Since M is a trigonometric polynomial, it is nonzero almost everywhere. Therefore $f(t-a) = M(t-a)^{-1} f(t)$ a.e. Iterating this, we obtain

$$f(t-an) \;=\; f(t) \prod_{j=1}^{n} M(t-ja)^{-1} \;=\; f(t)\, Q_n(t), \qquad n \geq 0. \tag{6}$$

Therefore, from equalities (5) and (6) we conclude that $f(t)\, Q_n(t) = f(t-an) = f(t)\, P_n(t-an)^{-1}$, so

$$Q_n(t) \;=\; P_n(t-an)^{-1}, \qquad t \in \operatorname{supp}(f). \tag{7}$$

Applying (5), we compute

$$\begin{aligned} \infty \;>\; \int_0^\infty |f(t)|^2\, dt \;&=\; \sum_{n=0}^\infty \int_0^a |f(t+an)|^2\, dt \\ &=\; \int_0^a \sum_{n=0}^\infty |f(t)|^2\, |P_n(t)|^2\, dt \;\geq\; \int_A |f(t)|^2 \sum_{n=0}^\infty |P_n(t)|^2\, dt. \end{aligned}$$

After a similar calculation based on (6) and the fact that $\int_{-\infty}^a |f(t)|^2\, dt < \infty$, we conclude that

$$\lim_{n\to\infty} P_n(t) \;=\; \lim_{n\to\infty} Q_n(t) \;=\; 0, \qquad \text{a.e. } t \in A. \tag{8}$$

If a is rational, say $a = p/q$, then $aq = p$ is an integer, so from (7) we have $Q_{qn}(t) = P_{qn}(t-aqn)^{-1} = P_{qn}(t)^{-1}$ for $t \in \operatorname{supp}(f)$. This contradicts (8).

Assume then that a is irrational. By Egorov's Theorem applied to (8), we can find a set $E \subset A$ with positive measure such that $P_n(t) \to 0$ uniformly for $t \in E$. Similarly, there is a set $F \subset A$ with positive measure such that $Q_n(t) \to 0$ uniformly for $t \in F$.

Now fix $0 < \varepsilon < 1$. Then there exists an integer N such that $|P_n(t)| < \varepsilon$ for $t \in E$ and $|Q_n(t)| < \varepsilon$ for $t \in F$ if $n > N$. Because a is irrational, the mapping

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$\tau t = t - a \bmod 1$ is a measure-preserving, ergodic map of $[0,1)$ onto itself. The Ergodic Theorem [W, Corollary 1.14.2] therefore implies that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} |\tau^{-n}E \cap F| \;=\; |E|\,|F| \;>\; 0.$$

So, there must be an $n > N$ such that $|\tau^{-n}E \cap F| > 0$. However, if $t \in \tau^{-n}E \cap F$ then $t - na \bmod 1 \in E$ and $t \in F$. Therefore $|P_n(t-na)| < \varepsilon$ and $|Q_n(t)| < \varepsilon$ for every $t \in \tau^{-n}E \cap F$. Since $Q_n(t) = P_n(t-na)^{-1}$ for each such t, this contradicts (8). □

4. Independence of arbitrary time-frequency translates

In this section we investigate whether $S(f,\Lambda)$ is independent for arbitrary f and Λ. First, Propositions 3 and 4 establish that $S(f,\Lambda)$ is linearly independent if Λ is an arbitrary finite set and f is an element of specific dense subsets of $L^2(\mathbf{R})$.

Proposition 3. *Fix any finite $\Lambda \subset \mathbf{R}^2$. If $f \in L^2(\mathbf{R})$ is supported in a half-line then $S(f,\Lambda)$ is linearly independent.*

Proof. By translating and time-reversing f if necessary (i.e., by applying a metaplectic transform to phase space), we may assume that f is supported in $[0,\infty)$ and that f is not supported in $[R,\infty)$ if $R > 0$. Write $\Lambda = \{(a_k, b_{kj}) : j = 1,\dots,M_k;\ k = 1,\dots,N\}$, where $a_1 < \cdots < a_N$, and set

$$\begin{aligned} s(t) \;&=\; \sum_{k=1}^{N}\sum_{j=1}^{M_k} c_{kj}\,\rho(a_k,b_{kj})f(t) \;=\; \sum_{k=1}^{N}\left(\sum_{j=1}^{M_k} c_{kj}\,e^{2\pi i b_{kj}t}\right) f(t+a_j) \\ &=\; \sum_{k=1}^{N} E_k(t)\,f(t+a_j). \end{aligned} \tag{9}$$

