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Linear Models of Nonlinear Systems

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To Ann-Christine

Abstract

Linear time-invariant approximations of nonlinear systems are used in many applications and can be obtained in several ways. For example, using system identification and the prediction-error method, it is always possible to estimate a linear model without considering the fact that the input and output measurements in many cases come from a nonlinear system. One of the main objectives of this thesis is to explain some properties of such approximate models.

More specifically, linear time-invariant models that are optimal approximations in the sense that they minimize a mean-square error criterion are considered. Linear models, both with and without a noise description, are studied. Some interesting, but in applications usually undesirable, properties of such optimal models are pointed out. It is shown that the optimal linear model can be very sensitive to small nonlinearities. Hence, the linear approximation of an almost linear system can be useless for some applications, such as robust control design. Furthermore, it is shown that standard validation methods, designed for identification of linear systems, cannot always be used to validate an optimal linear approximation of a nonlinear system.

In order to improve the models, conditions on the input signal that imply various useful properties of the linear approximations are given. It is shown, for instance, that minimum phase filtered white noise in many senses is a good choice of input signal. Furthermore, the class of separable signals is studied in detail. This class contains Gaussian signals and it turns out that these signals are especially useful for obtaining approximations of generalized Wiener-Hammerstein systems. It is also shown that some random multisine signals are separable. In addition, some theoretical results about almost linear systems are presented.

In standard methods for robust control design, the size of the model error is assumed to be known for all input signals. However, in many situations, this is not a realistic assumption when a nonlinear system is approximated with a linear model. In this thesis, it is described how robust control design of some nonlinear systems can be performed based on a discrete-time linear model and a model error model valid only for bounded inputs.

It is sometimes undesirable that small nonlinearities in a system influence the linear approximation of it. In some cases, this influence can be reduced if a small nonlinearity is included in the model. In this thesis, an identification method with this option is presented for nonlinear autoregressive systems with external inputs. Using this method, models with a parametric linear part and a nonparametric Lipschitz continuous nonlinear part can be estimated by solving a convex optimization problem.

Sammanfattning

Linjära tidsinvarianta approximationer av olinjära system har många användningsområden och kan tas fram på flera sätt. Om man har mätningar av in- och utsignalerna från ett olinjärt system kan man till exempel använda systemidentifiering och prediktionsfelsmetoden för att skatta en linjär modell utan att ta hänsyn till att systemet egentligen är olinjärt. Ett av huvudmålen med den här avhandlingen är att beskriva egenskaper för sådana approximativa modeller.

Framförallt studeras linjära tidsinvarianta modeller som är optimala approximationer i meningen att de minimerar ett kriterium baserat på medelkvadratfelet. Brusmodeller kan inkluderas i dessa modelltyper och både fallet med och utan brusmodell studeras här. Modeller som är optimala i medelkvadratfelsmening visar sig kunna uppvisa ett antal intressanta, men ibland oönskade, egenskaper. Bland annat visas det att en optimal linjär modell kan vara mycket känslig för små olinjäriteter. Denna känslighet är inte önskvärd i de flesta tillämpningar och innebär att en linjär approximation av ett nästan linjärt system kan vara oanvändbar för till exempel robust reglerdesign. Vidare visas det att en del valideringsmetoder som är framtagna för linjära system inte alltid kan användas för validering av linjära approximationer av olinjära system.

Man kan dock göra de optimala linjära modellerna mer användbara genom att välja lämpliga insignaler. Bland annat visas det att minfasfiltrerat vitt brus i många avseenden är ett bra val av insignal. Klassen av separabla signaler detaljstuderas också. Denna klass innehåller till exempel alla gaussiska signaler och just dessa signaler visar sig vara speciellt användbara för att ta fram approximationer av generaliserade wiener-hammersteinsystem. Dessutom visas det att en viss typ av slumpmässiga multisinussignaler är separabel. Några teoretiska resultat om nästan linjära system presenteras också.

De flesta metoder för robust reglerdesign kan bara användas om storleken på modellfelet är känd för alla tänkbara insignaler. Detta är emellertid ofta inte realistiskt när ett olinjärt system approximeras med en linjär modell. I denna avhandling beskrivs därför ett alternativt sätt att göra en robust reglerdesign baserat på en tidsdiskret modell och en modellfelsmodell som bara är giltig för begränsade insignaler.

Ibland skulle det vara önskvärt om en linjär modell av ett system inte påverkades av förekomsten av små olinjäriteter i systemet. Denna oönskade påverkan kan i vissa fall reduceras om en liten olinjär term tas med i modellen. En identifieringsmetod för olinjära autoregressiva system med externa insignaler där denna möjlighet finns beskrivs här. Med hjälp av denna metod kan modeller som består av en parametrisk linjär del och en ickeparametrisk lipschitzkontinuerlig olinjär del skattas genom att man löser ett konvext optimeringsproblem.

Acknowledgments

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I would also like to thank Dr. Jacob Roll, Professor Torkel Glad and Professor Anders Helmersson for our fruitful cooperation on some of the publications that this thesis is based on.

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Martin Enqvist

Linköping, October 2005

Contents

Notation

Symbols, Operators and Functions

$\mathbb{N}$	the set of natural numbers ($0 \in \mathbb{N}$)
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of positive integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$L^1(\mathbb{R}^n)$	the space of functions f such that $\int_{\mathbb{R}^n} \lvert f(x)\rvert \, dx < \infty$
$\in$	belongs to
$A \subset B$	A is a subset of B
$A \setminus B$	set difference, $\{x \mid x \in A \wedge x \notin B\}$
$\mathrm{card}(A)$	the cardinality of the set A
$\triangleq$	equal by definition
$\preccurlyeq$	component-wise inequality (for vectors), negative semidefiniteness (for a matrix A with $A \preccurlyeq 0$)
$\arg\min f(x)$	value of x that minimizes $f(x)$
$\lvert v\rvert$	$\sqrt{v^T v}$
q	the shift operator, $qu(t) = u(t+1)$
$(x(t))_{t=0}^{M}$	the sequence $x(0), x(1), \ldots, x(M)$
$\lVert x\rVert$	$\sqrt{\sum_{t=0}^{\infty} x(t)^T x(t)}$
$\lVert x\rVert_N$	$\sqrt{\sum_{t=0}^{N} x(t)^T x(t)}$
$[G(z)]_{\mathrm{causal}}$	the causal part of the transfer function $G(z)$
$\mathrm{E}(X)$	expected value of the random variable X

$\mathrm{E}(X\|Y)$	conditional expectation of X given Y
$R_u(\tau)$	covariance function of the signal u
$R_{yu}(\tau)$	cross-covariance function of the signals y and u
$\Phi_u(z)$	z-spectrum of the signal u
$\Phi_u(e^{i\omega})$	spectral density function of the signal u
$\Phi_{yu}(z)$	z-cross-spectrum of the signals y and u
$\Phi_{yu}(e^{i\omega})$	cross-spectral density function of the signals y and u
θ	parameter vector
$r(t)$	reference signal at time t
$u(t)$	input signal at time t
$y(t)$	output signal at time t
$y_{nf}(t)$	noise-free output signal at time t
$w(t)$	noise signal at time t
$\eta_0(t)$	OE-LTI-SOE residual at time t
$\varepsilon_0(t)$	GE-LTI-SOE residual at time t
$G_{0,OE}(z)$	Output Error Linear Time-Invariant Second Order Equivalent
$G_{0,GE}(z)$	system model part of the General Error Linear Time-Invariant Second Order Equivalent
$H_{0,GE}(z)$	noise model part of the General Error Linear Time-Invariant Second Order Equivalent

Abbreviations and Acronyms

ARX (system)	AutoRegressive (system) with eXternal input
DFT	Discrete Fourier Transform
FIR	Finite Impulse Response
GE	General Error
LTI	Linear Time-Invariant
NARX (system)	Nonlinear AutoRegressive (system) with eXternal input
NFIR	Nonlinear Finite Impulse Response
NOE	Nonlinear Output Error
OE	Output Error
QP	Quadratic Programming
SOE	Second Order Equivalent
w.p.1	with probability one

Assumptions

A1	Standard assumptions on the input (see page 36)
A2	Standard assumptions on the output (see page 36)
A3	Assumptions on the input and output signals for periodic inputs (see page 36)
A4	Assumption used in the definition of GE-LTI-SOEs (see page 37)
A5	Assumptions on the noise (see page 64)
A6	Assumptions on Gaussian probability density functions (see page 102)
A7	Assumptions on the input and output signals for Gaussian inputs (see page 102)

1 Introduction

Mathematical modeling of real-life systems is a very common methodology in science and engineering. It is used both as a means for achieving deeper knowledge about a system and as an engineering tool, e.g., as a basis for simulations or for design of controllers. Sometimes, it is possible to construct a model of a system from physical laws and principles. However, in other cases this is not possible, either because of a lack of knowledge of the studied system or because physical modeling is considered too time consuming. In these cases, system identification can be a way of solving the modeling problem.

System identification deals with the problem of how to estimate a model of a system from measured input and output signals. Usually, only estimation problems for dynamic systems, i.e., systems with some kind of memory, are called system identification. A system can be linear or nonlinear and, depending on the type of system, linear or nonlinear models can be estimated. In practice, linear models are very common and they are often used also when the true system is nonlinear. In these cases, the model can only give an approximate description of the system. It is therefore interesting to understand in what way an estimated linear model can approximate a nonlinear system and how this approximation depends on the properties of the true nonlinear system and of the input signal used. The main objective of this thesis is to give some answers to this problem. Furthermore, some robustness issues concerning system identification and automatic control are also discussed.

The field of automatic control concerns methods for changing the behavior of a dynamic system and a device designed for this purpose is called a controller. For example, controllers can be used to stabilize a system, to make a system less sensitive to disturbances or to change the response of a system to an external signal. Usually, a control method is designed for a particular class of control problems, which is often defined by a number of mathematical assumptions about the system. However, many methods are applied also to real-life systems that do not satisfy all these assumptions. Hence, it is important to investigate the robustness of a control method with respect to erroneous assumptions about the system. When a controller is designed based on a mathematical model, it is said to be robust towards model errors if the differences between the model and the true system cannot cause instability. This type of robustness of a controller is discussed here.

This chapter contains a brief discussion about systems and models and some motivating examples. These examples describe some of the phenomena or problems that can occur when a linear model of a nonlinear system is estimated. Furthermore, an outline of the thesis is given and finally, the main contributions are listed.

1.1 Systems and Models

A very important notion in system identification is the difference between a *system* and a *model*. In a wide sense, a system is any kind of physically or conceptually bounded object. Examples of systems are the solar system, a human brain cell and an electrical motor. A system is usually affected by external signals. For example, the solar system is affected by the gravity of other stars, a human brain cell is affected by neighboring cells and by the composition of the blood, and an electrical motor is affected by the voltage over its winding. External signals with a desired effect on the system are called *inputs*, while other, undesired signals that affect the system are called *disturbances*. Measurable signals that describe some property of the system are called *outputs*. Note that also disturbances can be measurable and that the classification of external signals as either inputs or disturbances is somewhat arbitrary. However, for a particular application it is usually easy to distinguish inputs and disturbances.

From a control engineering perspective, a system is some device whose behavior we would like to make more intelligent in some way. This can be done by designing a controller that measures the outputs from the system and then alters the input signals in order to achieve the desired behavior. From a control scientist's point of view, the only systems of interest are those with both input and output signals. In most parts of this thesis, only scalar systems, i.e., systems with one input and one output, are considered.

A mathematical description of a system is called a model. Whenever a system corresponds to a real-life object, it cannot be described exactly and any model of it will thus contain errors. Only in constructed examples, it is possible to give an exact description of the true system. However, any reference to the system always concerns the actual true system. Hence we can talk about model errors but not about system errors.

The common practice to approximate nonlinear systems using linear models can be done in many ways. For example, differentiation can be used to linearize a nonlinear system description locally, or some kind of linear equivalent of a nonlinear system can be derived for a particular input. In this thesis, we will investigate the latter of these two approaches. More specifically, we will study the behavior of linear model estimates obtained by system identification using input and output data from nonlinear systems. The system identification method that will be used here is the well-known *prediction-error method* (see Section 2.3), and we will only investigate its behavior when the number of measurements tends to infinity.

The prediction-error method can be used to compute estimates of some parameters θ

in a general linear model

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t)$$

of a system with input $u(t)$ and output $y(t)$. Here, q denotes the shift operator, $qu(t) = u(t+1)$, $G(q,\theta)$ is the linear model from input to output and $H(q,\theta)$ is a model of how the noise $e(t)$ affects the output. Both these models are parameterized by the vector θ. It will be assumed that both $H^{-1}(q,\theta)G(q,\theta)$ and $H^{-1}(q,\theta)$, with $H^{-1}(q,\theta) = 1/H(q,\theta)$, are stable models (see Section 2.1). It can be shown (Ljung, 1978) that the prediction-error parameter estimates under rather general conditions will converge to the parameters that minimize a *mean-square error criterion* $\mathrm{E}((H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)))^2)$, where $\mathrm{E}(x)$ denotes the expected value of the random value x.

In the special case when the noise model is equal to one, the mean-square error optimal model $G(q)$ will here be called the *Output Error Linear Time-Invariant Second Order Equivalent* (OE-LTI-SOE) and it will be denoted $G_{0,OE}(q)$. The corresponding mean-square error optimal model for a general noise model will be called the *General Error Linear Time-Invariant Second Order Equivalent* (GE-LTI-SOE) and it will be denoted $(G_{0,GE}(q), H_{0,GE}(q))$. In the next section, some motivating examples that illustrate the properties of OE-LTI-SOEs and GE-LTI-SOEs will be presented.

1.2 Motivating Examples

Although the use of linear models of nonlinear systems is straightforward in some cases, it can sometimes give rise to rather nonintuitive phenomena. This is shown in the following examples.

Example 1.1

Consider the simple static nonlinear system

$$y(t) = u(t)^3. \tag{1.1}$$

Intuitively, the best linear approximation of this system would be a static linear system $y(t) = c_0 u(t)$, where c_0 is some constant. However, this is not always the case. Let the input to the system (1.1) be

$$u(t) = e(t) + \frac{1}{2}e(t-1),$$

where $e(t)$ is a sequence of independent random variables with uniform distribution over the interval $[-1, 1]$. In this case, it turns out that the OE-LTI-SOE of the system (1.1) is

$$G_{0,OE}(q) = \frac{0.85 + 0.575q^{-1}}{1 + 0.5q^{-1}}.$$

Hence, a static nonlinear system can have a nonstatic OE-LTI-SOE .

Example 1.2

Let the input signal to (1.1) be generated in a different way according to

$$u(t) = \frac{1}{2}e(t) + e(t-1), \tag{1.2}$$

where $e(t)$ is the same signal as in Example 1.1. In this way, this input will have the same spectral density $\Phi_u(e^{i\omega})$ as the one in the previous example. However, the OE-LTI-SOE of the system (1.1) for the input (1.2) is

$$G_{0,OE}(q) = \frac{0.925 + 0.425q^{-1}}{1 + 0.5q^{-1}}.$$

Hence, a nonlinear system can have different OE-LTI-SOEs for two input signals with equal spectral densities.

Example 1.3

Consider the static nonlinear system

$$y(t) = u(t)^2 - 3$$

with the input

$$u(t) = e(t) + e(t-1)^2 - 1,$$

where $e(t)$ here is a white Gaussian process with zero mean and unit variance. The OE-LTI-SOE of this system is

$$G_{0,OE}(q) = \frac{8}{3} \approx 2.6667$$

while the GE-LTI-SOE is

$$G_{0,GE}(q) = \frac{\sqrt{4161} - 33}{12} \approx 2.6255,$$
$$H_{0,GE}(q) = 1 + \frac{65 - \sqrt{4161}}{8}q^{-1}.$$

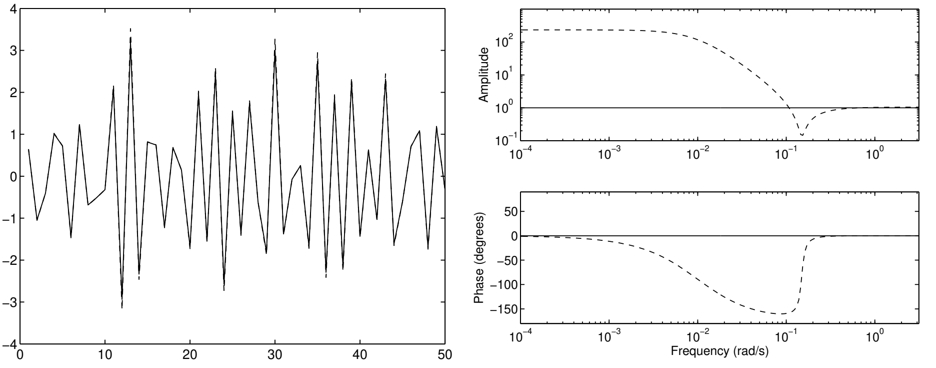
As can be seen from these expressions, $G_{0,OE}(q) \neq G_{0,GE}(q)$ despite the fact that the system operates in open loop.

Hence, the OE-LTI-SOE $G_{0,OE}(q)$ of an open-loop nonlinear system can be different from $G_{0,GE}(q)$ in the corresponding GE-LTI-SOE .

Example 1.4

Consider the nonlinear system

$$y(t) = y_l(t) + 0.01y_n(t),$$
$$y_l(t) = u(t),$$
$$y_n(t) = u(t)^3.$$



(a) *The output $y(t)$ (dashed) of the nonlinear system in Example 1.4 and the output $y_l(t) = u(t)$ (solid) of the linear part of that system for a particular realization of the input signal.*

(b) *The frequency response of the OE-LTI-SOE (dashed) and of the linear part (solid) of the nonlinear system in Example 1.4*

Figure 1.1: *The frequency response of the OE-LTI-SOE can be far from the response of the linear part of the system also when the nonlinear contributions to the output are small.*

The output from this system consists of a linear part, $y_l(t) = G_l(q)u(t)$ with $G_l(q) = 1$, and a nonlinear part, $0.01y_n(t) = 0.01u(t)^3$. Let the input signal be

$$u(t) = (1 - 2cq^{-1} + c^2q^{-2})e(t),$$

where $c = 0.99$ and where $e(t)$ is a white noise process with uniform distribution over the interval $[-1, 1]$. For this input, it is hard to distinguish the output $y(t)$ of the nonlinear system from the output $y_l(t)$ of the linear part of the system. This can be seen in Figure 1.1a for a particular realization of the input signal. However, the small differences between the output signals $y(t)$ and $y_l(t)$ will make the OE-LTI-SOE very different from the linear part G_l of the system. This difference can be seen in Figure 1.1b.

Hence, the distance between the OE-LTI-SOE and the linear part of the true system can be large also when the nonlinearities are small.

As can be seen in the previous examples, the OE-LTI-SOE of a nonlinear system is input dependent. Furthermore, there is no guarantee that the OE-LTI-SOE and the GE-LTI-SOE will be equal even for an open-loop nonlinear system. Neither will the OE-LTI-SOE always be close to the linear part of the system.

In particular, this last property can in some circumstances be undesirable, for example if the OE-LTI-SOE is supposed to be used as a basis for robust control design. Such a design puts restrictions on the control laws in order to guarantee the stability of the resulting true closed-loop system, despite the presence of model errors. A drawback with a model that is far from the linear part of an almost linear system is that the gain of the model errors might be unnecessarily large.

A robust control design based on a model with large model errors usually implies that the restrictions on the control laws will be rather hard. Hence, the use of an OE-LTI-SOE with large model errors can result in a rather poor control performance for the true system. It is thus interesting to understand under which circumstances the OE-LTI-SOE will be close to the linear part of the system when the nonlinearities are small. Furthermore, it would be interesting to have an identification method such that small nonlinearities can be ignored when a linear model is desired. A method with this option is discussed in this thesis.

Examples 1.1 and 1.2 show that the OE-LTI-SOE of a nonlinear system will be input dependent. One could argue that this input dependency is a problem and that an LTI approximation of a nonlinear system should not be derived for a particular input or class of inputs. However, as the following example illustrates, it is not realistic to believe that a linear model can be a good approximation of a nonlinear system for *all* inputs.

Example 1.5

Consider the nonlinear system

$$y(t) = \begin{cases} 1, & u(t) > 1, \\ u(t), & |u(t)| \leq 1, \\ -1, & u(t) < -1 \end{cases}$$

and assume that a linear approximation

$$\hat{y}(t) = b_0 u(t)$$

of this system is desired. Assume that we want this approximation to be optimal in the sense that

$$\sup_{u(t)\in\mathbb{R}} |y(t) - \hat{y}(t)|$$

is minimized. In this case, it is easy to see that the optimal model is $b_0 = 0$. Of course, this is not a very useful model.

Hence, a linear model of a nonlinear system should typically be derived and used only for a restricted class of inputs.

Examples 1.1-1.5 will be discussed in more detail later in this thesis (see Examples 4.2, 5.1, 4.3 and 8.1 and Chapter 10, respectively). Furthermore, some conditions and methods that prevent the behaviors shown in these examples will be presented.

1.3 Outline of the Thesis

Most of the results in this thesis concern systems with stationary stochastic input and output signals. Some background material about such signals and about linear and nonlinear systems can be found in Chapter 2. This chapter contains also a brief description of system identification using the prediction-error method and an introduction to separable processes. An overview of some existing methods for linearization of nonlinear systems can be found in Chapter 3.

The rest of this thesis is divided into two parts. The first part comprises Chapters 4 to 9 and concerns analysis of LTI-SOEs of nonlinear systems. The second part comprises Chapters 10 and 11 and concerns robustness issues for control design and system identification using linear models.

The linearization approach used in this thesis is based on minimization of the mean-square error. The LTI-SOEs obtained by this approach are described in Chapter 4 and some basic properties of these models are discussed in Chapter 5. There, it is also shown that a minimum phase filtered white noise input implies useful properties for the LTI-SOE of a nonlinear system.

Furthermore, it turns out that the class of separable inputs is especially useful for LTI approximations of nonlinear systems. Some results for these inputs are described in Chapter 6 while Gaussian inputs, which belong to the class of separable inputs, are discussed in Chapter 7. This chapter contains also some results about LTI-SOEs of generalized Wiener-Hammerstein systems. Furthermore, LTI approximations of almost linear systems are studied in Chapter 8 and the first part of the thesis is summarized with a discussion about different input signals in Chapter 9.

The second part of the thesis is more focused on methods. An approach to robust control using realistic model error models is described in Chapter 10. Furthermore, an identification method that sometimes can reduce the sensitivity of an estimated LTI model to small nonlinearities is discussed in Chapter 11.

Some final conclusions concerning the previously presented topics are given in Chapter 12.

1.4 Contributions

The main objective of this thesis is to explain some of the behavior of LTI-SOEs of nonlinear systems and to investigate some robustness issues concerning control and identification using linear models of nonlinear systems. For example, the phenomena shown in Examples 1.1-1.4 are discussed.

From a practical point of view, there are four contributions in this thesis that probably are more important than the others. The first one is the observation described in Sections 5.4 and 5.5 that minimum phase filtered white noise in many senses is a good choice of input signal for LTI approximations of nonlinear systems. Furthermore, the result in Lemma 6.1 that some random multisines are separable has direct practical implications and can be viewed as a theoretical motivation for an input signal that is already commonly used. The third contribution of practical interest is the result in Corollary 7.1 about generalized Wiener-Hammerstein systems with Gaussian inputs. This result implies that the linear parts of such a system will be factors in the OE-LTI-SOE of it. Finally, the mixed parametric and nonparametric identification method in Chapter 11 might be useful in some applications.

Some other results can also be viewed as main contributions. For example, the result about higher order separability in Theorem 6.3 is a generalization of a classic theoretical result. With the behavior of LTI-SOEs shown in Examples 1.1-1.4 in mind, also the result in Theorem 8.1 about uniform convergence of the linear approximations when the size of the nonlinearities tends to zero can be viewed as a main contribution.

Most of the material of this thesis has previously been published. With exception for the discussion about LTI-SOEs for periodic inputs and the separability of random multisines, the results in Chapters 4 to 9 have been published previously in

> M. Enqvist. Some results on linear models of nonlinear systems. Licentiate thesis no. 1046. Department of Electrical Engineering, Linköpings universitet, Linköping, Sweden, 2003.

The results about higher order separability and LTI-SOEs for Gaussian inputs in Section 6.3 and Chapter 7 can also be found in

> M. Enqvist and L. Ljung. Linear approximations of nonlinear FIR systems for separable input processes. *Automatica*, 41(3):459–473, 2005.

Early versions of the results in Chapter 8 about approximations of almost linear systems can be found in

> M. Enqvist and L. Ljung. Estimating nonlinear systems in a neighborhood of LTI-approximants. In *Proceedings of the 41st IEEE Conference on Decision and Control*, pages 1005–1010, Las Vegas, Nevada, December 2002.

The material in Chapter 7 about Gaussian inputs has previously been published also in

> M. Enqvist and L. Ljung. Linear models of nonlinear FIR systems with Gaussian inputs. In *Proceedings of the 13th IFAC Symposium on System Identification*, pages 1910–1915, Rotterdam, The Netherlands, August 2003.

Some of the examples and results in Chapter 8 can also be found in

> M. Enqvist and L. Ljung. LTI approximations of slightly nonlinear systems: Some intriguing examples. In *Proceedings of the 6th IFAC Symposium on Nonlinear Control Systems*, pages 639–644, Stuttgart, Germany, September 2004.

The approach to robust control in Chapter 10 has also been studied in

> S. T. Glad, A. Helmersson, M. Enqvist, and L. Ljung. Controllers for amplitude limited model error models. In *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.

Some of the results about minimum phase filtered white noise inputs in Chapter 5 are described in

> M. Enqvist. Benefits of the input minimum phase property for linearization of nonlinear systems. In *Proceedings of the International Symposium on Nonlinear Theory and its Applications*, pages 618–621, Bruges, Belgium, October 2005.

Most of the results about the mixed parametric and nonparametric method for system identification in Chapter 11 are published in

J. Roll, M. Enqvist, and L. Ljung. Consistent nonparametric estimation of NARX systems using convex optimization. In *Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference*, Seville, Spain, December 2005a. (To appear).

The results about separability of random multisines in Section 6.2 can be found also in

M. Enqvist. Identification of Hammerstein systems using separable random multisines. Submitted to the 14th IFAC Symposium on System Identification, Newcastle, Australia, March 2006.

2

Preliminaries

In this chapter, some background material about linear and nonlinear systems will be presented and the notation that will be used throughout this thesis will be introduced. Furthermore, a brief description of the basic ideas of system identification based on prediction-error methods and an introduction to separable processes will be given.

2.1 Linear Systems and Stochastic Processes

Linear time-invariant (LTI) dynamic systems and models are the foundation of control theory and system identification and are described in many textbooks (see, for example, Kailath, 1980; Rugh, 1996). Any discrete-time LTI system with input $u(t)$ and output $y(t)$ can be written as a convolution

$$y(t) = \sum_{k=-\infty}^{\infty} g(k)u(t-k).$$

The sequence $(g(k))_{k=-\infty}^{\infty}$ is called the *impulse response* of the system. An LTI system can also be represented by a *transfer function* $G(z)$, which is obtained by taking the z-transform of the impulse response, i.e.,

$$G(z) = \sum_{k=-\infty}^{\infty} g(k)z^{-k}.$$

Similarly, the function $G(q)$, where q is the shift operator $qu(t) = u(t+1)$, will be called the *transfer operator* of the system. A third way to represent a discrete-time LTI system is to write it as a *state-space description* or a *state equation*

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t)$ is a *state vector* and where A, B, C and D are matrices.

Although a transfer function sometimes can be written more compactly as a rational function of z, it should always be thought of as a certain series expansion in order to avoid any ambiguities. These ambiguities can occur due to the fact that a rational function corresponds to different series expansions in different regions of convergence. However, the series expansion is unique if the region of convergence is specified (Brown and Churchill, 1996, Sec. 50). Sometimes, this specification will be done using the following terminology.

Definition 2.1. A sequence $(m(k))_{k=-\infty}^{\infty}$ is *causal* if $m(k) = 0$ for all $k < 0$ and *strictly causal* if $m(k) = 0$ for all $k \leq 0$. The sequence is *anticausal* if $m(k) = 0$ for all $k > 0$ and *strictly anticausal* if $m(k) = 0$ for all $k \geq 0$.

The notion of causality can be used also for LTI systems.

Definition 2.2. An LTI system is *(strictly) causal* if its impulse response is (strictly) causal. Similarly, an LTI system is *(strictly) anticausal* if its impulse response is (strictly) anticausal.

In some cases, we will need to extract the causal part of a noncausal system. This will be done using the notation

$$[G(z)]_{\text{causal}} = \left[\sum_{k=-\infty}^{\infty} g(k) z^{-k} \right]_{\text{causal}} = \sum_{k=0}^{\infty} g(k) z^{-k}.$$

Causality of an LTI system implies that the system output only depends on past and present values of the input signal. Hence, all real-life systems are causal. Another important property of LTI systems is stability. In this thesis, we will only use the type of stability called *bounded input bounded output stability*, which is defined as follows.

Definition 2.3. An LTI system with impulse response $g(k)$ is *stable* if

$$\sum_{k=-\infty}^{\infty} |g(k)| < +\infty.$$

If a transfer function is said to be stable, it should always be viewed as coming from the series expansion, causal or noncausal, whose region of convergence contains the unit circle. On the other hand if a transfer function is said to be causal it should be viewed as coming from a, possibly unstable, causal series expansion.

Furthermore, an LTI system $G(z)$ is said to be *static* if only $g(0)$ is nonzero and nonstatic if there exists a $k \in \mathbb{Z} \setminus \{0\}$ such that $g(k) \neq 0$. If $g(k)$ is nonzero only for a finite number of k:s, the system is said to be a *finite impulse response* (FIR) system. An LTI system is said to be *monic* if $g(0) = 1$. In some cases, we will use the notation

$$G^{-1}(z) = \frac{1}{G(z)} = \sum_{k=0}^{\infty} \tilde{g}(k) z^{-k}$$

for the inverse system of a causal LTI system. As indicated above, $G^{-1}(z)$ should always be viewed as a causal series expansion. An important notion for control theory, and also for the discussion later in this thesis, is the concept of *minimum phase* systems.

Definition 2.4. An LTI system is *minimum phase* if both $G(z)$ and $G^{-1}(z) = 1/G(z)$ are stable and causal transfer functions.

The definitions that have been introduced so far have concerned LTI *systems* but do of course hold for LTI *models* and *filters* as well. The word *filter* will here be used as an alternative name for an LTI system whose main purpose is to change a signal in some way. The signals that will be discussed in this thesis are discrete-time stationary stochastic processes (see, for example, Gardner, 1986; Jazwinski, 1970).

Formally, a discrete-time stochastic process $(u(t))_{t=-\infty}^{\infty}$ is an indexed sequence of random variables where the parameter t corresponds to time. The processes that will be studied in this thesis will be real and *stationary*. Stationarity of a process means that the simultaneous probability density function of any set of variables $\{u(t+\tau), \tau \in D \subset \mathbb{Z}\}$ is independent of t. Furthermore, all processes will have zero mean, i.e., $\mathrm{E}(u(t)) = 0$ for all $t \in \mathbb{Z}$, and well-defined *covariance functions* $R_u(\tau)$. The covariance function of a process with zero mean is defined as

$$R_u(\tau) = \mathrm{E}(u(t)u(t-\tau)).$$

Furthermore, it will be assumed that the covariance function has a well-defined z-transform $\Phi_u(z)$ whose region of convergence contains the unit circle. The function $\Phi_u(z)$ can be written

$$\Phi_u(z) = \sum_{\tau=-\infty}^{\infty} R_u(\tau) z^{-\tau}$$

and it will, using the terminology in Kailath et al. (2000), be called the *z-spectrum* of the process. Properties like stability and causality that hold for LTI systems can be used also about z-spectra. Note that $\Phi_u(z^{-1}) = \Phi_u(z)$ since $R_u(-\tau) = R_u(\tau)$. The real-valued function $\Phi_u(e^{i\omega})$ of $\omega \in [-\pi, \pi]$ that is obtained when $z = e^{i\omega}$ will be called the *spectral density function* of the process.

If two processes $(u(t))_{t=-\infty}^{\infty}$ and $(y(t))_{t=-\infty}^{\infty}$ are considered, it will be assumed that they are jointly stationary and that the *cross-covariance function* $R_{yu}(\tau)$ between these processes exists. The cross-covariance function is defined as

$$R_{yu}(\tau) = \mathrm{E}(y(t)u(t-\tau)).$$

Furthermore, it will be assumed that also this function has a z-transform $\Phi_{yu}(z)$ whose region of convergence contains the unit circle. The function $\Phi_{yu}(z)$ can be written

$$\Phi_{yu}(z) = \sum_{\tau=-\infty}^{\infty} R_{yu}(\tau) z^{-\tau}$$

and will be called the *z-cross-spectrum*. Note that $\Phi_{yu}(z^{-1}) = \Phi_{uy}(z)$ and that all z-spectra and z-cross-spectra should always be interpreted as the series expansion whose region of convergence contains the unit circle.

A very important class of processes is *white noise processes*, which have the property that all $u(t)$, $t \in \mathbb{Z}$, are independent. Hence, for white processes only $R_u(0)$ is nonzero. Using white processes as inputs to LTI filters, it is possible to construct processes with arbitrary z-spectra. This follows from the next lemma about LTI filtering of stationary stochastic processes. This lemma has been taken from Kailath et al. (2000, p. 195).

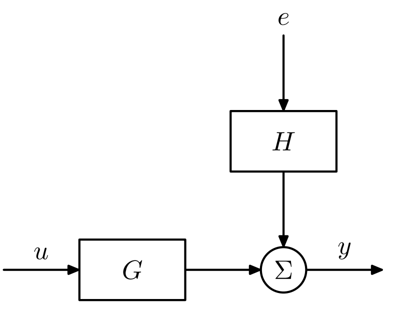


Figure 2.1: *The general LTI model.*

Lemma 2.1 (Filtering of Stationary Processes)
Let $(y(t))_{t=-\infty}^{\infty}$ be the stationary process that is obtained by passing a stationary process $(u(t))_{t=-\infty}^{\infty}$ with zero mean through a stable LTI system with transfer function $H(z)$. Then the relations

$$\Phi_y(z) = H(z)\Phi_u(z)H(z^{-1}),$$
$$\Phi_{yu}(z) = H(z)\Phi_u(z)$$

hold. Furthermore, if $(x(t))_{t=-\infty}^{\infty}$ is jointly stationary with $(y(t))_{t=-\infty}^{\infty}$ and $(u(t))_{t=-\infty}^{\infty}$ as just defined, then

$$\Phi_{xy}(z) = \Phi_{xu}(z)H(z^{-1}).$$

Proof: See Kailath et al. (2000, pp. 195-197). □

LTI models and stochastic processes will in this thesis be used to model arbitrary systems. Usually, it will be assumed that these systems contain some noise and hence we need models that include some kind of noise description. One model with this property is the following general LTI model of a system with input $u(t)$ and output $y(t)$,

$$y(t) = G(q)u(t) + H(q)e(t), \tag{2.1}$$

where $H(q)$ is a monic transfer operator that describes how the output depends on the white noise $e(t)$. The structure of the model (2.1) is illustrated in Figure 2.1.

The LTI model (2.1) can be used to define the optimal predictor $\hat{y}(t)$ of $y(t)$ given past output values $(y(t-k))_{k=1}^{\infty}$ and past and present input values $(u(t-k))_{k=0}^{\infty}$ (see, for example, Ljung, 1999, Chap. 3). This predictor can be written as

$$\hat{y}(t) = H^{-1}(q)G(q)u(t) + (1 - H^{-1}(q))y(t). \tag{2.2}$$

The predictor (2.2) is optimal in the sense that if (2.1) is an accurate description of the true system, it minimizes the *mean-square error* $\mathrm{E}((y(t) - \hat{y}(t))^2)$ and is equal to the conditional expectation of $y(t)$ given past output and past and present input values (Ljung, 1999, Chap. 3). Predictors of this kind are used in the prediction-error method, which will be described in Section 2.3. First, however, we will give a brief overview of some types of nonlinear systems that will be discussed later in this thesis.

2.2 Nonlinear Systems

In some of the results that will be presented in this thesis, there will be no explicit assumptions on the true nonlinear system that is modeled. Hence, the system can often be viewed as a black box that for a given stationary input signal $(u(t))_{t=-\infty}^{\infty}$ produces the output $(y(t))_{t=-\infty}^{\infty}$. However, in some other results, we will assume that the system belongs to certain classes of nonlinear systems and these classes will be defined here.

We will only consider nonlinear systems in discrete time in this thesis. Similarly to the LTI case, a nonlinear system will be said to be static if its output $y(t)$ can be written as a function of $u(t)$ only, i.e., if $y(t) = f(u(t))$, and the system is said to be nonstatic if $y(t)$ depends on any other $u(t-k)$, $k \in \mathbb{Z} \setminus \{0\}$. A class of systems that will be discussed later in this thesis is *nonlinear finite impulse response* (NFIR) systems. An NFIR system can for some $M \in \mathbb{N}$ be written as

$$y(t) = f\big((u(t-k))_{k=0}^{M}\big).$$

Here, the compact notation $f\big((u(t-k))_{k=0}^{M}\big)$ simply means

$$f(u(t), u(t-1), \dots, u(t-M)),$$

i.e., a function of a finite number of input components. An NFIR system is a special case of a *nonlinear autoregressive system with external input* (NARX system) (Sjöberg et al., 1995). Such a system can be written as

$$y(t) = f(\varphi(t)) + e(t),$$

where the vector $\varphi(t)$ has signal components of u and y as elements and where $e(t)$ is a white noise process. An NFIR system is also a special case of a *nonlinear output error* (NOE) system

$$y(t) = f\big((u(t-k))_{k=0}^{\infty}\big) + e(t),$$

where $e(t)$ is white noise.

Two other system classes that will be discussed later are *Wiener* and *Hammerstein* systems. A Wiener system consists of an LTI model followed by a static nonlinearity, i.e.,

$$\begin{aligned} y(t) &= f(v(t)), \\ v(t) &= G(q)u(t), \end{aligned}$$

while a Hammerstein system has these linear and nonlinear subsystems in the opposite order, i.e.,

$$\begin{aligned} y(t) &= G(q)v(t), \\ v(t) &= f(u(t)). \end{aligned}$$

Actually, Wiener and Hammerstein systems can be viewed as special cases of a more general system class known as *Wiener-Hammerstein* systems. Such a system consists of a Wiener system followed by an LTI system.

Just like LTI systems, many nonlinear systems can be written also in state-space form

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= h(x(t), u(t)), \end{aligned}$$

where $x(t)$ is a state vector.

A detailed description and characterization of many other types of nonlinear systems and models can be found in Pearson (1999) and Sjöberg et al. (1995). In the next section, we will describe some of the basic ideas in system identification.

