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Linear Maps on ⋆-Algebras Acting on Orthogonal Elements Like Derivations or Anti-Derivations

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Abstract. Let $\mathcal{U}$ be a unital ⋆-algebra and $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map behaving like a derivation or an anti-derivation at the following orthogonality conditions on elements of $\mathcal{U}$: $xy = 0$, $xy^\star = 0$, $xy = yx = 0$ and $xy^\star = y^\star x = 0$. We characterize the map δ when $\mathcal{U}$ is a zero product determined algebra. Special characterizations are obtained when our results are applied to properly infinite $W^\star$-algebras and unital simple $C^\star$-algebras with a non-trivial idempotent.

1. Introduction

Several authors studied linear (additive) maps that behave like homomorphisms, derivations or (right, left) centralizers of (Banach) algebras when acting on special products. We refer the reader to [1, 2, 4, 11] for a full account of the topic and a list of references. An interesting question is concerned with derivations. Over the last few years considerable attention has been paid to characterizations of derivations through zero products (for instance, see [1, 2, 4, 8, 11, 13] and the references therein). Motivated by these research, in this paper we consider the problem of characterizing linear maps on ⋆-algebras behaving like derivations or anti-derivations at orthogonal elements for several types of orthogonality conditions.

Throughout this paper all algebras and linear spaces will be over the complex field $\mathbb{C}$. Let $\mathcal{U}$ be an algebra and $\mathcal{M}$ be a $\mathcal{U}$-bimodule. Recall that a linear map $d : \mathcal{U} \to \mathcal{M}$ is called a *derivation* if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{U}$. Each map of the form $x \mapsto x\mu - \mu x$, where $\mu \in \mathcal{M}$, is a derivation which will be called an *inner derivation*. Also d is called an *anti-derivation* if $d(xy) = yd(x) + d(y)x$ for all $x, y \in \mathcal{U}$.

It was shown in [4] and [7] that every additive map δ behaving like a derivation at zero product elements on a unital prime ring $\mathcal{A}$ containing a nontrivial idempotent must have the form $\delta(x) = d(x) + \xi x$, where $d : \mathcal{A} \to \mathcal{A}$ is an additive derivation and ξ is a central element of $\mathcal{A}$. Note that nest algebras are important operator algebras that are not prime. Jing et al. in [17] showed that, for nest algebras on a Hilbert space and standard operator algebras on a Banach space, the set of linear maps acting on zero products like derivations with $\delta(1) = 0$ coincides with the set of inner derivations. Li et al. showed in [18] that every linear map δ behaving like a derivation on zero products with $\delta(I) = 0$ on a nest subalgebra of a factor von Neumann algebra is a derivation. In [13] additive maps on some operator algebras behaving like (α, β)-derivations are characterized. In [8] additive maps on a prime ring acting on some orthogonality condition are described

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where the ring has an involution and nontrivial idempotents. For other related references, see [13, 14] and the references therein.

In this paper we consider the problem of characterizing linear maps behaving like derivations or anti-derivations at orthogonal elements for several types of orthogonality conditions. In particular, we consider the subsequent conditions on a linear map $\delta : \mathcal{U} \to \mathcal{U}$ where $\mathcal{U}$ is a zero product determined $\star$-algebra with unity:

(i) *derivations through one-sided orthogonality conditions*

$$xy = 0 \Longrightarrow x\delta(y) + \delta(x)y = 0 \quad [\mathbb{P}_1];$$

$$xy^\star = 0 \Longrightarrow x\delta(y)^\star + \delta(x)y^\star = 0 \quad [\mathbb{P}_2];$$

(ii) *anti-derivations through one-sided orthogonality conditions*

$$xy = 0 \Longrightarrow y\delta(x) + \delta(y)x = 0 \quad [\mathbb{P}_3];$$

$$xy^\star = 0 \Longrightarrow \delta(y)^\star x + y^\star\delta(x) = 0 \quad [\mathbb{P}_4];$$

(iii) *derivations through two-sided orthogonality conditions*

$$xy = yx = 0 \Longrightarrow x\delta(y) + \delta(x)y = y\delta(x) + \delta(y)x = 0 \quad [\mathbb{P}_5];$$

$$xy^\star = y^\star x = 0 \Longrightarrow x\delta(y)^\star + \delta(x)y^\star = \delta(y)^\star x + y^\star\delta(x) = 0 \quad [\mathbb{P}_6];$$

where $x, y \in \mathcal{U}$. Our purpose is to investigate whether the above conditions characterize derivations ($\star$-derivations) or anti-derivations ($\star$-anti-derivations) on zero product determined $\star$-algebras with unity. Also we give applications of our results for some $C^\star$-algebras. Particularly, we characterize linear maps behaving like derivations or anti-derivations at orthogonal elements for several types of orthogonality conditions on properly infinite $W^\star$-algebras or unital simple $C^\star$-algebras with a non-trivial idempotent, which includes $B(\mathcal{H})$, the set of all bounded operators on a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geq 2$.

2. Primary tools

We denote the center of an algebra $\mathcal{U}$ by $Z(\mathcal{U})$ and the Lie bracket defined by $[x, y] = xy - yx$ for all $x, y \in \mathcal{U}$. Let $\mathcal{U}$ be a $\star$-algebra with unity 1 and $d : \mathcal{U} \to \mathcal{U}$ be a map. We say that d is a $\star$-map if $d(x^\star) = d(x)^\star$ for all $x \in \mathcal{U}$. Note that if d is a derivation or an anti-derivation, then $d(1) = 0$.

Remark 2.1. *Let $d : \mathcal{U} \to \mathcal{U}$ be an inner derivation where $d(x) = x\mu - \mu x$ for some $\mu \in \mathcal{U}$. If d is a $\star$-map, then $x^\star\mu - \mu x^\star = \mu^\star x^\star - x^\star\mu^\star$ for all $x \in \mathcal{U}$. So $Re\mu = \frac{1}{2}(\mu + \mu^\star) \in Z(\mathcal{U})$. Conversely for $\mu \in \mathcal{U}$ with $Re\mu \in Z(\mathcal{U})$, the map $d : \mathcal{U} \to \mathcal{U}$ defined by $d(x) = x\mu - \mu x$ is a $\star$-inner derivation.*

Let $\mathcal{U}$ be an algebra and $\mathcal{M}$ be a $\mathcal{U}$-bimodule. Recall that a linear map $d : \mathcal{U} \to \mathcal{M}$ is called a *Jordan derivation* if $d(xy + yx) = xd(y) + d(x)y + yd(x) + d(y)x$ for all $x, y \in \mathcal{U}$, or equivalently, $d(x^2) = xd(x) + d(x)x$ for all $x \in \mathcal{U}$. Clearly, each derivation is a Jordan derivation. The converse is, in general, not true. From the classical result of Brešar [3], each Jordan derivation on semiprime algebras is a derivation. Since every semisimple Banach algebra is a semiprime Banach algebra and all $C^\star$-algebras are semisimple Banach algebras, it follows that any Jordan derivation on a $C^\star$-algebra is a derivation.