Each E_k is a trigonometric polynomial. If $-a_N \le t < -a_{N-1}$ then there is only a single nonzero term in the last summation in (9), i.e., $s(t) = E_N(t)\,f(t+a_N) = 0$ for $t \in [-a_N, -a_{N-1})$. Therefore, if $s(t) = 0$ a.e., then f vanishes almost everywhere on $[0, a_N - a_{N-1})$, a contradiction. □

Proposition 4. *Fix any finite $\Lambda \subset \mathbf{R}^2$. If f is any finite linear combination of Hermite functions, then $S(f,\Lambda)$ is linearly independent.*

Proof. By dilating f if necessary, we can write $f(t) = p(t)\,e^{-t^2}$ with p a polynomial. Write $\Lambda = \{(a_k,b_{kj}) : j = 1,\dots,M_k;\ k = 1,\dots,N\}$, where $a_1 < \cdots < a_N$. If $N = 1$, then the elements of Λ are collinear. So, assume $N > 1$, and set

$$\begin{aligned} s(t) &= \sum_{k=1}^{N}\sum_{j=1}^{M_k} c_{kj}\,\rho(a_k,b_{kj})f(t) = e^{-t^2}\sum_{k=1}^{N}\left(\sum_{j=1}^{M_k} c_{kj}\,e^{-a_k^2}\,e^{2\pi i b_{kj}t}\right)e^{-2ta_k}\,p(t+a_k) \\ &= e^{-t^2}\sum_{k=1}^{N} E_k(t)\,e^{-2ta_k}\,p(t+a_k). \end{aligned}$$

Since $N > 1$ we must have either $a_1 < 0$ or $a_N > 0$. Suppose that $a_1 < 0$. Then since $a_1 < a_2,\dots,a_N$ and p is polynomial, $|e^{-2ta_1}\,p(t+a_1)|$ increases as $t \to \infty$ exponentially faster than $|e^{-2ta_k}\,p(t+a_k)|$ for $k = 2,\dots,N$. Moreover, each E_k is a trigonometric polynomial. In particular, E_1 is almost periodic and $E_2,\dots,E_N$ are bounded. Hence, if E_1 is nontrivial then we can find a sequence $\{t_n\}$ with

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$t_n \to \infty$ such that $|E_1(t_n)\, e^{-2t_n a_1}\, p(t_n + a_1)|$ increases exponentially faster than $|E_k(t_n)\, e^{-2t_n a_k}\, p(t_n + a_k)|$ for any $k = 2, \ldots, N$. Therefore, $s(t_n) \neq 0$ for large enough n. Since s is continuous, we conclude that $S(f, \Lambda)$ is independent. A similar argument applies if $a_N > 0$. □

Next, the following result implies that for each Λ, the set of all $f \in L^2(\mathbf{R})$ such that $S(f, \Lambda)$ is linearly independent is an open subset of $L^2(\mathbf{R})$. From Propositions 3 and 4, we know that this subset is dense in $L^2(\mathbf{R})$.

Proposition 5. *Fix any finite $\Lambda \subset L^2(\mathbf{R})$. Assume $f \in L^2(\mathbf{R})$ is such that $S(f, \Lambda)$ is linearly independent. Then there exists an $\varepsilon > 0$ such that $S(g, \Lambda)$ is linearly independent for any $g \in L^2(\mathbf{R})$ with $\|f - g\| < \varepsilon$.*

Proof. Write $\Lambda = \{(a_k, b_k)\}_{k=1}^N$, and define the continuous, linear mapping $T\colon \mathbf{C}^N \to L^2(\mathbf{R})$ by $T(c_1, \ldots, c_N) = \sum c_k\, \rho(a_k, b_k) f$. Note that T is injective since $S(f, \Lambda)$ is independent. Therefore T is continuously invertible on its range. In particular, there exist A, $B > 0$ such that

$$A \sum_{k=1}^N |c_k| \;\leq\; \Big\| \sum_{k=1}^N c_k\, \rho(a_k, b_k) f \Big\| \;\leq\; B \sum_{k=1}^N |c_k| \quad \text{for each } (c_1, \ldots, c_N) \in \mathbf{C}^N.$$

Therefore, if $\|f - g\| < A$ and $(c_1, \ldots, c_N) \in \mathbf{C}^N$, then

$$\begin{aligned} \Big\| \sum_{k=1}^N c_k\, \rho(a_k, b_k) g \Big\| \;&\geq\; \Big\| \sum_{k=1}^N c_k\, \rho(a_k, b_k) f \Big\| \;-\; \Big\| \sum_{k=1}^N c_k\, \rho(a_k, b_k)(g - f) \Big\| \\ &\geq\; A \sum_{k=1}^N |c_k| \;-\; \sum_{k=1}^N |c_k|\, \|\rho(a_k, b_k)(f - g)\| \\ &=\; (A - \|f - g\|) \sum_{k=1}^N |c_k|. \end{aligned}$$

□

Any finite collection of independent vectors is a Riesz basis for its linear span. The proof of Proposition 5, with the ℓ^2 norm on $\mathbf{C}^N$ replacing the ℓ^1 norm, therefore implies that a perturbation of a finite Riesz basis remains a Riesz basis for its span. The problem of perturbing bases is classical. The problem of perturbing general frames in Hilbert and Banach spaces has been explored recently in [CH].