2.3 System Identification

As was mentioned in the introduction to this thesis, *system identification* can be viewed as a synonym for mathematical modeling of dynamic systems using measurements of the input and output signals. Various identification methods can be found in the literature, but here we will only discuss one family of methods, namely *prediction-error methods*. We will discuss the general idea behind this method, some of its properties and also a special version of it designed for a class of input signals called *random multisines*.

2.3.1 Prediction-Error Methods

Prediction-error methods are based on the observation that predictors like (2.2) can be used to compare how well different LTI models can predict the output $y(t)$. The main idea is to use some kind of measure of the distance between the predicted output and the true output and to minimize this distance by adjusting some parameters in the model. Typically, a prediction-error method works with a finite data set $Z^N = (u(t), y(t))_{t=1}^N$ that contains simultaneous measurements of the input and output signals and a parameterized version of the general LTI model (2.1). This parameterized model can be written as

$$y(t, \theta) = G(q, \theta)u(t) + H(q, \theta)e(t), \tag{2.3}$$

where θ is a d-dimensional vector of parameters. For example, θ can be the coefficients of the numerator and denominator polynomials of G and H, provided that these transfer functions are chosen as rational functions.

Different *model structures* can be obtained by imposing some restrictions on the rational functions G and H. For example, the *autoregressive with external input* (ARX) model structure is obtained by letting

$$\begin{aligned} G(q, \theta) &= \frac{B(q, \theta)}{A(q, \theta)}, \\ H(q, \theta) &= \frac{1}{A(q, \theta)}, \end{aligned}$$

where A and B are polynomials. Similarly, the *output error* (OE) model structure is acquired if $H(q, \theta) = 1$. A family of LTI model structures is described in Ljung (1999, pp. 81-88).

If the model (2.3) would be a perfect description of the true system for some white noise process $e(t)$, the mean-square error optimal predictor $\hat{y}(t,\theta)$ of $y(t)$ would be

$$\hat{y}(t,\theta) = H^{-1}(q,\theta)G(q,\theta)u(t) + (1 - H^{-1}(q,\theta))y(t). \tag{2.4}$$

When a model structure has been selected in the prediction-error method, the corresponding predictor (2.4) is used to compute θ-dependent predictions $\hat{y}(t,\theta)$ based on the data in Z^N. A parameter estimate $\hat{\theta}_N$ can then be computed by minimizing a criterion $V_N(\theta, Z^N)$. For example, this criterion can be chosen to be quadratic such that

$$\hat{\theta}_N = \arg\min_{\theta \in D_M} V_N(\theta, Z^N) = \arg\min_{\theta \in D_M} \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t,\theta))^2. \tag{2.5}$$

Here, θ is restricted to some pre-specified set $D_M \subset \mathbb{R}^d$. Usually, D_M is the set of parameters that make the predictor (2.4) stable. In general, the minimization of $V_N(\theta, Z^N)$ has to be performed using some kind of numerical method. A common choice is to use a *Gauss-Newton* or a *damped Gauss-Newton* method. These methods use the gradient and an approximation of the Hessian of $V_N(\theta, Z^N)$ and have good convergence properties, especially in the vicinity of the optimum (see Ljung, 1999).

Detailed studies of the properties of the prediction-error estimate when the number of measurements tends to infinity have been made (Ljung, 1978, 1999). In Ljung (1978) it is shown that under rather weak conditions on the true system and on the input and output signals, the following convergence result holds with probability one.

$$\hat{\theta}_N \to \theta^* = \arg\min_{\theta \in D_M} \mathrm{E}((y(t) - \hat{y}(t,\theta))^2), \quad \text{w.p.1 as } N \to \infty \tag{2.6}$$

With some abuse of notation, $y(t)$ and $\hat{y}(t,\theta)$ here denote the stochastic signals while they previously in this section have denoted realizations of these signals. An important necessary condition on the input and output signals for (2.6) to hold concerns the dependency between signal components over time. Intuitively, this condition requires that the remote past of the process should be forgotten at an exponential rate. This condition is satisfied for many random input signals of practical interest, e.g., most filtered white noise signals and random binary signals. For such inputs, the properties of a model estimated using (2.5) can often be understood by studying the model that minimizes the mean-square error in (2.6). However, there is at least one important class of input signals for which the result (2.6) is not applicable, namely random multisines. Usually, a modified version of the prediction-error method is used for these input signals. This will be discussed in the next section.

The convergence result (2.6) holds also for many nonlinear systems. Since (2.6) shows that the prediction-error estimate with probability one will converge to the mean-square error optimal estimate θ^*, it is interesting to investigate what can be said about the LTI models that are defined by θ^* when the true system is nonlinear. This is the main objective of this thesis.

A convergence result that is similar to (2.6) can be shown under less restrictive assumptions on the input signal if the system is linear (Ljung, 1999, Chap. 8). In this result, the expectation in (2.6) is replaced with both an average over time and an expectation, and the convergence of the parameter estimates is obtained also for *quasi-stationary* signals (Ljung, 1999, Chap. 2), e.g., pseudo-random binary signals and deterministic multisine signals. However, this result cannot be applied to nonlinear systems. Furthermore, it is not obvious that all quasi-stationary signals are suitable for estimation of LTI models of nonlinear systems since some averaging due to the randomness of the input signal might be beneficial in this type of identification problem. Intuitively, the reason why a random input signal might be useful is that it often will give a linear model that approximates the average behavior of the system rather than just the system behavior for one fixed input.

However, also when a nonlinear system is approximated by an LTI model using the prediction-error method and a realization of a stochastic process as input, there is a risk that the obtained model will be adjusted too much to the particular realization of the input used in the identification experiment. For example, consider an identification experiment where a realization of a stochastic process with zero mean is used to model the average behavior of a nonlinear system in an interval around zero. If the realization of the input is short, there is often a significant probability that, for example, all signal components will have equal signs. In this case, an estimated model will usually not be able to describe the desired system behavior accurately. However, if the dependency between two input components $u(t)$ and $u(t-\tau)$ decreases as $|\tau|$ increases, the probability to get a realization where all components have equal signs tends to zero when the number of signal components in the realization goes to infinity. Hence, an estimated model will typically describe the average system behavior better if a large data set is used in the identification procedure. For many systems, such an input signal will guarantee also that (2.6) holds.

The method described here is not the only prediction-error method, but rather a commonly used member of a family of methods. The main differences between these methods are due to different choices of criterion in (2.5) and to the fact that a prefilter is used in some methods. It should also be noted that prediction-error methods can be used for other model structures than the ones based on (2.1). For example, both linear state-space models and general nonlinear black-box models can be used.

2.3.2 A Prediction-Error Method for Random Multisines

The use of periodic input signals in identification experiments is common in many applications and can be motivated in several ways (Pintelon and Schoukens, 2001; Ljung, 1999). For example, if the modeling is done using a frequency domain criterion, a periodic signal will remove the undesirable leakage effects that are usually present. Furthermore, it is easy to calculate good estimates of the noise level with such an input.

A discrete-time signal $u(t)$ is periodic if there is a positive integer P such that

$$u(t+P) = u(t), \quad \forall t \in \mathbb{Z}.$$

Consider a P-periodic input, with $P \in \mathbb{Z}_+$, to a system with the output

$$y(t) = y_{nf}(t) + w(t),$$

where $y_{nf}(t)$ is the noise-free output and $w(t)$ is measurement noise. Assume that measurements from M periods have been collected and that all transient effects have disappeared such that $y_{nf}(t)$ is P-periodic too. In this case, an average $\bar{y}(t)$ of the output signal over the periods can be calculated as

$$\bar{y}(t) = \frac{1}{M}\sum_{k=0}^{M-1} y(t+kP), \quad 1 \le t \le P.$$

In this way, a shorter signal with a higher signal-to-noise ratio is obtained and an estimate $\hat{\lambda}_w$ of the variance of the measurement noise can be calculated as

$$\hat{\lambda}_w = \frac{1}{(M-1)P}\sum_{k=0}^{M-1}\sum_{t=1}^{P}\left(y(t+kP) - \bar{y}(t)\right)^2.$$

A disadvantage with a P-periodic signal is that it can only be persistently exciting of, at most, order P (Ljung, 1999, Chap. 13). Hence, models with arbitrarily many parameters cannot be uniquely determined if a periodic input has been used. Further properties of periodic excitation signals can be found in Pintelon and Schoukens (2001) and Ljung (1999).

A particular class of periodic signals that has turned out to be useful for identification of linear and nonlinear systems is *random multisines* (Pintelon and Schoukens, 2001).

Definition 2.5. A *random multisine* signal is a stationary stochastic process $u(t)$ that can be written

$$u(t) = \sum_{k=1}^{Q} A_k \cos(\omega_k t + \psi_k), \tag{2.7}$$

where both A_k and ψ_k can be random variables and where all ω_k are constants that satisfy $|\omega_k| \le \pi$.

Here, the phases ψ_k will usually be independent random variables with uniform distribution on the interval $[0, 2\pi)$ and the amplitudes A_k will usually be constants. Furthermore, we will only consider periodic random multisines such that the period P is an integer, i.e., such that all ω_k can be written $\omega_k = \pi p_k$ for some $p_k \in \{x \in \mathbb{Q} \mid |x| \le 1\}$.

A linear system can be identified from one realization of a random multisine, but for a nonlinear system it is important to use several realizations in order to get a model that is not too adjusted to the signal shape of one realization only. Since a random multisine under certain assumptions on ω_k is periodic, the dependency between the signal components does not decrease over time. Hence, the convergence result (2.6) will not hold in general. This implies that for a nonlinear system, parameter estimates that give models that are good approximations of the mean-square error optimal model cannot be obtained by collecting many measurement from one single identification experiment. Instead, several experiments have to be performed.

With data sets from N_E experiments where different realizations of the input signal have been used and $N = MP$ measurements in each data set, with $M \in \mathbb{Z}_+$, a model can be estimated by minimizing the cost function

$$V_{N_E,N}(\theta, Z_{N_E}^N) = \frac{1}{N_E}\sum_{s=1}^{N_E}\frac{1}{N}\sum_{t=0}^{N-1}\left(y_s(t) - G(q,\theta)u_s(t)\right)^2 \tag{2.8}$$

with respect to the parameters θ. Here, $u_s(t)$ and $y_s(t)$ are the input and output signals from experiment s, respectively, and $Z_{N_E}^N$ is the combined data set with measurements from all experiments. Intuitively, this cost function can be viewed as an approximation of the mean-square error $\mathrm{E}((y(t) - G(q,\theta)u(t))^2)$, just like the cost function V_N in (2.5) can be viewed as an approximation of the mean-square error. However, $V_{N_E,N}$ will typically approach the mean-square error when the number of experiments N_E tends to infinity while V_N approaches it when the number of measurements N in one experiment tends to infinity according to (2.6).

With a periodic input, it is very natural to consider the modeling problem in the frequency domain. Applying the *Discrete Fourier Transform* (DFT) to the input and output signals gives the transforms

$$U_{s,N}(n) = \sum_{t=0}^{N-1} u_s(t) e^{-i2\pi nt/N}, \tag{2.9a}$$

$$Y_{s,N}(n) = \sum_{t=0}^{N-1} y_s(t) e^{-i2\pi nt/N}. \tag{2.9b}$$

Let $\hat{y}_s(t,\theta)$ denote the output from the stable model $G(q,\theta)$ for the input $u_s(t)$ and assume that the input has been applied at $t = -\infty$ such that all transients have disappeared at $t \geq 0$, i.e., that $\hat{y}_s(t,\theta)$ is P-periodic in the interval $0 \leq t \leq N-1$. Furthermore, let $\hat{Y}_{s,N}(n,\theta)$ denote the DFT of $\hat{y}_s(t,\theta)$, i.e.,

$$\hat{Y}_{s,N}(n,\theta) = \sum_{t=0}^{N-1} \hat{y}_s(t,\theta) e^{-i2\pi nt/N}.$$

The frequency response of the stable model $G(z,\theta)$ is obtained for $z = e^{i\omega}$. In particular, since $v(k) \triangleq e^{-i2\pi nk/N}$ is an N-periodic signal, it follows that

$$\begin{aligned} G(e^{i2\pi n/N},\theta) &= \sum_{k=0}^{\infty} g(k,\theta) e^{-i2\pi nk/N} \\ &= \sum_{t=0}^{N-1} \underbrace{\left(\sum_{l=0}^{\infty} g(t+lN,\theta) \right)}_{\triangleq \tilde{g}_N(t,\theta)} e^{-i2\pi nt/N} \triangleq \tilde{G}_N(n,\theta). \end{aligned} \tag{2.10}$$

Furthermore, since $u_s(t)$ is a P-periodic signal and $N = MP$, with $M \in \mathbb{Z}_+$, $u_s(t)$ is also N-periodic. Hence,

$$\hat{y}_s(t,\theta) = G(q,\theta)u_s(t) = \sum_{k=0}^{N-1} \tilde{g}(k,\theta) u_s(t-k)$$

and this implies that

$$\hat{Y}_{s,N}(n,\theta) = \tilde{G}_N(n,\theta) U_{s,N}(n) = G(e^{i2\pi n/N},\theta) U_{s,N}(n), \tag{2.11}$$

where we have used (2.10) in the last equality.

Using Parseval's formula, the cost function can be rewritten as

$$
\begin{aligned}
V_{N_E,N}(\theta, Z_{N_E}^N) &= \frac{1}{N_E}\sum_{s=1}^{N_E}\frac{1}{N}\sum_{t=0}^{N-1}(y_s(t) - \underbrace{G(q,\theta)u_s(t)}_{=\hat{y}_s(t,\theta)})^2 \\
&= \frac{1}{N_E}\sum_{s=1}^{N_E}\frac{1}{N^2}\sum_{n=0}^{N-1}\left|Y_{s,N}(n) - \hat{Y}_{s,N}(n,\theta)\right|^2 \\
&= \frac{1}{N_E}\sum_{s=1}^{N_E}\frac{1}{N^2}\sum_{n=0}^{N-1}\left|Y_{s,N}(n) - G(e^{i2\pi n/N},\theta)U_{s,N}(n)\right|^2.
\end{aligned}
$$

From this expression, it is obvious that two linear models will give the same value of the cost function if their frequency responses are equal at the frequencies where $U_{s,N}(n)$ is nonzero. Assume that the input is a random multisine such that $U_{s,N}(n)$ is nonzero at the frequencies where $n \in \Omega \subset \{0, 1, \ldots, N-1\}$ and zero otherwise and consider a nonparametric frequency response model $G_{np}(n) = G(e^{i2\pi n/N}, \theta)$. In this case, minimizing $V_{N_E,N}$ is equivalent to solving a least squares problem for each $n \in \Omega$. The resulting nonparametric estimate can be written

$$
\hat{G}_{np}(n) = \frac{\sum_{s=1}^{N_E} Y_{s,N}(n)\overline{U_{s,N}(n)}}{\sum_{s=1}^{N_E} |U_{s,N}(n)|^2}, \quad n \in \Omega. \tag{2.12}
$$

In particular, if $|U_{s,N}(n)|$ are equal for all s, (2.12) can be simplified to

$$
\hat{G}_{np}(n) = \frac{1}{N_E}\sum_{s=1}^{N_E}\frac{Y_{s,N}(n)}{U_{s,N}(n)}, \quad n \in \Omega. \tag{2.13}
$$

For example, this expression can be used when the input is a random multisine where all amplitudes A_k are constants and all frequencies ω_k are separate. More results about random multisines and frequency domain identification can be found in, for example, Pintelon and Schoukens (2001).

2.4 Separable Processes

Some of the results in this thesis concern processes that are separable in Nuttall's sense (Nuttall, 1958a), i.e., processes which satisfy the condition described in the following definition.

Definition 2.6 (Separability). A stationary stochastic process $u(t)$ with $\mathrm{E}(u(t)) = 0$ is *separable* (in Nuttall's sense) if

$$
\mathrm{E}(u(t-\tau)|u(t)) = a(\tau)u(t). \tag{2.14}
$$

for some function $a(\tau)$.

In this section, some of the main results for separable processes will be presented. These results can all be found in Nuttall (1958a,b) but they are here rewritten with the notation used in this thesis. Note that the technical report Nuttall (1958a) is an almost identical copy of the thesis Nuttall (1958b). Here, the first of these works will be used as the main reference. Furthermore, note that some of the proofs in this section are slightly different from the corresponding ones in Nuttall (1958a).

It is easy to show that the function $a(\tau)$ in (2.14) can be expressed using the covariance function of $u(t)$.

Lemma 2.2
Consider a separable stationary stochastic process $u(t)$ with $\mathrm{E}(u(t)) = 0$. The function $a(\tau)$ from (2.14) can then be written

$$a(\tau) = \frac{R_u(\tau)}{R_u(0)}. \tag{2.15}$$

Proof: The result follows immediately from the fact that

$$R_u(\tau) = \mathrm{E}(u(t)u(t-\tau)) = \mathrm{E}\big(u(t)\mathrm{E}\big(u(t-\tau)|u(t)\big)\big) = a(\tau)\mathrm{E}(u(t)^2) = a(\tau)R_u(0)$$

if $u(t)$ is separable. Here, we have used the facts that

$$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|X)), \tag{2.16a}$$

$$\mathrm{E}(g(X)Y|X) = g(X)\mathrm{E}(Y|X) \tag{2.16b}$$

for two random variables X and Y (see, for example, Gut, 1995, Chap. 2). □

Separability can be expressed also using characteristic functions. Hence, the following definition is useful.

Definition 2.7. Consider a stationary stochastic process $u(t)$ with $\mathrm{E}(u(t)) = 0$ and with first and second order characteristic functions

$$f_{u,1}(\xi_1) = \mathrm{E}(e^{i\xi_1 u(t)}), \tag{2.17a}$$

$$f_{u,2}(\xi_1, \xi_2, \tau) = \mathrm{E}(e^{i\xi_1 u(t) + i\xi_2 u(t-\tau)}), \tag{2.17b}$$

respectively. Then the *G-function* $G_u(\xi_1, \tau)$ of this process is defined as

$$G_u(\xi_1, \tau) = \left.\frac{\partial f_{u,2}(\xi_1, \xi_2, \tau)}{\partial \xi_2}\right|_{\xi_2 = 0} = \mathrm{E}(iu(t-\tau)e^{i\xi_1 u(t)}). \tag{2.18}$$

In Nuttall (1958a), a number of separable signals are listed, e.g., Gaussian processes, random binary processes and several types of modulated processes. For example, a single sinusoid with random phase is separable according to the following lemma.

Lemma 2.3
A random sine process

$$u(t) = A\cos(\omega t + \psi), \tag{2.19}$$

where ψ is a random variable with uniform distribution on the interval $[0, 2\pi)$ and where A and ω are constants, is a separable process. Furthermore, this process has the properties

$$R_u(\tau) = \frac{A^2}{2}\cos(\omega\tau), \tag{2.20a}$$

$$f_{u,1}(\xi_1) = J_0(A\xi_1), \tag{2.20b}$$

where J_0 is the zeroth order Bessel function.

Proof: Using basic properties of trigonometric functions, we have

$$\begin{aligned} u(t-\tau) &= A\cos(\omega t - \omega\tau + \psi) = A\cos(\omega t + \psi)\cos(\omega\tau) + A\sin(\omega t + \psi)\sin(\omega\tau) \\ &= u(t)\cos(\omega\tau) + A\sin(\omega t + \psi)\sin(\omega\tau). \end{aligned}$$

Since $A\sin(\omega t + \psi)$ equals $\sqrt{A^2 - u(t)^2}$ or $-\sqrt{A^2 - u(t)^2}$ with equal probabilities if $u(t)$ is given,

$$\mathrm{E}(A\sin(\omega t + \psi)|u(t)) = 0.$$

Hence, it follows that

$$\mathrm{E}(u(t-\tau)|u(t)) = \cos(\omega\tau)u(t),$$

i.e., $u(t)$ is separable. Furthermore, Lemma 2.2 implies that the covariance function of $u(t)$ is

$$R_u(\tau) = \cos(\omega\tau)\mathrm{E}(u(t)^2) = A^2\cos(\omega\tau)\frac{1}{2\pi}\int_0^{2\pi}\cos(\omega t + \psi)^2\,d\psi = \frac{A^2}{2}\cos(\omega\tau)$$

and the characteristic function is

$$\begin{aligned} f_{u,1}(\xi_1) &= \mathrm{E}(e^{i\xi_1 u(t)}) = \frac{1}{2\pi}\int_0^{2\pi} e^{iA\xi_1\cos(\omega t + \psi)}\,d\psi = \\ &= \left\{\tilde{\psi} = \omega t + \psi + \frac{\pi}{2}\right\} = \frac{1}{2\pi}\int_{\omega t + \pi/2}^{\omega t + 5\pi/2} e^{iA\xi_1\cos(\tilde{\psi} - \pi/2)}\,d\tilde{\psi} = \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{iA\xi_1\sin(\tilde{\psi})}\,d\tilde{\psi} = J_0(A\xi_1). \end{aligned}$$

□

In the previous lemma, separability of a single sinusoid with random phase was proved directly using Definition 2.6. However, in many cases it is more convenient to show separability of a signal using characteristic functions. This is possible according to the following theorem.

Theorem 2.1
Consider a stationary stochastic process $u(t)$ *with* $\mathrm{E}(u(t)) = 0$. *This process is separable if and only if*

$$G_u(\xi_1, \tau) = a(\tau) f'_{u,1}(\xi_1), \tag{2.21}$$

where $a(\tau) = R_u(\tau)/R_u(0)$ *and where* $f_{u,1}$ *and* G_u *are defined in (2.17a) and (2.18), respectively.*

Proof: IF: Assume that (2.21) holds. Then, using the definition of G_u in (2.18), it follows that

$$\mathrm{E}\Big(ie^{i\xi_1 u(t)}\Big(\mathrm{E}\big(u(t-\tau)|u(t)\big) - a(\tau)u(t)\Big)\Big) = G_u(\xi_1, \tau) - a(\tau) f'_{u,1}(\xi_1) = 0.$$

From the uniqueness of Fourier transforms it thus follows that (2.14) holds. Hence, $u(t)$ is separable if (2.21) holds.
ONLY IF: Assume that $u(t)$ is separable, i.e., that (2.14) holds. This implies that

$$\begin{aligned} G_u(\xi_1, \tau) &= \mathrm{E}(iu(t-\tau)e^{i\xi_1 u(t)}) = \mathrm{E}\big(ie^{i\xi_1 u(t)}\mathrm{E}\big(u(t-\tau)|u(t)\big)\big) \\ &= a(\tau)\mathrm{E}(iu(t)e^{i\xi_1 u(t)}) = a(\tau) f'_{u,1}(\xi_1), \end{aligned}$$

where we have used (2.14) in the second equality. Hence, (2.21) holds if $u(t)$ is separable.
□

In the next theorem from Nuttall (1958a), it is shown that the sum of Q independent separable processes is separable if the characteristic functions satisfy a certain condition.

Theorem 2.2
Consider Q *independent and separable stationary stochastic processes* $u_k(t)$ *with*

$$\mathrm{E}(u_k(t)) = 0$$

for $k = 1, \ldots, Q$ *and let*

$$u(t) = \sum_{k=1}^{Q} u_k(t). \tag{2.22}$$

Assume that the characteristic functions satisfy

$$f_{u_k,1}(\xi_1)^{1/\sigma_k^2} = f_{u_l,1}(\xi_1)^{1/\sigma_l^2}, \quad \forall k, l \in \{1, 2, \ldots, Q\}, \tag{2.23}$$

where $\sigma_m^2 = R_{u_m}(0)$. *Then* $u(t)$ *is separable.*

Proof: Since the signals $u_k(t)$ are independent, we have

$$f_{u,1}(\xi_1) = \prod_{k=1}^{Q} f_{u_k,1}(\xi_1) = f_{u_1,1}(\xi_1)^{\sum_{k=1}^{Q} \sigma_k^2/\sigma_1^2}, \tag{2.24}$$

$$f_{u,2}(\xi_1, \xi_2, \tau) = \prod_{k=1}^{Q} f_{u_k,2}(\xi_1, \xi_2, \tau), \tag{2.25}$$

$$R_u(\tau) = \sum_{k=1}^{Q} R_{u_k}(\tau), \tag{2.26}$$

where the last equality in (2.24) follows from (2.23). Furthermore, (2.25) implies that

$$G_u(\xi_1,\tau) = \left.\frac{\partial f_{u,2}(\xi_1,\xi_2,\tau)}{\partial \xi_2}\right|_{\xi_2=0} = \sum_{k=1}^{Q} G_{u_k}(\xi_1,\tau) \prod_{l=1,l\neq k}^{Q} f_{u_l,2}(\xi_1,0,\tau)$$
$$= \sum_{k=1}^{Q} G_{u_k}(\xi_1,\tau) \prod_{l=1,l\neq k}^{Q} f_{u_l,1}(\xi_1), \tag{2.27}$$

where we have used that $f_{u_l,2}(\xi_1,0,\tau) = f_{u_l,1}(\xi_1)$ in the last equality. From (2.23) it follows that

$$f'_{u_k,1}(\xi_1) = \frac{\sigma_k^2}{\sigma_1^2} f_{u_1,1}(\xi_1)^{\sigma_k^2/\sigma_1^2-1} f'_{u_1,1}(\xi_1). \tag{2.28}$$

Since all $u_k(t)$ are separable, (2.21) holds and by inserting (2.28) we obtain

$$G_{u_k}(\xi_1,\tau) = \frac{R_{u_k}(\tau)}{\sigma_1^2} f_{u_1,1}(\xi_1)^{\sigma_k^2/\sigma_1^2-1} f'_{u_1,1}(\xi_1). \tag{2.29}$$

Inserting (2.23) and (2.29) in (2.27) gives

$$G_u(\xi_1,\tau) = \sum_{k=1}^{Q} \frac{R_{u_k}(\tau)}{\sigma_1^2} f_{u_1,1}(\xi_1)^{\sum_{l=1}^{Q}\sigma_l^2/\sigma_1^2-1} f'_{u_1,1}(\xi_1)$$
$$= \frac{d}{d\xi_1}\left(f_{u_1,1}(\xi_1)^{\sum_{l=1}^{Q}\sigma_l^2/\sigma_1^2}\right) \frac{\sum_{k=1}^{Q} R_{u_k}(\tau)}{\sum_{l=1}^{Q}\sigma_l^2} = f'_{u,1}(\xi_1)\frac{R_u(\tau)}{R_u(0)},$$

where we have used (2.24) and (2.26) in the last equality. Hence, Theorem 2.1 gives that $u(t)$ is separable. □

In Nuttall (1958a), two different results for sums of separable processes are presented. Both these results concern sufficient conditions for the separability of the sum of a finite number of independent separable processes. The first of these conditions is that the individual correlation functions should be equal, while the second is the condition on the characteristic functions restated here in Theorem 2.2. Furthermore, it is shown in Nuttall (1958a) that the product of two independent separable processes with zero mean always will be separable.

The reason why separable processes are useful for identification experiments is that they are the most general class of input signals for which a certain *invariance property* holds.

Definition 2.8. Consider a stationary stochastic process $u(t)$ with $\mathrm{E}(u(t)) = 0$ and $R_u(\tau) < \infty$ for all $\tau \in \mathbb{Z}$ and a static nonlinearity $y(t) = f(u(t))$ such that $\mathrm{E}(y(t)) = 0$ and $R_{yu}(\tau) < \infty$ for all $\tau \in \mathbb{Z}$. The *invariance property* holds if

$$R_{yu}(\tau) = b_0 R_u(\tau), \quad \forall \tau \in \mathbb{Z}, \tag{2.30}$$

for some constant b_0.

It is easy to show that the separability of $u(t)$ is a sufficient condition for the invariance property (2.30) to hold. Consider a separable process $u(t)$ with zero mean and a static nonlinearity such that $y(t)$ has zero mean too. Then it follows that

$$\begin{aligned} R_{yu}(\tau) &= \mathrm{E}\big(f(u(t))u(t-\tau)\big) = \mathrm{E}\big(\mathrm{E}\big(f(u(t))u(t-\tau)|u(t)\big)\big) \\ &= \mathrm{E}\big(f(u(t))\mathrm{E}\big(u(t-\tau)|u(t)\big)\big) = a(\tau)\mathrm{E}\big(f(u(t))u(t)\big) = b_0 R_u(\tau), \end{aligned} \tag{2.31}$$

where $b_0 = \mathrm{E}\big(f(u(t))u(t)\big)/R_u(0)$ and where (2.15) has been used in the last equality.

In a certain sense, separability is also a necessary condition for (2.30) to hold. Consider an arbitrary stationary stochastic process $u(t)$ with zero mean and let D_u be a class of Lebesgue integrable functions such that

$$\begin{aligned} D_u = \{f : \mathbb{R} \to \mathbb{R} \mid \mathrm{E}\big(f(u(t))\big) = 0, \mathrm{E}\big(f(u(t))^2\big) < \infty, \\ R_{yu}(\tau) = \mathrm{E}\big(f(u(t))u(t-\tau)\big) \text{ exists } \forall \tau \in \mathbb{Z}\}. \end{aligned} \tag{2.32}$$

The following result shows a certain equivalence between the invariance property and separability of the input signal. In Section 6.3.2, this result will be extended to a more general type of nonlinear systems and thus the proof of Theorem 2.3 is omitted here.

Theorem 2.3
Consider a stationary stochastic process $u(t)$ with $\mathrm{E}(u(t)) = 0$ and $R_u(\tau) < \infty$ for all $\tau \in \mathbb{Z}$. The invariance property (2.30) holds for all $f \in D_u$ if and only if $u(t)$ is separable.

Proof: See Nuttall (1958a). □

For a separable process, it is easy to show that the mean-square error optimal LTI model of a static nonlinearity is a constant. However, this important property will not be described in this section but later in Chapter 6 when the LTI approximations based on the mean-square error have been properly defined. Several other results about separable processes are also presented in Nuttall (1958a), but since these results are not used in this thesis, they are not restated here.

After this discussion about the properties of LTI systems and some nonlinear systems and the brief introductions to the prediction-error method, random multisines and separable processes, we are now ready to move on to the main part of this thesis. First, however, we will in the next chapter give an overview of some of the linearization approaches that can be found in the literature.

3

Methods for Linearization of Nonlinear Systems

Since there are many different circumstances when LTI models of nonlinear systems are useful, various linearization approaches have been proposed. These approaches differ in many aspects, for example in the types of signals they are defined for, in their optimality properties and in which mathematical tools that are used. In this chapter, a brief overview of some of the linearization frameworks that can be found in the literature will be given. First, we will in the next section consider linearizations in a deterministic framework while we in the second part of this chapter will discuss LTI approximations for stochastic signals.

3.1 Deterministic Approaches

The most straightforward linearization approach is perhaps to use some kind of local linearization based on a truncated Taylor series expansion. For example, if the true system is an NFIR system (see Section 2.2)

$$y(t) = f((u(t-k))_{k=0}^{M}),$$

it can for small inputs be approximated with an LTI system

$$G_0(z) = \sum_{k=0}^{M} g_0(k) z^{-k},$$

where $g_0(k) = f'_{u(t-k)}(0)$. Of course, this approximation is only well-defined if f is differentiable at 0. Similar LTI approximations can be calculated also for nonlinear state-space systems (see, for example, Ljung and Glad, 1994, pp. 347-349).

The idea of deriving an LTI approximation by differentiation of a nonlinear system is used also in Mäkilä and Partington (2003). There, LTI approximations of some classes of nonlinear systems with l^∞ input and output signals are studied. It is shown that the *Fréchet derivative* of a Wiener, an NFIR or a bi-gain system is an optimal approximation in the sense that the induced gain of the model error is minimized. Furthermore, some relations to controller design and identification are discussed. Induced gains of linear and nonlinear systems for l^p signals are studied also in Partington and Mäkilä (2002). In particular, this paper contains some results about the accuracy of LTI approximations of piecewise linear systems.

LTI models that are optimal in the sense that they minimize an average squared error condition are characterized in Mäkilä (2004a) using a deterministic quasi-stationary signal framework. Actually, LTI approximations can be viewed as a special case of a general approximation problem for nonlinear systems studied in Mäkilä (2003a,b). These papers concern also approximations in an absolute error framework. In Mäkilä and Partington (2004), the least-squares LTI approximation problem from Mäkilä (2004a) is studied for a more general class of input signals. Furthermore, this paper presents conditions that guarantee quasi-stationarity of the output signal for some classes of nonlinear systems. Some of these results have also been published in the thesis Mäkilä (2004b).

LTI approximations of NFIR systems are considered in Mäkilä (2005). There, the concept of *nearly linear systems* is introduced. A nonlinear system is nearly linear if there exists an LTI system such that the l^∞ norm of the difference between the outputs of the nonlinear and the linear systems is bounded for all input signals. An LTI system with this property is called an *LTI companion* of the nonlinear system. For a nearly linear NFIR system, it turns out that a controller stabilizes the system if and only if it stabilizes its LTI companion. The importance of a careful input design for least-squares LTI approximations of NFIR systems is also discussed in Mäkilä (2005).

A different approach is used in Horowitz (1993). There, the *LTI equivalent* $P(s)$ of a continuous-time nonlinear system that has the output $y(t)$ for a particular input $u(t)$ is defined as the ratio

$$P(s) = \frac{Y(s)}{U(s)},$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output and input signal, respectively. Furthermore, it is shown that a set of such LTI equivalents can be used for controller design.

LTI approximations for a class of deterministic signals are also discussed in Sastry (1999). There, the existence of an optimal LTI model is mentioned and it is related to the theory of *describing functions*. A describing function is a parameterized linear approximation of a nonlinearity derived for a sinusoidal input signal. Describing functions are often used for analysis of closed-loop nonlinear systems (see, for example, Atherton, 1982; Sastry, 1999). LTI approximations for deterministic periodic signals are also investigated in Evans and Rees (2000a,b).

The concept of LTI-SOEs used in this thesis is, for example, discussed in Ljung (2001). However, a deterministic quasi-stationary framework is used in this paper while we here will study LTI-SOEs for stochastic signals. Despite this, the fundamental approximation results in Ljung (2001) are completely analogous to the ones that will be presented in Chapter 4.

3.2 Stochastic Approaches

Since the work of Wiener (1949), there has been a great activity in estimation and filtering using random signals. We will here present some of the existing approaches to linearization using stochastic signals and we begin with linearizations of static nonlinearities.

3.2.1 Results for Static Nonlinearities

Many problems concerning the interplay between stochastic signals and nonlinear systems are difficult to solve. However, some nonlinear systems can be written as combinations of LTI subsystems and static nonlinear functions. For example, LTI systems with input and/or output saturation turn out to be very common in applications.

It is usually easier to analyze a static nonlinearity than a dynamic one. Hence, it is no surprise that there has been a wide interest in understanding how a static nonlinearity can be linearized for a stochastic input signal. Many results in this area are directly or indirectly related to Bussgang's classic theorem about Gaussian signals (see Bussgang (1952) for the original report and, for example, Papoulis (1984) for a more recent reference).

In Bussgang (1952), it is shown that the cross-covariance function between the output and the input of a static nonlinear function is a scaled version of the input covariance function if the input is Gaussian, i.e., that the invariance property in Definition 2.8 holds in this case. This result is summarized in the following theorem.

Theorem 3.1 (Bussgang)
Let $y(t)$ be the output from a differentiable static nonlinearity f with a Gaussian input $u(t)$, i.e., $y(t) = f(u(t))$. Assume that the expectations $\mathrm{E}(y(t)) = \mathrm{E}(u(t)) = 0$, that the cross-covariance function $R_{yu}(\tau)$ and the covariance function $R_u(\tau)$ are well-defined for all $\tau \in \mathbb{Z}$ and that $\mathrm{E}(f'(u(t)))$ exists. Then it holds that

$$R_{yu}(\tau) = b_0 R_u(\tau),$$

where $b_0 = \mathrm{E}(f'(u(t)))$.

A direct implication of this result is that the mean-square error optimal LTI approximation of a static nonlinearity always will be a constant $b_0 = \mathrm{E}(f'(u(t)))$ when the input to the nonlinearity is Gaussian (cf. Section 6.1). It will be shown later in this thesis that this is not always true in the non-Gaussian case (see, for example, Example 4.2). The constant b_0 is called *equivalent gain* in Booton (1954) an can be viewed as a describing function for a random input signal. Just like ordinary describing functions, it can be used to analyze nonlinear closed-loop systems (Atherton, 1982, Chap. 8). The relation between Bussgang's theorem and some other results about Gaussian processes is discussed in Gorman and Zaborszky (1968).

Furthermore, Bussgang's theorem has been generalized to other classes of signals than Gaussian in Barrett and Lampard (1955), Brown (1957) and Nuttall (1958a). Nuttall's generalization is to the class of separable processes defined in Section 2.4. This class of signals is especially interesting since it contains the other two, and since it gives a useful necessary and sufficient condition on the input signal for the invariance property in Definition 2.8 to hold for any static nonlinearity in a wide class of functions (cf. Theorem 2.3). As was mentioned in Section 2.4, a number of separable signals are listed in Nuttall (1958a), e.g., Gaussian processes, random binary processes and several types of modulated processes. In addition, it is shown in McGraw and Wagner (1968) that signals with *elliptically symmetric distributions* are separable and they have also characterized these signals further.

A result related to separable processes can be found in Balakrishnan (1960). There, it is shown that processes for which conditional expectations like $\mathrm{E}(u(t-\sigma)|u(t))$ are of a specified form have characteristic functions that satisfy certain equations.

3.2.2 Results for Hammerstein and Wiener Systems

Separable processes in general, and Gaussian processes in particular, have turned out to be very useful for system identification of Wiener-Hammerstein systems (cf. Section 2.2). The reason for this is that the use of a separable input signal makes it possible to estimate the linear part of a Hammerstein system without compensating for the nonlinearity at the input. The corresponding result holds also for Wiener-Hammerstein systems with Gaussian inputs. Some results about identification of Wiener, Hammerstein or Wiener-Hammerstein systems using Gaussian signals can, for example, be found in Billings and Fakhouri (1978, 1982), Korenberg (1985), Hunter and Korenberg (1986) and Bendat (1998). Gaussian signals are not the only signals that have been studied in this context. For example, it is also possible to estimate the linear and nonlinear parts of a Wiener or Hammerstein system separately when the input is sinusoidal (Bai, 2003a,b). The case of a Hammerstein system with a pseudo-random binary input signal is studied in Bai (2004).

In this thesis, we will discuss linearization of Hammerstein systems in some detail (see Chapter 6). This system structure is common in many real-life applications and it is thus natural that identification of Hammerstein systems has been an active research field for quite some time. A brief overview of some of the existing methods can be found in Bai and Li (2004).

One popular method for identification of Hammerstein systems was originally proposed in Narendra and Gallman (1966) and has been studied in many papers (see, for example, Stoica, 1981; Bai and Li, 2004). In this method, a Hammerstein model with independent parameterizations of the linear and the nonlinear submodels is estimated by minimizing a quadratic cost function iteratively starting from an initial estimate, or guess, of the parameter values. In each step, the cost function is minimized with respect only to one of its arguments while using the previous value of the other one. Often, an LTI approximation of the system is first estimated without considering the nonlinearity at the input and the parameters from this approximate model are used as parameters of the linear submodel when the first estimate of the nonlinearity is computed in the next step. For example, the initial LTI model can be obtained by estimating an OE model using the prediction-error method (see Section 2.3.1).