The question of characterizing linear maps through zero products, Jordan products, etc. on algebras sometimes can be effectively solved by considering bilinear maps that preserve certain zero product properties (for instance, see [1, 2, 6, 10]). Motivated by these, Brešar et al. [5] introduced the concept of zero product (Jordan product) determined algebras, which can be used to study linear maps preserving zero products (Jordan products) and derivable (Jordan derivable) maps at zero point.

An algebra $\mathcal{U}$ is called *zero product determined* if for every linear space $\mathcal{X}$ and every bilinear map $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{X}$, the following holds: If $\phi(x, y) = 0$ whenever $xy = 0$, then there exists a linear map

$T : \mathcal{U} \to \mathcal{X}$ such that $\phi(x, y) = T(xy)$ for all $x, y \in \mathcal{U}$. If $\mathcal{U}$ has unity 1, then $\mathcal{U}$ is zero product determined if and only if for every linear space $\mathcal{X}$ and every bilinear map $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{X}$, the following holds: If $\phi(x, y) = 0$ whenever $xy = 0$, then $\phi(x, y) = \phi(xy, 1)$ (see [12]). Also in this case $\phi(x, 1) = \phi(1, x)$ for all $x \in \mathcal{U}$.

Recall that a $W^\star$-algebra is called *properly infinite* if it contains no nonzero finite central projection and a unital algebra $\mathcal{U}$ is called *simple* if 0 and $\mathcal{U}$ are the only ideals of $\mathcal{U}$.

Remark 2.2. *Every algebra which is generated by its idempotents is zero product determined [6]. So the following algebras are zero product determined:*

(i) Any algebra which is linearly spanned by its idempotents.
By [16, Lemma 3. 2] and [23, Theorem 1], $B(\mathcal{H})$ is linearly spanned by its idempotents. By [23, Theorem 4], every element in a properly infinite $W^\star$-algebra $\mathcal{U}$ is a sum of at most five idempotents. In [20] several classes of simple $C^\star$-algebras are given which are linearly spanned by their projections.

(ii) Any simple unital algebra containing a non-trivial idempotent, since these algebras are generated by their idempotents [4].

The ideas of the proof of the next lemma come from [2].

Lemma 2.3. *Let $\mathcal{U}$ be a zero product determined algebra with unity 1. Suppose that $\mathcal{X}$ is a linear space and $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{X}$ is a bilinear map satisfying*

$$xy = yx = 0 \Longrightarrow \phi(x, y) = 0 \quad (x, y \in \mathcal{U}).$$

Then

$$\phi(x, y) + \phi(y, x) = \phi(xy, 1) + \phi(1, yx) \quad \textit{and} \quad \phi([x, y], 1) = \phi(1, [x, y]),$$

for all $x, y \in \mathcal{U}$.

Proof. Fix $s, t \in \mathcal{U}$ such that $st = 0$. Define a bilinear map $\psi : \mathcal{U} \times \mathcal{U} \to \mathcal{X}$ by

$$\psi(x, y) = \phi(tx, ys).$$

It is easily checked that $\psi(x, y) = 0$ whenever $xy = 0$. Since $\mathcal{U}$ is a zero product determined algebra, it follows that $\psi(x, y) = \psi(xy, 1)$ for all $x, y \in \mathcal{U}$. Hence

$$\phi(tx, ys) = \phi(txy, s),$$

for all $x, y \in \mathcal{U}$, when $st = 0$. Now fix arbitrary elements $x, y \in \mathcal{U}$ and define $\varphi : \mathcal{U} \times \mathcal{U} \to \mathcal{X}$ by

$$\varphi(s, t) = \phi(tx, ys) - \phi(txy, s).$$

From the above, we see that $\varphi(s, t) = 0$ whenever $st = 0$. So $\varphi(s, t) = \varphi(st, 1)$ for all $s, t \in \mathcal{U}$. Thus

$$\phi(tx, ys) - \phi(txy, s) = \phi(x, yst) - \phi(xy, st),$$

for all $x, y, s, t \in \mathcal{U}$. Setting $x = s = 1$, we have

$$\phi(t, y) + \phi(y, t) = \phi(1, yt) + \phi(ty, 1),$$

for all $t, y \in \mathcal{U}$.

Now for any $x, y \in \mathcal{U}$ we have

$$\phi(x, y) + \phi(y, x) = \phi(xy, 1) + \phi(1, yx)$$

and

$$\phi(y, x) + \phi(x, y) = \phi(yx, 1) + \phi(1, xy).$$

By comparing these equations, we get

$$\phi([x, y], 1) = \phi(1, [x, y]),$$

for all $x, y \in \mathcal{U}$. □

The condition $\phi(x,1) = \phi(1,x)$ (for all $x \in \mathcal{U}$) does not seem to follow from Lemma 2.3. But we have the next lemma which has been proved in [14, Theorem 3.5].

Lemma 2.4. *Let $\mathcal{U}$ be an algebra with unity 1. Suppose that $\mathcal{X}$ is a linear space and $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{X}$ is a bilinear map satisfying*

$$xy = yx = 0 \Longrightarrow \phi(x,y) = 0 \quad (x, y \in \mathcal{U}).$$

Then

$$\phi(x,p) + \phi(p,x) = \phi(xp,1) + \phi(1,px) \quad \text{and} \quad \phi(p,1) = \phi(1,p),$$

for all $x \in \mathcal{U}$ and any idempotent $p \in \mathcal{U}$. If $\mathcal{U}$ is linearly spanned by its idempotents, then

$$\phi(x,y) + \phi(y,x) = \phi(xy,1) + \phi(1,yx) \quad \text{and} \quad \phi(x,1) = \phi(1,x),$$

for all $x, y \in \mathcal{U}$.

3. Main results

From this point on, unless specified otherwise, we assume $\mathcal{U}$ is a zero product determined $\star$-algebra with unity 1.