The following final result, on perturbing of the elements of Λ while keeping f fixed, can be proved with a technique similar to that used in the proof of Proposition 5, using the fact that translation and modulation are continuous in the L^2-norm.

Proposition 6. *Fix any finite $\Lambda = \{(a_k, b_k)\}_{k=1}^N$. Assume $f \in L^2(\mathbf{R})$ is such that $S(f, \Lambda)$ is linearly independent. Then there exists an $\varepsilon > 0$ such that $S(f, \Lambda')$ is linearly independent for any $\Lambda' = \{(a'_k, b'_k)\}_{k=1}^N$ such that $|a_k - a'_k|$, $|b_k - b'_k| < \varepsilon$ for $k = 1, \ldots, N$.*

Acknowledgments

We thank Ole Christensen, Palle Jorgensen, and David Walnut for several valuable discussions and suggestions. We also had helpful conversations with John Benedetto, Gerald Folland, A.J.E.M. Janssen, and Gilbert Strang.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

References

[BHW] J. Benedetto, C. Heil, and D. Walnut, *Differentiation and the Balian–Low Theorem*, J. Fourier Anal. Appl. **1** (1995), 355–402.

[CDM] A. Cavaretta, W. Dahmen, and C. A. Micchelli, *Stationary Subdivision*, Mem. Amer. Math. Soc. **93** (1991), 1–186. MR **92h:**65017

[C] O. Christensen, *Frames containing a Riesz basis and approximation of the frame coefficients using finite-dimensional methods*, preprint (1995).

[CH] O. Christensen and C. Heil, *Perturbations of frames and atomic decompositions*, Math. Nachr. (to appear).

[D1] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory **39** (1990), 961–1005. MR **91e:**42038

[D2] ———, *Ten Lectures on Wavelets*, SIAM Press, Philadelphia, 1992. MR **93e:**42045

[FG1] H. G. Feichtinger and K. Gröchenig, *A unified approach to atomic decompositions through integrable group representations*, Function Spaces and Applications, Lecture Notes in Math. (M. Cwikel et al., eds.), vol. 1302, Springer–Verlag, New York, 1988, pp. 52–73. MR **89h:**46035

[FG2] ———, *Banach spaces related to integrable group representations and their atomic decompositions,* I, J. Funct. Anal. **86** (1989), 307–340; *Banach spaces related to integrable group representations and their atomic decompositions,* II, Monatshefte für Mathematik **108** (1989), 129–148. MR **91g:**43012

[F] G. B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, Princeton, NJ, 1989. MR **92k:**22017

[HW] C. Heil and D. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review **31** (1989), 628–666. MR **91c:**42032

[J] A. J. E. M. Janssen, *The Zak transform: a signal transform for sampled time-continuous signals*, Philips J. Res. **43** (1988), 23–69. MR **89g:**94005

[K] G. Kaiser, *Deformations of Gabor frames*, J. Math. Phys. **35** (1994), 1372–1376. MR **95a:**42047

[L] H. J. Landau, *A sparse regular sequence of exponentials closed on large sets*, Bull. Amer. Math. Soc. **70** (1964), 566–569. MR **34:**6433

[MBHJ] S. Mann, R. G. Baraniuk, S. Haykin, and D. J. Jones, *The chirplet transform: mathematical considerations, and some of its applications*, IEEE Trans. Signal Proc. (to appear).

[RS] J. Ramanathan and T. Steger, *Incompleteness of sparse coherent states*, Appl. Comp. Harm. Anal. **2** (1995), 148–153.

[R] M. Rieffel, *Von Neumann algebras associated with pairs of lattices in Lie groups*, Math. Ann. **257** (1981), 403–418. MR **84f:**22010

[W] P. Walters, *An Introduction to Ergodic Theory*, Springer–Verlag, New York, 1982. MR **84e:**28017

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160 and The MITRE Corporation, Bedford, Massachusetts 01730

E-mail address: `heil@math.gatech.edu`

Department of Mathematics, Eastern Michigan University, Ypsilanti, Michigan 48197

E-mail address: `ramanath@emunix.emich.edu`

The MITRE Corporation, Bedford, Massachusetts 01730

E-mail address: `pnt@linus.mitre.org`

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use