The iterative approach guarantees that the cost function will be monotonically decreasing over the iterations. Furthermore, the convergence of the parameter estimates to the true parameter values can be shown in some special cases (Bai and Li, 2004). However, there are no such convergence results available for the general case. Hence, it is important that the initial LTI approximation is as good as possible in the sense that it resembles the true LTI subsystem as well as possible.

For a separable input signal, the iterative approach can be motivated by the fact that the invariance property holds, since this property implies that the LTI model that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$ will be equal to a scaled version of the LTI part of the system.

For nonseparable input signals, the estimation of the initial LTI model requires more attention. Some results concerning this problem for Wiener and Hammerstein systems with random multisine inputs can be found in Crama and Schoukens (2001, 2004). The approach has also been generalized to Hammerstein-Wiener and Wiener-Hammerstein systems in Crama and Schoukens (2004) and Crama and Schoukens (2005), respectively. All these papers concern random multisine inputs and use a nonparametric frequency response function estimate as the very first LTI model estimate. Some related material can also be found in Vandersteen et al. (1997) and Vandersteen and Schoukens (1999).

3.2.3 Results for General Nonlinear Systems

Bussgang's theorem has been generalized to functions of several variables (Atalik and Utku, 1976; Scarano et al., 1993; Lutes and Sarkani, 1997, Chap. 9). This generalization can be used to characterize optimal linear approximations of NFIR systems, as will be shown later in Chapter 7. However, the generalized version of Bussgang's theorem has been used mainly for linear approximations of continuous-time nonlinear state-space systems in the field of stochastic mechanical vibrations. This linearization approach is there known under names such as equivalent linearization, statistical linearization and stochastic linearization and it can, for example, be used to compute an approximation of the variance of the output from a nonlinear system.

LTI approximations of nonlinear systems for stochastic signals are often studied in a mean-square error framework. In this case, the optimal LTI model of a nonlinear system with input $u(t)$ and output $y(t)$ can be defined as the stable model $G_0(q)$ that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$. It is a well-known result that this optimal approximation can be written

$$G_0(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

and it is sometimes called the *equivalent linear model* (see, for example, Gardner, 1986, p. 382). The model $G_0(z)$ can also be called the *noncausal Wiener filter*, since it in general will be a stable but noncausal LTI system. In this thesis, we will also call it the *noncausal LTI-SOE* (cf. Chapter 5).

If the LTI approximation is assumed to be causal, it will in general not be equal to the ratio between $\Phi_{yu}(z)$ and $\Phi_u(z)$. Instead, it can be derived using techniques for causal Wiener filtering (see, for example, Kailath et al., 2000, Chap. 7). The mean-square error optimal stable and *causal* LTI approximation of a nonlinear system has been discussed in Schetzen (1980, p. 330) and is also the main topic of this thesis.

Optimal LTI approximations are also discussed in Pintelon and Schoukens (2001). There, the term *related dynamic system* is used for the mean-square error optimal LTI model and the part of the output signal that this model cannot explain is viewed as a *nonlinear distortion*. This name is natural since the effects of unmodeled nonlinearities on nonparametric frequency response estimates can be similar to the effects of measurement noise. For some special classes of input signals, including random multisines, a number of interesting properties of related linear systems can be derived. For example, the asymptotic behavior of the related dynamic system when the number of excited frequencies for a random multisine input signal tends to infinity has been studied. It turns out that such a signal in many ways is similar to a Gaussian signal when the number of frequency components is large.

Fundamental results about, for example, related dynamic systems, nonparametric frequency response estimates and random multisines can be found in Schoukens et al. (1998), Schoukens et al. (2001), Pintelon et al. (2001), Schoukens et al. (2002), Pintelon and Schoukens (2002), Pintelon et al. (2003) and Schoukens et al. (2005a). These results have been applied to various identification problems. In Schoukens et al. (2004a), the risk for instability in nonlinear closed-loop systems is investigated, while closed-loop identification is discussed in Schoukens et al. (2005b). An approach where approximate models consisting of an LTI part, a Wiener part, a Hammerstein part and a Wiener-Hammerstein part are estimated is described in Schoukens et al. (2003). Furthermore, some input design issues are discussed in Vanhoenacker et al. (2001) and Vanhoenacker and Schoukens (2003). Related results can also be found in Dobrowiecki and Schoukens (2001), Vandersteen et al. (2001) and Schoukens et al. (2004b).

In this chapter, an overview of some existing linearization approaches has been given. These approaches deal either with deterministic or stochastic signals and have rather different properties. In this thesis, we will consider only LTI approximations for stochastic signals and these approximations will be described in the next chapter.

Part I

LTI-SOEs

4

The Notion of LTI Second Order Equivalents

The previous chapter contained an overview of the various linearization frameworks that can be found in the literature. All these frameworks have their benefits and drawbacks depending on which type of linearization is desired. In this thesis, the main objective is to understand the behavior of the prediction-error method when the measured input and output signals come from a nonlinear system.

Hence, with the discussion from Section 2.3 about the asymptotic properties of the prediction-error method in mind, it is here natural to study linear approximations of nonlinear systems that are optimal in the mean-square error sense. Such approximations will be called *LTI Second Order Equivalents* (LTI-SOEs) of the nonlinear systems. This chapter contains both detailed derivations of two types of LTI-SOEs and some interpretations of these approximations. First, some restrictions on the input and output signals will be imposed in order to make the LTI-SOEs well-defined.

4.1 Assumptions on the Input and Output Signals

Since the class of nonlinear systems literally contains all kinds of systems, it is too general to be studied as a whole. In many cases, explicit restrictions on the considered types of nonlinear systems are introduced in order to enable further analyses of the properties of these systems. Examples of explicit restrictions are that the nonlinear systems should have finite gain, finite memory or some kind of stability property.

Also in this thesis, the class of nonlinear systems has to be restricted. However, we will not impose any explicit restrictions but instead assume that the input and output signals of the nonlinear systems have certain properties. These signal assumptions are listed here and impose implicit restrictions on the class of nonlinear systems that will be studied in the sequel.

Assumption A1. Assume that

(i) The input $u(t)$ is a real-valued stationary stochastic process with

$$\mathrm{E}(u(t)) = 0.$$

(ii) There exist $K > 0$ and α, $0 < \alpha < 1$ such that the second order moment $R_u(\tau) = \mathrm{E}(u(t)u(t-\tau))$, satisfies

$$|R_u(\tau)| < K\alpha^{|\tau|}, \quad \forall \tau \in \mathbb{Z}.$$

(iii) The z-spectrum $\Phi_u(z)$ (i.e., the z-transform of $R_u(\tau)$) has a unique canonical spectral factorization

$$\Phi_u(z) = L(z) r_u L(z^{-1}), \tag{4.1}$$

where $L(z)$ and $1/L(z)$ are causal transfer functions that are analytic in the set $\{z \in \mathbb{C} \mid |z| \geq 1\}$, $L(\infty) \triangleq \lim_{|z|\to\infty} L(z) = 1$ and r_u is a positive constant.

Assumption A2. Assume that

(i) The output $y(t)$ is a real-valued stationary stochastic process with

$$\mathrm{E}(y(t)) = 0.$$

(ii) There exist $K > 0$ and α, $0 < \alpha < 1$ such that the second order moments $R_{yu}(\tau) = \mathrm{E}(y(t)u(t-\tau))$ and $R_y(\tau) = \mathrm{E}(y(t)y(t-\tau))$ satisfy

$$|R_{yu}(\tau)| < K\alpha^{|\tau|}, \quad \forall \tau \in \mathbb{Z},$$
$$|R_y(\tau)| < K\alpha^{|\tau|}, \quad \forall \tau \in \mathbb{Z}.$$

In this thesis, Assumptions A1 and A2 are the standard assumptions on the input and output signals. However, some signals that do not satisfy these assumptions will be discussed too. For these signals, the following assumptions will be used.

Assumption A3 (Periodic input). Assume that

(i) The input $u(t)$ and output $y(t)$ are real-valued stationary stochastic processes with

$$\mathrm{E}(u(t)) = \mathrm{E}(y(t)) = 0.$$

(ii) Assume that $u(t)$ is P-periodic for some $P \in \mathbb{Z}_+$, i.e., that

$$u(t+P) = u(t), \quad \forall t \in \mathbb{Z}.$$

(iii) Assume that the second order moments $R_u(\tau) = \mathrm{E}(u(t)u(t-\tau))$ and $R_{yu}(\tau) = \mathrm{E}(y(t)u(t-\tau))$ exist and are P-periodic.

In Assumptions A1(i), A2(i) and A3(i) it is required that both the input and the output signal have zero mean. In practice, this assumption does not exclude systems with input and output signals that vary around a nonzero set point from being analyzed using the results in this thesis. For such systems, it is always possible to define new input and output signals that describe the deviations from the set point by subtracting the corresponding means of the two signals. By their construction, these new signals will have zero mean and they will hence satisfy the zero mean assumption in this thesis.

Assumptions A1(ii) and A2(ii) imply that the second order moments of $u(t)$ and $y(t)$ are bounded and that the z-spectra $\Phi_u(z)$ and $\Phi_y(z)$ and the z-cross-spectrum $\Phi_{yu}(z)$ converge absolutely and are analytic in the annulus $\{z \in \mathbb{C} \mid \alpha < |z| < \frac{1}{\alpha}\}$. It should be mentioned that the canonical spectral factorization in Assumption A1(iii) exists for all rational $\Phi_u(z)$ without zeros on the unit circle (Kailath et al., 2000, pp. 198-199). For a rational $\Phi_u(z)$, the conditions on $L(z)$ in Assumption A1(iii) mean just that it should be a monic minimum phase filter. However, we will not restrict ourselves only to rational spectra here. The following example shows that also a nonrational z-spectrum can have a canonical spectral factorization.

Example 4.1

The z-spectrum

$$\Phi_u(z) = e^{1/z+z}$$

can be factorized as $\Phi_u(z) = L(z)r_uL(z^{-1})$ with $L(z) = e^{1/z}$ and $r_u = 1$. The causal series expansions

$$L(z) = e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k},$$

$$L^{-1}(z) = e^{-1/z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^{-k}$$

converge absolutely and are analytic for $\{z \in \mathbb{C} \mid |z| > 0\}$ and $L(\infty) = 1$. Hence, $L(z)r_uL(z^{-1})$ is the unique canonical spectral factorization of the nonrational z-spectrum $\Phi_u(z)$.

Assumptions A1 and A2 give sufficient conditions for the type of LTI-SOE that will be defined in Section 4.2 to be well-defined. However, for the second type of LTI-SOE that will be introduced in Section 4.4 we will need another assumption. Let $\zeta(t) = (u(t)\ y(t-1))^T$. Then it follows that

$$R_\zeta(\tau) = \begin{pmatrix} R_u(\tau) & R_{uy}(\tau+1) \\ R_{yu}(\tau-1) & R_y(\tau) \end{pmatrix}, \tag{4.2a}$$

$$\Phi_\zeta(z) = \begin{pmatrix} \Phi_u(z) & z\Phi_{uy}(z) \\ z^{-1}\Phi_{yu}(z) & \Phi_y(z) \end{pmatrix}. \tag{4.2b}$$

Assumption A4. Assume that the signals $u(t)$ and $y(t)$ fulfill Assumptions A1 and A2 and that they also are such that $\Phi_\zeta(z)$ in (4.2b) has a unique canonical spectral factorization

$$\Phi_\zeta(z) = T(z)Q_\zeta T^T(z^{-1}),$$

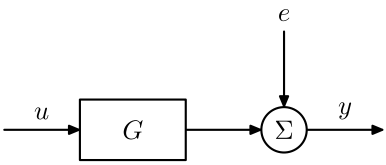


Figure 4.1: *The output error model.*

where $T(z)$ and $T^{-1}(z)$ are analytic in $\{z \in \mathbb{C} \mid |z| \geq 1\}$, $T(\infty) = I$ and Q_ζ is a positive definite matrix, i.e., $Q_\zeta \succ 0$. (Here, T^{-1} denotes the *matrix inverse*.)

Also the existence of a canonical factorization of a matrix-valued z-spectrum is guaranteed if, for example, $\Phi_\zeta(z)$ is a rational matrix without unit circle zeros (Kailath et al., 2000, p. 205). Assumptions A1, A2 and A4 will be used in the derivations of LTI-SOEs in the next section and in Section 4.4 and will be the standard assumptions throughout this thesis.

4.2 The Output Error Model Type

In Section 2.1 it was mentioned that a general LTI model can be written as

$$y(t) = G(q)u(t) + H(q)e(t).$$

In general, both G and H in this model can be transfer functions of any order. Here, however, at first we will only consider models where H is fixed to 1, i.e., *output error* models (Ljung, 1999). If G is causal this implies that only the input components

$$u(t), u(t-1), u(t-2), \dots$$

are used to predict the output $y(t)$ according to (2.2). The structure of an output error model is shown in Figure 4.1.

Using only output error models, the mean-square error optimal LTI approximation of a certain nonlinear system is simply the stable and causal LTI model $G_{0,OE}$ that minimizes $\mathrm{E}((y(t) - G(q)u(t))^2)$. This model is often called the Wiener filter for prediction of $y(t)$ from $(u(t-k))_{k=0}^{\infty}$ (Wiener, 1949). However, we will not use the term Wiener filter here, but instead call $G_{0,OE}$ the *Output Error LTI Second Order Equivalent* (OE-LTI-SOE) of the nonlinear system. Hence, we have the following definition.

Definition 4.1. Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. The *Output Error LTI Second Order Equivalent* (OE-LTI-SOE) of this system is the stable and causal LTI model $G_{0,OE}(q)$ that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$, i.e.,

$$G_{0,OE}(q) = \arg\min_{G \in \mathcal{G}} \mathrm{E}((y(t) - G(q)u(t))^2),$$

where $\mathcal{G}$ denotes the set of all stable and causal LTI models.

There are two main reasons for the change of name from the commonly used Wiener filter to OE-LTI-SOE. First, we want to avoid any ambiguities. Many different Wiener filters can be constructed for a given pair of input and output signals. Here, however, we are only interested in the Wiener filter that predicts $y(t)$ from $(u(t-k))_{k=0}^{\infty}$.

The second reason for the change of name is that we want to emphasize that the OE-LTI-SOE is an equivalent to the nonlinear system in the sense that it can explain the causal part of the cross-covariance function $R_{yu}(\tau)$ between the input and the output of the system. This observation, which is rather obvious for OE-LTI-SOEs (see Corollary 4.2), becomes more interesting if we study LTI models that contain a general error description, i.e., models with $H \neq 1$. LTI-SOEs for this case will be defined later in Section 4.4 and will be called *General Error LTI Second Order Equivalent* (GE-LTI-SOE). In Section 4.4, it will be shown that GE-LTI-SOEs can explain both the covariance function $R_y(\tau)$ and the cross-covariance function $R_{yu}(\tau)$. Hence, such a model is an equivalent to the nonlinear system in the sense that it is impossible to distinguish it from the true system if only second order properties of the input, output and model residuals are considered. This will be discussed in more detail in Section 4.4 (see the comments to Corollary 4.4 on page 56).

It should be noted that we are not only interested in the filtering and prediction capabilities of the OE-LTI-SOE (and the GE-LTI-SOE), but also in the model itself. For example, we are not only interested in how good an estimate of $y(t)$ the model can produce, but also in issues like how the model order and model coefficients depend on the nonlinear system and on the input signal.

It should immediately be pointed out that the OE-LTI-SOE of a nonlinear system depends on which input signal is used. Hence, we can only talk about the OE-LTI-SOE of a nonlinear system with respect to a particular input signal. The fact that the OE-LTI-SOE is input-dependent is natural if we view it as an example of undermodeling. Undermodeling occurs when a system is approximated by a model of lower complexity. An often studied example of undermodeling is when an LTI system is approximated with an LTI model of lower order than the true system. Actually, also in this case of linear undermodeling the mean-square error optimal approximation is input-dependent (Ljung, 1999, Sec. 8.5).

Another property of OE-LTI-SOEs that should be pointed out, besides their input-dependency, is that they are assumed to be initialized at $t = -\infty$ such that all transients have died out. In Definition 4.1 it is assumed that the complete infinite sequence

$$(u(t-k))_{k=0}^{\infty}$$

of random variables is available for the computation of an estimate of $y(t)$.

The following theorem is a direct consequence of classic Wiener filter theory, and the proof of the theorem is almost identical to the derivation of the scalar Wiener filter in Kailath et al. (2000, pp. 231-234). It is included here for the sake of completeness.

Theorem 4.1 (OE-LTI-SOEs)
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Then the OE-LTI-SOE $G_{0,OE}$ of this system is

$$G_{0,OE}(z) = \frac{1}{r_u L(z)} \left[\frac{\Phi_{yu}(z)}{L(z^{-1})} \right]_{\text{causal}}, \tag{4.3}$$

where $[\ldots]_{\text{causal}}$ *denotes taking the causal part (see Section 2.1), and where* $L(z)$ *is the canonical spectral factor of* $\Phi_u(z)$ *from (4.1).*

Proof: The criterion $\mathrm{E}((y(t) - G(q)u(t))^2)$ that $G_{0,OE}$ should minimize is equivalent to the *Wiener-Hopf condition*

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_{0,OE}(k) R_u(\tau - k) = 0, \quad \tau \geq 0 \tag{4.4}$$

or, alternatively, to

$$\Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) = K_{OE}(z), \tag{4.5}$$

where $K_{OE}(z)$ is a stable and strictly anticausal transfer function. The Wiener-Hopf criterion follows from the fact that for the optimal model, the error

$$y(t) - G_{0,OE}(q)u(t)$$

should be orthogonal to $\{u(t-k)\}_{k=0}^{\infty}$ (using the inner-product $\langle u, v\rangle = \mathrm{E}(uv)$). Using the spectral factorization according to (4.1) and multiplying by $L^{-1}(z^{-1})$ now gives

$$\tilde{K}_{OE}(z) \triangleq K_{OE}(z)L^{-1}(z^{-1}) = \Phi_{yu}(z)L^{-1}(z^{-1}) - G_{0,OE}(z)L(z)r_u, \tag{4.6}$$

where $\tilde{K}_{OE}(z)$ is a stable and strictly anticausal transfer function due to the fact that it is a product of the stable and strictly anticausal transfer function $K_{OE}(z)$ and the stable and anticausal transfer function $L^{-1}(z^{-1})$. For (4.6) to hold it is necessary that the second right hand term, which is a stable and causal transfer function, is equal to the causal part of the first, i.e.,

$$G_{0,OE}(z)L(z)r_u = \left[\frac{\Phi_{yu}(z)}{L(z^{-1})}\right]_{\text{causal}} \tag{4.7}$$

and (4.3) follows. □

In general, the OE-LTI-SOE has to be calculated as in (4.3), which means that the canonical spectral factor $L(z)$ of the input z-spectrum has to be obtained. However, in some cases this is not necessary and the OE-LTI-SOE can be calculated using a simplified expression. This is shown in the following corollary.

Corollary 4.1
Consider a nonlinear system with input $u(t)$ *and output* $y(t)$ *such that Assumptions A1 and A2 are fulfilled, and assume that the ratio* $\Phi_{yu}(z)/\Phi_u(z)$ *defines a stable and causal LTI system. Then*

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}. \tag{4.8}$$

Proof: Assume that

$$C(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

is a stable and causal transfer function. Then

$$\Phi_{yu}(z) = C(z)\Phi_u(z) = C(z)L(z)r_uL(z^{-1})$$

and (4.3) gives

$$G_{0,OE}(z) = \frac{1}{r_u L(z)} \left[\frac{C(z)L(z)r_u L(z^{-1})}{L(z^{-1})} \right]_{\text{causal}} = C(z),$$

since $C(z)L(z)r_u$ is a stable and causal transfer function. □

The OE-LTI-SOE of a system will be called *regular* if (4.8) holds. Hence, we have the following definition.

Definition 4.2. An OE-LTI-SOE $G_{0,OE}(z)$ is *regular* if it can be written

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}. \tag{4.9}$$

The Wiener-Hopf condition (4.4) implies that the model residuals for the OE-LTI-SOE are uncorrelated with past and current inputs. This is stated more clearly in the following corollary.

Corollary 4.2
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Let the residuals be defined by

$$\eta_0(t) = y(t) - G_{0,OE}(q)u(t). \tag{4.10}$$

Then

$$\Phi_{\eta_0 u}(z) = \Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) \quad \textit{is strictly anticausal} \tag{4.11}$$

and

$$R_{\eta_0}(0) = R_y(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega})\, d\omega. \tag{4.12}$$

Proof: The result (4.11) follows directly from (4.5). Furthermore, we can write

$$\begin{aligned}
\Phi_{\eta_0}(z) &= \Phi_y(z) - \Phi_{yu}(z)G_{0,OE}(z^{-1}) - G_{0,OE}(z)\Phi_{uy}(z) \\
&\quad + G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) \\
&= \Phi_y(z) - \left(\Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z)\right) G_{0,OE}(z^{-1}) \\
&\quad - G_{0,OE}(z)\left(\Phi_{uy}(z) - \Phi_u(z)G_{0,OE}(z^{-1})\right) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}).
\end{aligned}$$

Since $G_{0,OE}(z^{-1})$ is anticausal and since $\Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z)$ by (4.11) is strictly anticausal, it follows that

$$\int_{-\pi}^{\pi} \left(\Phi_{yu}(e^{i\omega}) - G_{0,OE}(e^{i\omega})\Phi_u(e^{i\omega})\right) G_{0,OE}(e^{-i\omega})\, d\omega = 0$$

and that

$$\int_{-\pi}^{\pi} G_{0,OE}(e^{i\omega})\left(\Phi_{uy}(e^{i\omega}) - \Phi_u(e^{i\omega})G_{0,OE}(e^{-i\omega})\right)\, d\omega = 0.$$

Because

$$R_{\eta_0}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta_0}(e^{i\omega})\, d\omega$$

we get

$$R_{\eta_0}(0) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(e^{i\omega})\, d\omega}_{=R_y(0)} - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega})\, d\omega.$$

□

Corollary 4.2 shows that the OE-LTI-SOE really is the best output error model of the true system as the remaining residuals $\eta_0(t)$ are uncorrelated with past and present input signal components.

An example of the OE-LTI-SOE of a simple nonlinear system can be found below. Note that the OE-LTI-SOE is nonstatic in this example although the nonlinear system is static.

Example 4.2

Consider the static nonlinear system

$$y(t) = u(t)^3 \tag{4.13}$$

with the input

$$u(t) = e(t) + \frac{1}{2}e(t-1),$$

where $e(t)$ is a sequence of independent random variables with uniform distribution over the interval $[-1, 1]$. Hence, $\mathrm{E}(e(t)^2) = \frac{1}{3}$ and $\mathrm{E}(e(t)^4) = \frac{1}{5}$ and, using the fact that $e(t)$ and $e(t-1)$ are independent, we get

$$\begin{aligned}
R_{yu}(0) &= \mathrm{E}(u(t)^4) = \mathrm{E}(e(t)^4) + \frac{6}{4}\mathrm{E}(e(t)^2 e(t-1)^2) + \frac{1}{16}\mathrm{E}(e(t-1)^4) \\
&= \frac{1}{5} + \frac{6}{4}\cdot\frac{1}{9} + \frac{1}{16}\cdot\frac{1}{5} = \frac{91}{240}, \\
R_{yu}(1) &= \mathrm{E}(u(t)^3 u(t-1)) \\
&= \mathrm{E}\Big(\Big(e(t)^3 + \frac{3}{2}e(t)^2 e(t-1) + \frac{3}{4}e(t)e(t-1)^2 + \frac{1}{8}e(t-1)^3\Big)\cdot \\
&\quad \cdot\Big(e(t-1) + \frac{1}{2}e(t-2)\Big)\Big) = \frac{3}{2}\mathrm{E}(e(t)^2 e(t-1)^2) + \frac{1}{8}\mathrm{E}(e(t-1)^4) \\
&= \frac{3}{2}\cdot\frac{1}{9} + \frac{1}{8}\cdot\frac{1}{5} = \frac{46}{240},
\end{aligned}$$

$$
\begin{aligned}
R_{yu}(-1) &= \mathrm{E}(u(t)^3 u(t+1)) \\
&= \mathrm{E}(\left(e(t)^3 + \frac{3}{2}e(t)^2 e(t-1) + \frac{3}{4}e(t)e(t-1)^2 + \frac{1}{8}e(t-1)^3\right) \cdot \\
&\quad \cdot \left(e(t+1) + \frac{1}{2}e(t)\right)) = \frac{1}{2}\mathrm{E}(e(t)^4) + \frac{3}{8}\mathrm{E}(e(t)^2 e(t-1)^2) \\
&= \frac{1}{2}\cdot\frac{1}{5} + \frac{3}{8}\cdot\frac{1}{9} = \frac{34}{240}, \\
R_{yu}(\tau) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}.
\end{aligned}
$$

This gives

$$\Phi_{yu}(z) = \frac{1}{240}(34z + 91 + 46z^{-1}) = \frac{1}{240}\left(1 + \frac{1}{2}z\right)\left(68 + 46z^{-1}\right).$$

Furthermore, Lemma 2.1 gives

$$\Phi_u(z) = \left(1 + \frac{1}{2}z^{-1}\right) \cdot \frac{1}{3} \cdot \left(1 + \frac{1}{2}z\right) = \frac{1}{12}\left(2z + 5 + 2z^{-1}\right)$$

and hence

$$\frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{1}{40} \cdot \frac{34 + 23z^{-1}}{1 + \frac{1}{2}z^{-1}}.$$

Since the ratio $\Phi_{yu}(z)/\Phi_u(z)$ here is stable and causal, the OE-LTI-SOE of the system (4.13) for this input is

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{1}{40} \cdot \frac{34 + 23z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{0.85 + 0.575z^{-1}}{1 + 0.5z^{-1}}.$$

Note that although the nonlinear system is static the OE-LTI-SOE is not.

The fact that a static nonlinear system can have a nonstatic OE-LTI-SOE is a first indication that LTI approximations of nonlinear systems in a mean-square error framework are not as straightforward as one might expect. More examples of this will be presented later in this thesis. In Chapters 6 and 7 we will also discuss classes of input signals which guarantee that the output of the OE-LTI-SOE will depend on the same number of input signal components as the nonlinear system.

For most systems, the order of the OE-LTI-SOE is unknown. In practice, this implies that several output error models have to be estimated and that a validation procedure has to be used in order to find the best model. Naturally, there is no guarantee that the correct order of the OE-LTI-SOE will be found. As a matter of fact, the OE-LTI-SOE can sometimes be an infinite order model. Hence, it is interesting to characterize in what sense an output error model with lower order than the OE-LTI-SOE approximates the OE-LTI-SOE.

This is a relevant question also when the true system is an LTI system. In that case, it can be shown that a low order model will approximate the true system mainly for frequencies where $\Phi_u(e^{i\omega})$ is large (Ljung, 1999, p. 266). As a matter of fact, this result holds also when the true system is nonlinear. In this case, a low order output error model will approximate the OE-LTI-SOE instead of the true system as well as possible for frequencies where $\Phi_u(e^{i\omega})$ is large according to the following theorem. This theorem is basically a special case of Theorem 4.1 in Ljung (2001) and the proof is very similar to the outlined proof in Problem 8G.5 in Ljung (1999).

Theorem 4.2
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Let $G_{0,OE}$ be the corresponding OE-LTI-SOE according to Theorem 4.1. Suppose that a parameterized stable and causal output error model $G(q,\theta)$ is fitted to the signals u and y according to

$$\hat{\theta} = \arg\min_{\theta} \mathrm{E}(\eta(t,\theta)^2), \tag{4.14}$$

where

$$\eta(t,\theta) = y(t) - G(q,\theta)u(t). \tag{4.15}$$

Then it follows that

$$\hat{\theta} = \arg\min_{\theta} \int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta)|^2 \Phi_u(e^{i\omega})\, d\omega. \tag{4.16}$$

Proof: The z-spectrum of $\eta(t,\theta)$ is

$$\begin{aligned}
\Phi_\eta(z,\theta) &= \begin{pmatrix} -G(z,\theta) & 1 \end{pmatrix} \begin{pmatrix} \Phi_u(z) & \Phi_{uy}(z) \\ \Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} \begin{pmatrix} -G(z^{-1},\theta) \\ 1 \end{pmatrix} \\
&= \Phi_y(z) - G(z,\theta)\Phi_{uy}(z) - G(z^{-1},\theta)\Phi_{yu}(z) + G(z,\theta)\Phi_u(z)G(z^{-1},\theta) \\
&= \left(G(z,\theta) - \frac{\Phi_{yu}(z)}{\Phi_u(z)} \right) \Phi_u(z) \left(G(z^{-1},\theta) - \frac{\Phi_{yu}(z^{-1})}{\Phi_u(z^{-1})} \right) \\
&\quad - \frac{\Phi_{yu}(z)\Phi_{yu}(z^{-1})}{\Phi_u(z)} + \Phi_y(z).
\end{aligned}$$

Let

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\Phi_y(e^{i\omega}) - \frac{|\Phi_{yu}(e^{i\omega})|^2}{\Phi_u(e^{i\omega})} \right) d\omega, \\
B_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\Phi_{\eta_0 u}(e^{i\omega})|^2}{\Phi_u(e^{i\omega})}\, d\omega,
\end{aligned}$$

where η_0 is the residual signal defined in (4.10). Parseval's relation gives

$$\mathrm{E}(\eta(t,\theta)^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_\eta(e^{i\omega},\theta)\, d\omega$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{\Phi_{yu}(e^{i\omega})}{\Phi_u(e^{i\omega})} - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega})\,d\omega + A_0 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| G_{0,OE}(e^{i\omega}) + \frac{\Phi_{\eta_0 u}(e^{i\omega})}{\Phi_u(e^{i\omega})} - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega})\,d\omega + A_0 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega})\,d\omega \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{i\omega})(G_{0,OE}(e^{-i\omega}) - G(e^{-i\omega},\theta))\,d\omega \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{-i\omega})(G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta))\,d\omega + A_0 + B_0,
\end{aligned}
$$

where we have used (4.11) in the third equality. Since $\Phi_{\eta_0 u}(z)$ by (4.11) is strictly anti-causal and since $G_{0,OE}(z)$ and $G(z,\theta)$ both are causal, a term-by-term integration shows that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{i\omega})(G_{0,OE}(e^{-i\omega}) - G(e^{-i\omega},\theta))\,d\omega = 0, \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{-i\omega})(G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta))\,d\omega = 0.
\end{aligned}
$$

Thus

$$
\mathrm{E}(\eta(t,\theta)^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega})\,d\omega + A_0 + B_0
$$

and (4.16) follows as A_0 and B_0 are independent of θ. □

Theorem 4.2 shows that a low order output error model approximation of an OE-LTI-SOE results in the same kind of approximation as a low order approximation of an LTI system. More specifically, (4.16) shows that if $\Phi_u(e^{i\omega})$ is large in a certain frequency region, the parameter vector θ will be chosen such that

$$
|G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta)|
$$

is small in that frequency region.

However, it is important to remember that there is a major difference between the linear and the nonlinear cases. If the true system is an LTI system, it is always desirable to approximate it as well as possible, at least for some frequencies. On the other hand, if the system is nonlinear, there is no guarantee that the OE-LTI-SOE is a good model of the system for any other input signals than the one that it was defined for. Actually, it might be a bad model also for this signal.

If, for example, a second order output error model is estimated and the input power is focused in a certain frequency region, the model will in general approximate a *different* OE-LTI-SOE than if, for example, a white input signal had been used. Actually, input signals with equal $\Phi_u(z)$ but different distributions will in general give rise to different OE-LTI-SOEs of a nonlinear system. A simple example of this will be shown later in Section 5.4. These observations make it much harder to design the input such that it is suitable for low order LTI approximations when the system is nonlinear.

4.3 The Output Error Model Type for Periodic Inputs

For nonlinear systems with periodic inputs satisfying Assumption A3, OE-LTI-SOEs can be defined using the following alternative definition.

Definition 4.3. Consider a nonlinear system with a P-periodic stochastic input $u(t)$ and an output $y(t)$ such that Assumption A3 is fulfilled. An OE-LTI-SOE of this system is a stable and causal LTI model $G_{0,OE}(q)$ that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$, i.e.,

$$G_{0,OE}(q) = \arg\min_{G \in \mathcal{G}} \mathrm{E}((y(t) - G(q)u(t))^2),$$

where $\mathcal{G}$ denotes the set of all stable and causal LTI models. Let $\mathcal{G}_{0,OE}$ denote the set of all OE-LTI-SOEs for this particular pair of input and output signals, i.e.,

$$\mathcal{G}_{0,OE} = \{G_{0,OE}(q)\}.$$

Note that $\mathcal{G}_{0,OE}$ always will contain more than one model. For example, consider a system with an OE-LTI-SOE $G_{0,OE,0}(q)$ for a particular P-periodic input $u(t)$. Then the models $G_{0,OE,k}(q) = (1 + q^{-kP})G_{0,OE,0}(q)/2$, $k \in \mathbb{N}$, are OE-LTI-SOEs too, since they will produce the same stationary output as $G_{0,OE,0}(q)$. Obviously, the impulse responses from these OE-LTI-SOEs are quite different, but this does not matter here since the transient response of a model is not considered in the definition of the OE-LTI-SOE.

The fact that the OE-LTI-SOE of a nonlinear system with a periodic input is not unique does not imply that there are not any uniquely determined features in these models. These unique features are most easy to describe in the frequency domain. Hence, we will now rewrite the mean-square error in the frequency domain in a similar way as the cost function was rewritten in Section 2.3.2. Consider a positive integer M and a P-periodic input signal and an output such that Assumption A3 is satisfied. Applying the Discrete Fourier Transform (DFT) to the input and output signal gives

$$U_N(n) = \sum_{t=0}^{N-1} u(t)e^{-i2\pi nt/N}, \tag{4.17a}$$

$$Y_N(n) = \sum_{t=0}^{N-1} y(t)e^{-i2\pi nt/N}, \tag{4.17b}$$

where $N = MP$. Note that we here consider stochastic processes, i.e., $U_N(n)$ and $Y_N(n)$ are random variables. Let $\hat{y}_G(t)$ denote the output from the stable model $G(q)$ for the input $u(t)$ and assume that the input has been applied at $t = -\infty$ such that all transients are gone at $t \geq 0$, i.e., that $\hat{y}_G(t)$ is P-periodic in the interval $0 \leq t \leq N-1$. Furthermore, let $\hat{Y}_{G,N}(n)$ denote the DFT of $\hat{y}_G(t)$, i.e.,

$$\hat{Y}_{G,N}(n) = \sum_{t=0}^{N-1} \hat{y}_G(t) e^{-i2\pi nt/N}.$$

Using the same reasoning as in (2.10), we obtain that

$$G(e^{i2\pi n/N}) = \tilde{G}_N(n), \tag{4.18}$$

where $\tilde{G}_N(n)$ is the DFT of

$$\tilde{g}_N(t) = \sum_{l=0}^{\infty} g(t+lN), \quad 0 \leq t \leq N-1.$$

Since $u(t)$ is a P-periodic signal and $N = MP$, with $M \in \mathbb{Z}_+$, $u(t)$ is also N-periodic. Hence,

$$\hat{y}_G(t) = G(q)u(t) = \sum_{k=0}^{N-1} \tilde{g}(k)u(t-k)$$

and this implies that

$$\hat{Y}_{G,N}(n) = \tilde{G}_N(n)U_N(n) = G(e^{i2\pi n/N})U_N(n), \tag{4.19}$$

where we have used (4.18) in the last equality.

Since both the input and the output are stationary stochastic processes, the mean-square error can be rewritten as

$$\begin{aligned}
\mathrm{E}((y(t) - G(q)u(t))^2) &= \mathrm{E}(\frac{1}{N}\sum_{t=0}^{N-1}(y(t) - G(q)u(t))^2) \\
&= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}\Big|Y_N(n) - \hat{Y}_{G,N}(n)\Big|^2) \\
&= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}\Big|Y_N(n) - G(e^{i2\pi n/N})U_N(n)\Big|^2),
\end{aligned} \tag{4.20}$$

where we have used Parseval's formula in the second equality and (4.19) in the third. Using this expression for the mean-square error, it is easy to show how the frequency response of the OE-LTI-SOE can be calculated.

Theorem 4.3 (OE-LTI-SOEs for Periodic Inputs)
Consider a nonlinear system with a P-periodic input $u(t)$ and output $y(t)$ such that Assumption A3 is fulfilled and a positive integer M. Then an OE-LTI-SOE $G_{0,OE}$ of this system has a frequency response with the property

$$G_{0,OE}(e^{i2\pi n/N}) = \frac{\mathrm{E}(Y_N(n)\overline{U_N(n)})}{\mathrm{E}(|U_N(n)|^2)}, \quad n \in \Omega, \tag{4.21}$$

where $\Omega \subset \{0, 1, \ldots, N-1\} \triangleq \mathbb{Z}_N$ *is the set of integers for which* $\mathrm{E}(|U_N(n)|^2) \neq 0$ *and where* $N = MP$.

Proof: Schwarz inequality for random variables (Råde and Westergren, 1995, p. 407) can be used to show that $\mathrm{E}(|U_N(n)|^2) = 0$ implies that

$$\mathrm{E}(Y_N(n)\overline{U_N(n)}) = \mathrm{E}(U_N(n)\overline{Y_N(n)}) = 0.$$

Let Ω_c denote the set of integers in $\mathbb{Z}_N$ that does not belong to Ω, i.e., the complement of Ω in $\mathbb{Z}_N$. The mean-square error (4.20) can be rewritten as

$$\begin{aligned}\mathrm{E}((y(t) - G(q)u(t))^2) &= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}\left|Y_N(n) - G(e^{i2\pi n/N})U_N(n)\right|^2) \\ = \mathrm{E}(\frac{1}{N^2}\sum_{n\in\Omega}\left|Y_N(n) - G(e^{i2\pi n/N})U_N(n)\right|^2) &+ \mathrm{E}(\frac{1}{N^2}\sum_{n\in\Omega_c}|Y_N(n)|^2).\end{aligned}$$

The condition that $G_{0,OE}$ should minimize this criterion is equivalent to that $G_{0,OE}$ should satisfy the Wiener-Hopf condition

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_{0,OE}(k)R_u(\tau - k) = 0, \quad \tau \geq 0 \tag{4.22}$$

or, alternatively, that

$$\mathrm{E}(Y_N(n)\overline{U_N(n)}) - G_{0,OE}(e^{i2\pi n/N})\mathrm{E}(|U_N(n)|^2) = 0, \quad n \in \Omega. \tag{4.23}$$

This equivalence follows from a similar orthogonality argument as the Wiener-Hopf condition followed from in the proof of Theorem 4.1. The expression for the OE-LTI-SOE in (4.21) follows readily from (4.23). □

Note that since $R_{yu}(\tau)$ and $R_u(\tau)$ are P-periodic, the Wiener-Hopf criterion (4.22) is satisfied if

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_{0,OE}(k)R_u(\tau - k) = 0, \quad 0 \leq \tau \leq P - 1.$$

Actually, the periodicity implies that (4.22) is equivalent to the criterion

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_{0,OE}(k)R_u(\tau - k) = 0, \quad \forall \tau \in \mathbb{Z}. \tag{4.24}$$

Theorem 4.3 shows that the frequency response of an OE-LTI-SOE of a nonlinear system with a periodic input signal is uniquely determined only at the excited frequencies $\omega_n = 2\pi n/N$, $n \in \Omega$. The frequency response of the OE-LTI-SOE is arbitrary at all other frequencies. Of course, this is not a surprising observation but merely a version of a classic result from system identification literature (Ljung, 1999, Chap. 8).