First, we characterize derivations on $\mathcal{U}$ through one-sided orthogonality conditions.

Theorem 3.1. *Let $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Then*

(i) δ satisfies

$$xy = 0 \Longrightarrow x\delta(y) + \delta(x)y = 0 \quad (x, y \in \mathcal{U}) \quad [\mathbb{P}_1]$$

if and only if there is a derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in Z(\mathcal{U})$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$.

(ii) δ satisfies

$$xy^\star = 0 \Longrightarrow x\delta(y)^\star + \delta(x)y^\star = 0 \quad (x, y \in \mathcal{U}) \quad [\mathbb{P}_2]$$

if and only if there is a $\star$-derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in \mathcal{U}$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$.

Proof. (*i*) Suppose δ satisfies $[\mathbb{P}_1]$. Define a bilinear map $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by $\phi(x,y) = x\delta(y) + \delta(x)y$. If $x, y \in \mathcal{U}$ such that $xy = 0$, then $\phi(x,y) = 0$. Since $\mathcal{U}$ is a zero product determined algebra, it follows that $\phi(x,y) = \phi(xy,1)$ and $\phi(x,1) = \phi(1,x)$ for all $x, y \in \mathcal{U}$. So

$$x\delta(y) + \delta(x)y = xy\delta(1) + \delta(xy) \quad \text{and} \quad x\delta(1) = \delta(1)x \tag{1}$$

for all $x, y \in \mathcal{U}$. Let $\xi = \delta(1)$, then $\xi \in Z(\mathcal{U})$. Define $d : \mathcal{U} \to \mathcal{U}$ by $d(x) = \delta(x) - \xi x$. By (1), it is easily checked that d is a derivation.

The converse is proved easily.

(*ii*) Suppose δ satisfies $[\mathbb{P}_2]$. Set $\xi = \delta(1)$. Define $d : \mathcal{U} \to \mathcal{U}$ by $d(x) = \delta(x) - \xi x$. Then d is a linear map which satisfies

$$xy^\star = 0 \Longrightarrow xd(y)^\star + d(x)y^\star = 0 \quad (x, y \in \mathcal{U}) \tag{2}$$

and $d(1) = 0$. We will show that d is a $\star$-derivation. To this end, we consider the bilinear map $\phi : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ by $\phi(x,y) = xd(y^\star)^\star + d(x)y$. If $x, y \in \mathcal{U}$ such that $xy = 0$, then $x(y^\star)^\star = 0$ and (2) gives $\phi(x,y) = 0$. Since $\mathcal{U}$

is a zero product determined algebra, we get $\phi(x, y) = \phi(xy, 1)$ and $\phi(1, x) = \phi(x, 1)$ for all $x, y \in \mathcal{U}$. From equations $\phi(1, x) = \phi(x, 1)$ and $d(1) = 0$, we have

$$d(x^\star)^\star = d(x),$$

for all $x \in \mathcal{U}$. So d is a $\star$-map. Now, by equation $\phi(x, y) = \phi(xy, 1)$,

$$xd(y^\star)^\star + d(x)y = xyd(1^\star)^\star + d(xy),$$

for all $x, y \in \mathcal{U}$. Since d is a $\star$-map, it follows that d is a derivation.

The converse is proved easily. □

Note that in part (*ii*) of the above theorem, it is not necessarily true that $\xi \in Z(\mathcal{U})$. For example, take any $\xi \in \mathcal{U}$ but not in $Z(\mathcal{U})$ and define $\delta : \mathcal{U} \to \mathcal{U}$ by $\delta(x) = \xi x$. Then δ satisfies $[\mathbb{P}_2]$ and δ is the sum of the zero derivation and ξx, but $\xi \notin Z(\mathcal{U})$.

Remark 3.2. *Let $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Then*

(i) δ satisfies $[\mathbb{P}_1]$ if and only if

$$\delta(xy) = x\delta(y) + \delta(x)y - x\delta(1)y \quad (x, y \in \mathcal{U})$$

with $\delta(1) \in Z(\mathcal{U})$. Part (i) follows from Theorem 3.1 directly.

(ii) δ satisfies $[\mathbb{P}_2]$ if and only if

$$\delta(xy) = x\delta(y^\star)^\star + \delta(x)y - xy\delta(1)^\star \quad (x, y \in \mathcal{U}).$$

To see part (ii), suppose that δ satisfies $[\mathbb{P}_2]$. By Theorem 3.1, there is a $\star$-derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi = \delta(1) \in \mathcal{U}$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$. Since $d(x) = \delta(x) - \xi x$ is a $\star$-map, $(\delta(x) - \xi x) = \delta(x^\star)^\star - x\xi^\star$ for all $x \in \mathcal{U}$. So

$$\begin{aligned}\delta(xy) &= xd(y) + d(x)y + \xi xy \\ &= x\delta(y) + \delta(x)y - x\xi y \\ &= x(\delta(y) - \xi y) + \delta(x)y \\ &= x\delta(y^\star)^\star + \delta(x)y - xy\delta(1)^\star\end{aligned}$$

for all $x, y \in \mathcal{U}$. The converse is immediate.

By [24, Theorem 4.1.6] and [24, Theorem 4.1.11] respectively, every derivation on a $W^\star$-algebra and every derivation on a simple $C^\star$-algebra with unity is an inner derivation. In view of these results and Theorem 3.1, we have the next corollary.

Corollary 3.3. *Let $\mathcal{U}$ be a properly infinite $W^\star$-algebra or a unital simple $C^\star$-algebra with a non-trivial idempotent. If $\delta : \mathcal{U} \to \mathcal{U}$ is a linear map, then*

(i) δ satisfies $[\mathbb{P}_1]$ if and only if there are $\mu, \nu \in \mathcal{U}$ such that $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$ and $\mu - \nu \in Z(\mathcal{U})$.

(ii) δ satisfies $[\mathbb{P}_2]$ if and only if there are $\mu, \nu \in \mathcal{U}$ such that $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$ and $Re\mu \in Z(\mathcal{U})$.

Proof. In our proof we use the fact that $\mathcal{U}$ is a zero product determined algebra. (*i*) Suppose δ satisfies $[\mathbb{P}_1]$. By Theorem 3.1-(*i*), there is a derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in Z(\mathcal{U})$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$. Since every derivation on $\mathcal{U}$ is inner, it follows that $d(x) = x\mu - \mu x$ for all $x \in \mathcal{U}$, where $\mu \in \mathcal{U}$. Setting $\nu = \mu - \xi$. So $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$ and $\mu - \nu \in Z(\mathcal{U})$.