However, Theorem 4.3 gives a theoretical motivation to why a nonparametric frequency response estimate can be particularly useful when the input is periodic. For example, if N_E experiments have been performed with different realizations of the input signal, the nonparametric frequency response estimate

$$\hat{G}_{np}(n) = \frac{\sum_{s=1}^{N_E} Y_{s,N}(n)\overline{U_{s,N}(n)}}{\sum_{s=1}^{N_E} |U_{s,N}(n)|^2} = \frac{\frac{1}{N_E}\sum_{s=1}^{N_E} Y_{s,N}(n)\overline{U_{s,N}(n)}}{\frac{1}{N_E}\sum_{s=1}^{N_E} |U_{s,N}(n)|^2} \tag{4.25}$$

from (2.12) can be calculated at the excited frequencies. If N_E is large, this estimate will typically be a good approximation of the expression for the OE-LTI-SOE in (4.21). Hence, by calculating a nonparametric frequency response estimate based on data from several experiments, approximations of the properties that are common to all OE-LTI-SOEs of the system can be obtained.

In the previous section, it was shown that the model residuals for the OE-LTI-SOE are uncorrelated with past and current inputs but in general not with future input components. However, for a periodic input, the residuals will actually be uncorrelated with all input components. This observation is summarized in the following corollary.

Corollary 4.3
Consider a nonlinear system with a P-periodic input $u(t)$ and output $y(t)$ such that Assumption A3 is fulfilled, an OE-LTI-SOE $G_{0,OE}$ of this system and a positive integer M. Let the residuals be defined by

$$\eta_0(t) = y(t) - G_{0,OE}(q)u(t) \tag{4.26}$$

Then

$$\begin{aligned}\mathrm{E}(H_{0,N}(n)\overline{U_N(n)}) &= \mathrm{E}(Y_N(n)\overline{U_N(n)}) - G_{0,OE}(e^{i2\pi n/N})\mathrm{E}(|U_N(n)|^2)\\ &= 0, \quad \forall n \in \Omega, &(4.27a)\\ R_{\eta_0 u}(\tau) &= R_{yu}(\tau) - G_{0,OE}(q)R_u(\tau) = 0, &(4.27b)\end{aligned}$$

where $H_{0,N}(n)$ is the DFT of η_0, $N = MP$ and Ω defines the excited frequencies. Furthermore, the variance of the residuals can be calculated as

$$R_{\eta_0}(0) = R_y(0) - \frac{1}{N^2}\sum_{n=0}^{N-1} |G_{0,OE}(e^{i2\pi n/N})|^2 \mathrm{E}(|U_N(n)|^2). \tag{4.28}$$

Proof: The results (4.27a) and (4.27b) follow directly from (4.23) and (4.24), respectively. The expression for the variance of η_0 can be shown using (4.21). Using Parseval's identity, we get

$$\begin{aligned}R_{\eta_0}(0) &= \mathrm{E}(\frac{1}{N}\sum_{t=0}^{N-1}(y(t) - G_{0,OE}(q)u(t))^2)\\ &= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}|Y_N(n) - G_{0,OE}(e^{i2\pi n/N})U_N(n)|^2)\end{aligned}$$

$$= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}(|Y_N(n)|^2 - Y_N(n)\overline{U_N(n)G_{0,OE}(e^{i2\pi n/N})}$$

$$- G_{0,OE}(e^{i2\pi n/N})U_N(n)\overline{Y_N(n)} + |G_{0,OE}(e^{i2\pi n/N})U_N(n)|^2))$$

$$= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}|Y_N(n)|^2) - \frac{1}{N^2}\sum_{n=0}^{N-1}|G_{0,OE}(e^{i2\pi n/N})|^2\mathrm{E}(|U_N(n)|^2),$$

where we have used (4.21) and the fact that $\mathrm{E}(|U_N(n)|^2) = 0$ implies that

$$\mathrm{E}(Y_N(n)\overline{U_N(n)}) = \mathrm{E}(U_N(n)\overline{Y_N(n)}) = 0$$

in the last equality. Since

$$R_y(0) = \mathrm{E}(\frac{1}{N}\sum_{t=0}^{N-1}y(t)^2) = \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}|Y_N(n)|^2),$$

(4.28) has been shown. □

Although the results (4.27) in the previous corollary are rather obvious, they show an important difference between OE-LTI-SOEs for nonperiodic inputs and OE-LTI-SOEs for periodic inputs. However, it will be shown later that many nonperiodic signals also give OE-LTI-SOEs with residual signals that are uncorrelated with the input signal.

If a parametric model is calculated by minimizing the variance of the residuals, it will approximate the frequency response of the OE-LTI-SOE according to the following theorem.

Theorem 4.4
Consider a nonlinear system with a P-periodic input $u(t)$ and output $y(t)$ such that Assumption A3 is fulfilled, an OE-LTI-SOE $G_{0,OE}$ of this system and a positive integer M. Suppose that a parameterized stable and causal output error model $G(q,\theta)$ is fitted to the signals u and y according to

$$\hat{\theta} = \arg\min_{\theta} \mathrm{E}(\eta(t,\theta)^2), \tag{4.29}$$

where

$$\eta(t,\theta) = y(t) - G(q,\theta)u(t) \tag{4.30}$$

Then it follows that

$$\hat{\theta} = \arg\min_{\theta}\sum_{n=0}^{N-1}|G_{0,OE}(e^{i2\pi n/N}) - G(e^{i2\pi n/N},\theta)|^2\mathrm{E}(|U_N(n)|^2). \tag{4.31}$$

Proof: The variance of the model residual can be written

$$\begin{aligned}
\mathrm{E}(\eta(t,\theta)^2) &= \mathrm{E}(\frac{1}{N}\sum_{t=0}^{N-1}(y(t)-G(q,\theta)u(t))^2) \\
&= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}|Y_N(n)-G(e^{i2\pi n/N},\theta)U_N(n)|^2) \\
&= \mathrm{E}(\frac{1}{N^2}\sum_{n=0}^{N-1}|H_{0,N}(n)+(G_{0,OE}(e^{i2\pi n/N})-G(e^{i2\pi n/N},\theta))U_N(n)|^2) \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\mathrm{E}(|H_{0,N}(n)|^2) \\
&\quad + \frac{1}{N^2}\sum_{n=0}^{N-1}|G_{0,OE}(e^{i2\pi n/N})-G(e^{i2\pi n/N},\theta)|^2\mathrm{E}(|U_N(n)|^2),
\end{aligned}$$

where $H_{0,N}(n)$ is the DFT of $\eta_0(t)$ in (4.26) and where we have used (4.27a) in the last equality. The result (4.31) follows since

$$\frac{1}{N^2}\sum_{n=0}^{N-1}\mathrm{E}(|H_{0,N}(n)|^2)$$

is independent of θ. □

Theorem 4.4 shows that if a parametric model is calculated by minimizing the variance of the model residuals, this model will try to approximate the frequency response of the OE-LTI-SOE at the excited frequencies. Note that there is no guarantee that $\hat{\theta}$ will be unique. If the order of the parametric model is higher than the lowest order OE-LTI-SOE, there will be many optimal values for $\hat{\theta}$. However, in this sense, there is no difference between the results for nonperiodic and periodic inputs. No matter the type of input, if the order of the parametric model is higher than for the OE-LTI-SOE, the optimal parametric model will not be unique.

More properties of OE-LTI-SOEs will be presented in Chapter 5. First, we will turn our attention to LTI-SOEs that contain a noise model.

4.4 The General Error Model Type

Consider once again the general LTI model (2.1) from Section 2.1. For this model, the optimal predictor can be written as in (2.2). Predictors with this structure are used in the prediction-error method to compute parameter dependent predictions of the output signal of the system (cf. Section 2.3). The predicted output is compared with the measured output and the parameters are selected such that a quadratic criterion is minimized.

With this in mind, it is natural to define the best general LTI model in the mean-square error sense as the LTI model whose predictor (2.2) minimizes the mean-square error. The optimal LTI model according to this definition will here be called the *General Error LTI*

Second Order Equivalent (GE-LTI-SOE). It is described more clearly in the following definition.

Definition 4.4. Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A4 are fulfilled. The *General Error LTI Second Order Equivalent* (GE-LTI-SOE) of this system is the pair of transfer operators $(G_{0,GE}(q),\ H_{0,GE}(q))$ that are formed as

$$G_{0,GE}(q) = (1 - q^{-1}\tilde{W}_y(q))^{-1}\tilde{W}_u(q),$$
$$H_{0,GE}(q) = (1 - q^{-1}\tilde{W}_y(q))^{-1}.$$

Here, $\tilde{W}_u(q)$ and $\tilde{W}_y(q)$ are the stable and causal LTI filters that minimize the mean-square error $\mathrm{E}((y(t) - W_u(q)u(t) - W_y(q)y(t-1))^2)$, i.e.,

$$(\tilde{W}_u(q),\ \tilde{W}_y(q)) = \underset{W_u, W_y \in \mathcal{G}}{\arg\min}\ \mathrm{E}((y(t) - W_u(q)u(t) - W_y(q)y(t-1))^2),$$

where $\mathcal{G}$ denotes the set of all stable and causal LTI models.

GE-LTI-SOEs are, just like OE-LTI-SOEs, input-dependent and hence it is only possible to talk about the GE-LTI-SOE of a nonlinear system with respect to a particular input signal. Another similarity between GE-LTI-SOEs and OE-LTI-SOEs is that $\tilde{W}_u(q)$ and $\tilde{W}_y(q)$ are assumed to be initialized at $t = -\infty$.

In the next theorem, expressions for the GE-LTI-SOE similar to the ones in Ljung (2001) will be derived. The only differences between these two versions of GE-LTI-SOE expressions are that the GE-LTI-SOE here is allowed to contain a direct term from the input and that it is expressed explicitly using components of the canonical spectral factor $T(z)$ from Assumption A4. The proof of the theorem is very similar to the proof of Theorem 4.1 and hence to the proof of the scalar Wiener filter in Kailath et al. (2000).

Theorem 4.5 (GE-LTI-SOEs)
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A4 are fulfilled. Then the GE-LTI-SOE $(G_{0,GE}(z),\ H_{0,GE}(z))$ of this system is

$$G_{0,GE}(z) = \frac{zT_{21}(z)}{T_{11}(z)}, \tag{4.32a}$$
$$H_{0,GE}(z) = \frac{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}{T_{11}(z)}, \tag{4.32b}$$

where $T_{11}(z)$, $T_{12}(z)$, $T_{21}(z)$ and $T_{22}(z)$ are elements of the canonical spectral factor of the z-spectrum for $\zeta(t) = (u(t),\ y(t-1))^T$, i.e.,

$$\Phi_\zeta(z) = \begin{pmatrix} \Phi_u(z) & z\Phi_{uy}(z) \\ z^{-1}\Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} = T(z)Q_\zeta T^T(z^{-1}), \tag{4.33a}$$
$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{pmatrix}. \tag{4.33b}$$

Proof: The GE-LTI-SOE is defined by means of the Wiener filter $(\tilde{W}_u(q),\ \tilde{W}_y(q))$ that predicts $y(t)$ from $(y(t-k))_{k=1}^{\infty}$ and $(u(t-k))_{k=0}^{\infty}$ using

$$\hat{y}(t) = \tilde{W}_u(q)u(t) + \tilde{W}_y(q)y(t-1). \tag{4.34}$$

The filter components $\tilde{W}_u(q)$ and $\tilde{W}_y(q)$ are defined as the stable and causal LTI filters that minimize $\mathrm{E}((y(t) - W_u(q)u(t) - W_y(q)y(t-1))^2)$ or, equivalently, the filters that satisfy the *Wiener-Hopf conditions*

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} \tilde{w}_y(k)R_{yu}(\tau-k-1) - \sum_{k=0}^{\infty} \tilde{w}_u(k)R_u(\tau-k) = 0, \quad \tau \geq 0, \tag{4.35a}$$

$$R_y(\tau+1) - \sum_{k=0}^{\infty} \tilde{w}_y(k)R_y(\tau-k) - \sum_{k=0}^{\infty} \tilde{w}_u(k)R_{uy}(\tau+1-k) = 0, \quad \tau \geq 0. \tag{4.35b}$$

Using the z-transform, these conditions can be rewritten as

$$\Phi_{yu}(z) - \tilde{W}_y(z)z^{-1}\Phi_{yu}(z) - \tilde{W}_u(z)\Phi_u(z) = K_1(z), \tag{4.36a}$$

$$z\Phi_y(z) - \tilde{W}_y(z)\Phi_y(z) - \tilde{W}_u(z)z\Phi_{uy}(z) = K_2(z), \tag{4.36b}$$

where $K_1(z)$ and $K_2(z)$ are stable and strictly anticausal transfer functions. Equations (4.36) and (4.2b) give

$$\begin{pmatrix} K_1(z) & K_2(z) \end{pmatrix} = \underbrace{\begin{pmatrix} \Phi_{yu}(z) & z\Phi_y(z) \end{pmatrix}}_{=(0\ z)\Phi_\zeta(z)} - \begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} \Phi_\zeta(z). \tag{4.37}$$

Using the spectral factorization of $\Phi_\zeta(z)$ and multiplying by $T^{-T}(z^{-1})$ now gives

$$\tilde{K}(z) \triangleq \begin{pmatrix} K_1(z) & K_2(z) \end{pmatrix} T^{-T}(z^{-1}) = \begin{pmatrix} 0 & z \end{pmatrix} T(z)Q_\zeta - \begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} T(z)Q_\zeta, \tag{4.38}$$

where $\tilde{K}(z)$ is a stable and strictly anticausal transfer matrix, due to the fact that it is a product of the stable and strictly anticausal transfer matrix $(K_1(z)\ K_2(z))$ and the stable and anticausal transfer matrix $T^{-T}(z^{-1})$. For (4.38) to hold it is necessary that the second right hand term, which is a stable and causal transfer matrix, is equal to the causal part of the first, i.e.,

$$\begin{aligned}
&\begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} T(z)Q_\zeta = \left[\begin{pmatrix} 0 & z \end{pmatrix} T(z)Q_\zeta\right]_{\text{causal}} \\
&= \left[\begin{pmatrix} zT_{21}(z)Q_{\zeta 11} + zT_{22}(z)Q_{\zeta 21} & zT_{21}(z)Q_{\zeta 12} + zT_{22}(z)Q_{\zeta 22} \end{pmatrix}\right]_{\text{causal}} \\
&= \begin{pmatrix} zT_{21}(z)Q_{\zeta 11} + z(T_{22}(z)-1)Q_{\zeta 21} & zT_{21}(z)Q_{\zeta 12} + z(T_{22}(z)-1)Q_{\zeta 22} \end{pmatrix} \\
&= \left(\begin{pmatrix} 0 & z \end{pmatrix} T(z) - \begin{pmatrix} 0 & z \end{pmatrix}\right) Q_\zeta,
\end{aligned}$$

where the third equality follows since $T_{21}(z)$ is a stable and strictly causal transfer function while $T_{22}(z)$ is a monic stable and causal transfer function. This gives

$$\begin{aligned}
\begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} &= \left(\begin{pmatrix} 0 & z \end{pmatrix} T(z) - \begin{pmatrix} 0 & z \end{pmatrix}\right) T^{-1}(z) \\
&= \begin{pmatrix} \frac{zT_{21}(z)}{T_{11}(z)T_{22}(z)-T_{12}(z)T_{21}(z)} & z\left(1 - \frac{T_{11}(z)}{T_{11}(z)T_{22}(z)-T_{12}(z)T_{21}(z)}\right) \end{pmatrix}.
\end{aligned} \tag{4.39}$$

Let

$$\varepsilon_0(t) = y(t) - \hat{y}(t) = (1 - q^{-1}\tilde{W}_y(q))y(t) - \tilde{W}_u(q)u(t) \tag{4.40}$$

and rewrite this expression in analogy with (2.1) and (2.2) as

$$y(t) = (1 - q^{-1}\tilde{W}_y(q))^{-1}\tilde{W}_u(q)u(t) + (1 - q^{-1}\tilde{W}_y(q))^{-1}\varepsilon_0(t). \tag{4.41}$$

With

$$G_{0,GE}(q) = (1 - q^{-1}\tilde{W}_y(q))^{-1}\tilde{W}_u(q),$$
$$H_{0,GE}(q) = (1 - q^{-1}\tilde{W}_y(q))^{-1},$$

(4.41) can be written as

$$y(t) = G_{0,GE}(q)u(t) + H_{0,GE}(q)\varepsilon_0(t). \tag{4.42}$$

Hence, using (4.39) the GE-LTI-SOE of the nonlinear system turns out to be

$$G_{0,GE}(z) = \frac{zT_{21}(z)}{T_{11}(z)},$$
$$H_{0,GE}(z) = \frac{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}{T_{11}(z)}.$$

□

From (4.32) we see that the calculation of a GE-LTI-SOE requires knowledge of the canonical spectral factorization of the matrix $\Phi_\zeta(z)$. Although there exists a number of methods for making this factorization, it is definitely more complicated to perform than the canonical factorization of the input z-spectrum that is required for the calculation of the OE-LTI-SOE in Theorem 4.1. In Section 5.6, we will describe some classes of input signals that simplify the calculation of GE-LTI-SOEs significantly.

It should be noted that the factor $\frac{1}{T_{11}(z)}$ in (4.32) should as usual be interpreted as a causal series expansion. This is always possible because $T_{11}(z)$ is analytic on and outside the unit circle and $T_{11}(\infty) = 1$. Let $H_{0,GE}^{-1}$ denote the transfer function of the inverse model of $H_{0,GE}$, i.e.,

$$H_{0,GE}^{-1}(z) = \frac{1}{H_{0,GE}(z)}. \tag{4.44}$$

By the construction of the GE-LTI-SOE, $H_{0,GE}^{-1}(z)G_{0,GE}(z)$ and $H_{0,GE}^{-1}(z)$ will be stable transfer functions. However, neither $G_{0,GE}(z)$ nor $H_{0,GE}(z)$ needs to be stable since $T_{11}(z)$ might have zeros outside the unit circle. The only thing that is guaranteed is that all unstable poles to $G_{0,GE}(z)$ are also poles to $H_{0,GE}(z)$ and vice versa.

In the Section 4.2, it was shown that the OE-LTI-SOE of a nonlinear system with a nonperiodic input can explain all correlations between the output and past and present input signal components (see Corollary 4.2). As the inclusion of a noise model in the GE-LTI-SOE makes this model more flexible than the OE-LTI-SOE, one might expect that the GE-LTI-SOE should be able to explain more correlations than the OE-LTI-SOE. In the following theorem it will be shown that this is also the case.

Corollary 4.4
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A4 are fulfilled. Let $(G_{0,GE}, H_{0,GE})$ be the corresponding GE-LTI-SOE according to Theorem 4.5 and let $\varepsilon_0(t)$ be defined by (4.40). Then it holds that

$$\begin{aligned}
\Phi_{\varepsilon_0 u}(z) &= H_{0,GE}^{-1}(z)(\Phi_{yu}(z) - G_{0,GE}(z)\Phi_u(z)) \\
&= zQ_{\zeta 21}T_{11}(z^{-1}) + zQ_{\zeta 22}T_{12}(z^{-1}), && (4.45a)\\
\Phi_{\varepsilon_0 y}(z) &= H_{0,GE}^{-1}(z)(\Phi_y(z) - G_{0,GE}(z)\Phi_{uy}(z)) \\
&= Q_{\zeta 21}T_{21}(z^{-1}) + Q_{\zeta 22}T_{22}(z^{-1}), && (4.45b)\\
\Phi_{\varepsilon_0}(z) &= H_{0,GE}^{-1}(z)(\Phi_{y\varepsilon_0}(z) - G_{0,GE}(z)\Phi_{u\varepsilon_0}(z)) = Q_{\zeta 22} \triangleq \lambda_0. && (4.45c)
\end{aligned}$$

Furthermore, an alternative way to describe the relations between the z-spectra of u, y and ε_0 by the GE-LTI-SOE is

$$\begin{aligned}
\Phi_{yu}(z) &= G_{0,GE}(z)\Phi_u(z) + H_{0,GE}(z)\Phi_{\varepsilon_0 u}(z), && (4.46a)\\
\Phi_{y\varepsilon_0}(z) &= G_{0,GE}(z)\Phi_{u\varepsilon_0}(z) + H_{0,GE}(z)\lambda_0, && (4.46b)\\
\Phi_y(z) &= \begin{pmatrix} G_{0,GE}(z) & H_{0,GE}(z) \end{pmatrix} \begin{pmatrix} \Phi_u(z) & \Phi_{u\varepsilon_0}(z) \\ \Phi_{\varepsilon_0 u}(z) & \lambda_0 \end{pmatrix} \begin{pmatrix} G_{0,GE}(z^{-1}) \\ H_{0,GE}(z^{-1}) \end{pmatrix}. && (4.46c)
\end{aligned}$$

Proof: Note that (4.40) can be written

$$\varepsilon_0(t) = H_{0,GE}^{-1}(q)(y(t) - G_{0,GE}(q)u(t)). \qquad (4.47)$$

This gives

$$\begin{aligned}
\Phi_{\varepsilon_0 u}(z) &= H_{0,GE}^{-1}(z)(\Phi_{yu}(z) - G_{0,GE}(z)\Phi_u(z)),\\
\Phi_{\varepsilon_0 y}(z) &= H_{0,GE}^{-1}(z)(\Phi_y(z) - G_{0,GE}(z)\Phi_{uy}(z)).
\end{aligned}$$

Together with (4.32) and (4.33a) these expressions can be rewritten as

$$\begin{aligned}
&\begin{pmatrix} z^{-1}\Phi_{\varepsilon_0 u}(z) & \Phi_{\varepsilon_0 y}(z) \end{pmatrix}\\
&= H_{0,GE}^{-1}(z) \begin{pmatrix} z^{-1}\Phi_{yu}(z) - z^{-1}G_{0,GE}(z)\Phi_u(z) & \Phi_y(z) - G_{0,GE}(z)\Phi_{uy}(z) \end{pmatrix}\\
&= \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)} \begin{pmatrix} -\frac{T_{21}(z)}{T_{11}(z)} & 1 \end{pmatrix} \Phi_\zeta(z) = \begin{pmatrix} 0 & 1 \end{pmatrix} T^{-1}(z)\Phi_\zeta(z)\\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} Q_\zeta T^T(z^{-1}) = \begin{pmatrix} Q_{\zeta 21} & Q_{\zeta 22} \end{pmatrix} T^T(z^{-1})\\
&= \begin{pmatrix} Q_{\zeta 21}T_{11}(z^{-1}) + Q_{\zeta 22}T_{12}(z^{-1}) & Q_{\zeta 21}T_{21}(z^{-1}) + Q_{\zeta 22}T_{22}(z^{-1}) \end{pmatrix}.
\end{aligned}$$

Hence, (4.45a) and (4.45b) have been shown. Furthermore, (4.47) gives, using (4.32), (4.45a) and (4.45b), the following expression

$$\begin{aligned}
\Phi_{\varepsilon_0}(z) &= H_{0,GE}^{-1}(z)(\Phi_{y\varepsilon_0}(z) - G_{0,GE}(z)\Phi_{u\varepsilon_0}(z)) \\
&= \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}(Q_{\zeta 21}T_{21}(z) + Q_{\zeta 22}T_{22}(z) \\
&\quad - \frac{zT_{21}(z)}{T_{11}(z)}z^{-1}\left(Q_{\zeta 21}T_{11}(z) + Q_{\zeta 22}T_{12}(z)\right)) \\
&= \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}\left(Q_{\zeta 22}T_{22}(z) - Q_{\zeta 22}\frac{T_{12}(z)T_{21}(z)}{T_{11}(z)}\right) \\
&= Q_{\zeta 22}
\end{aligned}$$

and (4.45c) follows.

The expressions (4.45a) and (4.45c) can be rewritten as (4.46a) and (4.46b), respectively. Finally, (4.46c) follows if (4.46a) and (4.46b) are inserted in (4.45b). □

The main result of Corollary 4.4 is that $\Phi_{\varepsilon_0 u}(z)$ and $\Phi_{\varepsilon_0 y}(z)$ are strictly anticausal and anticausal, respectively, and that $\Phi_{\varepsilon_0}(z)$ is a constant. Hence, (4.45) illustrates that the GE-LTI-SOE really is the best possible LTI model of the nonlinear system as the remaining residuals $\varepsilon_0(t)$ are uncorrelated with past outputs, past and present inputs and with residuals at all other time instants.

In addition, Corollary 4.4 explains why the name *LTI Second Order Equivalent* is natural. The alternative version of (4.45) in (4.46) emphasizes the filtering capabilities of $G_{0,GE}$ and $H_{0,GE}$. As a matter of fact, (4.46) shows that the GE-LTI-SOE is impossible to distinguish from the true nonlinear system only by looking at second order properties of y, u and ε_0. The LTI system $(G_{0,GE}(q),\ H_{0,GE}(q))$ is thus equivalent to the nonlinear system for the input in question if only second order properties are considered, hence the name GE-LTI-SOE. The additional *General Error* in the name GE-LTI-SOE is added in order to distinguish this type of LTI model from the previously described output error model type, which does not include a noise model.

A fundamental observation about LTI-SOEs is that the OE-LTI-SOE $G_{0,OE}$ and the GE-LTI-SOE $G_{0,GE}$ are not always equal. This is shown in Example 4.3. The particular system and input signal used in this example have been taken from Forssell and Ljung (2000), where it is used to show that $\Phi_{yu}(z)/\Phi_u(z)$ can be noncausal. Here, however, we will also derive the GE-LTI-SOE of this system.

Example 4.3

Consider the static nonlinear system

$$y(t) = u(t)^2 - 3 \tag{4.48}$$

with the input

$$u(t) = e(t) + e(t-1)^2 - 1,$$

where $e(t)$ here is a white Gaussian process with zero mean and unit variance. Straightforward calculations (see Appendix A), which are similar to the ones in Example 4.2, give

$$\Phi_u(z) = 3,$$
$$\Phi_{yu}(z) = 2z + 8,$$
$$\Phi_y(z) = 8z + 66 + 8z^{-1}.$$

This gives

$$\Phi_\zeta(z) = \begin{pmatrix} 3 & 2+8z \\ 2+8z^{-1} & 8z+66+8z^{-1} \end{pmatrix} = T(z)Q_\zeta T^T(z^{-1}),$$
$$T(z) = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{4161}-33}{12}z^{-1} & 1+\frac{65-\sqrt{4161}}{8}z^{-1} \end{pmatrix},$$
$$Q_\zeta = \begin{pmatrix} 3 & 2 \\ 2 & 23+\frac{\sqrt{4161}}{3} \end{pmatrix}.$$

The spectral factor $T(z)$ has been computed by a diagonalization of $\Phi_\zeta(z)$ followed by a factorization of the derived diagonal matrix and an adjustment in order to achieve $T(\infty) = I$. The complete derivation of $T(z)$ can be found in Appendix A. From the derived transform expressions, the OE-LTI-SOE of the system (4.48) for this input is found to be

$$G_{0,OE}(z) = \frac{8}{3} \approx 2.6667,$$

while the GE-LTI-SOE is

$$G_{0,GE}(z) = \frac{\sqrt{4161}-33}{12} \approx 2.6255,$$
$$H_{0,GE}(z) = 1 + \frac{65-\sqrt{4161}}{8}z^{-1}.$$

Note that $G_{0,OE}(z) \neq G_{0,GE}(z)$ despite the fact that the system operates in open loop.

The fact that $G_{0,OE}(z) \neq G_{0,GE}(z)$ for some open-loop nonlinear systems and inputs is the reason why a matrix-valued spectral factorization in general has to be performed when the GE-LTI-SOE is calculated. If $G_{0,OE}(z)$ and $G_{0,GE}(z)$ always would have been equal, the GE-LTI-SOE could have been calculated using only scalar spectral factorizations. In Section 5.6 it will be shown that such a simplified calculation of the GE-LTI-SOE is possible for some classes of input signals.

Just like in the case of OE-LTI-SOEs, it is in practice often hard to know the correct order of the GE-LTI-SOE. As a matter of fact, it might actually be infinite dimensional. Hence, it is also here interesting to understand in what sense a low order model can approximate the GE-LTI-SOE of a nonlinear system. Also for GE-LTI-SOEs, this approximation is similar to a low order approximation of an LTI system, something which is shown in the next theorem. Apart from the fact that a GE-LTI-SOE is here allowed to contain a direct term from the input, this theorem is identical to Theorem 4.1 in Ljung (2001). The proof is, just like for Theorem 4.2, similar to the outlined proof in Problem 8G.5 in Ljung (1999).

Theorem 4.6
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A4 are fulfilled. Let $(G_{0,GE}, H_{0,GE})$ be the corresponding GE-LTI-SOE according to Theorem 4.5, let $\varepsilon_0(t)$ be defined by (4.40) and assume that both $G_{0,GE}$ and $H_{0,GE}$ are stable. Consider a parameterized stable and causal model $(G(q,\theta),\ H(q,\theta))$, where $H(q,\theta)$ is monic and where $H^{-1}(q,\theta)$ is stable and causal. Suppose that this model is fit to the signals u and y according to

$$\hat{\theta} = \arg\min_{\theta} \mathrm{E}(\varepsilon(t,\theta)^2), \tag{4.49}$$

where

$$\begin{aligned}\varepsilon(t,\theta) &= y(t) - H^{-1}(q,\theta)G(q,\theta)u(t) - (1 - H^{-1}(q,\theta))y(t)\\ &= H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)).\end{aligned} \tag{4.50}$$

Then it follows that

$$\hat{\theta} = \arg\min_{\theta} \int_{-\pi}^{\pi} \Delta_{GH}(e^{i\omega},\theta)^T \begin{pmatrix} \Phi_u(e^{i\omega}) & \Phi_{u\varepsilon_0}(e^{i\omega}) \\ \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0 \end{pmatrix} \Delta_{GH}(e^{-i\omega},\theta)\, d\omega, \tag{4.51}$$

where

$$\Delta_{GH}(z,\theta) = \frac{1}{H(z,\theta)} \begin{pmatrix} G_{0,GE}(z) - G(z,\theta) \\ H_{0,GE}(z) - H(z,\theta) \end{pmatrix}. \tag{4.52}$$

Proof: Equation (4.42) can be used to rewrite (4.50) as

$$\begin{aligned}\varepsilon(t,\theta) &= H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t))\\ &= H^{-1}(q,\theta)(G_{0,GE}(q)u(t) + H_{0,GE}(q)\varepsilon_0(t) - G(q,\theta)u(t))\\ &= H^{-1}(q,\theta)((G_{0,GE}(q) - G(q,\theta))u(t) + (H_{0,GE}(q) - H(q,\theta))\varepsilon_0(t))\\ &\quad + \varepsilon_0(t) = \left(\Delta_{GH}(q,\theta)^T + \begin{pmatrix} 0 & 1 \end{pmatrix}\right) \begin{pmatrix} u(t) \\ \varepsilon_0(t) \end{pmatrix}.\end{aligned}$$

This gives

$$\begin{aligned}\Phi_\varepsilon(z,\theta) &= \left(\Delta_{GH}(z,\theta)^T + \begin{pmatrix} 0 & 1 \end{pmatrix}\right) \begin{pmatrix} \Phi_u(z) & \Phi_{u\varepsilon_0}(z) \\ \Phi_{\varepsilon_0 u}(z) & \lambda_0 \end{pmatrix} \left(\Delta_{GH}(z^{-1},\theta) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\\ &= \Delta_{GH}(z,\theta)^T \begin{pmatrix} \Phi_u(z) & \Phi_{u\varepsilon_0}(z) \\ \Phi_{\varepsilon_0 u}(z) & \lambda_0 \end{pmatrix} \Delta_{GH}(z^{-1},\theta)\\ &\quad + \Delta_{GH}(z,\theta)^T \begin{pmatrix} \Phi_{u\varepsilon_0}(z) \\ \lambda_0 \end{pmatrix} + \begin{pmatrix} \Phi_{\varepsilon_0 u}(z) & \lambda_0 \end{pmatrix} \Delta_{GH}(z^{-1},\theta) + \lambda_0.\end{aligned}$$

Parseval's relation gives

$$
\begin{aligned}
\mathrm{E}(\varepsilon(t,\theta)^2) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_\varepsilon(e^{i\omega},\theta)d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_{GH}(e^{i\omega},\theta)^T \begin{pmatrix} \Phi_u(e^{i\omega}) & \Phi_{u\varepsilon_0}(e^{i\omega}) \\ \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0 \end{pmatrix} \Delta_{GH}(e^{-i\omega},\theta)\, d\omega \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_{GH}(e^{i\omega},\theta)^T \begin{pmatrix} \Phi_{u\varepsilon_0}(e^{i\omega}) \\ \lambda_0 \end{pmatrix} d\omega \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \begin{pmatrix} \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0 \end{pmatrix} \Delta_{GH}(e^{-i\omega},\theta)\, d\omega + \lambda_0.
\end{aligned}
$$

The transfer function $H^{-1}(z,\theta)(G_{0,GE}(z)-G(z,\theta))$ is causal and, since both $H_{0,GE}(z)$ and $H(z,\theta)$ are monic, the transfer function

$$H^{-1}(z,\theta)(H_{0,GE}(z)-H(z,\theta))$$

is strictly causal. Since $\Phi_{u\varepsilon_0}(z)$ is strictly causal (cf. (4.45a)), this implies that a term-by-term integration gives

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_{GH}(e^{i\omega},\theta)^T \begin{pmatrix} \Phi_{u\varepsilon_0}(e^{i\omega}) \\ \lambda_0 \end{pmatrix} d\omega = 0, \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \begin{pmatrix} \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0 \end{pmatrix} \Delta_{GH}(e^{-i\omega},\theta)\, d\omega = 0.
\end{aligned}
$$

Hence,

$$\mathrm{E}(\varepsilon(t,\theta)^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_{GH}(e^{i\omega},\theta)^T \begin{pmatrix} \Phi_u(e^{i\omega}) & \Phi_{u\varepsilon_0}(e^{i\omega}) \\ \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0 \end{pmatrix} \Delta_{GH}(e^{-i\omega},\theta)\, d\omega + \lambda_0$$

and (4.51) follows since λ_0 is independent of θ. □

The previous theorem shows that the GE-LTI-SOE can be approximated by an LTI model of lower order than the GE-LTI-SOE according to (4.51). Just like for OE-LTI-SOEs, it is here important to remember that the GE-LTI-SOE is in general input-dependent. Hence, different input signals will make a parameterized LTI model close to *different* GE-LTI-SOEs.

Theorem 4.6 can be simplified if the true system actually is an open-loop LTI system. In this case, there is no correlation between the residuals and the input signal, i.e., $\Phi_{u\varepsilon_0}(z)$ is identically equal to zero (see Ljung, 1999, p. 265). However, for an open-loop nonlinear system this is not true in general.

Consider once again the system and input signal in Example 4.3. Using (4.45a) and the expressions for $T(z)$ and Q_ζ from Appendix A it is easy to verify that $\Phi_{u\varepsilon_0}(z) = 2z^{-1}$ in this case, despite the fact that the system in Example 4.3 is an open-loop system.

Without any prior knowledge about the structure of this system and using an intuitive reasoning based on the linear case, a nonzero strictly causal $\Phi_{u\varepsilon_0}(z)$ might have been taken as an indication that the system actually was a closed-loop system. Hence, it is a justified question whether such a closed-loop interpretation of a GE-LTI-SOE can always be made. This is the topic of the next section.

4.5 Interpretations of the GE-LTI-SOE

The fact that $\Phi_{u\varepsilon_0}(z)$ in general is nonzero in (4.46) has some implications on the LTI interpretation. If we would like to interpret all second order correlations between u, y and ε_0 as results of linear filter connections we have to allow the overall linear system to include some kind of feedback or feedforward connections. This is natural as we have not imposed any restrictions on the true system that exclude the possibility that it in fact is a linear closed-loop system.

It turns out that when the GE-LTI-SOE is well-defined according to Theorem 4.5 it is always possible to compose a stable linear closed-loop system that fulfills (4.46) and that also explains the correlation between $u(t)$ and past $\varepsilon_0(t-k)$, i.e., it explains why $\Phi_{u\varepsilon_0}(z) \neq 0$. Consider the closed-loop system in Figure 4.2. This system is described by

$$\begin{aligned} u(t) &= r_0(t) - F_0(q)N_0(q)\tilde{H}_0(q)\varepsilon_0(t) - F_0(q)N_0(q)\tilde{G}_0(q)u(t) \\ \Rightarrow u(t) &= \frac{1}{1+F_0(q)N_0(q)\tilde{G}_0(q)} r_0(t) + \frac{-F_0(q)N_0(q)\tilde{H}_0(q)}{1+F_0(q)N_0(q)\tilde{G}_0(q)} \varepsilon_0(t), \end{aligned} \tag{4.53a}$$

$$\begin{aligned} y(t) &= N_0(q)\tilde{H}_0(q)\varepsilon_0(t) + N_0(q)\tilde{G}_0(q)r_0(t) - N_0(q)\tilde{G}_0(q)F_0(q)y(t) \\ \Rightarrow y(t) &= \frac{N_0(q)\tilde{G}_0(q)}{1+F_0(q)N_0(q)\tilde{G}_0(q)} r_0(t) + \frac{N_0(q)\tilde{H}_0(q)}{1+F_0(q)N_0(q)\tilde{G}_0(q)} \varepsilon_0(t). \end{aligned} \tag{4.53b}$$

Let $N_0(z)$ be a causal, possibly unstable, LTI system such that both $G_{0,GE}(z)$ and $H_{0,GE}(z)$ can be factorized as

$$G_{0,GE}(z) = \tilde{G}_0(z)N_0(z), \tag{4.54a}$$

$$H_{0,GE}(z) = \tilde{H}_0(z)N_0(z), \tag{4.54b}$$

where $\tilde{G}_0$ and $\tilde{H}_0$ are causal and stable LTI systems. Furthermore, let

$$F_0(z) = -\frac{\Phi_{u\varepsilon_0}(z)}{\Phi_{y\varepsilon_0}(z)} \tag{4.55}$$

and, as usual, interpret the factor $\frac{1}{\Phi_{y\varepsilon_0}(z)}$ in this expression as a causal series expansion. This is always possible because $\Phi_{y\varepsilon_0}(z) = Q_{\zeta 21}T_{21}(z) + Q_{\zeta 22}T_{22}(z)$ is analytic on and outside the unit circle and $\Phi_{y\varepsilon_0}(\infty) = Q_{\zeta 22} > 0$ (see (4.45b)). Since $\Phi_{u\varepsilon_0}(z)$ is strictly

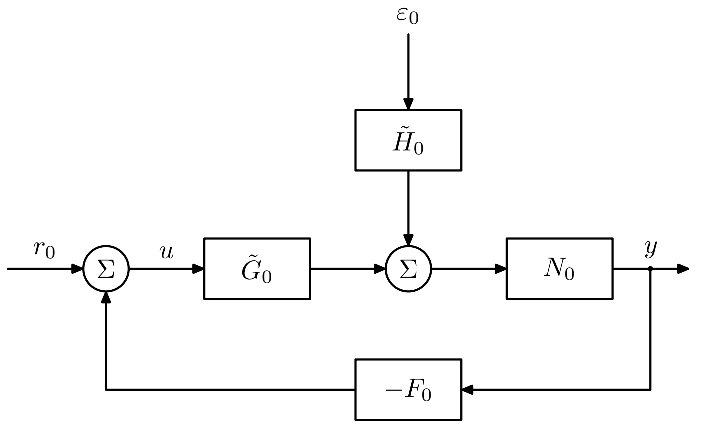


Figure 4.2: *The GE-LTI-SOE can be interpreted as being part of a linear closed-loop system.*

causal (see (4.45a)), also $F_0(z)$ will be strictly causal. Note, however, that $F_0(z)$ might be unstable.

The interpretation of the GE-LTI-SOE as a part of a linear closed-loop system is meaningless if this system is unstable. Hence, it is crucial to check whether the transfer functions in (4.53) are stable when the definitions (4.54) and (4.55) are used. First, (4.46b), (4.54a) and (4.55) give

$$1 + F_0(z)\underbrace{N_0(z)\tilde{G}_0(z)}_{=G_{0,GE}(z)} = \frac{\Phi_{y\varepsilon_0}(z) - G_{0,GE}(z)\Phi_{u\varepsilon_0}(z)}{\Phi_{y\varepsilon_0}(z)} = \frac{H_{0,GE}(z)\lambda_0}{\Phi_{y\varepsilon_0}(z)}.$$

Thus

$$\frac{1}{1 + F_0(z)N_0(z)\tilde{G}_0(z)} = \frac{\Phi_{y\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0} \tag{4.56}$$

is stable since $H_{0,GE}^{-1}(z)$ is stable and $\Phi_{y\varepsilon_0}(z)$ by its construction (4.45b) is analytic and hence absolutely convergent on the unit circle. Since $H_{0,GE}^{-1}(z)G_{0,GE}(z)$ is stable as well, the stability property of $\Phi_{y\varepsilon_0}(z)$ also imply that

$$\frac{N_0(z)\tilde{G}_0(z)}{1 + F_0(z)N_0(z)\tilde{G}_0(z)} = \frac{G_{0,GE}(z)\Phi_{y\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0},$$
$$\frac{N_0(z)\tilde{H}_0(z)}{1 + F_0(z)N_0(z)\tilde{G}_0(z)} = \frac{\Phi_{y\varepsilon_0}(z)}{\lambda_0}$$

are stable. Finally,

$$\frac{-F_0(z)N_0(z)\tilde{H}_0(z)}{1 + F_0(z)N_0(z)\tilde{G}_0(z)} = \frac{\Phi_{u\varepsilon_0}(z)}{\lambda_0} \tag{4.57}$$

is stable since $\Phi_{u\varepsilon_0}(z)$ by (4.45a) is analytic and hence absolutely convergent on the unit circle. Hence, all transfer functions in (4.53) are stable, and since

$$\frac{F_0(z)}{1 + F_0(z)N_0(z)\tilde{G}_0(z)} = -\frac{\Phi_{u\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0}$$

is also stable, the closed-loop system is internally stable (Glad and Ljung, 2000, Def. 6.1).

As indicated in Figure 4.2, a reference signal r_0 might have to be included as an external input signal to the closed-loop model in order to explain the part of u that does not originate from ε_0. Unlike $G_{0,GE}$ and $H_{0,GE}$, which are defined such that (4.42) holds, F_0 can only explain the correlation between u and ε_0 and not the complete signal u. If (4.54) and (4.55) are inserted in (4.53a) and the result is used to express $\Phi_{u\varepsilon_0}(z)$ in $\Phi_{r_0\varepsilon_0}(z)$ and λ_0 we get

$$\begin{aligned}
\Phi_{u\varepsilon_0}(z) &= \frac{1}{1 + F_0(z)N_0(z)\tilde{G}_0(z)}\Phi_{r_0\varepsilon_0}(z) + \frac{-F_0(z)N_0(z)\tilde{H}_0(z)}{1 + F_0(z)N_0(z)\tilde{G}_0(z)}\lambda_0 \\
\Rightarrow \Phi_{u\varepsilon_0}(z) &= \frac{\Phi_{y\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0}\Phi_{r_0\varepsilon_0}(z) + \frac{\Phi_{u\varepsilon_0}(z)}{\lambda_0}\lambda_0 \\
\Rightarrow \Phi_{r_0\varepsilon_0}(z) &= 0,
\end{aligned} \tag{4.58}$$

where (4.56) and (4.57) have been used to rewrite the closed-loop transfer functions. Equation (4.58) shows that r_0 is uncorrelated with ε_0, which is natural if we want a closed-loop interpretation.

Since the linear closed-loop system in Figure 4.2 is internally stable when F_0 is defined as in (4.55), it can always be used to show how the signals u and y could have been generated from an LTI model. More specifically, it is impossible to disprove that the linear closed-loop system has not generated the signals merely by looking at second order properties of u, y and ε_0. Hence, the closed-loop model in Figure 4.2 with the definitions in (4.54) and (4.55) might be called the complete second order equivalent LTI description of the true system. Of course, since the GE-LTI-SOE is input-dependent, this description is too.

Actually, it is sometimes possible to directly judge the quality of the GE-LTI-SOE if some additional information about the system is available. For example, assume that the input signal u has been generated in such a way that there can be no correlation between $u(t)$ and any system disturbances. If the GE-LTI-SOE of the true system for this u indeed results in a nonzero $\Phi_{u\varepsilon_0}(z)$, then the GE-LTI-SOE cannot be a correct description of the true system.

The interpretation of the GE-LTI-SOE can be somewhat simplified if both $G_{0,GE}$ and $H_{0,GE}$ turn out to be stable. If this is the case, N_0 can be set to 1 in (4.54). This makes it possible to draw a simplified version of the closed-loop model from Figure 4.2. This simplified closed-loop model is shown in Figure 4.3.

However, the closed-loop model in Figure 4.3 is not the only interpretation of a stable GE-LTI-SOE. An alternative explanation of the cross-correlation between u and ε_0 can be given by the feedforward model in Figure 4.4. In this model, let

$$F_{f0}(z) = \frac{\Phi_{u\varepsilon_0}(z)}{\lambda_0}, \tag{4.59a}$$

$$r_{f0}(t) = u(t) - F_{f0}(q)\varepsilon_0(t). \tag{4.59b}$$

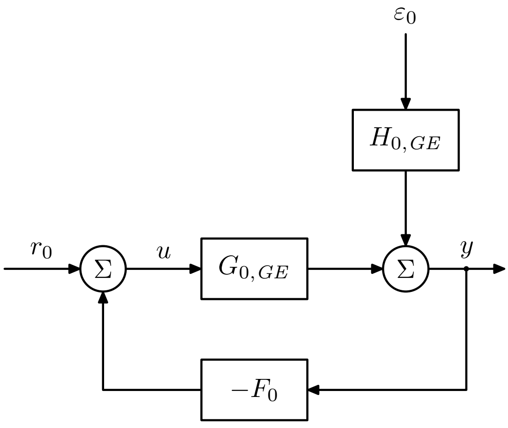


Figure 4.3: *The GE-LTI-SOE can be interpreted as being part of a simplified linear closed-loop system if both $G_{0,GE}$ and $H_{0,GE}$ are stable.*

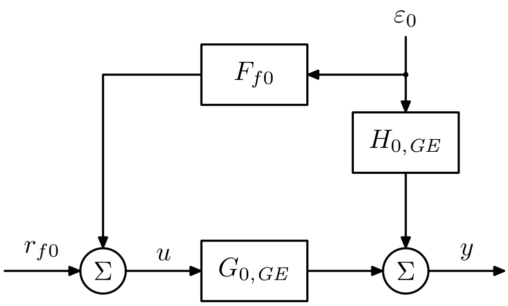


Figure 4.4: *The interpretation of the GE-LTI-SOE as a linear feedforward structure when $G_{0,GE}$ and $H_{0,GE}$ are stable.*

With this definition, the transfer function $F_{f0}(z)$ is stable since $\Phi_{u\varepsilon_0}(z)$ by (4.45a) is absolutely convergent on the unit circle. Furthermore, it follows that

$$\Phi_{r_{f0}\varepsilon_0}(z) = \Phi_{u\varepsilon_0}(z) - F_{f0}(z)\lambda_0 = 0 \tag{4.60}$$

and r_{f0} and ε_0 are thus uncorrelated.

The conclusion that can be drawn from the discussion in this section is that a GE-LTI-SOE always can be interpreted as being a part of an internally stable feedback or, in the case of a stable GE-LTI-SOE, feedforward system. Hence, by looking only at second order properties, it is impossible to disprove that any data set, with input and output measurements that fulfill Assumptions A1, A2 and A4, might have been generated by this closed-loop system. However, in some cases additional prior knowledge about the structure of the nonlinear system is available, and this knowledge can influence the interpretation of the LTI-SOE. In the next section, additional knowledge about the noise in the system will be discussed.

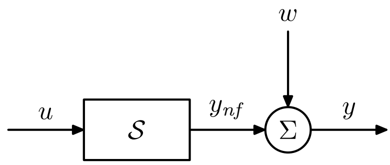


Figure 4.5: *In some cases, the noise is assumed to be output additive.*

4.6 Assumptions on the Noise

In the derivation of the OE-LTI-SOE and GE-LTI-SOE in the previous sections, no assumptions were made about the structure of the nonlinear system. Although structural assumptions are not necessary for the existence of the OE-LTI-SOE and the GE-LTI-SOE, it is hard to draw any conclusions about the properties and usefulness of these second order equivalents without any further information about the nonlinear system.

One important structural property of a system is how the noise enters. For most of the results in this thesis we will need the following assumption that says that the noise is additive and uncorrelated with the input and the noise-free output.

Assumption A5. Assume that the output $y(t)$ can be written

$$y(t) = y_{nf}(t) + w(t), \tag{4.61}$$

where y_{nf} is the noise-free response of the nonlinear system and not influenced by any other external signals than u, and where w is a noise term which is uncorrelated with u and y_{nf}.

In most cases, this assumption does not hold for a closed-loop system. This is due to the fact that the measured output that is fed back to the controller in such a system usually contains some noise. The input is thus in general correlated with the noise when it comes from a closed-loop system. Hence, Assumption A5 can be viewed as an assumption that the system has the structure shown in Figure 4.5.

However, the fact that Assumption A5 is needed for most of the results in this thesis is not the only reason why LTI approximations for systems in closed loop are not studied much here. Most of the results later in this thesis hold only for input signals with special distributions, e.g., Gaussian inputs or LTI filtered white noise inputs. For an open-loop system, this limitation is often not a problem since the input signal in many cases can be designed rather freely by the user. On the other hand, for a nonlinear closed-loop system, it is usually very hard to guarantee that the distribution of the input signal belongs to a certain class. Hence, most parts of the remaining chapters in this thesis deal with open-loop systems.

As has been mentioned previously, the fact that a certain system operates in open loop can be used to disprove that the GE-LTI-SOE represents the true system. This can be done because it may be necessary to view the GE-LTI-SOE as being a part of a closed-loop system in order to explain a nonzero $\Phi_{u\varepsilon_0}(z)$ (cf. Example 4.3).

This concludes our introductory discussion about the notion of LTI Second Order Equivalents. In this chapter, it has been shown that the OE-LTI-SOE and GE-LTI-SOE of a nonlinear system are well-defined for rather general classes of input and output signals. Furthermore, properties and interpretations of these LTI approximations that hold for all signals in these classes have been discussed. In the Chapters 5 to 8, we will consider more restricted and specialized classes of input signals for which further properties of the OE-LTI-SOEs and GE-LTI-SOEs can be shown.

5

Basic Properties of LTI-SOEs

In Example 4.2 in the previous chapter, it was shown that the OE-LTI-SOE of a static nonlinear system can be nonstatic. This observation can be viewed as an indication that some caution is needed when conclusions are drawn about the behavior of OE-LTI-SOEs. A behavior that intuitively seems correct, for example that the OE-LTI-SOE of a static nonlinear system should be static, can actually be erroneous.

In this chapter, we will make some rather straightforward assumptions about the input signals and investigate what these assumptions imply. More specifically, we will assume that the input signal is symmetrically distributed and show that this implies that the OE-LTI-SOE only depends on the odd part of the system. We will show that if a nonlinear system can be written as a linear combination of several subsystems with the same input, its OE-LTI-SOE can be written as a linear combination too. We will also consider input signals generated by filtering white noise through a minimum phase filter. It turns out that for such an input signal, spectral and residual analysis can be used for validation just like in the LTI case. Furthermore, minimum phase filtered white noise can be useful if an LTI system is identified in closed-loop and a nonlinear controller is used. This signal type guarantees also that the $G_{0,OE}(z)$ and $G_{0,GE}(z)$ will be equal. This will be discussed at the end of this chapter. First, however, we will investigate how output additive noise affects the OE-LTI-SOE.

5.1 Additive Noise

Noise can affect a nonlinear system in many ways. For example, the noise can be added to or multiplied with the input before it enters the actual system, or affect the output of the system. Although a large variation of noise dependency can be found in applications, we will here mainly study one type of noise, namely output additive noise that is uncorrelated with the input and the noise-free output as in Assumption A5. Actually, the following lemma shows that this type of noise does not affect the OE-LTI-SOE of a nonlinear system.

Lemma 5.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A5 are fulfilled. This means that the additive noise $w(t)$ in (4.61) is uncorrelated with the input and the noise-free output y_{nf}. Then the OE-LTI-SOE is not influenced by $w(t)$.

Proof: Since the noise w by Assumption A5 is uncorrelated with u and y_{nf}, we have for any stable LTI model $G(q)$ that

$$\begin{aligned}\mathrm{E}((y(t) - G(q)u(t))^2) &= \mathrm{E}((y_{nf}(t) + w(t) - G(q)u(t))^2) \\ &= \mathrm{E}(y_{nf}(t) - G(q)u(t)^2) + \mathrm{E}(w(t)^2).\end{aligned} \tag{5.1}$$

Hence, the criterion in Definition 4.1 can be rewritten

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y(t) - G(q)u(t))^2) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y_{nf}(t) - G(q)u(t))^2)$$

and this shows that the OE-LTI-SOE is not influenced by the additive noise. □

The result of the previous lemma is the rather convenient fact that the OE-LTI-SOE is independent of the noise if Assumption A5 holds. Although this assumption might be considered rather restrictive, it will be used frequently in this thesis. Some results about OE-LTI-SOEs where Assumption A5 is not required will be presented later in this chapter (see Sections 5.4 and 5.5). First, however, we will draw some conclusions about the different influence odd and even nonlinearities have on the OE-LTI-SOE.

5.2 Even and Odd Nonlinearities

Assume that Assumption A5 holds for a certain system and that the noise-free output y_{nf} can be written as $y_{nf}(t) = f((u(t-k))_{k=0}^{M})$ for some nonnegative integer M. This means that the system is an NFIR system (see Section 2.2), and in this case, we can divide the real-valued function f in an even part f_e and an odd part f_o

$$\begin{aligned}f((u(t-k))_{k=0}^{M}) = &\underbrace{\frac{f((u(t-k))_{k=0}^{M}) + f(-(u(t-k))_{k=0}^{M})}{2}}_{=f_e((u(t-k))_{k=0}^{M})} \\ &+ \underbrace{\frac{f((u(t-k))_{k=0}^{M}) - f(-(u(t-k))_{k=0}^{M})}{2}}_{=f_o((u(t-k))_{k=0}^{M})}\end{aligned} \tag{5.2}$$

such that

$$\begin{aligned}f_e(-(u(t-k))_{k=0}^{M}) &= f_e((u(t-k))_{k=0}^{M}), \\ f_o(-(u(t-k))_{k=0}^{M}) &= -f_o((u(t-k))_{k=0}^{M}).\end{aligned}$$

If all simultaneous probability density functions for the process u are even functions, the OE-LTI-SOE of an NFIR system will only depend on the odd part f_o of the system. Hence, we have the following lemma.

Lemma 5.2
Consider an NFIR system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A5 are fulfilled. The noise-free output y_{nf} from (4.61) can then, for some nonnegative integer M, be written

$$y_{nf}(t) = f((u(t-k))_{k=0}^{M}) = f_e((u(t-k))_{k=0}^{M}) + f_o((u(t-k))_{k=0}^{M}), \tag{5.3}$$

where f_e and f_o are even and odd functions, respectively. Assume that all simultaneous probability density functions for the process u are even functions. Then the OE-LTI-SOE depends only on the odd part f_o of the system, i.e.,

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}\big(\big(f_o((u(t-k))_{k=0}^{M}) - G(q)u(t)\big)^2\big). \tag{5.4}$$

Proof: From Lemma 5.1 we have

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y_{nf}(t) - G(q)u(t))^2).$$

Using (5.3), this mean-square error criterion can be rewritten

$$\begin{aligned}
&\mathrm{E}((y_{nf}(t) - G(q)u(t))^2) = \mathrm{E}\big(\big(f_o((u(t-k))_{k=0}^{M}) - G(q)u(t)\big)^2\big)\\
&+ \mathrm{E}\big(\big(f_e((u(t-k))_{k=0}^{M})\big)^2\big) + 2\mathrm{E}(f_o((u(t-k))_{k=0}^{M})f_e((u(t-k))_{k=0}^{M}))\\
&- 2\sum_{j=0}^{\infty} g(j)\mathrm{E}(u(t-j)f_e((u(t-k))_{k=0}^{M})).
\end{aligned}$$

The cross-terms in this expansion are equal to zero since

$$f_o((u(t-k))_{k=0}^{M})f_e((u(t-k))_{k=0}^{M})$$

and

$$u(t-j)f_e((u(t-k))_{k=0}^{M}),$$

$j \in \mathbb{N}$, are odd functions of $(u(t-k))_{k=0}^{\max(M,j)}$ and since all probability density functions of u are even functions. Hence

$$\begin{aligned}
\mathrm{E}((y_{nf}(t) - G(q)u(t))^2) &= \mathrm{E}\big(\big(f_o((u(t-k))_{k=0}^{M}) - G(q)u(t)\big)^2\big)\\
&\quad + \mathrm{E}\big(\big(f_e((u(t-k))_{k=0}^{M})\big)^2\big)
\end{aligned} \tag{5.5}$$

and (5.4) follows. □

The fact that the OE-LTI-SOE is independent of even nonlinearities when a symmetrically distributed input signal is used implies that if two nonlinear systems only differ by even nonlinearities, they will have the same OE-LTI-SOE. However, the variance of the model residuals will of course be larger if a system contains large even nonlinearities. The influence of the even and odd parts of a system on the variance and spectral density of the model residuals is shown by the following lemma.

Lemma 5.3
Consider an NFIR system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A5 are fulfilled. The noise-free output $y_{nf}(t)$ from (4.61) can then, for some nonnegative integer M, be written as in (5.3). Assume that all simultaneous probability density functions for the process u are even functions. Then the z-spectrum of the OE-LTI-SOE residuals $\eta_0(t)$ can be written

$$\Phi_{\eta_0}(z) = \Phi_{d_o}(z) + \Phi_{y_e}(z) + \Phi_w(z), \tag{5.6}$$

where $d_o(t) = f_o((u(t-k))_{k=0}^M) - G_{0,OE}(q)u(t)$ *and* $y_e(t) = f_e((u(t-k))_{k=0}^M)$.

Proof: Assumption A5 gives that

$$\begin{aligned} R_{\eta_0}(\tau) = \mathrm{E}((y_{nf}(t) - G_{0,OE}(q)u(t))(y_{nf}(t-\tau) - G_{0,OE}(q)u(t-\tau))) \\ + R_w(\tau), \quad \forall \tau \in \mathbb{Z} \end{aligned}$$

and using (5.3), we get

$$\begin{aligned} R_{\eta_0}(\tau) &= \mathrm{E}((d_o(t) + y_e(t))(d_o(t-\tau) + y_e(t-\tau))) + R_w(\tau) \\ &= R_{d_o}(\tau) + R_{d_o y_e}(\tau) + R_{y_e d_o}(\tau) + R_{y_e}(\tau) + R_w(\tau), \quad \forall \tau \in \mathbb{Z}. \end{aligned}$$

Since

$$\begin{aligned} R_{d_o y_e}(\tau) = \mathrm{E}(f_o((u(t-k))_{k=0}^M) f_e((u(t-k-\tau))_{k=0}^M)) \\ - \sum_{j=0}^{\infty} g_{0,OE}(j) \mathrm{E}(u(t-j) f_e((u(t-k-\tau))_{k=0}^M)) \end{aligned}$$

contains only expectations of odd functions and since all probability density functions of u are even, we get $R_{d_o y_e}(\tau) = R_{y_e d_o}(\tau) = 0$ for all $\tau \in \mathbb{Z}$. Hence,

$$R_{\eta_0}(\tau) = R_{d_o}(\tau) + R_{y_e}(\tau) + R_w(\tau), \quad \forall \tau \in \mathbb{Z} \tag{5.7}$$

and (5.6) follows. □

In particular, (5.7) shows that the variance of the OE-LTI-SOE residuals $\eta_0(t)$ can be written

$$\begin{aligned} R_{\eta_0}(0) = \mathrm{E}(\eta_0(t)^2) = \mathrm{E}(\big(f_o((u(t-k))_{k=0}^M) - G(q)u(t)\big)^2) \\ + \mathrm{E}(\big(f_e((u(t-k))_{k=0}^M)\big)^2) + \mathrm{E}(w(t)^2). \end{aligned} \tag{5.8}$$

This expression is valid when Assumption A5 holds and when the input signal has even probability density functions, and it shows that there are three conceptually different contributions to $R_{\eta_0}(0)$. The first term in (5.8) is the variance of the unmodeled part of the odd nonlinearities, while the second and third terms are the variance of the even part of the system and of the noise, respectively.

Usually, it is not obvious how the input signal should be designed in order to minimize the variance of the residuals. In Section 5.4, we will consider inputs that have been generated by filtering white noise through a minimum phase filter. Later in Section 5.5.1, it will be shown that these inputs reduce the variance of the residuals. First, we will study OE-LTI-SOEs for nonlinear systems with parallel subsystems.

5.3 Sums of Nonlinear Systems

If the output from a nonlinear system can be written as the sum of the outputs from several nonlinear subsystems with the same input, the OE-LTI-SOE of the complete system will be the sum of the OE-LTI-SOEs of the corresponding subsystems according to the following theorem.

Theorem 5.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A5 are satisfied. Assume also that the system can be written as a sum of several nonlinear subsystems, i.e., as

$$y(t) = \sum_{i=1}^{M} y_i(t) + w(t), \tag{5.9a}$$

$$y_i(t) = f_i((u(t-k))_{k=0}^{\infty}) \tag{5.9b}$$

and that each nonlinearity f_i has an OE-LTI-SOE $G_{0,OE,i}(z)$. Then the OE-LTI-SOE of the complete nonlinear system (from u to y) is

$$G_{0,OE}(z) = \sum_{i=1}^{M} G_{0,OE,i}(z). \tag{5.10}$$

Proof: The z-cross-spectrum can be written as

$$\Phi_{yu}(z) = \sum_{i=1}^{M} \Phi_{y_i u}(z).$$

Hence, the OE-LTI-SOE of the complete system can be written

$$\begin{aligned} G_{0,OE}(z) &= \frac{1}{r_u L(z)} \left[\frac{\sum_{i=1}^{M} \Phi_{y_i u}(z)}{L(z^{-1})} \right]_{\text{causal}} = \sum_{i=1}^{M} \frac{1}{r_u L(z)} \left[\frac{\Phi_{y_i u}(z)}{L(z^{-1})} \right]_{\text{causal}} \\ &= \sum_{i=1}^{M} G_{0,OE,i}(z), \end{aligned}$$

where we have used (4.3) and the fact that the causality operator is linear. □

Theorem 5.1 implies that OE-LTI-SOEs of parallel connected nonlinear systems can be calculated by studying the subsystems. Hence, if a nonlinear system can be written as a linear combination of a number of nonlinear subsystems with the same input, the OE-LTI-SOE of this complete system can be written as the same linear combination of the OE-LTI-SOEs of the subsystems. The OE-LTI-SOE can thus be viewed as a linear operator on the set of all nonlinear systems.

In the next section, we will consider OE-LTI-SOEs for inputs generated by filtering white noise through a minimum phase filter. It turns out that these inputs imply some useful properties of the OE-LTI-SOEs.

5.4 Minimum Phase Input Filters

A common way to generate a signal u such that its spectral density is equal to some predefined function is to filter white noise e through an LTI filter $L(z)$. Since, by Lemma 2.1, the result of this procedure will be a signal with spectral density

$$\Phi_u(e^{i\omega}) = |L(e^{i\omega})|^2 R_e(0),$$

it is often convenient to consider only $|L(e^{i\omega})|$ when the filter $L(z)$ is designed and let the phase $\arg(L(e^{i\omega}))$ become whatever it becomes. For example, this works well if u is going to be used as input to an LTI system in a linear identification experiment.

However, if the signal u is to be used for an LTI approximation of a nonlinear system, the phase of the prefilter is crucial for the behavior of this approximation. In the following theorem, it will be shown that in this case, it is beneficial to generate the input signal by filtering white noise through an LTI filter which has the minimum phase property (see Definition 2.4).

Theorem 5.2
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Assume that the input signal has been generated by filtering white, possibly non-Gaussian, noise $e(t)$ through a minimum phase filter $L_m(z)$. Assume also that any other external signals that affect the output are independent of u. Then the OE-LTI-SOE is regular (see Definition 4.2) and can be written

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{\Phi_{ye}(z)}{L_m(z)R_e(0)}. \tag{5.11}$$

Proof: The canonical spectral factorization of $\Phi_u(z)$ (cf. (4.1)) is

$$\begin{aligned} L(z) &= \frac{L_m(z)}{l_m(0)}, \\ r_u &= l_m(0)^2 R_e(0). \end{aligned}$$

Using (4.3) from Theorem 4.1, this gives

$$\begin{aligned} G_{0,OE}(z) &= \frac{l_m(0)}{l_m(0)^2 R_e(0) L_m(z)} \left[\frac{l_m(0)\Phi_{yu}(z)}{L_m(z^{-1})}\right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\frac{\Phi_{yu}(z)}{L_m(z^{-1})}\right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\frac{\Phi_{ye}(z)L_m(z^{-1})}{L_m(z^{-1})}\right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\Phi_{ye}(z)\right]_{\text{causal}}, \end{aligned}$$

where we have used Lemma 2.1 in the third equality. The nonlinear system is, by our standard assumption, causal and u is independent of all other external signals that affect the output. This, together with the fact that e is a white noise process, implies that $y(t)$ is independent of $e(t-\tau)$ for all $\tau < 0$. Hence $R_{ye}(\tau) = 0$ for all $\tau < 0$ and

$$\Phi_{ye}(z) = \sum_{\tau=0}^{\infty} R_{ye}(\tau) z^{-\tau}.$$

Since the series $\Phi_{ye}(z)$ contains no positive powers of z, taking the causal part does not remove anything. Hence, we have

$$G_{0,OE}(z) = \frac{1}{R_e(0)L_m(z)} \left[\Phi_{ye}(z)\right]_{\text{causal}} = \frac{\Phi_{ye}(z)}{L_m(z)R_e(0)} = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

and we have shown (5.11). □

The assumption in Theorem 5.2 that any other external signals that affect the output should be independent of the input u implies that this theorem usually cannot be applied if the system is a closed-loop LTI system. However, for open-loop systems, the conditions in Theorem 5.2 are not very restrictive.

Actually, the reason why the OE-LTI-SOE in Example 4.2 on page 42 was regular is that the input signal in that example was generated by filtering white noise through a minimum phase filter. It is interesting to see what happens with the OE-LTI-SOE if the input filter in Example 4.2 is replaced by a non-minimum phase filter giving the same $\Phi_u(z)$. This is done in the following example.

Example 5.1

Consider the static nonlinear system

$$y(t) = u(t)^3 \tag{5.12}$$

with the input

$$u(t) = \frac{1}{2}e(t) + e(t-1),$$

where $e(t)$ is a sequence of independent random variables with uniform distribution over the interval $[-1, 1]$. For the moment, let $\tilde{R}_{yu}(\tau)$ denote the cross-covariance function in Example 4.2. Then

$$\begin{aligned}
R_{yu}(0) &= \mathrm{E}(u(t)^4) = \mathrm{E}(\left(\frac{1}{2}e(t) + e(t-1)\right)^4) = \mathrm{E}(\left(e(t) + \frac{1}{2}e(t-1)\right)^4) \\
&= \tilde{R}_{yu}(0) = \frac{91}{240}, \\
R_{yu}(1) &= \mathrm{E}(u(t)^3 u(t-1)) \\
&= \mathrm{E}(\left(\frac{1}{2}e(t) + e(t-1)\right)^3 \cdot \left(\frac{1}{2}e(t-1) + e(t-2)\right)) \\
&= \mathrm{E}(\left(e(t) + \frac{1}{2}e(t-1)\right)^3 \cdot \left(e(t+1) + \frac{1}{2}e(t)\right)) \\
&= \tilde{R}_{yu}(-1) = \frac{34}{240},
\end{aligned}$$

$$
\begin{aligned}
R_{yu}(-1) &= \mathrm{E}(u(t)^3 u(t+1)) \\
&= \mathrm{E}(\left(\frac{1}{2}e(t) + e(t-1)\right)^3 \cdot \left(\frac{1}{2}e(t+1) + e(t)\right)) \\
&= \mathrm{E}(\left(e(t) + \frac{1}{2}e(t-1)\right)^3 \cdot \left(e(t-1) + \frac{1}{2}e(t-2)\right)) \\
&= \tilde{R}_{yu}(1) = \frac{46}{240}, \\
R_{yu}(\tau) &= \tilde{R}_{yu}(\tau) = 0, \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}.
\end{aligned}
$$

This gives

$$
\Phi_{yu}(z) = \frac{1}{240}(46z + 91 + 34z^{-1}) = \frac{1}{240}\left(1 + \frac{1}{2}z^{-1}\right)(68 + 46z)\,.
$$

Furthermore, Lemma 2.1 gives

$$
\Phi_u(z) = \left(\frac{1}{2} + z^{-1}\right) \cdot \frac{1}{3} \cdot \left(\frac{1}{2} + z\right) = \frac{1}{12}\left(2z + 5 + 2z^{-1}\right).
$$

The canonical spectral factor of $\Phi_u(z)$ is $L(z) = 1 + \frac{1}{2}z^{-1}$ and $r_u = \frac{1}{3}$. Theorem 4.1 gives

$$
\begin{aligned}
G_{0,OE}(z) &= \frac{1}{r_u L(z)} \left[\frac{\Phi_{yu}(z)}{L(z^{-1})}\right]_{\mathrm{causal}} = \frac{3}{1 + \frac{1}{2}z^{-1}} \left[\frac{46z + 91 + 34z^{-1}}{240(1 + \frac{1}{2}z)}\right]_{\mathrm{causal}} \\
&= \frac{1}{80} \cdot \frac{1}{1 + \frac{1}{2}z^{-1}} \left[\frac{9z}{1 + \frac{1}{2}z} + 74 + 34z^{-1}\right]_{\mathrm{causal}} \\
&= \frac{1}{80} \cdot \frac{74 + 34z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{1}{40} \cdot \frac{37 + 17z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{0.925 + 0.425z^{-1}}{1 + 0.5z^{-1}}.
\end{aligned}
$$

Here, just like in Example 4.2, the OE-LTI-SOE of the static nonlinear system $y(t) = u(t)^3$ is nonstatic. However, the OE-LTI-SOE in Example 4.2 is not equal to the OE-LTI-SOE here, since the two input signals have different distributions.

The input signals in Examples 4.2 and 5.1 are similar in the sense that they have equal z-spectra. Furthermore, the probability density function for one input signal component, (or the amplitude distribution of every single $u(t)$), is the same in both examples. Since the nonlinear system is static, this also implies that the probability density functions of a single $y(t)$ are equal in these examples.

Despite these similarities between the input signals in Example 4.2 and 5.1, these inputs generate different OE-LTI-SOEs because the simultaneous probability density functions of $u(t)$ and $u(t-1)$ are different in the two examples. As we have not calculated $R_{\eta_0}(0)$ in these examples, it is not obvious which OE-LTI-SOE that is most successful in approximating the true system. However, in the next section we will show that it is always better to use an input generated by a minimum phase filter than a non-minimum phase filter.

5.5 Properties for Regular OE-LTI-SOEs

The fact that the OE-LTI-SOE is regular for, for example, input signals generated by filtering white noise through a minimum phase filter is convenient, since it makes it possible to calculate the OE-LTI-SOE without spectral factorization of the input z-spectrum. In addition, OE-LTI-SOEs of this kind exhibit a number of interesting properties.

5.5.1 Optimality Properties

The perhaps most obvious property that holds for a regular OE-LTI-SOE concerns the residuals $\eta_0(t)$. In the following lemma it will be shown that for such an OE-LTI-SOE the residuals will be uncorrelated with *all* input signal components (cf. Corollary 4.2).

Lemma 5.4
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Assume that the OE-LTI-SOE is regular, i.e., that it can be written as

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}, \tag{5.13}$$

and let

$$\eta_0(t) = y(t) - G_{0,OE}(q)u(t). \tag{5.14}$$

Then it follows that

$$\Phi_{\eta_0 u}(z) = \Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) = 0, \tag{5.15a}$$

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}). \tag{5.15b}$$

Proof: The expression for $\Phi_{\eta_0 u}(z)$ in (5.15a) follows directly from (5.13) and (5.14). Furthermore, using Lemma 2.1, (5.13) and (5.14) also give

$$\begin{aligned}\Phi_{\eta_0}(z) &= \Phi_y(z) - G_{0,OE}(z)\Phi_{uy}(z) - \Phi_{yu}(z)G_{0,OE}(z^{-1}) \\ &\quad + G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})\end{aligned}$$

and hence (5.15b) has been shown. □

The fact that (5.15a) holds when $G_{0,OE}(z)$ is regular shows that the OE-LTI-SOE in this case really is the best noncausal LTI model since the model residuals are uncorrelated with all input signal components. This is no surprise since it can be shown that the ratio $\Phi_{yu}(z)/\Phi_u(z)$ is always the mean-square error optimal noncausal LTI model, i.e., the noncausal LTI-SOE (see Section 3.2.3). If this ratio is causal, it is of course equal to the OE-LTI-SOE.

Intuitively, it seems that it should always be a good idea to use input signals for which the OE-LTI-SOE is equal to the noncausal LTI-SOE. As a matter of fact, input signals for which the OE-LTI-SOE of a nonlinear system is regular exhibit the following optimality property.

Theorem 5.3
Consider a nonlinear system with input $u_1(t)$ and output $y_1(t)$ such that Assumptions A1 and A2 are fulfilled. Let $G_{0,OE,1}(z)$ denote the OE-LTI-SOE of the nonlinear system with respect to u_1 and assume that it is regular, i.e., that it can be written as

$$G_{0,OE,1}(z) = \frac{\Phi_{y_1u_1}(z)}{\Phi_{u_1}(z)}.$$

Furthermore, let $\eta_{0,1}(t) = y_1(t) - G_{0,OE,1}(q)u_1(t)$.

Consider also another input signal $u_2(t)$ to the same nonlinear system. Assume that this signal generates the output $y_2(t)$ and that $u_2(t)$ and $y_2(t)$ satisfy Assumptions A1 and A2. Let $G_{0,OE,2}(z)$ denote the OE-LTI-SOE of the nonlinear system with respect to u_2 and let $\eta_{0,2}(t) = y_2(t) - G_{0,OE,2}(q)u_2(t)$. Assume that

$$\begin{aligned}\Phi_{u_2}(e^{i\omega}) &= \Phi_{u_1}(e^{i\omega}), \quad \forall\omega \in [-\pi, \pi],\\ |\Phi_{y_2u_2}(e^{i\omega})| &= |\Phi_{y_1u_1}(e^{i\omega})|, \quad \forall\omega \in [-\pi, \pi],\\ R_{y_2}(0) &= R_{y_1}(0).\end{aligned}$$

Then the model residual variance for the OE-LTI-SOE corresponding to u_2 cannot be smaller than it is for the one corresponding to u_1, i.e.,

$$R_{\eta_{0,2}}(0) \geq R_{\eta_{0,1}}(0). \tag{5.16}$$

Proof: From (4.12) we have for any OE-LTI-SOE that

$$R_{\eta_0}(0) = R_y(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega})\, d\omega.$$

For any input signal, the noncausal LTI-SOE is always $\frac{\Phi_{yu}(z)}{\Phi_u(z)}$. It is easy to verify that (4.12) holds also for the noncausal LTI-SOE if $G_{0,OE}(e^{i\omega})$ is replaced by $\frac{\Phi_{yu}(e^{i\omega})}{\Phi_u(e^{i\omega})}$. As the stable and causal LTI systems are a subset of the stable and noncausal, it follows that the OE-LTI-SOE will always have a minimum mean-square error that is greater than or equal to the minimum mean-square error that is obtained for the noncausal LTI-SOE. Hence,

$$\begin{aligned}R_{\eta_{0,2}}(0) &\geq R_{y_2}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\Phi_{y_2u_2}(e^{i\omega})}{\Phi_{u_2}(e^{i\omega})}\right|^2 \Phi_{u_2}(e^{i\omega})\, d\omega\\ &= R_{y_1}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\Phi_{y_1u_1}(e^{i\omega})}{\Phi_{u_1}(e^{i\omega})}\right|^2 \Phi_{u_1}(e^{i\omega})\, d\omega = R_{\eta_{0,1}}(0)\end{aligned}$$

since $G_{0,OE,1}(z) = \frac{\Phi_{y_1u_1}(z)}{\Phi_{u_1}(z)}$. □

Theorem 5.3 shows that, for example, a given minimum phase generated input signal is optimal over a set of other inputs in the sense that it minimizes the variance of the model residuals. Usually, it is not easy to describe this set of input signals and in some cases it might actually be empty. However, the input signals in Examples 4.2 and 5.1 fulfill the assumptions in Theorem 5.3. Hence, it follows from (5.16) that the variance of the model residual in Example 4.2 is less than or equal to the corresponding variance in Example 5.1. Actually, it can be shown that the variance in the first example is strictly less than the variance in the second one.

It should be noted that there is no guarantee that a regular OE-LTI-SOE will be a useful model of a nonlinear system. The only thing that is guaranteed is that it will be easier to see if an estimated model is close to the OE-LTI-SOE since some useful validation methods will work in this case. This will be discussed more in the next section.