The converse is clear.

(*ii*) Suppose δ satisfies $[\mathbb{P}_2]$. By Theorem 3.1-(*ii*), there is a $\star$-derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in \mathcal{U}$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$. Now d is a $\star$-derivation which is inner. So by Remark 2.1, there is a $\mu \in \mathcal{U}$ with $Re\mu \in Z(\mathcal{U})$, such that $d(x) = x\mu - \mu x$ for all $x \in \mathcal{U}$. Setting $\nu = \mu - \xi$. It follows that $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$.

The converse is clear. □

In the next theorem we characterize anti-derivations through one-sided orthogonality conditions.

Theorem 3.4. *Let $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map.*

(i) Suppose that

$$xy = 0 \Longrightarrow y\delta(x) + \delta(y)x = 0 \quad (x, y \in \mathcal{U}) \quad [\mathbb{P}_3].$$

Then there is a Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in Z(\mathcal{U})$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$.

(ii) Suppose that

$$xy^\star = 0 \Longrightarrow \delta(y)^\star x + y^\star \delta(x) = 0 \quad (x, y \in \mathcal{U}) \quad [\mathbb{P}_4].$$

Then there is a $\star$-Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in \mathcal{U}$ such that $\delta(x) = d(x) + x\xi$ for all $x \in \mathcal{U}$.

Proof. (i) Define a bilinear map $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by $\phi(x, y) = y\delta(x) + \delta(y)x$. Then $\phi(x, y) = 0$ for all $x, y \in \mathcal{U}$ with $xy = 0$. Since $\mathcal{U}$ is a zero product determined algebra, we obtain $\phi(x, y) = \phi(xy, 1)$ and $\phi(1, x) = \phi(x, 1)$ for all $x, y \in \mathcal{U}$. So

$$y\delta(x) + \delta(y)x = \delta(1)xy + \delta(xy) \quad \text{and} \quad \delta(1)x = x\delta(1), \tag{3}$$

for all $x, y \in \mathcal{U}$. Let $\xi = \delta(1)$, then $\xi \in Z(\mathcal{U})$. Define $d : \mathcal{U} \to \mathcal{U}$ by $d(x) = \delta(x) - \xi x$. By (3) and the fact that $\xi \in Z(\mathcal{U})$, it follows that d is a Jordan derivation.

(ii) Consider the bilinear map $\phi : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by $\phi(x, y) = \delta(y^\star)^\star x + y\delta(x)$. If $x, y \in \mathcal{U}$ such that $xy = 0$, then $\phi(x, y) = 0$. Since $\mathcal{U}$ is a zero product determined algebra, we get $\phi(x, y) = \phi(xy, 1)$ and $\phi(1, x) = \phi(x, 1)$ for all $x, y \in \mathcal{U}$. Define $d : \mathcal{U} \to \mathcal{U}$ by $d(x) = \delta(x) - x\xi$, where $\xi = \delta(1)$. By $\phi(1, x) = \phi(x, 1)$,

$$\delta(x^\star)^\star + x\xi = \xi^\star x + \delta(x), \tag{4}$$

for all $x \in \mathcal{U}$. So $d(x^\star) = d(x)^\star$, for all $x \in \mathcal{U}$ and hence d is a $\star$-map. By $\phi(x, y) = \phi(xy, 1)$,

$$\delta(xy) = y\delta(x) + \delta(y^\star)^\star x - \xi^\star xy, \tag{5}$$

for all $x, y \in \mathcal{U}$. From (4) and (5), it follows that

$$\begin{aligned} d(x^2) &= \delta(x^2) - x^2\xi \\ &= x\delta(x) + \delta(x^\star)^\star x - \xi^\star x^2 - x^2\xi \\ &= x\delta(x) + \delta(x)x - x\xi x - x^2\xi \\ &= xd(x) + d(x)x, \end{aligned}$$

for all $x \in \mathcal{U}$. Thus d is a $\star$-Jordan derivation. □

Remark 3.5. *In Corollary 3.8 below, we will see that the converse of Theorem 3.4 (both parts) is not true and in part (ii) of Theorem 3.4, ξ is not always in $Z(\mathcal{U})$. Also the following equivalent conditions are contained in the proof of Theorem 3.4.*

Let $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Then

(i) δ satisfies $[\mathbb{P}_3]$ if and only if

$$\delta(xy) = \delta(y)x + y\delta(x) - x\delta(1)y \quad (x, y \in \mathcal{U})$$

with $\delta(1) \in Z(\mathcal{U})$.

(ii) δ satisfies $[\mathbb{P}_4]$ if and only if

$$\delta(xy) = \delta(y^\star)^\star x + y\delta(x) - \delta(1)^\star xy \quad (x, y \in \mathcal{U}).$$

If a unital algebra $\mathcal{U}$ contains a non-trivial idempotent e, let $e_1 = e$ and $e_2 = 1 - e$. Define the Peirce decomposition with respect to e by $\mathcal{U}_{ij} = e_i\mathcal{U}e_j, i, j = 1, 2$. Let $\mathcal{M}$ be a unital $\mathcal{U}$-bimodule. We say $\mathcal{U}_{ij}$ is *faithful* to $\mathcal{M}$ if $\mathcal{U}_{ij}m = \{0\}$ implies $e_jm = 0$ and $m\mathcal{U}_{ij} = \{0\}$ implies $me_i = 0$, for all $m \in \mathcal{M}$. Clearly, $\mathcal{U}_{11}$ and $\mathcal{U}_{22}$ are always faithful to $\mathcal{M}$. Peirce decomposition can be useful for characterizing derivable maps, see [19, 21, 22].