5.5.2 Spectral and Residual Analysis

A common way to validate an estimated model of an open-loop LTI system is to compare the frequency response of the model with a nonparametric frequency response estimate obtained by spectral analysis. If these frequency responses are similar this indicates that the order of the parametric model is sufficiently high and that the numerical computation of the estimate has been successful. In Ljung (1999, Sec. 6.4), it is shown that the spectral analysis frequency response estimate $\hat{G}_N(e^{i\omega_0})$ based on N measurements can be written

$$\hat{G}_N(e^{i\omega_0}) = \frac{\hat{\Phi}_{yu}^N(e^{i\omega_0})}{\hat{\Phi}_u^N(e^{i\omega_0})},$$

where $\hat{\Phi}_u^N(e^{i\omega_0})$ and $\hat{\Phi}_{yu}^N(e^{i\omega_0})$ are estimates of the spectral and cross-spectral densities that can be written as smoothed periodograms.

If an LTI model is estimated for an open-loop nonlinear system, it might be tempting to use spectral analysis as a validation method also in this case. However, the spectral analysis frequency response estimate can be quite different from the frequency response of the OE-LTI-SOE and is thus in general useless for validation purposes. Only when the OE-LTI-SOE is regular, spectral analysis can be used as a validation method.

Example 5.2

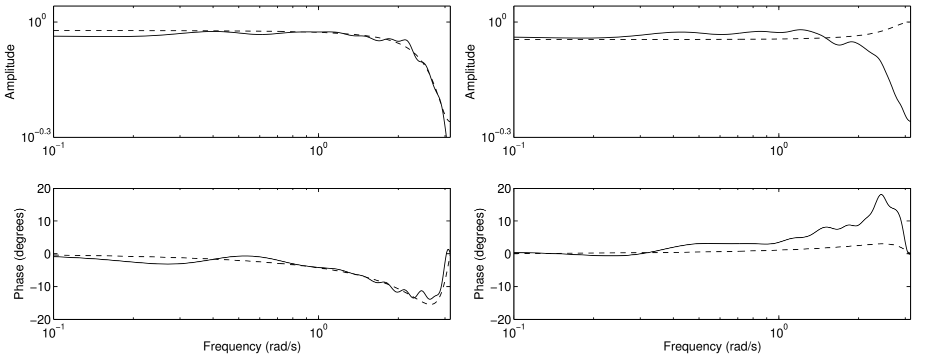
Consider once again the system and input signals from Examples 4.2 and 5.1. In these examples, it was shown that the OE-LTI-SOEs of this system are

$$G_{0,OE,1}(z) = \frac{0.85 + 0.575z^{-1}}{1 + 0.5z^{-1}}$$

and

$$G_{0,OE,2}(z) = \frac{0.925 + 0.425z^{-1}}{1 + 0.5z^{-1}},$$

respectively. Here, two data sets with 10 000 noise-free input and output measurements have been generated. The first of these data sets was generated with a realization of the minimum phase filtered signal from Example 4.2 as input while the second data set was generated with a realization of the non-minimum phase filtered signal from Example 5.1 as input.



(a) The OE-LTI-SOE (dashed) and the spectral analysis estimate (solid) for an input generated by a minimum phase filter.

(b) The OE-LTI-SOE (dashed) and the spectral analysis estimate (solid) for an input generated by a non-minimum phase filter.

Figure 5.1: *A nonparametric frequency response estimate will be a good approximation of the OE-LTI-SOE only when $G_{0,OE}(z)$ is regular.*

Nonparametric frequency response estimates have been computed from these data sets using spectral analysis with a Hamming window of lag size 30. These estimates are shown in Figure 5.1 together with the frequency responses of the corresponding OE-LTI-SOEs. The MATLAB code that has been used to generate this figure is available in Appendix B.1.

In Figure 5.1 it can be seen that there is a close match between the OE-LTI-SOE and the nonparametric frequency response estimate when the input has been generated by a minimum phase filter. However, when the input has been generated by a non-minimum phase filter, the OE-LTI-SOE is quite different from the nonparametric estimate.

The conclusion that can be drawn from Example 5.2 is that for LTI approximations of nonlinear systems, spectral analysis can be used as a validation method only when an input signal that guarantees that $G_{0,OE}(z)$ is regular has been used. An additional property of such input signals is that they make the result of another validation method, residual analysis, easier to interpret. Residual analysis can be used to check if there is remaining correlation between the input and the residuals for a certain model. Such remaining correlation indicates that the model order might be too low. If an accurate model of an open-loop LTI system has been found, the residuals will be uncorrelated with the input signal.

However, if the OE-LTI-SOE of a nonlinear system has been estimated, it will according to (4.11) in general only have residuals that are uncorrelated with past and present input signal components. If it is not known that the true system is nonlinear, the remaining correlation between the residuals and future input components might be taken as an indication that the system actually is a closed-loop system. However, if an input that guarantees that the OE-LTI-SOE is regular has been used, this cannot happen since the residuals then by (5.15a) will be uncorrelated with all input components. In the next section, this property will be used for another purpose.

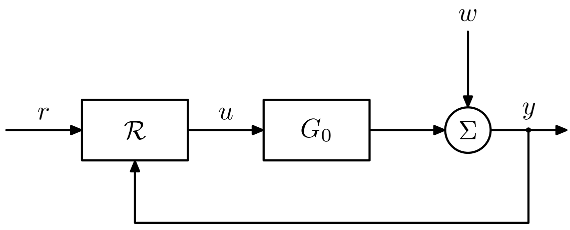


Figure 5.2: *A nonlinear closed-loop system.*

5.5.3 Closed-loop Identification

One application where LTI approximations of nonlinear systems are useful is closed-loop identification of LTI systems which operate under nonlinear feedback. Consider the closed loop system in Figure 5.2. This system consists of an unknown LTI plant G_0 and a nonlinear controller $\mathcal{R}$, and its output can be described as

$$y(t) = G_0(q)u(t) + w(t), \tag{5.17}$$

where u and w are correlated due to the feedback loop. The reference signal r is here assumed to be independent of the process noise w.

Suppose that a model of G_0 is desired and that measurements of r, u and y are available. In this case, the most natural way to estimate such a model is often to use the *direct prediction-error method* (Ljung, 1999, Sec. 13.5). Provided that the model structure is flexible enough to contain the true system, including the true noise description, this approach defines a consistent estimator of G_0.

However, if the model of G_0 is to be used for controller design, there might be one drawback with the direct approach. When an approximate model is used for controller design, it is usually appropriate that it is as accurate as possible in a frequency interval around the desired crossover frequency. In the open-loop case, this accuracy can be increased by the use of a frequency weighting. Unfortunately, in the closed-loop case, such a frequency weighting cannot be used in the direct approach without making the estimator of G_0 biased.

A solution to this problem when the complete closed-loop system is linear is given by the *two-step method* (Van den Hof and Schrama, 1993) or, when a nonlinear controller is present, a version of this method called the *projection method* (Forssell and Ljung, 2000). Both these methods can be viewed as ways to translate the closed-loop identification problem to an open-loop problem where frequency weighting can be used. The main idea used in the two-step and projection methods is to first estimate an LTI model $S(z)$ from r to u and then to use this model to construct a simulated input signal $\hat{u}(t) = S(q)r(t)$. Finally, the identification of G_0 is performed using measurements of y and of the simulated input $\hat{u}$ instead of the true input u.

The reason why it is beneficial to use $\hat{u}$ instead of u in the identification of G_0 is that this enables the use of frequency weighting since $\hat{u}$ and the noise will be uncorrelated, just like for an open-loop system. This is due to the fact that if an output error model

$$u(t) = S(q)r(t) + \eta(t)$$

is used to describe the mapping from r to u and if the identification of $S(z)$ is successful, the result will, in the case of linear feedback, be that

$$S(z) = \frac{\Phi_{ur}(z)}{\Phi_r(z)}. \tag{5.18}$$

Let $\tilde{u}(t) = u(t) - \hat{u}(t)$. Then (5.18) implies that

$$\Phi_{\tilde{u}\hat{u}}(z) = \Phi_{u\hat{u}}(z) - \Phi_{\hat{u}}(z) = \Phi_{ur}(z)S(z^{-1}) - S(z)\Phi_r(z)S(z^{-1}) = 0$$

and (5.17) can thus be rewritten as

$$y(t) = G_0(q)\hat{u}(t) + \tilde{w}(t), \tag{5.19}$$

where $\tilde{w}(t) = w(t) + G_0(q)\tilde{u}(t)$ is uncorrelated with $\hat{u}$ since both w and $\tilde{u}$ are uncorrelated with $\hat{u}$. Hence, the system (5.19) can be viewed as an open-loop system and frequency weighting can thus be used.

For a linear closed-loop system, $S(z)$ is simply the ordinary sensitivity function, which of course is causal. However, when a nonlinear controller is present in the closed-loop system, the mapping from r to u is nonlinear. In that case, the ratio $\frac{\Phi_{ur}(z)}{\Phi_r(z)}$ might be noncausal if we want a stable interpretation of it. This is also pointed out in Forssell and Ljung (2000). The main difference between the projection method and the two-step method is that in the former, a noncausal FIR model is used to model the mapping from r to u, while in the latter only a causal $S(z)$ is used. Hence, the projection method is applicable to closed-loop systems with nonlinear controllers while the two-step method in general is not.

However, from the discussion in Section 5.4 we know that if, for example, the reference signal has been generated by filtering white noise through a minimum phase filter, then $\frac{\Phi_{ur}(z)}{\Phi_r(z)}$ will be causal since the nonlinear mapping from r to u is causal and r and w are independent. This means that if a minimum phase generated input signal is used, there is no need to use a noncausal $S(z)$, and hence the two-step method can be applied instead of the projection method.

One advantage of the two-step method is that $S(z)$ can be a rational function. In the projection method in Forssell and Ljung (2000), $S(z)$ is a noncausal FIR model, which means that the true sensitivity function usually cannot be modeled exactly even when the controller is linear. With this observation in mind, it seems that the two-step method is at least an as good alternative for closed-loop identification of an LTI system with nonlinear feedback as the projection method, provided that the reference signal has been designed such that $\frac{\Phi_{ur}(z)}{\Phi_r(z)}$ will be stable and causal independently of the structure of the controller. One example of a class of such reference signals is minimum phase filtered white noise.

Finally, it should be emphasized that the discussion here is based on properties of OE-LTI-SOEs and that it hence is valid mostly for large data sets. As a matter of fact, the use of a noncausal $S(z)$ in the projection method might be useful for smaller data sets also in cases where $S(z)$ asymptotically will be causal (Forssell and Ljung, 2000).

5.6 LTI-SOEs with a General Error Model

In the previous section, it was shown that regular OE-LTI-SOEs exhibit a number of interesting properties. It is thus a legitimate question whether these input signals also generate GE-LTI-SOEs with special properties.

Hence, we will now shift focus and discuss GE-LTI-SOEs of nonlinear systems with input signals such that the OE-LTI-SOE is regular. In Example 4.3, it was shown that in general $G_{0,OE}(z)$ and $G_{0,GE}(z)$ are not equal even for open-loop nonlinear systems. However, when the OE-LTI-SOE is regular, it follows that $G_{0,GE}(z)$ will be equal to $G_{0,OE}(z)$. This is shown in the following theorem.

Theorem 5.4
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A4 are fulfilled. Assume that the input signal is such that the OE-LTI-SOE is regular, i.e., that it can be written as

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}. \tag{5.20}$$

Assume also that

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})$$

from (5.15b) has a canonical spectral factorization

$$\Phi_{\eta_0}(z) = L_{\eta_0}(z) r_{\eta_0} L_{\eta_0}(z^{-1}) \tag{5.21}$$

with $r_{\eta_0} > 0$. Then the GE-LTI-SOE is

$$G_{0,GE}(z) = G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}, \tag{5.22a}$$

$$H_{0,GE}(z) = L_{\eta_0}(z). \tag{5.22b}$$

Proof: Let

$$T(z) = \begin{pmatrix} L(z) & 0 \\ z^{-1}G_{0,OE}(z)L(z) & L_{\eta_0}(z) \end{pmatrix},$$

$$Q_\zeta = \begin{pmatrix} r_u & 0 \\ 0 & r_{\eta_0} \end{pmatrix},$$

where $L(z)$ and r_u are factors in the canonical spectral factorization of $\Phi_u(z)$ according to (4.1). Then

$$T(z)Q_\zeta T^T(z^{-1}) = \begin{pmatrix} L(z) & 0 \\ z^{-1}G_{0,OE}(z)L(z) & L_{\eta_0}(z) \end{pmatrix} \begin{pmatrix} r_u & 0 \\ 0 & r_{\eta_0} \end{pmatrix}$$

$$\cdot \begin{pmatrix} L(z^{-1}) & zG_{0,OE}(z^{-1})L(z^{-1}) \\ 0 & L_{\eta_0}(z^{-1}) \end{pmatrix} = \begin{pmatrix} \Phi_u(z) & z\Phi_{uy}(z) \\ z^{-1}\Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} = \Phi_\zeta(z),$$

where we have used (5.20) in the second equality. Since $T(z)$ and

$$T^{-1}(z) = \begin{pmatrix} \frac{1}{L(z)} & 0 \\ -\frac{z^{-1}G_{0,OE}(z)}{L_{\eta_0}(z)} & \frac{1}{L_{\eta_0}(z)} \end{pmatrix}$$

both are analytic in $\{z \in \mathbb{C} \mid |z| \geq 1\}$, $T(\infty) = I$ and $Q_\zeta \succ 0$, we have found the canonical spectral factorization of $\Phi_\zeta(z)$, and from (4.32) in Theorem 4.5 we obtain

$$G_{0,GE}(z) = \frac{zT_{21}(z)}{T_{11}(z)} = G_{0,OE}(z),$$
$$H_{0,GE}(z) = \frac{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}{T_{11}(z)} = L_{\eta_0}(z).$$

□

Theorem 5.4 shows that $G_{0,OE}(z)$ and $G_{0,GE}(z)$ will be equal if the OE-LTI-SOE is regular and that $H_{0,GE}(z)$ in this case will be equal to the canonical spectral factor of $\Phi_{\eta_0}(z)$. Hence, the GE-LTI-SOE will be stable when the OE-LTI-SOE is regular and a canonical spectral factorization of a matrix-valued z-spectrum will not be needed for the calculation of the GE-LTI-SOE. Furthermore, Theorem 5.4 can be used to describe how even and odd nonlinearities will affect the GE-LTI-SOE.

Consider an NFIR system with an input signal that has even probability density functions and is such that Lemmas 5.2 and 5.3 and Theorem 5.4 can be applied. In this case, $G_{0,GE}(z) = G_{0,OE}(z)$ will, by Lemma 5.2, depend only on the odd nonlinearities. Furthermore, (5.6) shows that

$$\Phi_{\eta_0}(z) = \Phi_{d_o}(z) + \Phi_{y_e}(z) + \Phi_w(z).$$

Hence, (5.22b) implies that there will be three different contributions to $H_{0,GE}(z)$. The first, $\Phi_{d_o}(z)$, is the z-spectrum of the unmodeled odd nonlinear part of the system output, while $\Phi_{y_e}(z)$ and $\Phi_w(z)$ are the z-spectra of the contributions to the output from the even part and the noise, respectively. Furthermore, Theorem 5.4 also gives the following corollary.

Corollary 5.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1, A2 and A4 are satisfied. Assume that the input signal is such that the OE-LTI-SOE is regular. Assume also that

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})$$

from (5.15b) has a canonical spectral factorization

$$\Phi_{\eta_0}(z) = L_{\eta_0}(z)r_{\eta_0}L_{\eta_0}(z^{-1})$$

with $r_{\eta_0} > 0$. Then it follows that $\varepsilon_0(t)$ (see (4.47)) has the following properties

$$\Phi_{\varepsilon_0 u}(z) = 0, \tag{5.23a}$$
$$\Phi_{\varepsilon_0}(z) = r_{\eta_0}. \tag{5.23b}$$

Proof: The fact that $G_{0,GE}(z) = G_{0,OE}(z)$ according to Theorem 5.4 implies that (4.47) can be rewritten as

$$\begin{aligned}\varepsilon_0(t) &= H_{0,GE}^{-1}(q)(y(t) - G_{0,GE}(q)u(t)) \\ &= H_{0,GE}^{-1}(q)(y(t) - G_{0,OE}(q)u(t)) = H_{0,GE}^{-1}(q)\eta_0(t).\end{aligned}$$

Using Lemmas 2.1 and 5.4, this gives

$$\begin{aligned}\Phi_{\varepsilon_0 u}(z) &= H_{0,GE}^{-1}(z)\Phi_{\eta_0 u}(z) = 0, \\ \Phi_{\varepsilon_0}(z) &= H_{0,GE}^{-1}(z)\Phi_{\eta_0}(z)H_{0,GE}^{-1}(z^{-1}) = r_{\eta_0},\end{aligned}$$

where (5.21) and (5.22b) have been used in the last equality. The results in (5.23) have thus been shown. □

Corollary 5.1 shows that residual analysis without any further considerations can be used as a validation method also for GE-LTI-SOEs when the corresponding OE-LTI-SOEs are regular since there will be no spurious correlations between $u(t)$ and past $\varepsilon_0(t-k)$. In addition, this shows that there will be no need for a closed-loop interpretation of the GE-LTI-SOE in this case.

In this chapter, the basic properties of OE-LTI-SOEs have been discussed. It has been shown that an OE-LTI-SOE is independent of output additive noise that is uncorrelated with the input and the noise-free output. Furthermore, it has been shown that the OE-LTI-SOE often depends only on the odd nonlinearities in a system and that it can be viewed as a linear operator. A class of input signals guaranteeing that the OE-LTI-SOE will be regular, and some properties that the OE-LTI-SOE and GE-LTI-SOE exhibit in this case have also been discussed. In the next chapter, we will turn our attention to NFIR systems with separable inputs.

6

NFIR Systems with Separable Inputs

Nonlinear finite impulse response (NFIR) systems are systems that can be written as

$$y(t) = f((u(t-k))_{k=0}^{M}) + w(t) \tag{6.1}$$

for some $M \in \mathbb{N}$. The important special case of a static nonlinearity is obtained for $M = 0$. Subsystems consisting of static nonlinearities are common in applications. For example, many control systems contain various types of sensor and input nonlinearities that often can be described as static. Furthermore, many nonlinear systems can be approximated by NFIR models. However, we will not discuss such nonlinear approximations here but instead consider how an NFIR system can be approximated by an LTI model.

Intuitively, the natural LTI approximation of an NFIR system is an FIR model. However, the mean-square error optimal LTI approximation, i.e., the OE-LTI-SOE, of such a system will in general be an LTI system with an infinite impulse response. This might not be a problem if the impulse response length M of the NFIR system is known, since it is always possible to estimate an FIR model with the same impulse response length in this case. Although this model might not be the optimal LTI model, it will at least have a structure that probably can be viewed as reasonable compared to the structure of the nonlinear system.

However, in the more realistic case that M is unknown, the structure of the OE-LTI-SOE becomes important. If an NFIR system with impulse response length M has an OE-LTI-SOE that is an FIR model with impulse response length M, it will be rather easy to find an appropriate linear FIR model of this system. When the number of measurements tends to infinity, the parameters of a chosen FIR model will converge to the parameter values given by Theorem 4.2. The problem of finding the impulse response length M of the NFIR system can thus be solved by estimating linear FIR models with different impulse response lengths. If too large an impulse response length is chosen in the model, the parameters that correspond to the extra terms in the impulse response will simply approach zero asymptotically, just as if the NFIR system would have been a linear FIR system. Hence, it is possible to estimate M without more effort than if the true system would have been linear.

On the other hand, if an NFIR system with impulse response length M has an OE-LTI-SOE with an infinite impulse response length, it will be impossible to estimate M using only linear approximations. In this case, an increase of the impulse response length in an estimated FIR model will reduce the variance of the model residuals and make the model a better approximation of the OE-LTI-SOE according to Theorem 4.2. Since the OE-LTI-SOE has an infinite impulse response, no information about M can be derived from the FIR approximations of it.

With the previous discussion in mind, it seems that it often should be desirable to preserve the finite impulse response property when an NFIR system is approximated by its OE-LTI-SOE. This is the main topic of this chapter and it turns out that this issue is related to the notion of separable processes. An overview of the theory for separable processes was given in Section 2.4. In this chapter, it will be shown that some random multisines are separable. Since random multisines are a common choice of input signal in identification experiments, this is a theoretical result with practical consequences. Here, the usefulness of random multisines will be illustrated in the case of identification of Hammerstein systems. Furthermore, the notion of higher order separability will be introduced and it will be shown that this type of separability gives a necessary and sufficient condition on the input signal for the OE-LTI-SOE of an arbitrary NFIR system to be an FIR model.

6.1 Separability

As was shown in (2.31) in Section 2.4, it is easy to show that separability is a sufficient condition for the invariance property

$$R_{yu}(\tau) = b_0 R_u(\tau), \quad \forall \tau \in \mathbb{Z}$$

from (2.30) to hold. Using z-spectra, the invariance property can be written

$$\Phi_{yu}(z) = b_0 \Phi_u(z),$$

and thus, from Corollary 4.1, it follows that the OE-LTI-SOE of a static nonlinearity is regular and equal to the constant b_0. Hence, for a separable input, the OE-LTI-SOE of a static nonlinearity is static. This result was shown in Nuttall (1958a) and it will be generalized to NFIR systems in Section 6.3.

Although a number of separable processes can be found in literature (Nuttall, 1958a; McGraw and Wagner, 1968), it seems that the problem of how to construct separable processes has not been extensively studied. The generation of a particular nontrivial separable process is described in the following example.

Example 6.1

Consider a process u defined as

$$u(t) = e(t) + e(t-1),$$

where e is a white process with exponential distribution over the interval $[-1, +\infty)$ such that $\mathrm{E}(e(t)) = 0$ and $\mathrm{E}(e(t)^2) = 1$. These properties follow if each random variable $e(t)$ has the probability density function $p(x) = e^{-(x+1)}$ for $x \geq -1$.

Since the process e is white, $u(t+\tau)$ and $u(t)$ are independent if $|\tau| > 1$. Hence, $\mathrm{E}(u(t+\tau)|u(t)) = 0$ for $|\tau| > 1$. Furthermore, we have that

$$\begin{aligned}
\mathrm{E}(u(t+1)|u(t)) &= \underbrace{\mathrm{E}(e(t+1)|e(t)+e(t-1))}_{=0} + \mathrm{E}(e(t)|e(t)+e(t-1)) \\
&= \mathrm{E}(e(t)|e(t)+e(t-1)), \\
\mathrm{E}(u(t-1)|u(t)) &= \mathrm{E}(e(t-1)|e(t)+e(t-1)) + \underbrace{\mathrm{E}(e(t-2)|e(t)+e(t-1))}_{=0} \\
&= \mathrm{E}(e(t-1)|e(t)+e(t-1)).
\end{aligned}$$

From these expressions we see that u is separable if e is such that

$$\mathrm{E}(e(t)|e(t)+e(t-1)=c) = \mathrm{E}(e(t-1)|e(t)+e(t-1)=c) = b \cdot c \tag{6.2}$$

for some constant b that does not depend on c. We will now show that these equalities hold.

Let X and Y be two independent random variables with probability density functions

$$p_X(x) = \begin{cases} e^{-(x+1)}, & \text{if } x \geq -1, \\ 0, & \text{if } x < -1 \end{cases}$$

and

$$p_Y(y) = \begin{cases} e^{-(y+1)}, & \text{if } y \geq -1, \\ 0, & \text{if } y < -1, \end{cases}$$

and let $W = X + Y$. Then the joint probability density function for X and W is

$$p_{X,W}(x,w) = p_X(x)p_Y(w-x) = \begin{cases} e^{-(w+2)}, & \text{if } -1 \leq x \leq w+1, \\ 0, & \text{otherwise.} \end{cases}$$

For $w \geq -2$, it follows that

$$p_W(w) = \int_{-1}^{w+1} p_{X,W}(x,w)\,dx = \int_{-1}^{w+1} e^{-(w+2)}\,dx = (w+2)e^{-(w+2)}$$

such that

$$p_W(w) = \begin{cases} (w+2)e^{-(w+2)}, & \text{if } w \geq -2, \\ 0, & \text{if } w < -2. \end{cases}$$

This gives

$$p_{X|W=c}(x) = \begin{cases} \frac{p_{X,W}(x,c)}{p_W(c)} = \frac{1}{c+2}, & \text{if } -1 \leq x < c+1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathrm{E}(X|W=c) = \int_{-1}^{c+1} \frac{x}{c+2}\,dx = \frac{c}{2}. \tag{6.3}$$

Replacing W with $e(t) + e(t-1)$ and X with either $e(t)$ or $e(t-1)$ in (6.3) shows that (6.2) holds with $b = 1/2$. Hence,

$$\mathrm{E}(u(t+1)|u(t) = c) = \mathrm{E}(u(t-1)|u(t) = c) = \frac{c}{2}$$

and u is thus separable.

In the previous example, a separable process was obtained by passing a white noise process through a linear FIR filter, i.e., by linear filtering of a separable process. Alternatively, this example can be viewed as an example where the sum of two *dependent* separable processes is separable. Just like the sum of two *independent* separable processes is not separable in general (cf. Section 2.4 or Nuttall (1958a)), linear filtering of a separable process will often result in a nonseparable process. For example, the input signal in Example 4.2 is not separable since it gives a nonstatic OE-LTI-SOE of a static nonlinear system. However, this input signal was generated by linear filtering of a white, and thus separable, signal.

The observation that separability often is lost by linear filtering is an important result that must be considered when a separable process is used in an identification experiment. For example, most of the separable processes in Nuttall (1958a) will not be separable after they have passed through a linear subsystem. However, Gaussian processes, which are a subset of the class of separable processes, have the advantage that the Gaussianity, and thus the separability, will not be affected by linear filtering. This property makes these processes particularly useful for some types of identification problems, as will be described later in Chapter 7.

Separable processes are useful not only for LTI approximations of static nonlinearities but also for LTI approximations of Hammerstein systems. As has been mentioned previously in Section 3.2.2, the initial identification of the LTI part of a Hammerstein system without considering the nonlinearity is an important step in many methods. In the following theorem, it is shown that a scaled version of the LTI subsystem is an OE-LTI-SOE of the system if the input is separable.

Theorem 6.1
Consider a Hammerstein system

$$y(t) = G_L(q)v(t) + w(t), \tag{6.4a}$$

$$v(t) = f(u(t)), \tag{6.4b}$$

where $G_L(q)$ is a stable and causal LTI system and where $w(t)$ is measurement noise. Assume that Assumptions A1 and A2 hold if the input is nonperiodic and that Assumption A3 holds if it is periodic. Furthermore, assume that the input is separable and that Assumption A5 holds. Then $b_0 G_L(q)$ with $b_0 = \mathrm{E}\big(f(u(t))u(t)\big)/R_u(0)$ is an OE-LTI-SOE of this system.

Proof: The invariance property

$$\begin{aligned} R_{vu}(\tau) &= \mathrm{E}\big(f(u(t))u(t-\tau)\big) = \mathrm{E}\big(f(u(t))\mathrm{E}\big(u(t-\tau)|u(t)\big)\big) \\ &= \frac{R_u(\tau)}{R_u(0)}\mathrm{E}\big(f(u(t))u(t)\big) = b_0 R_u(\tau) \end{aligned} \tag{6.5}$$

holds since $u(t)$ is separable. Here, Lemma 2.2 has been used in the third equality. The condition that $G_{0,OE}$ should minimize

$$\mathrm{E}\big((y(t) - G(q)u(t))^2\big)$$

is equivalent to $G_{0,OE}$ satisfying the Wiener-Hopf condition

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_{0,OE}(k)R_u(\tau - k) = 0, \quad \tau \geq 0. \tag{6.6}$$

From the system description (6.4a), we have

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_L(k)R_{vu}(\tau - k) = 0, \quad \forall \tau \in \mathbb{Z}$$

and inserting (6.5) gives

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} b_0 g_L(k)R_u(\tau - k) = 0, \quad \forall \tau \in \mathbb{Z}.$$

Hence, $b_0 G_L(q)$ is an OE-LTI-SOE of the system. □

Note that for a nonperiodic input and output satisfying Assumptions A1 and A2, the model $b_0 G_L(q)$ is the unique OE-LTI-SOE of the system.

6.2 Separable Random Multisines

Although the separability of a single random phase sinusoid was shown in Nuttall (1958a), there does not seem to be any results about separability of random multisines in literature. However, it turns out that such signals are separable if all amplitudes are constant and equal. This result is proven in the following lemma.

Lemma 6.1
A random multisine

$$u(t) = \sum_{k=1}^{Q} A_k \cos(\omega_k t + \psi_k),$$

where all A_k *are constants,* $A_k = \bar{A}$*, and all* ψ_k *are independent random variables with uniform distribution on the interval* $[0, 2\pi)$*, is separable.*

Proof: The signals

$$u_k(t) = A_k \cos(\omega_k t + \psi_k), \quad k = 1, 2, \ldots, Q$$

are independent and

$$f_{u_k,1}(\xi_1) = f_{u_l,1}(\xi_1) = J_0(\bar{A}\xi_1),$$
$$\sigma_k^2 = \sigma_l^2 = \frac{\bar{A}^2}{2}$$

for all $k, l \in \{1, 2, \ldots, Q\}$ from Lemma 2.3. Hence, $u(t)$ is separable according to Theorem 2.2. □

Besides the fact that the separability of random multisines is theoretically interesting, it gives a mathematical explanation to why these signals have turned out to be so useful for identification of Hammerstein systems in previous works (Crama and Schoukens, 2001; Crama et al., 2004; Crama and Schoukens, 2004, 2005).

For a separable random multisine, Theorem 6.1 shows that the LTI part of the system can be estimated without compensating for or estimating the nonlinearity at the input of the system. Hence, the model that minimizes the cost function $V_{N_E,N}$ defined in (2.8) will usually be a good approximation of the LTI subsystem if the number of experiments N_E is large. Furthermore, Theorem 6.1 shows that the number of excited frequencies does not affect the fact that the frequency response of the LTI subsystem can be estimated consistently at these frequencies without considering the nonlinearity. This result is verified numerically in the following example.

Example 6.2

Consider the Hammerstein system

$$y(t) = G_L(q)v(t) = \frac{1.6 - 1.6q^{-1} + 0.4q^{-2}}{1 - 1.56q^{-1} + 0.96q^{-2}}v(t), \tag{6.7a}$$

$$v(t) = f(u(t)) = u(t)^3, \tag{6.7b}$$

with the input

$$u_1(t) = \sum_{k=1}^{6} \cos(\omega_k t + \psi_k), \tag{6.8}$$

where $\omega_k = 2\pi k/40$ and where ψ_k are independent random variables with uniform distribution on the interval $[0, 2\pi)$. 500 realizations of the phases have been generated and an input signal with 400 samples has been constructed for each realization. For each input, an identification experiment has been performed where the last periods (40 samples) of the input and output signals have been collected.

Based on the 500 data sets, each consisting of 40 input and output measurements, a nonparametric frequency response estimate $\hat{G}(e^{i\omega_k})$ has been computed using the least-squares solution (2.13) that minimizes the cost function $V_{N_E,N}$ in (2.8). A scaled version of this estimate is shown in Figure 6.1 together with the linear part of the system. As can be seen in this figure, the nonparametric frequency response estimate is very close to being a scaled version of $G_L(e^{i\omega})$. Actually, the relative errors

$$\rho_k = \frac{|\hat{G}(e^{i\omega_k})/\hat{c}_0 - G_L(e^{i\omega_k})|}{|G_L(e^{i\omega_k})|}, \quad k = 1, 2, \ldots, 6,$$

where

$$\hat{c}_0 = \frac{1}{6}\sum_{k=1}^{6} \frac{|\hat{G}(e^{i\omega_k})|}{|G_L(e^{i\omega_k})|},$$

are less than 4% here. An identical identification experiment has been performed with the input

$$u_2(t) = \sum_{k=1}^{6} 2^{5-k} \cos(\omega_k t + \psi_k).$$

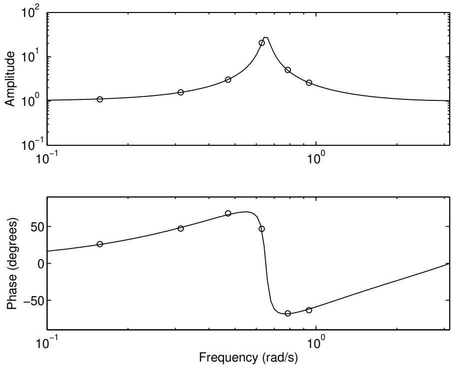


Figure 6.1: *The frequency response of the linear part $G_L(q)$ of the Hammerstein system from Example 6.2 (solid line) and a scaled version of the nonparametric frequency response estimate (circles).*

However, in this case the relative errors were 27%, 5%, 12%, 12%, 12% and 17%, respectively, i.e., significantly larger than for the separable random multisine. Furthermore, for $u_2(t)$, the relative errors do not decrease if the number of phase realizations is increased. Hence, it seems that for this input, the best LTI approximation is not a scaled version of $G_L(q)$.

Example 6.2 illustrates the main benefit of using separable random multisines for Hammerstein system identification since it shows that for such an input signal, it is rather easy to obtain good estimates of the LTI part of the system. Furthermore, the example indicates that not all random multisines are separable.

6.3 Higher Order Separability

Since separability of an input signal is such a useful property for the analysis of systems with static nonlinearities, it is interesting to investigate if this notion can be generalized. It turns out that it is actually rather straightforward to define a more general type of separability.

6.3.1 Definition and Basic Properties

As a matter of fact, the type of separability in Definition 2.6 should be denoted separability of order one to distinguish it from a more general type of separability. However, in case the order of separability is obvious from the context, we will continue to say that a process is separable whenever there is no risk for ambiguities.

Separability of a process means that certain conditional expectations are linear. A more general version of the previous definition of separability can be written as follows.

Definition 6.1 (Separability of order $M+1$). Consider an integer $M \geq 0$ and a stationary stochastic process u with zero mean. This process is *separable of order* $M+1$

if

$$\mathrm{E}\big(u(t-\tau)|u(t), u(t-1)\ldots, u(t-M)\big) = \sum_{i=0}^{M} a_{\tau,i} u(t-i), \quad \forall \tau \in \mathbb{Z}, \tag{6.9}$$

i.e., if the conditional expectation is linear in u.

Obviously, the previous definition of first order separability is a special case of Definition 6.1.

As was mentioned in Chapter 2, the notion of separability of order one is discussed in detail in Nuttall (1958a) and it is also mentioned briefly (on page 55) that this notion might be extended to separability of higher orders by considering integrals like

$$\int_{-\infty}^{\infty} x_t p(x_t, x_{t-\tau_1}, x_{t-\tau_2})\, dx_t.$$

However, no further conclusions are drawn in Nuttall (1958a) and to the author's knowledge, no such extension has been made elsewhere.

It is not obvious that all first order separable processes are separable of higher orders as well. However, there are at least a few such cases. For example, since (6.9) is a well-known property of Gaussian signals (see, for example, Brockwell and Davis, 1987, p. 64), it immediately follows that such signals are separable of order $M+1$ for any $M \in \mathbb{N}$. Furthermore, it is easy to see that white, possibly non-Gaussian, signals satisfy (6.9) too.

It is straightforward to show that higher order separability implies lower order separability.

Theorem 6.2 (Separability of different orders)
If a process u is separable of order $M+1$, it is separable of order $1, 2, \ldots, M$ too.

Proof: It holds that

$$\begin{aligned}
&\mathrm{E}\big(u(t-\tau)|u(t), u(t-1)\ldots, u(t-M+1)\big)\\
&= \mathrm{E}_{u(t-M)}\big(\mathrm{E}\big(u(t-\tau)|u(t), u(t-1)\ldots, u(t-M)\big)\big)\\
&= \mathrm{E}_{u(t-M)}\big(\sum_{i=0}^{M} a_{\tau,i} u(t-i)\big) = \sum_{i=0}^{M-1} a_{\tau,i} u(t-i), \quad \forall \tau \in \mathbb{Z}.
\end{aligned}$$

Hence, the process is separable of order M. The result follows by induction. □

6.3.2 Higher Order Separability and OE-LTI-SOEs

We will here consider NFIR systems (6.1) with input signals $u(t)$ that satisfy the conditions in Assumption A1, i.e., real-valued inputs with zero mean, an exponentially bounded covariance function and a z-spectrum with a canonical spectral factorization. For each choice of such a stochastic process u, let D_u be a class of Lebesgue integrable functions such that

$$\begin{aligned}
D_u = \{f : \mathbb{R}^{M+1} \to \mathbb{R} \mid &\mathrm{E}\big(f((u(t-k))_{k=0}^{M})\big) = 0,\\
&\mathrm{E}\big(f((u(t-k))_{k=0}^{M})^2\big) < \infty,\\
&R_{yu}(\tau) = \mathrm{E}\big(f((u(t-k))_{k=0}^{M}) u(t-\tau)\big) \text{ exists } \forall \tau \in \mathbb{Z}\}.
\end{aligned}$$

This class of functions is a straightforward generalization of the corresponding definition for functions on $\mathbb{R}$ in (2.32). Note that the conditions in the definition of D_u are weaker than the related conditions on the output signal in Assumption A2. Here, we will use the notation

$$\mathbf{R}_U = \begin{pmatrix} R_u(0) & R_u(1) & \dots & R_u(M) \\ R_u(1) & R_u(0) & \dots & R_u(M-1) \\ \vdots & & & \vdots \\ R_u(M) & R_u(M-1) & \dots & R_u(0) \end{pmatrix}, \tag{6.10}$$

$$\mathbf{R}_{YU} = \begin{pmatrix} R_{yu}(0) & R_{yu}(1) & \dots & R_{yu}(M) \end{pmatrix}^T.$$

We will here discuss under which conditions the OE-LTI-SOE of an NFIR system will be an FIR model. In this discussion, we will need the notion of the mean-square error optimal FIR model of a system. The following lemma is a classic result (see, for example, Kailath et al., 2000, Theorems 3.2.1 and 3.2.2) and holds for each fixed choice of u. It is included here for the sake of completeness.