Lemma 3.6. *Let $\mathcal{U}$ be a unital algebra with a non-trivial idempotent e, C be the set of all commutators in $\mathcal{U}$, and $\mathcal{M}$ be a unital $\mathcal{U}$-bimodule. Suppose $\mathcal{U}_{12}$ and $\mathcal{U}_{21}$ from the Peirce decomposition with respect to e are faithful to $\mathcal{M}$. If d is a derivation from $\mathcal{U}$ to $\mathcal{M}$ such that $d(c) = mc, \forall c \in C$, for some fixed $m \in \mathcal{M}$ then $d = m = 0$.*

Proof. Let $e_1 = e$ and $e_2 = 1 - e$. For any $a \in \mathcal{U}$ and $i, j = 1, 2$, let $a_{ij} = e_iae_j \in \mathcal{U}_{ij}$. Note $a_{12} = [e_1a, e_2]$ and $a_{21} = [e_2a, e_1]$, so $a_{12}, a_{21} \in C$. For any $a_{12}, b_{12} \in \mathcal{U}_{12}$ and $a_{21}, b_{21} \in \mathcal{U}_{21}$, $d(a_{12}b_{12}) = d(a_{12})b_{12} + a_{12}d(b_{12}) = ma_{12}b_{12} + a_{12}mb_{12}$. Thus $a_{12}mb_{12} = 0$. Since $\mathcal{U}_{12}$ is faithful, $e_2me_1 = 0$. Similarly, from $d(a_{21}b_{21}) = d(a_{21})b_{21} + a_{21}d(b_{21}) = ma_{21}b_{21} + a_{21}mb_{21}$, we get $e_1me_2 = 0$. From $d([a_{12}, b_{21}]) = m[a_{12}, b_{21}]$ and

$$\begin{aligned} d([a_{12}, b_{21}]) &= [d(a_{12}), b_{21}] + [a_{12}, d(b_{21})] = [ma_{12}, b_{21}] + [a_{12}, mb_{21}] \\ &= m[a_{12}, b_{21}] + a_{12}mb_{21} - b_{21}ma_{12}, \end{aligned}$$

we get $a_{12}mb_{21} - b_{21}ma_{12} = 0$. It follows $a_{12}mb_{21} = b_{21}ma_{12} = 0$. Since $\mathcal{U}_{12}$ and $\mathcal{U}_{21}$ are faithful, $e_1me_1 = e_2me_2 = 0$. Thus $m = 0$.

To see $d(a) = 0, \forall a \in \mathcal{U}$, write $a = a_{11} + a_{12} + a_{21} + a_{22}$ by the Peirce decomposition with respect to e. Since $d(a_{12}) = d(a_{21}) = 0$, we only need to show $d(a_{11}) = d(a_{22}) = 0$. For any $b_{12} \in \mathcal{U}_{12}$, $a_{11}b_{12} \in \mathcal{U}_{12}$. Thus $d(a_{11}b_{12}) = d(b_{12}) = 0$. So $d(a_{11}b_{12}) = d(a_{11})b_{12} + a_{11}d(b_{12})$ gives $d(a_{11})b_{12} = 0$. Since $\mathcal{U}_{12}$ is faithful, $d(a_{11})e_1 = 0$. For any $b_{21} \in \mathcal{U}_{21}$, $d(a_{11})b_{21} + a_{11}d(b_{21}) = d(a_{11}b_{21}) = 0$. Thus $d(a_{11})b_{21} = 0$. Since $\mathcal{U}_{21}$ is faithful, $d(a_{11})e_2 = 0$. Therefore $d(a_{11}) = 0$. Similarly we can show $d(a_{22}) = 0$. □

Corollary 3.7. *Let $\mathcal{U}$ be a simple unital algebra with a non-trivial idempotent e and C be the set of all commutators in $\mathcal{U}$. If d is a derivation from $\mathcal{U}$ to $\mathcal{U}$ such that $d(c) = mc, \forall c \in C$, for some fixed $m \in \mathcal{U}$ then $d = m = 0$.*

Proof. If $\mathcal{U}$ is a simple unital algebra with a non-trivial idempotent e, then the Peirce decomposition $\mathcal{U}_{ij}$ with respect to e are faithful to $\mathcal{U}$; indeed, to see $\mathcal{U}_{ij}$ is faithful, for any $m \in \mathcal{U}$, if $m\mathcal{U}_{ij} = \{0\}$, then $\mathcal{U}me_i\mathcal{U}e_j = \{0\}$. Thus $\mathcal{U}me_i\mathcal{U} \neq \mathcal{U}$. So $\mathcal{U}me_i\mathcal{U}$, as an ideal of $\mathcal{U}$, must be zero, and $me_i = 0$. Similarly we can show $\mathcal{U}_{ij}m = \{0\}$ implies $e_jm = 0$. The conclusion now follows from Lemma 3.6. □

Now we can easily obtain the next corollary as a consequence of Theorem 3.4.

Corollary 3.8. *Let $\mathcal{U}$ be either a properly infinite $W^\star$-algebra or a simple unital $C^\star$-algebra with a non-trivial idempotent. If $\delta : \mathcal{U} \to \mathcal{U}$ is a linear map, then*

(i) δ satisfies $[\mathbb{P}_3]$ if and only if $\delta(x) = 0$, for all $x \in \mathcal{U}$.

(ii) δ satisfies $[\mathbb{P}_4]$ if and only if $\xi = \delta(1)$ is skew-Hermitian and $\delta(x) = \xi x$, for all $x \in \mathcal{U}$.

Proof. In our proof we use the fact that $\mathcal{U}$ is a zero product determined algebra and every Jordan derivation on $\mathcal{U}$ is a derivation.

(*i*) Suppose δ satisfies $[\mathbb{P}_3]$. By Theorem 3.4-(*i*), there is a Jordan derivation (hence a derivation) $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi = \delta(1) \in Z(\mathcal{U})$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$. Putting this in Eq. 3, we get $d([x, y]) = -2\xi[x, y]$ for all $x, y \in \mathcal{U}$.

If $\mathcal{U}$ is a properly infinite $W^\star$-algebra then, by [15], every element of $\mathcal{U}$ is the sum of two commutators. In particular, the unity 1 can be written as a sum of two commutators. Thus $-2\xi \cdot 1 = d(1) = 0$, so $d = \xi = 0$.

If $\mathcal{U}$ is a simple unital $C^\star$-algebra with a non-trivial idempotent, $d = \xi = 0$ by Corollary 3.7

The converse is clear.

(*ii*) Suppose δ satisfies $[\mathbb{P}_4]$. By Theorem 3.4-(*ii*), there is a $\star$-Jordan derivation (hence a derivation) $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi = \delta(1) \in \mathcal{U}$ such that $\delta(x) = d(x) + x\xi$ for all $x \in \mathcal{U}$. Putting this in Eq. 5, we get $d([x, y]) = -\xi^\star[x, y] - [x, y]\xi$ for all $x, y \in \mathcal{U}$. Let $\delta_\xi(x) = \xi x - x\xi$, for all $x \in \mathcal{U}$. Then $(d - \delta_\xi)([x, y]) = -(\xi^\star + \xi)[x, y]$. Similar to (*i*), we get $d - \delta_\xi = 0$ and $\xi^\star + \xi = 0$, so ξ is skew-Hermitian and $\delta(x) = d(x) + x\xi = \delta_\xi(x) + x\xi = \xi x$, for all $x \in \mathcal{U}$.