Lemma 6.2 (FIR approximation)
Consider an input signal u that fulfills the conditions in Assumption A1 and for which $\mathbf{R}_U$ is positive definite ($\mathbf{R}_U \succ 0$). Then for each NFIR system f in the corresponding class D_u, there exists a unique linear FIR model

$$G_{0,FIR}(z) = \sum_{k=0}^{M} \bar{b}_f(k) z^{-k}$$

of length M that is an optimal FIR approximation of length M in the mean-square error sense. This FIR model has parameters

$$\bar{B}_f = \begin{pmatrix} \bar{b}_f(0) & \bar{b}_f(1) & \dots & \bar{b}_f(M) \end{pmatrix}^T = \mathbf{R}_U^{-1}\mathbf{R}_{YU} \tag{6.11}$$

and satisfies

$$R_{yu}(\tau) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\tau - k), \quad \tau = 0, 1, \dots, M. \tag{6.12}$$

Proof: The parameters $\bar{B}_f$ in the optimal FIR model of length M are obtained by minimizing

$$\mathrm{E}\left(\left(f((u(t-k))_{k=0}^{M}) - \sum_{k=0}^{M} b(k) u(t-k) \right)^2 \right)$$

with respect to B. Differentiating this expression with respect to $b(i)$ gives the following equations for the stationary points of the mean-square error criterion

$$-2\left(R_{yu}(i) - \sum_{k=0}^{M} \bar{b}_f(k) R_u(i-k) \right) = 0, \quad i = 0, 1, \dots, M.$$

This can also be written as

$$\mathbf{R}_U \bar{B}_f = \mathbf{R}_{YU}$$

and (6.11) and (6.12) follows readily from this expression. □

From (6.12) we see that $G_{0,FIR}$ can explain the cross-covariance function $R_{yu}(\tau)$ for $\tau = 0, 1, \ldots, M$. However, sometimes it can actually explain the complete cross-covariance function, i.e.,

$$R_{yu}(\tau) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\tau - k), \quad \forall \tau \in \mathbb{Z} \tag{6.13}$$

or, equivalently,

$$\Phi_{yu}(z) = G_{0,FIR}(z)\Phi_u(z).$$

In this case, we know from Corollary 4.1 that $G_{0,FIR}$ is not only the mean-square error optimal FIR approximation of length M of the system, but also the OE-LTI-SOE of the system. It turns out that this will always be true if the input process is separable of order $M + 1$ and $f \in D_u$.

First, we will here consider noise-free NFIR systems, i.e., nonlinear systems with impulse response lengths $M \geq 0$ that can be written as

$$y(t) = f((u(t-k))_{k=0}^{M}).$$

We will use the following notation

$$\mathbf{R}_{U,\tau} = \begin{pmatrix} R_u(\tau) & R_u(\tau - 1) & \ldots & R_u(\tau - M) \end{pmatrix}^T,$$

and we will assume that $\mathbf{R}_U$ (see (6.10)) is a positive definite matrix ($\mathbf{R}_U \succ 0$) such that the vector

$$C_\tau = \begin{pmatrix} c_{\tau,0} & c_{\tau,1} & \ldots & c_{\tau,M} \end{pmatrix}^T = \mathbf{R}_U^{-1}\mathbf{R}_{U,\tau} \tag{6.14}$$

is well-defined.

We will now show that the definition of separability implies that the constants $a_{\tau,i}$ from Definition 6.1 satisfy $a_{\tau,i} = c_{\tau,i}$.

Lemma 6.3
Consider a signal u which is separable of order $M + 1$, i.e., that (6.9) holds for some constants $a_{\tau,i}$. Then it holds that

$$A_\tau = C_\tau = \mathbf{R}_U^{-1}\mathbf{R}_{U,\tau}, \tag{6.15}$$

where

$$A_\tau = \begin{pmatrix} a_{\tau,0} & a_{\tau,1} & \ldots & a_{\tau,M} \end{pmatrix}^T.$$

Proof: For $k = 0, 1, \ldots, M$, Definition 6.1 gives

$$\begin{aligned} R_u(\tau - k) &= \mathrm{E}\big(u(t-k)u(t-\tau)\big) \\ &= \mathrm{E}\Big(\mathrm{E}\big(u(t-k)u(t-\tau)|u(t), u(t-1)\ldots, u(t-M)\big)\Big) \\ &= \mathrm{E}\Big(u(t-k)\mathrm{E}\big(u(t-\tau)|u(t), u(t-1)\ldots, u(t-M)\big)\Big) \\ &= \sum_{i=0}^{M} a_{\tau,i}\mathrm{E}\big(u(t-k)u(t-i)\big) = \sum_{i=0}^{M} a_{\tau,i} R_u(k-i). \end{aligned}$$

The previous expression can also be written as

$$\mathbf{R}_U A_\tau = \mathbf{R}_{U,\tau}.$$

This shows that (6.15) holds. □

The previous lemma shows that separability of order $M+1$ is equivalent to that the property

$$\mathrm{E}(u(t-\tau)|u(t), u(t-1)\ldots, u(t-M)) = \sum_{i=0}^{M} c_{\tau,i} u(t-i), \quad \forall \tau \in \mathbb{Z} \tag{6.16}$$

holds. In the next lemma, we will show that separability of u is a necessary and sufficient condition for the equality (6.13) to hold for all $\tau \in \mathbb{Z}$ and for all $f \in D_u$.

Lemma 6.4 (Separability of order $M+1$)
Consider a fixed $M \geq 0$ and a certain choice of input signal u that fulfills the conditions in Assumption A1, and for which $\mathbf{R}_U \succ 0$ and $\mathrm{E}(|u(t)|) < \infty$. Let $\bar{B}_f$ denote the parameters of the mean-square error optimal FIR approximation of length M of each $f \in D_u$, i.e., $\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{YU}$ according to Lemma 6.2. Then

$$R_{yu}(\tau) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\tau-k), \quad \forall \tau \in \mathbb{Z} \text{ and } \forall f \in D_u \tag{6.17}$$

if and only if u is separable of order $M+1$.

Proof: Using (6.14) and (6.11), it follows that

$$\sum_{k=0}^{M} \bar{b}_f(k) R_u(\tau-k) = \bar{B}_f^T \mathbf{R}_{U,\tau} = \mathbf{R}_{YU}^T C_\tau = \sum_{i=0}^{M} c_{\tau,i} R_{yu}(i). \tag{6.18}$$

IF: Assume that u is separable of order $M+1$, i.e., that (6.16) holds. By the construction of $\bar{B}_f$, the equality (6.17) already holds for $\tau = 0, 1, \ldots, M$ for all $f \in D_u$ (cf. (6.11)). Take an arbitrary $f \in D_u$ and let $y(t) = f((u(t-k))_{k=0}^M)$. Furthermore, take an arbitrary $\tau > M$ or $\tau < 0$. Then it follows that

$$\begin{aligned} R_{yu}(\tau) &= \mathrm{E}\big(y(t)u(t-\tau)\big) = \mathrm{E}\big(\mathrm{E}\big(y(t)u(t-\tau)|u(t), u(t-1), \ldots, u(t-M)\big)\big) \\ &= \mathrm{E}\big(y(t)\mathrm{E}\big(u(t-\tau)|u(t), u(t-1), \ldots, u(t-M)\big)\big) \\ &= \sum_{i=0}^{M} c_{\tau,i} \mathrm{E}(y(t)u(t-i)) = \sum_{i=0}^{M} c_{\tau,i} R_{yu}(i) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\tau-k), \end{aligned}$$

where the third equality follows from the fact that $y(t)$ depends only on $u(t), u(t-1), \ldots, u(t-M)$ while the fourth equality follows from (6.16) and the last from (6.18). Since both f and τ were arbitrary, (6.17) holds for all $\tau \in \mathbb{Z}$ and for all $f \in D_u$.

ONLY IF: Assume that (6.17) holds for a particular u. Take an arbitrary $\tau > M$ or $\tau < 0$. Using (6.18), (6.17) gives the equality

$$\int_{\mathbb{R}^{M+1}} f(x_t, \ldots, x_{t-M}) \Big(\int_{-\infty}^{\infty} x_{t-\tau} p_\tau(x_t, \ldots, x_{t-M}, x_{t-\tau})\, dx_{t-\tau} - \sum_{i=0}^{M} c_{\tau,i} x_{t-i} p(x_t, \ldots, x_{t-M}) \Big)\, dx_t \ldots dx_{t-M} = 0, \quad \forall f \in D_u, \tag{6.19}$$

where p and p_τ are the joint probability density functions of

$$(u(t), u(t-1), \ldots, u(t-M))^T$$

and

$$(u(t), u(t-1), \ldots, u(t-M), u(t-\tau))^T,$$

respectively. Let

$$v_\tau(x_t, \ldots, x_{t-M}) = \int_{-\infty}^{\infty} x_{t-\tau} p_\tau(x_t, \ldots, x_{t-M}, x_{t-\tau})\, dx_{t-\tau} - \sum_{i=0}^{M} c_{\tau,i} x_{t-i} p(x_t, \ldots, x_{t-M})$$

and define a function

$$f_0(x_t, \ldots, x_{t-M}) = \mathrm{sign}(v_\tau(x_t, \ldots, x_{t-M})) - \mu_0,$$

where

$$\mu_0 = \mathrm{E}\big(\mathrm{sign}\big(v_\tau((u(t-k))_{k=0}^{M})\big)\big).$$

Since

$$\mathrm{E}\big(f_0((u(t-k))_{k=0}^{M})\big) = 0,$$
$$\mathrm{E}\big(f_0((u(t-k))_{k=0}^{M})^2\big) < \infty$$

and

$$\begin{aligned}
&|\mathrm{E}\big(f_0((u(t-k))_{k=0}^{M}) u(t-\sigma)\big)| \\
&= |\mathrm{E}\big(\mathrm{sign}\big(v_\tau((u(t-k))_{k=0}^{M})\big) u(t-\sigma)\big) - \mu_0 \underbrace{\mathrm{E}(u(t-\sigma))}_{=0}| \\
&= |\mathrm{E}\big(\mathrm{sign}\big(v_\tau((u(t-k))_{k=0}^{M})\big) u(t-\sigma)\big)| \\
&\le \mathrm{E}\big(|\mathrm{sign}\big(v_\tau((u(t-k))_{k=0}^{M})\big) u(t-\sigma)|\big) \\
&\le \mathrm{E}(|u(t-\sigma)|) < \infty, \quad \forall \sigma \in \mathbb{Z},
\end{aligned}$$

it follows that $f_0 \in D_u$. Hence, (6.19) holds for $f = f_0$ and this implies that

$$\int\limits_{\mathbb{R}^{M+1}} |v_\tau(x_t, \ldots, x_{t-M})| \, dx_t \ldots dx_{t-M}$$

$$- \mu_0 \underbrace{\mathrm{E}(u(t-\tau))}_{=0} + \mu_0 \sum_{i=0}^{M} c_{\tau,i} \underbrace{\mathrm{E}(u(t-i))}_{=0} = 0$$

$$\Rightarrow \int\limits_{\mathbb{R}^{M+1}} |v_\tau(x_t, \ldots, x_{t-M})| \, dx_t \ldots dx_{t-M} = 0$$

$$\Rightarrow v_\tau(x_t, \ldots, x_{t-M}) = 0 \quad \text{almost everywhere.}$$

The conditional probability density function of $u(t-\tau)$ given $u(t) = x_t, u(t-1) = x_{t-1}, \ldots, u(t-M) = x_{t-M}$ is

$$p_{\tau,c}(x_{t-\tau}) = \frac{p_\tau(x_t, \ldots, x_{t-M}, x_{t-\tau})}{p(x_t, \ldots, x_{t-M})}$$

if $p(x_t, \ldots, x_{t-M}) > 0$. Hence, the fact that

$$v_\tau(x_t, \ldots, x_{t-M}) = 0$$

implies that

$$\int\limits_{-\infty}^{\infty} x_{t-\tau} p_{\tau,c}(x_{t-\tau}) \, dx_{t-\tau} = \sum_{i=0}^{M} c_{\tau,i} x_{t-i}$$

or, equivalently, that (6.16) holds for the chosen τ. Since τ was arbitrary, (6.16) follows and u is thus separable of order $M+1$. □

Lemma 6.4 is an extension of the corresponding theorem about separability of order one in Nuttall (1958a). Just like this classic result can be used to study linear approximations of static nonlinearities, Lemma 6.4 can be used to characterize exactly for which input signals the OE-LTI-SOE of an arbitrary NFIR system will be an FIR model.

Besides the results about mean-square error optimal stable and causal LTI predictors, which here have been used to define OE-LTI-SOEs, classic Wiener filtering theory also contains results about mean-square error optimal stable noncausal LTI predictors, usually known as *Wiener smoothers* or noncausal Wiener filters (see, for example, Kailath et al., 2000, Theorem 7.3.1). For a nonlinear system with input u and output y that satisfy Assumptions A1 and A2, these results show that the best, in mean-square error sense, stable but possibly noncausal LTI approximation of this system is given by the ratio

$$\frac{\Phi_{yu}(z)}{\Phi_u(z)}. \tag{6.20}$$

A simple example of a causal nonlinear system with an input such that (6.20) becomes noncausal can be found in Forssell and Ljung (2000). However, for a separable input, this cannot happen since the following result holds.

Theorem 6.3
Consider a fixed $M \geq 0$ *and a certain input signal* u *that fulfills the conditions in Assumption A1, and for which* $\mathbf{R}_U \succ 0$ *and* $\mathrm{E}(|u(t)|) < \infty$*. Consider NFIR systems*

$$y(t) = y_{nf}(t) + w(t) = f((u(t-k))_{k=0}^{M}) + w(t),$$

where the noise $w(t)$ *is such that Assumption A5 is fulfilled for all* f*. Then the OE-LTI-SOE of such a system will be well-defined and equal to a linear FIR model*

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \sum_{k=0}^{M} \bar{b}_f(k) z^{-k}, \tag{6.21}$$

where $\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{YU}$ *for* all $f \in D_u$*, if and only if* u *is separable of order* $M+1$*.*

Proof: Assumption A5 gives $\Phi_{yu}(z) = \Phi_{y_{nf}u}(z)$. Hence, the OE-LTI-SOE is not influenced by the noise term $w(t)$. Since the input satisfies the conditions in Lemma 6.4, we have that

$$\Phi_{y_{nf}u}(z) = \sum_{k=0}^{M} \bar{b}_f(k) z^{-k} \Phi_u(z), \tag{6.22}$$

where $\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{YU}$ for all $f \in D_u$, if and only if u is separable of order $M+1$. If (6.22) holds for all $f \in D_u$, the NFIR systems that correspond to these functions have outputs that satisfy Assumption A2. Hence, the OE-LTI-SOEs of these NFIR systems are well-defined and Corollary 4.1 can be applied to show that

$$G_{0,OE}(z) = \frac{\Phi_{y_{nf}u}(z)}{\Phi_u(z)} = \sum_{k=0}^{M} \bar{b}_f(k) z^{-k}$$

for all $f \in D_u$. The theorem has thus been shown. □

Theorem 6.3 shows that separability of order $M+1$ is a necessary and sufficient condition for the OE-LTI-SOE to be equal to an FIR model of length M for all NFIR systems defined by functions in D_u. Furthermore, this theorem shows that even if we consider noncausal LTI models, a separable input will give an optimal model that is a causal FIR model.

In many cases, it is possible to shed some light on a theoretical result by interpreting it in a geometrical framework. This can be done also in our case. For a fixed t, we can view the output $y(t)$ and the components of the input signal $u(\tau)$, $\tau \in \mathbb{Z}$ as vectors in an infinite dimensional inner-product space with the inner product $\langle u, v\rangle = \mathrm{E}(uv)$ (see Brockwell and Davis, 1987).

The output from the OE-LTI-SOE of the NFIR system will in this framework be the orthogonal projection of $y(t)$ into the linear subspace that is spanned by

$$u(t), u(t-1), \ldots, u(t-\infty).$$

From (6.21) we can draw the conclusion that this projection actually lies in the finite dimensional linear subspace that is spanned by

$$u(t), u(t-1), \ldots, u(t-M)$$

if u is separable. This is a compact and intuitive way to describe the theoretical results presented previously in this section.

6.3.3 Identification of Generalized Hammerstein Systems

Just like first order separability can be used when a Hammerstein system is approximated by an LTI model, higher order separability can be used to characterize the OE-LTI-SOE of a more general class of systems that we will call generalized Hammerstein systems. These systems consist of an NFIR system followed by an LTI system and the OE-LTI-SOE of such a system with a separable input is described in the following theorem.

Theorem 6.4
Consider a generalized Hammerstein system

$$y(t) = G_L(q)v(t) + w(t), \tag{6.23a}$$

$$v(t) = f((u(t-k))_{k=0}^{M}), \tag{6.23b}$$

where $G_L(q)$ is a stable and causal LTI system and where $w(t)$ is measurement noise. Assume that the input to this system is separable of order $M+1$ and that Assumptions A1, A2 and A5 hold. Then

$$G_{0,OE}(q) = G_L(q)B(q), \tag{6.24}$$

where

$$B(q) = \sum_{k=0}^{M} \bar{b}_f(k) q^{-k}$$

and $\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{VU}$.

Proof: Assumption A5 implies that

$$\Phi_{yu}(z) = G_L(z)\Phi_{vu}(z)$$

and Lemma 6.4 gives

$$\Phi_{vu}(z) = B(z)\Phi_u(z).$$

Hence,

$$\Phi_{yu}(z) = G_L(z)B(z)\Phi_u(z)$$

and (6.24) follows from Corollary 4.1. □

Since $B(q)$ is an FIR model, the previous theorem shows that the denominator of $G_L(q)$ can be consistently estimated without compensating for the NFIR subsystem.

The focus of this chapter has been on FIR approximations of NFIR systems. The separability of a class of random multisines has been shown. Furthermore, it has been shown that separability of order $M+1$ is a necessary and sufficient condition on the input signal for the OE-LTI-SOE of any NFIR system in a rather wide class of systems to be an FIR model. In the next chapter, NFIR systems with Gaussian inputs will be studied and it will be shown that a couple of additional useful properties hold for Gaussian inputs.

7

NFIR Systems with Gaussian Inputs

Random variables and processes with Gaussian distributions play a rather special role in probability theory and applications. They exhibit a large number of properties that simplify many statistical problems or make the solutions to the problems more general. One interesting property that holds if two random variables have a simultaneous Gaussian distribution, is that these variables are independent if and only if they are uncorrelated.

This property implies that a Gaussian process u, whose z-spectrum has a canonical spectral factorization, can always be viewed as if it has been generated by filtering white Gaussian noise through a minimum phase LTI filter. This results follows from the fact that any signal can be viewed as if it has been generated by filtering an *uncorrelated* signal with a minimum phase filter. Hence, the OE-LTI-SOE of an arbitrary causal nonlinear system with a Gaussian input will be

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

provided that the input and output signals fulfill the conditions in Theorem 5.2. This implies that all properties from Sections 5.5 and 5.6 will hold also for Gaussian input signals. However, in this chapter we will show that OE-LTI-SOEs for Gaussian inputs also have properties that other minimum phase generated inputs do not have.

For example, a Gaussian process is separable of any order. This observation was made in the previous chapter and implies that for an NFIR system with a Gaussian input, the cross-covariance function between y and u can always be written as in (6.13). This result is a kind of generalization of Bussgang's theorem (Theorem 3.1) to NFIR systems. The reason why (6.13) is not a proper generalization of Bussgang's theorem is that it is not obvious that the coefficients $\bar{b}_f(k)$ can be calculated as expectations of derivatives of f in the same way as $b_0 = \mathrm{E}(f'(u(t)))$ in Theorem 3.1. In the next section, a generalized version of Bussgang's theorem will be shown.

This result implies that the OE-LTI-SOEs of NFIR systems with Gaussian inputs always will be linear FIR models and that the coefficients in these models will have a natural definition in terms of properties of the true system. This relation between the FIR model parameters and the true system and the fact that, unlike separability, Gaussianity of a process is preserved under linear filtering turn out to be useful in some applications.

It will be shown later in this chapter that these properties of Gaussian signals can be used for structure identification of NFIR systems and for identification of classes of systems that here will be called *generalized Wiener-Hammerstein systems*. However, first we will show the generalization of Bussgang's classic theorem.

7.1 OE-LTI-SOEs of NFIR Systems with Gaussian Input Processes

The generalization of Bussgang's theorem to NFIR systems can be found in, for example, Scarano et al. (1993) and has also previously been used in the research area of stochastic mechanical vibrations (see, for example, Atalik and Utku, 1976; Lutes and Sarkani, 1997). We will however restate the result here under the following technical assumptions.

Assumption A6. Assume that the real-valued functions $f(x)$ and $p(\tilde{x})$, where $x \in \mathbb{R}^N$ and $\tilde{x} = (x^T, x_{N+1})^T \in \mathbb{R}^{N+1}$, are such that $f \cdot p$, $f'_{x_i} \cdot p$ and $f \cdot \tilde{x}_i \cdot p$, $i = 1, \ldots, (N+1)$ all belong to $L^1(\mathbb{R}^{N+1})$ and that $f(x)p(\tilde{x}) \to 0$ when $|\tilde{x}| \to +\infty$. (Here, f'_{x_i} is the partial derivative of f with respect to x_i).

Assumption A7. Consider two stationary stochastic processes u and y such that $y(t) = f((u(t-k))_{k=0}^{M})$. Assume that u is a Gaussian process with zero mean and that $\mathrm{E}(y(t)) = 0$. Form random vectors

$$\omega_\sigma = (u(t), u(t-1), \ldots, u(t-M), u(t-\sigma))^T \tag{7.1}$$

with $\sigma < 0$ or $\sigma > M$. Let P_σ and p_σ denote the covariance matrices and joint probability density functions of these vectors, respectively. Assume that $\det P_\sigma \neq 0$ and that f and p_σ satisfy Assumption A6 for all $\sigma < 0$ or $\sigma > M$.

Assumptions A6 and A7 assure that the input is Gaussian and that the function $f(x)$ does not grow too fast. Assumption A6 holds if, for example, f is a polynomial and p is a Gaussian probability density function. This assumption is used in the following lemma.

Lemma 7.1
Let

$$\tilde{x} = (x^T, x_{N+1})^T = (x_1, x_2, \ldots, x_N, x_{N+1})^T \tag{7.2}$$

be a jointly Gaussian distributed random vector with zero mean and covariance matrix C *with* $\det C \neq 0$. *Let* $f : \mathbb{R}^N \to \mathbb{R}$ *be a differentiable function of* x *with* $\mathrm{E}(f(x)) = 0$ *and let* p *denote the probability density function of* $\tilde{x}$. *Furthermore, assume that* f *and* p *satisfy Assumption A6. Then*

$$\mathrm{E}(f(x)\tilde{x}) = Cw, \tag{7.3}$$

where

$$w = \begin{pmatrix} \mathrm{E}(f'_{x_1}(x)) \\ \mathrm{E}(f'_{x_2}(x)) \\ \vdots \\ \mathrm{E}(f'_{x_N}(x)) \\ 0 \end{pmatrix}.$$

Proof: Factorize C as $C = \tilde{Q}\tilde{Q}^T$ and define a new stochastic vector z as $z = \tilde{Q}^{-1}\tilde{x}$. Then z is jointly normally distributed with zero mean and a covariance matrix that is equal to the identity matrix. Let Q denote the matrix that is obtained from $\tilde{Q}$ by removing the last row. Then $x = Qz$ and we get

$$\begin{aligned} \mathrm{E}(f(x)\tilde{x}) &= \tilde{Q}\mathrm{E}(f(x)\tilde{Q}^{-1}\tilde{x}) = \tilde{Q}\mathrm{E}(f(Qz)z) \\ &= \tilde{Q}\begin{pmatrix} \mathrm{E}(\frac{\partial f(Qz)}{\partial z_1}) \\ \mathrm{E}(\frac{\partial f(Qz)}{\partial z_2}) \\ \vdots \\ \mathrm{E}(\frac{\partial f(Qz)}{\partial z_{N+1}}) \end{pmatrix} = \tilde{Q}\tilde{Q}^T \begin{pmatrix} \mathrm{E}(f'_{x_1}(x)) \\ \mathrm{E}(f'_{x_2}(x)) \\ \vdots \\ \mathrm{E}(f'_{x_N}(x)) \\ 0 \end{pmatrix} = Cw. \end{aligned}$$

The third equality follows from the fact that $\mathrm{E}(h(z)z_i) = \mathrm{E}(h'_{z_i}(z))$ when z has an $N(0, I)$ distribution. This equality holds since

$$\int_{-\infty}^{\infty} g(r)re^{-r^2/2}\,dr = \Big[-g(r)e^{-r^2/2}\Big]_{r=-\infty}^{\infty} + \int_{-\infty}^{\infty} g'(r)e^{-r^2/2}\,dr.$$

Furthermore, the fourth equality in the derivation above follows from the chain rule, which can be written here as

$$\frac{\partial f(Qz)}{\partial z_i} = \frac{\partial f(Qz)}{\partial x_1}Q_{1i} + \frac{\partial f(Qz)}{\partial x_2}Q_{2i} + \ldots + \frac{\partial f(Qz)}{\partial x_N}Q_{Ni}.$$

□

Lemma 7.1 is used in the following generalization of Bussgang's theorem.

Theorem 7.1
Let $y(t) = f((u(t-k))_{k=0}^M)$ be an NFIR system with a stationary Gaussian process u as input. Assume that u and y satisfy Assumption A7. Then it follows that

$$R_{yu}(\tau) = \sum_{k=0}^{M} b(k)R_u(\tau - k), \quad \forall \tau \in \mathbb{Z}, \tag{7.4}$$

where

$$b(k) = \mathrm{E}\big(f'_{u(t-k)}\big((u(t-j))_{j=0}^M\big)\big).$$

Proof: Choose an arbitrary $\sigma < 0$ or $\sigma > M$ and let

$$x = (u(t), u(t-1), \dots, u(t-M))^T$$

and $x_{N+1} = u(t-\sigma)$ in Lemma 7.1. Then Equation (7.3) gives

$$\mathrm{E}(y(t)\begin{pmatrix} u(t) \\ u(t-1) \\ \vdots \\ u(t-M) \\ u(t-\sigma) \end{pmatrix}) = \begin{pmatrix} R_u(0) & R_u(1) & \dots & R_u(M) & R_u(\sigma) \\ R_u(1) & R_u(0) & \dots & R_u(M-1) & R_u(\sigma-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_u(M) & R_u(M-1) & \dots & R_u(0) & R_u(\sigma-M) \\ R_u(\sigma) & R_u(\sigma-1) & \dots & R_u(\sigma-M) & R_u(0) \end{pmatrix} w, \quad (7.5)$$

where $w_{i+1} = \mathrm{E}\big(f'_{u(t-i)}\big((u(t-k))_{k=0}^M\big)\big)$ for $i = 0, \dots, M$ and $w_{M+2} = 0$. Equation (7.5) can be written more compactly as

$$R_{yu}(\tau) = \sum_{k=0}^{M} b(k) R_u(\tau - k), \quad \tau = 0, 1, \dots, M, \sigma,$$

where $b(k) = w_{k+1} = \mathrm{E}\big(f'_{u(t-k)}\big((u(t-j))_{j=0}^M\big)\big)$. As σ was chosen arbitrarily, this relation holds for all $\tau \in \mathbb{Z}$. □

Using z-transforms, the result (7.4) can also be written as

$$\Phi_{yu}(z) = B(z)\Phi_u(z), \quad (7.6)$$

where $B(z) = \sum_{k=0}^{M} b(k) z^{-k}$. This relation can be used to characterize the OE-LTI-SOE of an NFIR system with a Gaussian input. The OE-LTI-SOE is in general obtained by the Wiener filter construction in (4.3). However, from (7.6) we see that the ratio $\Phi_{yu}(z)/\Phi_u(z)$ is stable and causal if the nonlinear system is an NFIR system with a Gaussian input. Hence, with Corollary 4.1 in mind we can state the following theorem.

Theorem 7.2
Consider an NFIR system

$$y(t) = f((u(t-k))_{k=0}^M) + w(t)$$

with a Gaussian input $u(t)$ such that Assumptions A1, A2, A5 and A7 are satisfied. Then the OE-LTI-SOE of this system is the linear FIR model

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \sum_{k=0}^{M} b(k) z^{-k}, \quad (7.7)$$

where

$$b(k) = \mathrm{E}\big(f'_{u(t-k)}\big((u(t-j))_{j=0}^M\big)\big). \quad (7.8)$$

The fact that (7.8) holds for a Gaussian input but not for a general separable input might seem like a minor difference. However, it will be shown in the next section that (7.8) can be rather useful if the purpose of estimating a linear model is to obtain information about the structure of the nonlinear system. In this case, a Gaussian process is a more suitable choice of input signal than a general separable process. Furthermore, Gaussianity of a process is preserved under linear filtering while separability in general is not. An application where this fact is crucial will also be described in the next section.

7.2 Applications

The characterization (7.7) of the OE-LTI-SOE of an NFIR system with a Gaussian input is not only theoretically interesting, but can also be useful in some applications of system identification. We will here briefly discuss two such applied identification problems.

7.2.1 Structure Identification of NFIR Systems

The most obvious application of the result (7.7) is perhaps to use it for guidance when an NFIR system is going to be identified. However, linear models are not useful for all types of NFIR systems. Any NFIR system can be written as a sum of an even and an odd function. Since all Gaussian probability density functions with zero mean are even functions, the OE-LTI-SOE of an NFIR system is only influenced by the odd part of the system (cf. Lemma 5.2).

Hence, we will here only consider odd NFIR systems, i.e., NFIR systems $y(t) = f((u(t-n_k-j))_{j=0}^{M})$ where

$$f((-u(t-n_k-j))_{j=0}^{M}) = -f((u(t-n_k-j))_{j=0}^{M}).$$

When such an odd NFIR system is going to be identified, it is in general not obvious how the time delay n_k and order M should be estimated in an efficient way. However, if the input is Gaussian and sufficiently many measurements can be collected, n_k and M can both be obtained from an impulse response estimate. Such an estimate can be computed very efficiently by means of the least-squares method.

Furthermore, if only a few of the input terms

$$u(t-n_k), u(t-n_k-1), \dots, u(t-n_k-M)$$

enter the system in a nonlinear way, it might be interesting to know which these terms are. If a nonlinear model of the system is desired, this knowledge can be used to reduce the complexity of the proposed model. A coefficient $b(j)$ in (7.7) will be invariant of the input properties if the corresponding input term $u(t-j)$ only affects the system linearly, while an input term that affects the system in a nonlinear way will have an input dependent b-coefficient in (7.7).

This fact makes it possible to extract information about which nonlinear terms are present in the system simply by looking at the differences between FIR models that have been estimated with different Gaussian input signals. The coefficients that correspond to an input term that enters the system in a nonlinear way will be different in these estimates, provided that the covariance functions of the inputs are different. This idea is used in the following example.

Example 7.1

Consider the nonlinear system $y(t) = u(t) + u(t-1)^3$ and assume that the input to this system is Gaussian and such that the conditions in Theorem 7.2 are fulfilled. Then the OE-LTI-SOE of this system will be

$$G_{0,OE}(q) = b(0) + b(1)q^{-1},$$

Figure 7.1: *A generalized Wiener-Hammerstein system.*

where $b(0) = 1$ and $b(1) = 3R_u(0)$. If the variance of the input is changed, $b(1)$ will change too, while $b(0)$ will remain equal to one. Hence, it is easy to see which input signal component affects $y(t)$ in a nonlinear way.

An overview of other methods for structure identification of nonlinear systems can be found in Haber and Unbehauen (1990).

7.2.2 Identification of Generalized Wiener-Hammerstein Systems

It has been mentioned previously in Section 3.2.2 that Bussgang's theorem has been used to show important results concerning the identification of Hammerstein and Wiener systems (see, for example, Billings and Fakhouri, 1982). In principle, these results state that an estimated LTI model will converge to a scaled version of the linear part of a Hammerstein or Wiener system when the number of measurements tends to infinity, provided that the input is Gaussian. These results simplify the identification of Wiener and Hammerstein systems significantly.

Hence, it is interesting to investigate if the result (7.7) about the OE-LTI-SOEs of NFIR systems can be used to prove similar results for extended classes of systems. In this section, we will study a type of systems that we will call generalized Wiener-Hammerstein systems.

More specifically, we will call a nonlinear system a generalized Wiener-Hammerstein system if it consists of an LTI system $n(t) = G_1(q)u(t)$ followed by an NFIR system $v(t) = f((n(t-k))_{k=0}^{M})$ followed by an LTI system $y(t) = G_2(q)v(t)$ as is shown in Figure 7.1. The following corollary to Theorem 7.2 shows that the OE-LTI-SOE of such a system has a certain structure.

Corollary 7.1
Consider a generalized Wiener-Hammerstein system $y(t) = G_2(q)v(t) + w(t)$ where $v(t) = f((n(t-k))_{k=0}^{M})$ and $n(t) = G_1(q)u(t)$ and where $G_1(q)$ and $G_2(q)$ are stable and causal LTI systems. Assume that $u(t)$ is Gaussian and that $u(t)$ and $y(t)$ fulfill Assumptions A1, A2 and A5. Assume also that $n(t)$ and $v(t)$ fulfill Assumptions A1, A2 and A7. Then the OE-LTI-SOE of this system is

$$G_{0,OE}(z) = G_2(z)B(z)G_1(z), \tag{7.9}$$

where $B(z) = \sum_{k=0}^{M} b(k)z^{-k}$ and

$$b(k) = \mathrm{E}\big(f'_{n(t-k)}\big((n(t-j))_{j=0}^{M}\big)\big).$$

Proof: We have

$$\Phi_{yu}(z) = G_2(z)\Phi_{vu}(z), \tag{7.10a}$$
$$\Phi_{vn}(z) = \Phi_{vu}(z)G_1(z^{-1}), \tag{7.10b}$$
$$\Phi_n(z) = G_1(z)\Phi_u(z)G_1(z^{-1}). \tag{7.10c}$$

In addition, Theorem 7.2 gives that

$$\Phi_{vn}(z) = B(z)\Phi_n(z). \tag{7.11}$$

Inserting (7.10b) and (7.10c) in (7.11) gives

$$\Phi_{vu}(z) = B(z)G_1(z)\Phi_u(z), \tag{7.12}$$

and inserting (7.10a) in (7.12) gives

$$\Phi_{yu}(z) = G_2(z)B(z)G_1(z)\Phi_u(z).$$

Hence, (7.9) follows from Corollary 4.1. □

Corollary 7.1 shows that the OE-LTI-SOE of a generalized Wiener-Hammerstein system with a Gaussian input will be $G_2(z)B(z)G_1(z)$, and hence an estimated output error model will approach this model when the number of measurements tends to infinity. In particular, as $B(z)$ is an FIR model, this shows that the denominator of the estimated model will approach the product of the denominators of G_1 and G_2 if the degree of the model denominator polynomial is correct.

We will thus get consistent estimates of the poles of G_1 and G_2 despite the presence of the NFIR system. This is particularly useful if either G_1 or G_2 is equal to one, i.e., if we have either a generalized Hammerstein or a generalized Wiener system. The consistency of the pole estimates for a generalized Hammerstein system is verified numerically in Example 7.2.

Example 7.2

Consider a generalized Hammerstein system

$$y(t) = G(q)f(u(t), u(t-1)) + w(t),$$

where

$$G(q) = \frac{1}{1 + 0.6q^{-1} + 0.1q^{-2}},$$
$$f(u(t), u(t-1)) = \arctan(u(t)) \cdot u(t-1)^2$$

and where $w(t)$ is white Gaussian noise with $\mathrm{E}(w(t)) = 0$ and $\mathrm{E}(w(t)^2) = 1$.

Let the input $u(t)$ be generated by linear filtering of a white Gaussian process $e(t)$ with $\mathrm{E}(e(t)) = 0$ and $\mathrm{E}(e(t)^2) = 1$ such that

$$u(t) = \frac{1 - 0.8q^{-1} + 0.1q^{-2}}{1 - 0.2q^{-1}} e(t),$$

and assume that $e(t)$ and $w(s)$ are independent for all $t, s \in \mathbb{Z}$.

This input signal has been used in an identification experiment where a data set consisting of 100 000 measurements of $u(t)$ and $y(t)$ was collected. The large number of measurements has been chosen since the convergence towards the OE-LTI-SOE might be slow. A linear output error model $\hat{G}_{OE}$ with $n_b = n_f = 2$ and $n_k = 0$ has been estimated from this data set and the result was

$$\hat{G}_{OE}(q) = \frac{0.7481 - 0.7164q^{-1}}{1 + 0.6045q^{-1} + 0.1031q^{-2}}. \tag{7.13}$$

As can easily be seen from (7.13), the denominator of $\hat{G}_{OE}(q)$ is indeed close to the denominator of $G(q)$. This is exactly what one would expect as the previous theoretical discussion give that the OE-LTI-SOE of the generalized Hammerstein system is the product between $G(q)$ and an FIR model $B(q)$. The MATLAB code that has been used in this example is available in Appendix B.2.

The following example verifies Corollary 7.1 also for a particular generalized Wiener system.

Example 7.3

Consider a generalized Wiener system consisting of the same linear and nonlinear blocks as the generalized Hammerstein system in Example 7.2 but with the linear block before the nonlinear, i.e.,

$$\begin{aligned} y(t) &= f(n(t), n(t-1)) + w(t), \\ n(t) &= G(q)u(t), \end{aligned}$$

where

$$\begin{aligned} G(q) &= \frac{1}{1 + 0.6q^{-1} + 0.1q^{-2}}, \\ f(n(t), n(t-1)) &= \arctan(n(t)) \cdot n(t-1)^2, \end{aligned}$$

and where $w(t)$ is white Gaussian noise with $\mathrm{E}(w(t)) = 0$ and $\mathrm{E}(w(t)^2) = 1$.

Let the input $u(t)$ be generated in the same way as in Example 7.2, i.e.,

$$u(t) = \frac{1 - 0.8q^{-1} + 0.1q^{-2}}{1 - 0.2q^{-1}} e(t),$$

where $e(t)$ is a white Gaussian process with $\mathrm{E}(e(t)) = 0$ and $\mathrm{E}(e(t)^2) = 1$ such that $e(t)$ and $w(s)$ are independent for all $t, s \in \mathbb{Z}$.

An identification experiment has been performed on this generalized Wiener system with a realization of this $u(t)$ as input and 100 000 measurements of $u(t)$ and $y(t)$ have been collected. A linear output error model $\hat{G}_{OE}(q)$ with $n_b = n_f = 2$ and $n_k = 0$ has been estimated from the measurements and the result was

$$\hat{G}_{OE}(q) = \frac{0.9292 - 2.066q^{-1}}{1 + 0.5971q^{-1} + 0.09784q^{-2}}. \tag{7.14}$$

From (7.14) we can see that the denominator of $\hat{G}_{OE}(q)$ is close to the denominator of $G(q)$ also when the data has been generated by a generalized Wiener system. The MATLAB code that has been used in this example is available in Appendix B.3.

In this chapter we have studied OE-LTI-SOEs of NFIR systems with Gaussian inputs. We have shown that the OE-LTI-SOE of such a system is always an FIR model with certain coefficients and that this fact can be used for structure identification of NFIR systems. More specifically, it can be used to tell which input signal components that actually affect the output in a nonlinear way. Furthermore, we have shown that the OE-LTI-SOE of a generalized Wiener-Hammerstein system always will be the product of the LTI parts of the system and an FIR model. Hence, it is possible to estimate the denominator polynomials of the linear parts consistently without compensating for any nonlinearities.

In the next chapter, OE-LTI-SOEs of nonlinear systems that are almost linear will be studied. In particular, the sensitivity of the OE-LTI-SOE to small nonlinearities will be investigated and it will be shown that this sensitivity in some cases depends on how non-Gaussian the input signal is.