The converse is clear. □

It should be noted that there are linear maps on unital $\star$-algebras satisfying $[\mathbb{P}_3]$ and $[\mathbb{P}_4]$ that are not left multipliers, hence nonzero, as shown in the following example.

Example 3.9. *Let $\mathcal{A}$ be an abelian unital $\star$-algebra generated by idempotents. Define a new unital $\star$-algebra as follows: $\mathcal{U} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \in \mathcal{A}\}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} a^\star & c^\star \\ b^\star & d^\star \end{pmatrix}$. For any $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and $v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, $u + v$ is defined by the usual matrix addition and uv is defined by*

$$uv = \begin{pmatrix} u_{11}v_{11} & u_{11}v_{12} + u_{12}v_{22} \\ u_{21}v_{11} + u_{22}v_{21} & u_{22}v_{22} \end{pmatrix}.$$

Define a linear map δ on $\mathcal{U}$ by $\delta(u) = \begin{pmatrix} 0 & u_{21} \\ u_{12} & 0 \end{pmatrix}$. A direct computation shows δ is a $\star$-antiderivation, so δ satisfies both $[\mathbb{P}_3]$ and $[\mathbb{P}_4]$, but δ is not a left multiplier, as $\delta(1) = 0$.

Remark 3.10. *A linear map $\delta : \mathcal{U} \to \mathcal{U}$ behaving like a derivation or an anti-derivation at $x^\star y = 0$ can be characterized by setting $\tau(x) = \delta(x^\star)^\star$, it follows that τ behaves like a derivation or an anti-derivation at $xy^\star = 0$ and hence characterizations of δ can be obtained from $[\mathbb{P}_2]$ and $[\mathbb{P}_4]$, respectively.*

Next we will consider a linear map $\delta : \mathcal{U} \to \mathcal{U}$ behaving like a derivation at two-sided orthogonality conditions.

Remark 3.11. *Let $\mathcal{U}$ be a unital algebra and $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Then the following are equivalent.*

(i) For all $x, y \in \mathcal{U}$,

$$\delta(xy + yx) = x\delta(y) + \delta(x)y + y\delta(x) + \delta(y)x - xy\delta(1) - \delta(1)yx.$$

(ii) There are linear maps $d_1, d_2 : \mathcal{U} \to \mathcal{U}$ such that, for all $x, y \in \mathcal{U}$,

(a)

$$\delta(xy + yx) = \delta(x)y + xd_1(y) + y\delta(x) + d_2(y)x.$$

(b)

$$d_1(xy + yx) = d_1(x)y + xd_1(y) + d_1(y)x + yd_1(x) + [x, [\delta(1), y]]$$

and

$$d_2(xy + yx) = d_2(x)y + xd_2(y) + d_2(y)x + yd_2(x) + [x, [\delta(1), y]].$$

(c)

$$d_1(x) = d_2(x) + [\delta(1), x].$$

(iii) There are linear maps $d_1, d_2 : \mathcal{U} \to \mathcal{U}$ such that, for all $x, y \in \mathcal{U}$,

$$\delta(x) = d_1(x) + x\delta(1) = d_2(x) + \delta(1)x,$$

$$d_1(xy + yx) = d_1(x)y + xd_1(y) + d_1(y)x + yd_1(x) + [x, [\delta(1), y]]$$

and

$$d_2(xy + yx) = d_2(x)y + xd_2(y) + d_2(y)x + yd_2(x) + [x, [\delta(1), y]].$$

Proof. Let $\xi = \delta(1)$.

$(i) \Longrightarrow (ii)$: Define linear maps $d_1, d_2 : \mathcal{U} \to \mathcal{U}$ by

$$d_1(x) = \delta(x) - x\xi \quad \text{and} \quad d_2(x) = \delta(x) - \xi x.$$

It is clear that (a) and (c) hold for d_1 and d_2. From the definitions of d_1 and d_2, we have, for all $x, y \in \mathcal{U}$,

$$\begin{aligned} d_1(xy + yx) &= \delta(xy + yx) - xy\xi - yx\xi \\ &= x\delta(y) + \delta(x)y + y\delta(x) + \delta(y)x - xy\xi - \xi yx - xy\xi - \xi yx \\ &= xd_1(y) + d_2(y)x + yd_1(x) + d_1(x)y + x\xi y - xy\xi \\ &= xd_1(y) + d_1(y)x - [\xi, y]x + yd_1(x) + d_1(x)y + x[\xi, y] \\ &= xd_1(y) + d_1(y)x + yd_1(x) + d_1(x)y + [x, [\xi, y]]. \end{aligned}$$

By a similar method we get

$$d_2(xy + yx) = d_2(x)y + xd_2(y) + d_2(y)x + yd_2(x) + [x, [\xi, y]].$$

$(ii) \Longrightarrow (i)$: From (b), $d_1(1) = d_2(1) = 0$. Define linear maps $T_1, T_2 : \mathcal{U} \to \mathcal{U}$ by

$$T_1(x) = \delta(x) - d_1(x) \quad \text{and} \quad T_2(x) = \delta(x) - d_2(x).$$

Then $T_1(1) = T_2(1) = \delta(1)$ and for all $x, y \in \mathcal{U}$,

$$\begin{aligned} T_1(xy + yx) &= \delta(xy + yx) - d_1(xy + yx) \\ &= \delta(x)y + xd_1(y) + y\delta(x) + d_2(y)x \\ &- d_1(x)y - xd_1(y) - d_1(y)x - yd_1(x) - [x, [\xi, y]] \\ &= T_1(x)y + yT_1(x) + d_1(y)x - [\xi, y]x - d_1(y)x - [x, [\xi, y]] \\ &= T_1(x)y + yT_1(x) + x[y, \xi], \end{aligned}$$

Setting $x = 1$, we get $T_1(y) = y\xi$. Similarly we obtain $T_2(y) = \xi y$. Hence

$$d_1(y) = \delta(y) - y\xi \quad \text{and} \quad d_2(y) = \delta(y) - \xi y.$$

Replacing $d_1(y)$ and $d_2(y)$ with these identities in (a), we get (i).

$(i) \Longrightarrow (iii)$: The proof is similar to that of $(i) \Longrightarrow (ii)$.

$(iii) \Longrightarrow (i)$: This is straightforward. □

Theorem 3.12. *Let $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Suppose that*

$$xy = yx = 0 \Longrightarrow x\delta(y) + \delta(x)y = y\delta(x) + \delta(y)x = 0 \quad (x, y \in \mathcal{U}) \quad [\mathbb{P}_5].$$

Then, for all $x, y \in \mathcal{U}$,

$$\delta(xy + yx) = x\delta(y) + \delta(x)y + y\delta(x) + \delta(y)x - xy\delta(1) - \delta(1)yx$$

and

$$[x, y]\delta(1) = \delta(1)[x, y].$$

Also if $[x, [\delta(1), y]] = 0$ for all $x, y \in \mathcal{U}$, then there are Jordan derivations $d_1, d_2 : \mathcal{U} \to \mathcal{U}$ such that, for all $x \in \mathcal{U}$,

$$\delta(x) = d_1(x) + x\delta(1) = d_2(x) + \delta(1)x.$$

Proof. Define a bilinear map $\phi : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ by $\phi(x, y) = x\delta(y)+\delta(x)y$. So $\phi(x, y) = 0$, whenever $xy = yx = 0$. Hence by Lemma 2.3, we get, for all $x, y \in \mathcal{U}$, $\phi([x, y], 1) = \phi(1, [x, y])$ and

$$\phi(x, y) + \phi(y, x) = \phi(xy, 1) + \phi(1, yx),$$

Therefore $\delta(1)[x, y] = [x, y]\delta(1)$ and

$$x\delta(y) + \delta(x)y + y\delta(x) + \delta(y)x = \delta(xy) + xy\delta(1) + \delta(yx) + \delta(1)yx.$$

Now by Remark 3.11, the rest of theorem is given. □

If $\delta(1) \in Z(\mathcal{U})$, it is obvious that $[x, [\delta(1), y]] = 0$ $(x, y \in \mathcal{U})$ and $d_1 = d_2$ in the above theorem. Indeed, in this case there is a Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ such that $\delta(x) = d(x) + \delta(1)x$ for all $x \in \mathcal{U}$.

Theorem 3.13. *Let $\mathcal{U}$ be a unital $\star$-algebra which is generated by its idempotents and $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map satisfying* $[\mathbb{P}_5]$. *Then there is a Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ such that $\delta(x) = d(x) + \delta(1)x$ for all $x \in \mathcal{U}$, where $\delta(1) \in Z(\mathcal{U})$.*

Proof. Define a bilinear map $\phi : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ by $\phi(x, y) = x\delta(y)+\delta(x)y$. So $\phi(x, y) = 0$, whenever $xy = yx = 0$. By Lemma 2.4, we get $\phi(p, 1) = \phi(1, p)$ for all idempotent $p \in \mathcal{U}$. So $p\delta(1) = \delta(1)p$ for each idempotent $p \in \mathcal{U}$. Since $\mathcal{U}$ is generated by its idempotents, it follows that $\delta(1) \in Z(\mathcal{U})$. Since $\mathcal{U}$ is a zero product determined algebra, the conclusion follows from Theorem 3.12. □

In the next corollary we apply Theorem 3.13 to some $C^\star$-algebras which are generated by its idempotents.

Corollary 3.14. *Let $\mathcal{U}$ be either a properly infinite $W^\star$-algebra or a unital simple $C^\star$-algebra with a non-trivial idempotent. Suppose that $\delta : \mathcal{U} \to \mathcal{U}$ is a linear map. Then δ satisfies* $[\mathbb{P}_5]$ *if and only if there are $\mu, \nu \in \mathcal{U}$ such that $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$, where $\mu - \nu \in Z(\mathcal{U})$.*

Proof. Since $\mathcal{U}$ is generated by its idempotents, from Theorem 3.13 it follows that there is a Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ such that $\delta(x) = d(x) + \delta(1)x$ for all $x \in \mathcal{U}$, where $\delta(1) \in Z(\mathcal{U})$. The rest of proof is obtained by using a similar argument as that in the proof of Corollary 3.3-(*i*) . □

Theorem 3.15. *Let $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Suppose that*

$$xy^\star = y^\star x = 0 \Longrightarrow x\delta(y)^\star + \delta(x)y^\star = \delta(y)^\star x + y^\star\delta(x) = 0 \quad (x, y \in \mathcal{U}) \quad [\mathbb{P}_6].$$

Then, for all $x, y \in \mathcal{U}$,

$$\delta(xy) + \delta(x^\star y^\star)^\star + xy\delta(1)^\star + \delta(1)yx = \delta(x)y + x\delta(y^\star)^\star + \delta(y)x + y\delta(x^\star)^\star,$$

$$\delta(xy) + \delta(x^\star y^\star)^\star + \delta(1)^\star xy + yx\delta(1) = \delta(x^\star)^\star y + x\delta(y) + \delta(y^\star)^\star x + y\delta(x)$$

and

$$Re\delta(1)[x, y] = [x, y]Re\delta(1).$$

Moreover,

$$(\delta([x, y]) - \delta(1)[x, y])^\star = \delta([x, y]^\star) - \delta(1)[x, y]^\star$$

and

$$(\delta([x, y]) - [x, y]\delta(1))^\star = \delta([x, y]^\star) - [x, y]^\star\delta(1).$$

Proof. Define bilinear maps $\phi, \psi : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by

$$\phi(x, y) = x\delta(y^\star)^\star + \delta(x)y \text{ and } \psi(x, y) = y\delta(x) + \delta(y^\star)^\star x,$$

for all $x, y \in \mathcal{U}$. It follows that $\phi(x, y) = 0$ and $\psi(x, y) = 0$, whenever $xy = yx = 0$. By Lemma 2.3, we get

$$\phi(x, y) + \phi(y, x) = \phi(xy, 1) + \phi(1, yx) \quad \text{and} \quad \phi([x, y], 1) = \phi(1, [x, y]),$$

$$\psi(x, y) + \psi(y, x) = \psi(xy, 1) + \psi(1, yx) \quad \text{and} \quad \psi([x, y], 1) = \psi(1, [x, y]),$$

for all $x, y \in \mathcal{U}$. The desired conclusion now follows from these equations. □

Theorem 3.16. *Let $\mathcal{U}$ be a unital $\star$-algebra which is linearly spanned by its idempotents and $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map satisfying $[\mathbb{P}_6]$. Then there are $\star$-Jordan derivations $d_1, d_2 : \mathcal{U} \to \mathcal{U}$ such that*

$$\delta(x) = d_1(x) + \delta(1)x = d_2(x) + x\delta(1),$$

for all $x \in \mathcal{U}$, where $Re\delta(1) \in Z(\mathcal{U})$.

Proof. Define bilinear maps $\phi, \psi : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by

$$\phi(x, y) = x\delta(y^\star)^\star + \delta(x)y \text{ and } \psi(x, y) = y\delta(x) + \delta(y^\star)^\star x,$$

for all $x, y \in \mathcal{U}$. It follows that $\phi(x, y) = 0$ and $\psi(x, y) = 0$, whenever $xy = yx = 0$. By Lemma 2.4, we get $\phi(x, 1) = \phi(1, x)$ and $\psi(x, 1) = \psi(1, x)$ for all $x \in \mathcal{U}$. Set $\xi = \delta(1)$. Therefore

$$x\xi^\star + \delta(x) = \delta(x^\star)^\star + \xi x \tag{6}$$

and

$$x\xi + \delta(x^\star)^\star = \delta(x) + \xi^\star x, \tag{7}$$

for all $x \in \mathcal{U}$. By comparing equations (6) and (7), we get $x(\xi + \xi^\star) = (\xi + \xi^\star)x$ for all $x \in \mathcal{U}$. Hence $Re\xi \in Z(\mathcal{U})$.

Define $d_1 : \mathcal{U} \to \mathcal{U}$ by $d_1(x) = \delta(x) - \xi x$. Then d_1 is a linear map and by (6), we have $d_1(x^\star) = d_1(x)^\star$ for all $x \in \mathcal{U}$. Hence d_1 is a $\star$-map. If $xy = yx = 0$, then by hypothesis, definition of d_1 and the fact that d_1 is a $\star$-map and $Re\xi \in Z(\mathcal{U})$, we have

$$\begin{aligned} xd_1(y) + d_1(x)y &= xd_1(y^\star)^\star + d_1(x)y \\ &= x(\delta(y^\star) - \xi y^\star)^\star + (\delta(x) - \xi x)y = 0 \end{aligned}$$

and

$$\begin{aligned} yd_1(x) + d_1(y)x &= yd_1(x) + d_1(y^\star)^\star x \\ &= y(\delta(x) - \xi x) + (\delta(y^\star) - \xi y^\star)^\star x \\ &= -y\xi x - y\xi^\star x = -yx(\xi + \xi^\star) = 0. \end{aligned}$$

So d_1 satisfies $[\mathbb{P}_5]$ and $d_1(1) = 0$. Hence by Theorem 3.12, d_1 is a Jordan derivation, since $\mathcal{U}$ is a zero product determined algebra.

By a similar method as above, we can show that there is a $\star$-Jordan derivation $d_2 : \mathcal{U} \to \mathcal{U}$ such that $\delta(x) = d_2(x) + x\xi$ for all $x \in \mathcal{U}$. □

In Theorem 3.16, if $\delta(1) \in Z(\mathcal{U})$, it is obvious that $d_1 = d_2$. Indeed, in this case there is a Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ such that $\delta(x) = d(x) + \delta(1)x$ for all $x \in \mathcal{U}$. Note that in this theorem, it is not necessary that $\xi \in Z(\mathcal{U})$. For example, take any $\xi \in \mathcal{U}$ such that ξ is not in $Z(\mathcal{U})$ and $Re\xi \in Z(\mathcal{U})$. Then the linear map $\delta : \mathcal{U} \to \mathcal{U}$ defined by $\delta(x) = \xi x$ satisfies $[\mathbb{P}_6]$ and δ is the sum of the zero derivation and $x\xi$, but ξ is not in $Z(\mathcal{U})$.

In the next corollary we apply Theorem 3.15 to properly infinite $W^\star$-algebras and unital simple $C^\star$-algebras which are linearly spanned by its idempotents, several classes of such $C^\star$-algebras are given in [20].

Corollary 3.17. *Let $\mathcal{U}$ be a properly infinite $W^\star$-algebra or a unital simple $C^\star$-algebra which is linearly spanned by its idempotents (in particular $B(\mathcal{H})$) and $\delta : \mathcal{U} \to \mathcal{U}$ be a linear map. Then δ satisfies $[\mathbb{P}_6]$ if and only if there are $\mu, \nu \in \mathcal{U}$ such that $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$, where $Re\mu \in Z(\mathcal{U})$ and $Re(\mu - \nu) \in Z(\mathcal{U})$.*

Proof. Suppose δ satisfies $[\mathbb{P}_6]$. Since $\mathcal{U}$ is linearly spanned by its idempotents, by Theorem 3.16, there is a $\star$-Jordan derivation $d : \mathcal{U} \to \mathcal{U}$ and an element $\xi \in \mathcal{U}$ such that $\delta(x) = d(x) + \xi x$ for all $x \in \mathcal{U}$ and $Re\xi \in Z(\mathcal{U})$. Since every Jordan derivation on $\mathcal{U}$ is a derivation and any derivation on $\mathcal{U}$ is inner, it follows that d is an inner derivation. Hence there is a $\mu \in \mathcal{U}$ with $Re\mu \in Z(\mathcal{U})$, such that $d(x) = x\mu - \mu x$ for all $x \in \mathcal{U}$. Letting $\nu = \mu - \xi$. So $Re\xi = Re(\mu - \nu) \in Z(\mathcal{U})$, $Re\mu \in Z(\mathcal{U})$ and $\delta(x) = x\mu - \nu x$ for all $x \in \mathcal{U}$.

The converse is proved easily. □

It seems that the converse of Theorems 3.12, 3.13, 3.15 and 3.16 do not hold.

Remark 3.18. *The proofs of our results regarding conditions $[\mathbb{P}_1]$, $[\mathbb{P}_3]$ and $[\mathbb{P}_5]$ work in a more general setting for characterizing maps from a zero product determined unital algebra to its bimodules. More specifically, by a similar method we can obtain Theorem 3.1-(i), Theorem 3.4-(i) and Theorem 3.12 for a linear map $\delta : \mathcal{U} \to \mathcal{M}$ satisfying one of the conditions $[\mathbb{P}_1]$, $[\mathbb{P}_3]$ and $[\mathbb{P}_5]$, respectively, where $\mathcal{U}$ is a zero product determined unital algebra and $\mathcal{M}$ is a $\mathcal{U}$-bimodule.*

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