8

Almost Linear Systems

When system identification is used to model real-life systems, it is very common to neglect the presence of small nonlinearities in the true system. This works well in many cases but, as we will see in the beginning in this chapter, LTI approximations of almost linear systems can sometimes exhibit a rather strange behavior. Later in this chapter, a convergence result that holds when the nonlinearities tend to zero will also be shown. Finally, a bound on the distance between the OE-LTI-SOE of an almost linear NFIR system and the linear part of that system will be given.

8.1 Almost Linear Systems

The use of a linear model is very natural when the true system is close to being linear. In many cases, the behavior of an almost linear system can be understood, at least intuitively, from the theory of linear systems. Hence, it is a legitimate question to ask whether this linear intuition can be extended also to OE-LTI-SOEs for almost linear systems. An almost linear system will here be defined as a system that for a certain input signal can be written

$$y(t) = G_l(q)u(t) + \alpha y_n(t) + w(t),$$

where the linear term $G_l(q)u(t)$ has much larger variance than the nonlinear term $\alpha y_n(t)$. Here, the parameter α defines the size of the nonlinear part of the system and $w(t)$ is a noise term.

If the nonlinear contribution to the output is small for a certain input, one might assume that the corresponding OE-LTI-SOE would be close to the linear part of the system in some sense. However, as we will see in the following example, this is not always the case.

Example 8.1

Consider the nonlinear system

$$\begin{aligned} y(t) &= y_l(t) + \alpha y_n(t), \\ y_l(t) &= u(t), \\ y_n(t) &= u(t)^3. \end{aligned}$$

The output from this system consists of a linear part, $y_l(t)$, and a nonlinear part, $\alpha y_n(t)$, whose size is controlled by the parameter α. Here, the transfer function $G_l(q)$ of the linear part is equal to one. For bounded input signals, small values of α will give a system output that is close to the output from G_l. In this sense, α defines how close the nonlinear system is to the linear system $G_l(q)$. Let the input signal be

$$u(t) = L_m(q, c)e(t), \tag{8.1}$$

where

$$L_m(q, c) = \left(1 - cq^{-1}\right)^2 = 1 - 2cq^{-1} + c^2q^{-2}, \quad 0 < c < 1 \tag{8.2}$$

and where $e(t)$ is a white noise process with uniform distribution over the interval $[-1, 1]$. For all c with $0 < c < 1$, the input is bounded, $-4 < u(t) < 4$. For this input, a small value of α like, for example, $\alpha = 0.01$ will give an output that is very similar to the output from G_l, i.e., the output when $\alpha = 0$. This can be seen in Figure 8.1a for a particular realization of the input signal. However, the small differences between these output signals will sometimes give rise to totally different OE-LTI-SOEs.

Since the input is generated by filtering white noise through a minimum phase filter, Theorem 5.2 gives that the OE-LTI-SOE can be written

$$G_{0,OE}(z, \alpha, c) = \frac{\Phi_{yu}(z, \alpha, c)}{\Phi_u(z, c)} = \underbrace{G_l(z)}_{=1} + \alpha \frac{\Phi_{y_n e}(z, c)}{L_m(z, c)R_e(0)}. \tag{8.3}$$

If $y_n(t)$ is expanded we get

$$\begin{aligned} y_n(t) = e(t)^3 &- 6ce(t)^2e(t-1) + 3c^2e(t)^2e(t-2) + 12c^2e(t)e(t-1)^2 \\ &- 12c^3e(t)e(t-1)e(t-2) + 3c^4e(t)e(t-2)^2 - 8c^3e(t-1)^3 \\ &+ 12c^4e(t-1)^2e(t-2) - 6c^5e(t-1)e(t-2)^2 + c^6e(t-2)^3. \end{aligned}$$

Using the fact that $\mathrm{E}(e(t)^2) = \frac{1}{3}$ and $\mathrm{E}(e(t)^4) = \frac{1}{5}$ and that $e(t)$ and $e(t-k)$ are independent when $k \neq 0$, the cross-covariance function $R_{y_n e}(\tau, c)$ can be calculated as

$$\begin{aligned} R_{y_n e}(0, c) &= \mathrm{E}(y_n(t)e(t)) = \mathrm{E}(e(t)^4) + 12c^2\mathrm{E}(e(t)^2e(t-1)^2) \\ &\quad + 3c^4\mathrm{E}(e(t)^2e(t-2)^2) = \frac{1}{5} + 12c^2\frac{1}{9} + 3c^4\frac{1}{9} = \frac{1}{15}\left(3 + 20c^2 + 5c^4\right), \\ R_{y_n e}(1, c) &= \mathrm{E}(y_n(t)e(t-1)) = -6c\mathrm{E}(e(t)^2e(t-1)^2) - 8c^3\mathrm{E}(e(t-1)^4) \\ &\quad - 6c^5\mathrm{E}(e(t-1)^2e(t-2)^2) = -6c\frac{1}{9} - 8c^3\frac{1}{5} - 6c^5\frac{1}{9} \\ &= -\frac{1}{15}\left(10c + 24c^3 + 10c^5\right), \end{aligned}$$

$$R_{y_n e}(2,c) = \mathrm{E}(y_n(t)e(t-2)) = 3c^2\mathrm{E}(e(t)^2e(t-2)^2) + 12c^4\mathrm{E}(e(t-1)^2e(t-2)^2)$$
$$+ c^6\mathrm{E}(e(t-2)^4) = 3c^2\frac{1}{9} + 12c^4\frac{1}{9} + c^6\frac{1}{5} = \frac{1}{15}\left(5c^2+20c^4+3c^6\right),$$
$$R_{y_n e}(\tau,c) = 0 \quad \forall \tau \in \mathbb{Z} \setminus \{0,\ 1,\ 2\}.$$

Inserted in (8.3) this gives

$$G_{0,OE}(z,\alpha,c) =$$
$$= 1 + \frac{\alpha}{5}\cdot\frac{\left(3+20c^2+5c^4\right) - \left(10c+24c^3+10c^5\right)z^{-1} + \left(5c^2+20c^4+3c^6\right)z^{-2}}{1-2cz^{-1}+c^2z^{-2}}. \tag{8.4}$$

Let $\Delta_G(z,\alpha,c) \triangleq \sum_{k=0}^{\infty}\delta_G(k,\alpha,c)z^{-k} = G_{0,OE}(z,\alpha,c) - G_l(z)$. Then the static gain of Δ_G is

$$\Delta_G(1,\alpha,c) = \frac{\alpha}{5}\cdot\frac{3-10c+25c^2-24c^3+25c^4-10c^5+3c^6}{(1-c)^2}. \tag{8.5}$$

From (8.5) we see that the numerator of $\Delta_G(1,\alpha,c)$ approaches 12α when $c \to 1$, i.e., for c close to 1 we have $\Delta_G(1,\alpha,c) \approx \frac{12\alpha}{5(1-c)^2}$. This implies that no matter how small $\alpha > 0$ we select, we can always make $\Delta_G(1,\alpha,c)$ arbitrarily large by choosing a c sufficiently close to 1. That is, no matter how linear the system is, there is always a bounded input signal such that its OE-LTI-SOE is far from $G_l(e^{i\omega})$ for $\omega = 0$. The difference between $|G_{0,OE}(e^{i\omega}, 0.01, 0.99)|$ and $|G_l(e^{i\omega})|$ is shown in Figure 8.1b. Furthermore, since

$$|\Delta_G(1,\alpha,c)| = |\sum_{k=0}^{\infty}\delta_G(k,\alpha,c)| \le \sum_{k=0}^{\infty}|\delta_G(k,\alpha,c)|,$$

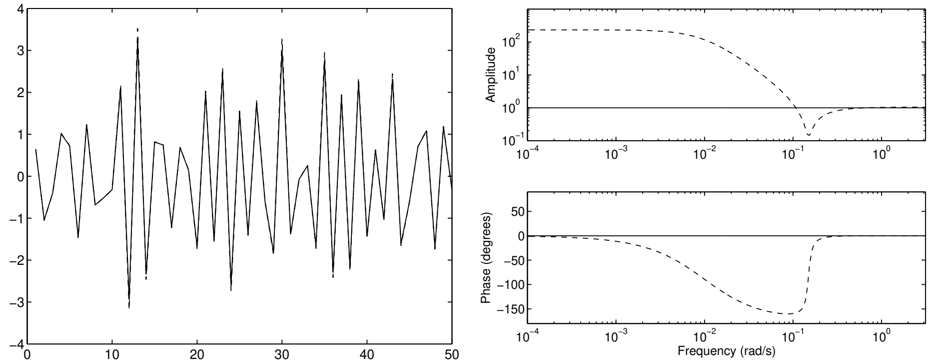
it follows that for any $\alpha > 0$, the l^1-norm of the impulse response of Δ_G can be made arbitrarily large by an appropriate choice of c. It can also be shown that, for any fixed $\alpha > 0$,

$$\sum_{k=0}^{\infty}\delta_G(k,\alpha,c)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_G(e^{i\omega},\alpha,c)|^2\,d\omega$$

can be made arbitrarily large by taking a c close to 1.

The previous example is a clear indication that the OE-LTI-SOE not always can be understood from linear theory. There is no corresponding behavior in the linear case, since a small linear, time-invariant deviation from G_l in Example 8.1 only would have given rise to exactly the same small deviation in the OE-LTI-SOE. It should be noted that for small α values, G_l is a good model of the true system and it can predict the output very well from the input signal. Despite this, the prediction error estimate will not converge to G_l but to $G_{0,OE}$ when the number of measurements tends to infinity.

In the case of linear undermodeling, i.e., when a linear system is approximated by a model of lower order, the optimal approximation depends only of the spectral density of the input Ljung (1999) and not on the distribution of the input signal components.



(a) It is hard to distinguish the output $y(t)$ (dashed) of the nonlinear system (with $\alpha = 0.01$ and $c = 0.99$) in Example 8.1 from the output $y_l(t) = u(t)$ (solid) of the linear part of that system for a particular realization of the input signal.

(b) The frequency response of the OE-LTI-SOE $G_{0,OE}(e^{i\omega}, 0.01, 0.99)$ (dashed) differs from the one of the linear part $G_l(e^{i\omega})$ (solid) of the system in Example 8.1.

Figure 8.1: *The frequency response of the OE-LTI-SOE can be far from the response of the linear part of the system also when the nonlinear contributions to the output are small.*

However, when a nonlinear system is approximated with a linear model, the distribution of the input affects the result significantly. For example, in Example 8.1 the input is generated from the white uniformly distributed signal $e(t)$. If this signal is replaced with a white Gaussian signal with the same variance, the corresponding OE-LTI-SOE will be completely different. This is shown in the next example.

Example 8.2

Consider once again the nonlinear system in Example 8.1 but now with an input $u(t)$ generated by filtering a white Gaussian process $e(t)$ with zero mean and variance $1/3$ through the filter $L_m(q, c)$ in (8.2). Since the system is static and the input is Gaussian, Theorem 3.1 gives that

$$\begin{aligned} R_{yu}(\tau, \alpha, c) &= R_u(\tau) + \alpha \mathrm{E}(3u(t)^2) R_u(\tau) \\ &= (1 + \alpha(1 + 4c^2 + c^4)) R_u(\tau), \end{aligned} \tag{8.6}$$

where we have used that $\mathrm{E}(u(t)^2) = (1 + 4c^2 + c^4)/3$. Since (8.6) implies that

$$\Phi_{yu}(z, \alpha, c) = (1 + \alpha(1 + 4c^2 + c^4)) \Phi_u(z),$$

the OE-LTI-SOE of the system is

$$G_{0,OE}(z, \alpha, c) = (1 + \alpha(1 + 4c^2 + c^4))$$

for this particular input. Hence, the fact that the input is Gaussian implies the OE-LTI-SOE is static. Furthermore, it is obvious that the deviation of the OE-LTI-SOE from the linear part (i.e., from 1) is bounded in this case. With $\Delta_G(z,\alpha,c) = G_{0,OE}(z,\alpha,c) - G_l(z) = \alpha(1+4c^2+c^4)$ and $0 < c < 1$, we have

$$|\Delta_G(z,\alpha,c)| < 6|\alpha|.$$

Here, unlike in Example 8.1, we cannot make Δ_G large for an arbitrarily small $\alpha > 0$ by selecting $c < 1$ sufficiently close to 1. For example, $\alpha = 0.01$ and $c = 0.99$ give

$$G_{0,OE}(z, 0.01, 0.99) \approx 1.0588 \tag{8.7}$$

and $|\Delta_G(z,\alpha,c)| \approx 0.0588$, which can be compared with the large Δ_G that is obtained when $e(t)$ is uniformly distributed (see Figure 8.1b).

From Example 8.2 we see that the fact that the input is Gaussian can, at least in some cases, prevent the OE-LTI-SOE from being far from the linear part of an almost linear system. Probably, this is often a desirable property and hence one could argue that Gaussian inputs should always be used in identification experiments where slightly nonlinear systems are approximated with LTI models. However, an input with a distribution that is similar to a Gaussian distribution might also work fine. This is illustrated in the next example.

Example 8.3

Again, consider the nonlinear system in Example 8.1 with an input $u(t)$ generated by filtering a white process $e(t)$ with zero mean and variance $1/3$ through the filter $L_m(q,c)$ in (8.2). Let $c = 0.99$ and $\alpha = 0.01$. In the two previous examples, it has been shown that this choice of parameters and a uniformly distributed $e(t)$ give an OE-LTI-SOE with the frequency response shown in Figure 8.1b while a Gaussian $e(t)$ gives the OE-LTI-SOE in (8.7). Here, it will be shown that for non-Gaussian choices of $e(t)$ with distributions that are more Gaussian than the uniform distribution, the OE-LTI-SOEs will have frequency responses that are closer to the linear part of the system. Let

$$e_M(t) = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \tilde{e}_k(t), \quad M \in \mathbb{Z}_+,$$

where $\tilde{e}_k(t)$ are independent white signals with uniform distribution over the interval $[-1, 1]$ and zero mean. In this way, $\mathrm{E}(e_M(t)^2) = 1/3$ for all M and the central limit theorem implies that $e_M(t)$ will become more Gaussian for larger M. Let

$$u_M(t) = L_m(q, 0.99)e_M(t)$$

be inputs to the nonlinear system and let the OE-LTI-SOE of the system for each of these inputs be denoted by $G_{0,OE,M}(z)$. In this way, $u_1(t)$ and $G_{0,OE,1}(z)$ correspond to the input and the OE-LTI-SOE in Example 8.1, respectively. Since all inputs have been generated in the same way as this input,

$$G_{0,OE,M}(z) = \frac{b_0(M) + b_1(M)z^{-1} + b_2(M)z^{-2}}{1 - 1.98z^{-1} + 0.9801z^{-2}}$$

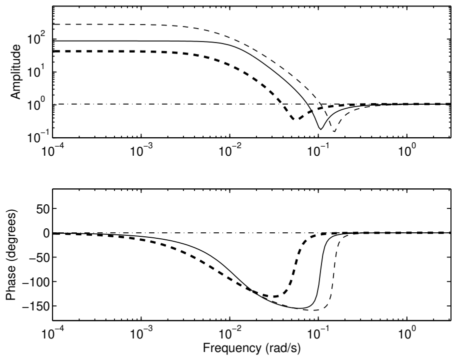


Figure 8.2: *The frequency responses of the estimated output error models $\hat{G}_{0,OE,G}(z)$ (dash-dotted), $\hat{G}_{0,OE,1}(z)$ (thin dashed), $\hat{G}_{0,OE,2}(z)$ (solid) and $\hat{G}_{0,OE,8}(z)$ (thick dashed) from Example 8.3.*

for all M. Furthermore, let $u_G(t)$ denote the Gaussian input in Example 8.2 that is obtained when $e(t)$ is a white Gaussian input with zero mean and variance $1/3$ and let $G_{0,OE,G}(z)$ be the corresponding OE-LTI-SOE (i.e., the OE-LTI-SOE in (8.7)).

Data sets with 50 000 measurements of $u_M(t)$ for $M = 1$, 2 and 8, $u_G(t)$ and of the corresponding system outputs have been generated in MATLAB and an output error model $\hat{G}_{0,OE,M}(z)$ with $n_b = 3$, $n_f = 2$ and $n_k = 0$ has been estimated for each data set. The frequency responses of these estimated models are shown in Figure 8.2.

By comparing this figure with Figure 8.1b, it can be seen that $\hat{G}_{0,OE,1}(z)$ indeed is close to the exact OE-LTI-SOE. Furthermore, Figure 8.2 shows that the OE-LTI-SOEs for the non-Gaussian inputs get closer to the OE-LTI-SOE for the Gaussian input if the input has a more Gaussian-like distribution.

The results in Example 8.3 indicate that the distance between the OE-LTI-SOE and the linear part of an almost linear system depends on how non-Gaussian the input is. With this in mind, it seems preferable to use inputs that are as Gaussian as possible in this case.

Examples 8.1 to 8.3 show that the OE-LTI-SOE in some cases can be far from the linear part of the system. This can in some circumstances be an undesirable property, e.g., if the OE-LTI-SOE is supposed to be used as a basis for robust control design. Such a design puts restrictions on the control laws in order to guarantee the stability of the resulting true closed-loop system, despite the presence of model errors.

Assume that the true system is almost linear in the sense that it deviates from an LTI system G_l by a nonlinearity with a small gain. In this case, G_l is a very good basis for robust control design, since the small nonlinearity often only gives rise to rather mild restrictions on the controller. However, if an OE-LTI-SOE that is far from G_l is used for the controller design, the restrictions on the controller might become much harder, since the gain of the model error now is large. Hence, it is interesting to investigate under what circumstances we can guarantee that the OE-LTI-SOE will be close to the linear part of the system when the nonlinearities are small. Some answers to this question will be given in this chapter.

Consider a system $y(t) = f((u(t-k))_{k=0}^{\infty}, \alpha)$ where α, just like in the previous examples, is a parameter that defines the size of the nonlinear part of f. Let S_A denote the set of all inputs such that they, and the system outputs they generate, fulfill Assumptions A1 and A2. Assume that f is continuous at $\alpha = 0$ and that

$$f((u(t-k))_{k=0}^{\infty}, 0)$$

is a stable and causal LTI system G_l, i.e.,

$$f((u(t-k))_{k=0}^{\infty}, 0) = \sum_{k=0}^{\infty} g_l(k)u(t-k) = G_l(q)u(t). \tag{8.8}$$

Let $G_{0,OE}(z, \alpha)$ denote the OE-LTI-SOE that is obtained for a particular input signal u and a particular α. The conclusion that we can draw from Example 8.1 is that we cannot in general assume that, for example,

$$\sup_{u \in S_A} |G_{0,OE}(e^{i\omega}, \alpha) - G_l(e^{i\omega})| \tag{8.9}$$

will approach 0 when $\alpha \to 0$ for a fixed $\omega \in [-\pi, \pi]$. For some systems we can, whenever there is a small nonlinear term in the system output, find a u for which the OE-LTI-SOE is far from G_l. This property of the OE-LTI-SOE makes it different from, for example, a linearization based on a Taylor series expansion. Since such a linearization is based only on local properties of the nonlinear system, e.g., for $u(t) \equiv 0$, it will converge uniformly when $\alpha \to 0$. However, as we will see in the next section, it is possible to show that also OE-LTI-SOEs exhibit some, less general, convergence properties.

8.2 A Convergence Result

Despite the fact that it even for an almost linear system is not true that the OE-LTI-SOEs will be close to the linear part of the system for all inputs that fulfill Assumptions A1 and A2, it is possible to say something about the behavior of the OE-LTI-SOE for a more restricted class of input signals, whose main feature is that their spectral densities are strictly positive. This is done in the following theorem.

Theorem 8.1
Consider a nonlinear system $y(t) = f((u(t-k))_{k=0}^{\infty}, \alpha) + w(t)$ such that the function $f((x(t-k))_{k=0}^{\infty}, \alpha) \to f((x(t-k))_{k=0}^{\infty}, 0) = \sum_{k=0}^{\infty} g_l(k)x(t-k)$ uniformly on the set of sequences $M_f = \{(x(t-k))_{k=0}^{\infty} \mid |x(t-k)| < u_{max}\ \forall k \in \mathbb{N}\}$ when $\alpha \to 0$. Assume that the limit $G_l(q)$ is a stable and causal LTI system. Let S_f denote the set of stochastic input signals that fulfill the following conditions

(i) $P(|u(t)| \geq u_{max}) = 0$,

(ii) $\mathrm{E}(|u(t)|) \leq m_c < \infty$,

(iii) $\Phi_u(e^{i\omega}) \geq \mu_c > 0$ *for all* $\omega \in [-\pi, \pi]$,

(iv) $u(t)$ *is such that* $G_{0,OE}(z, \alpha)$ *is regular for all* α *with* $|\alpha| < \alpha_{max}$,

(v) $u(t)$ *and* $y(t) = f((u(t-k))_{k=0}^{\infty}, \alpha) + w(t)$ *fulfill Assumptions A1, A2 and A5 for all* α *with* $|\alpha| < \alpha_{max}$*,*

(vi) $\exists M_c \in \mathbb{Z}_+,\ \lambda_c,\ 0 \le \lambda_c < 1,\ K_c > 0$ *such that* $|R_{yu}(\tau, \alpha)| < K_c \lambda_c^{|\tau|}$ *when* $|\tau| > M_c\ \forall \alpha$ *with* $|\alpha| < \alpha_{max}$*,*

where u_{max}*,* m_c*,* μ_c*,* α_{max}*,* M_c*,* λ_c *and* K_c *are given constants. Then*

$$\sup_{u \in S_f} \int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}, \alpha) - G_l(e^{i\omega})|^n \, d\omega \to 0, \quad \alpha \to 0, \quad n = 1, 2. \tag{8.10}$$

Proof: Take an arbitrary $\tau \in \mathbb{Z}$. Then

$$\begin{aligned}
&\sup_{u \in S_f} |R_{yu}(\tau, \alpha) - R_{yu}(\tau, 0)| \\
&= \sup_{u \in S_f} \left| \mathrm{E}\left(\left(f((u(t-k))_{k=0}^{\infty}, \alpha) - \sum_{k=0}^{\infty} g_l(k) u(t-k) \right) u(t-\tau) \right) \right| \\
&\le \sup_{u \in S_f} \mathrm{E}\left(\left| f((u(t-k))_{k=0}^{\infty}, \alpha) - \sum_{k=0}^{\infty} g_l(k) u(t-k) \right| |u(t-\tau)| \right) \\
&\le \sup_{u \in S_f} \mathrm{E}\left(\sup_{x \in M_f} \left| f((x(t-k))_{k=0}^{\infty}, \alpha) - \sum_{k=0}^{\infty} g_l(k) x(t-k) \right| |u(t-\tau)| \right) \\
&= \sup_{x \in M_f} \left| f((x(t-k))_{k=0}^{\infty}, \alpha) - \sum_{k=0}^{\infty} g_l(k) x(t-k) \right| \sup_{u \in S_f} \mathrm{E}(|u(t-\tau)|) \\
&\le \sup_{x \in M_f} \left| f((x(t-k))_{k=0}^{\infty}, \alpha) - \sum_{k=0}^{\infty} g_l(k) x(t-k) \right| m_c \to 0, \quad \alpha \to 0.
\end{aligned}$$

Here, we have used (i) and (ii) in the second and last inequality, respectively. Since τ was arbitrary it follows that

$$\sup_{u \in S_f} |R_{yu}(\tau, \alpha) - R_{yu}(\tau, 0)| \to 0, \quad \alpha \to 0, \quad \forall \tau \in \mathbb{Z}. \tag{8.11}$$

Now we need to show that $\sup_{u \in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}, \alpha) - \Phi_{yu}(e^{i\omega}, 0)|^2 \, d\omega \to 0$ when $\alpha \to 0$. Take an arbitrary $\varepsilon > 0$. By Parseval's identity, which holds for all α with $|\alpha| < \alpha_{max}$ according to (vi), we get

$$\begin{aligned}
&\sup_{u \in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}, \alpha) - \Phi_{yu}(e^{i\omega}, 0)|^2 \, d\omega \\
&= \sup_{u \in S_f} 2\pi \sum_{\tau=-\infty}^{\infty} |R_{yu}(\tau, \alpha) - R_{yu}(\tau, 0)|^2 \\
&= \sup_{u \in S_f} 2\pi \sum_{\tau=-C_0}^{C_0} |R_{yu}(\tau, \alpha) - R_{yu}(\tau, 0)|^2 + Q(C_0, \alpha).
\end{aligned}$$

Choose C_0 such that the term $Q(C_0, \alpha)$ is less than $\varepsilon/2$ for all α with $|\alpha| < \alpha_{max}$. (This is possible since the tails of the series above will be small for all α according to (vi)). Then, from (8.11) it follows that $\exists \delta_\varepsilon > 0$ such that

$$|\alpha| < \delta_\varepsilon \quad \Rightarrow \quad \sup_{u \in S_f} 2\pi \sum_{\tau=-C_0}^{C_0} |R_{yu}(\tau, \alpha) - R_{yu}(\tau, 0)|^2 < \varepsilon/2.$$

Thus $\sup_{u \in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}, \alpha) - \Phi_{yu}(e^{i\omega}, 0)|^2 \, d\omega < \varepsilon$ if $|\alpha| < \delta_\varepsilon$ and since ε was arbitrary we get

$$\sup_{u \in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}, \alpha) - \Phi_{yu}(e^{i\omega}, 0)|^2 \, d\omega \to 0, \quad \alpha \to 0. \tag{8.12}$$

Using (iii) and (iv) together with (8.12), we obtain

$$\begin{aligned}
&\sup_{u \in S_f} \int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}, \alpha) - G_l(e^{i\omega})|^2 \, d\omega \\
&= \sup_{u \in S_f} \int_{-\pi}^{\pi} \frac{|\Phi_{yu}(e^{i\omega}, \alpha) - \Phi_{yu}(e^{i\omega}, 0)|^2}{\Phi_u(e^{i\omega})^2} \, d\omega \\
&\leq \sup_{u \in S_f} \frac{1}{\mu_c^2} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}, \alpha) - \Phi_{yu}(e^{i\omega}, 0)|^2 \, d\omega \to 0, \quad \alpha \to 0
\end{aligned}$$

and thus we have shown (8.10) for $n = 2$. Finally, Schwarz inequality gives

$$\begin{aligned}
&\sup_{u \in S_f} \int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}, \alpha) - G_l(e^{i\omega})| \, d\omega \\
&\leq \sup_{u \in S_f} \sqrt{2\pi} \left(\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}, \alpha) - G_l(e^{i\omega})|^2 \, d\omega \right)^{1/2} \to 0, \quad \alpha \to 0.
\end{aligned}$$

Hence, we have shown (8.10) also for $n = 1$. □

Theorem 8.1 gives conditions on the system and set of input signals that guarantee a uniform convergence of the OE-LTI-SOEs when α tends to zero, i.e., when the system becomes linear. The reason why we cannot apply this theorem in Example 8.1 is that the set of input signals generated by (8.1) for all c with $0 < c < 1$ does not fulfill condition (iii) since $\Phi_u(1) = \frac{(1-c)^2}{3}$. Hence, there is no constant $\mu_c > 0$ such that *all* considered input signals fulfill $\Phi_u(e^{i\omega}) \geq \mu_c$ for all ω in the interval $[-\pi, \pi]$.

Furthermore, Theorem 8.1 shows that it does not matter if $G_{0,OE}(z)$ and $G_l(z)$ have different orders. For input signals in S_f, $G_{0,OE}(z)$ will still approach $G_l(z)$ in a well-behaved way according to (8.10) when the nonlinearities tend to zero.

8.3 Almost Linear NFIR Systems

The theorem in the previous section tells us that the OE-LTI-SOE will converge to the linear part of the system when the nonlinearities tend to zero if $u \in S_f$, but not how fast this convergence is. In order to be able to derive an upper bound on the distance between the OE-LTI-SOE and the linear part of a system with a nonzero nonlinearity of a certain size, we will have to make some new restrictions on the types of systems and excitation signals.

Hence, we will in this section only consider NFIR systems with white input signals that can be written like $y(t) = f((u(t-k))_{k=0}^{M}) + w(t)$ and that are close to a linear system $z(t) = \sum_{k=0}^{M} g_l(k)u(t-k) + w(t)$. The following theorem gives an upper bound on the distance between the OE-LTI-SOE and the linear part of such a nonlinear system.

Theorem 8.2
Consider an NFIR system

$$y(t) = f((u(t-k))_{k=0}^{M}) + w(t),$$

where $u(t)$ is a white input signal and where f satisfies

$$\left| f((u(t-k))_{k=0}^{M}) - \sum_{k=0}^{M} g_l(k)u(t-k) \right| < a, \quad \forall t \in \mathbb{Z} \tag{8.13}$$

for every realization of u. Assume that the output $y(t)$ together with $u(t)$ fulfill the conditions in Assumptions A1, A2 and A5. Then

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})| \, d\omega < a 2\pi \sqrt{(M+1)} \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right), \tag{8.14a}$$

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})|^2 \, d\omega < a^2 2\pi (M+1) \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right)^2. \tag{8.14b}$$

Proof: We start by proving the following inequality

$$\left| R_{yu}(\tau) - \sum_{k=0}^{M} g_l(k) R_u(\tau - k) \right| < a\mathrm{E}(|u(t-\tau)|), \quad \forall \tau \in \mathbb{Z}. \tag{8.15}$$

Take an arbitrary $\tau \in \mathbb{Z}$. Then Assumption A5 gives

$$\begin{aligned}
&\left| R_{yu}(\tau) - \sum_{k=0}^{M} g_l(k) R_u(\tau - k) \right| \\
&= \left| \mathrm{E}(y_{nf}(t)u(t-\tau)) - \sum_{k=0}^{M} g_l(k) \mathrm{E}(u(t-k)u(t-\tau)) \right| \\
&= \left| \mathrm{E}\Big(\big(y_{nf}(t) - \sum_{k=0}^{M} g_l(k)u(t-k) \big) u(t-\tau) \Big) \right|
\end{aligned}$$

$$\le \mathrm{E}(|y_{nf}(t) - \sum_{k=0}^{M} g_l(k)u(t-k)||u(t-\tau)|)$$
$$< a\mathrm{E}(|u(t-\tau)|).$$

Since τ was arbitrary, (8.15) follows. The assumption that $u(t)$ consists of independent random variables implies that $\Phi_u(e^{i\omega}) = R_u(0)$,

$$\sum_{k=0}^{M} g_l(k)R_u(\tau - k) = g_l(\tau)R_u(0), \quad 0 \le \tau \le M,$$

and that $R_{yu}(\tau) = 0$ when $\tau > M$ or $\tau < 0$. This, together with Parseval's identity and Equation (8.15) give

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})|^2 \, d\omega$$
$$= \frac{1}{R_u(0)^2} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}) - G_l(e^{i\omega})R_u(0)|^2 \, d\omega$$
$$= \frac{2\pi}{R_u(0)^2} \sum_{\tau=0}^{M} |R_{yu}(\tau) - g_l(\tau)R_u(0)|^2$$
$$< a^2 2\pi(M+1) \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right)^2 .$$

Finally, Schwarz inequality gives

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})| \, d\omega \le \left(\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})|^2 \, d\omega \right)^{1/2} \sqrt{2\pi}$$
$$< a2\pi\sqrt{(M+1)} \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right).$$

□

Theorem 8.2 tells us that the distance

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})| \, d\omega$$

between the OE-LTI-SOE and G_l is less than a bound that is proportional to a, i.e., to the upper bound on the size of the nonlinearity in (8.13). Furthermore, this theorem shows the effect on the OE-LTI-SOE of a scaling of the input signal.

The use of $\tilde{u} = \nu u$ as input instead of u will result in a new OE-LTI-SOE. The distance in (8.14a) between this new OE-LTI-SOE and G_l will have an upper bound that is $\frac{1}{|\nu|}$ times the original bound. When a white input signal is used, it is thus possible to reduce the distance between the OE-LTI-SOE and the linear part of a nonlinear FIR system that fulfill (8.13) simply by scaling the input signal. This is natural, since the relative error gets smaller for large inputs.

The main objective of this chapter has been to describe the behavior of OE-LTI-SOEs for almost linear systems. It has been shown that the OE-LTI-SOE sometimes can be far from the linear part of such a system and that the distribution of the input is very important for this behavior. Furthermore, a convergence result has been derived and, finally, a bound on the distance between the OE-LTI-SOE of an almost linear NFIR system with a white input signal has been presented.

9

Discussion

A number of results about LTI-SOEs have been shown in the previous chapters of this thesis. Since some of these results have practical implications, it might be appropriate to compile some guidelines for the user. This will be done in this chapter.

It should once again be pointed out that the conclusions that have been drawn in Chapters 4 to 8 concern *asymptotic* properties of prediction-error model estimates, i.e., properties when the number of measurements and, in some cases, the number of experiments tends to infinity. Hence, the results are usually only applicable to identification problems where large data sets are used.

Previously, we have seen that there can be remaining correlation between the input and the model residuals both for the OE-LTI-SOE and the GE-LTI-SOE. If no additional knowledge about the system structure is available, this correlation might be taken as an indication that the system operates in closed loop.

However, for some classes of input signals, this cannot happen for an open-loop nonlinear system. If the input has been generated by filtering white noise through a minimum phase filter, there will be no spurious correlation between the input and the residuals of the OE-LTI-SOE and the GE-LTI-SOE according to Lemma 5.4 and Corollary 5.1, respectively. Furthermore, these input signals have the following properties.

- Each minimum phase generated input signal is optimal over a class of other input signals in the sense that the variance of the residuals of the OE-LTI-SOE is minimized (see Theorem 5.3).

- Residual and spectral analysis can be used to validate an estimated model and to see if it is sufficiently close to the OE-LTI-SOE of the system (see Section 5.5.2).

- A minimum phase generated input signal is a good choice of reference signal for closed-loop identification using the two-step method since it implies that only a causal $S(z)$ has to be estimated (see Section 5.5.3).

If the nonlinear system is a generalized Hammerstein system, it might be desirable to use a separable input signal. These signals have the following properties.

- For a separable input signal there will be no extra dynamics in the OE-LTI-SOE. Hence, the OE-LTI-SOE of an NFIR system will be an FIR model of the same order and the OE-LTI-SOE of a generalized Hammerstein system will be the cascade product of an FIR model and the linear part of the system (see Theorems 6.3 and 6.4). Hence, it is possible to estimate the denominator polynomial of the linear part consistently without compensating for the nonlinearities.

- In particular, for a separable input, the OE-LTI-SOE of a Hammerstein system will be a scaled version of the linear part of the system (see Theorem 6.1).

One particular choice of a separable input signal is Gaussian noise with arbitrary color. This class of input signals is also a subset of the class of minimum phase generated signals and it has the following properties in addition to the ones above.

- For a Gaussian input signal, the OE-LTI-SOE of an NFIR system will be an FIR model with coefficients that are expectations of the partial derivatives of the nonlinear function (see Theorem 7.2). This simplifies structure identification of such a system significantly.

- Furthermore, the OE-LTI-SOE of a generalized Hammerstein, Wiener or Wiener-Hammerstein system will be the cascade product of the linear parts of that system and an FIR model (see Corollary 7.1). Again, this implies that it is possible to estimate the denominator polynomials of the linear parts consistently without compensating for the nonlinearities.

With the properties mentioned above in mind, it is possible to give a general advice concerning LTI approximations of nonlinear systems using the prediction-error method. Although there might be circumstances where this advice is not suitable, it is at least applicable to the identification problems studied in this thesis.

> *Use a Gaussian signal, a separable random multisine or any other separable signal as the input in the identification experiment. If this is not possible, use any other signal generated by filtering white noise through a minimum phase filter. Collect a large data set by using either one long realization of the input or a large number of different realizations, depending on the type of input signal used.*

Part II

Identification and Control

10

Robust Control

A number of properties of LTI approximations of nonlinear systems have been discussed in the previous chapters. In most cases, these properties have been considered from a system identification point of view, i.e., the focus has been more on theoretical approximation issues and on questions like under which circumstances the LTI model will inherit structural properties from the true system. However, it is important to remember that in most applications, the modeling of a system is done for a specific purpose. Different modeling issues are relevant depending on this purpose and on the type of application.

In this chapter, some aspects of robust control design based on LTI models of nonlinear systems will be discussed. In most modern control design methods, the controller is calculated based on a model of the true systems. Such a controller is *robust* to model errors if it can be guaranteed that the differences between the model and the system will not cause instability when the controller is used with the true system. Control design based on an approximate model has been studied for a long time and there are thus a large number of different robustness results.

For example, when both the true system and the model are linear, and thus also the model error, several methods exist where the robustness issues can be considered explicitly in the design process (Zhou et al., 1996). Furthermore, many engineering guidelines concerning controller tuning and classic methods, such as loop shaping and state feedback, have intrinsic robustness properties. Typically, controllers designed with any of these methods will be robust to at least some linear model errors.

Also the common case when the true system is nonlinear and the design is based on a linear model has been studied previously. In principle, it is possible to analyze this setup using the small gain theorem (Vidyasagar, 1993; Sastry, 1999; Khalil, 2002). However, such an analysis will often be rather pessimistic. Intuitively, the result in the small gain theorem is that a closed-loop system will be stable if the product of the gains of the subsystems in the loop is less than one (see, for example, Sastry, 1999, p. 147). The gain $\|\mathcal{S}\|$ of a system $\mathcal{S}$ with input $u(t)$ and output $y(t)$ can be defined as the smallest constant

β for which

$$\|y\| \leq \alpha + \beta\|u\|, \quad \forall u(t), \tag{10.1}$$

where α is a constant and where $\|\cdot\|$ denotes the signal norm

$$\|z\|^2 = \sum_{t=0}^{\infty} z(t)^T z(t).$$

The small gain theorem can be used to prove a robustness result that says that the true system is stable if the product of the gain of the model error and the gain from an output disturbance to u in the closed-loop system is less than one (Sastry, 1999, pp. 150-151). However, this result is most useful when the gain of the model error is small compared to the gain of the model.

As a simple example, consider a saturation system, i.e., the nonlinear system

$$y(t) = \begin{cases} 1, & u(t) > 1, \\ u(t), & |u(t)| \leq 1, \\ -1, & u(t) < -1 \end{cases}$$

and assume that a linear approximation

$$\hat{y}(t) = b_0 u(t)$$

of this system should be defined such that the gain of the model error is as small as possible. However, it is easy to see that for every choice of b_0, the gain of the model error is equal to b_0. Of course, a model of a system is not very useful if the model error is of the same size as the model.

The same problem as in this example occurs when most nonlinear systems are approximated with a linear model. However, there are many successful real-life applications of control design based on an LTI model of a nonlinear system and thus, the robustness result based on the small gain theorem seems to be too pessimistic. In this chapter, it will be shown that a more useful theoretical robustness result can be obtained if the definition of the gain of a system is modified. The method used here is closely related to optimal control (see, for example, Bryson and Ho, 1975; van der Schaft, 1992). Furthermore, different approaches to robust stability of linear systems with input saturation have been investigated previously (Kim and Bien, 1994; Henrion and Tarbouriech, 1999).

It should also be mentioned that there are many results about system identification for control purposes available in the literature (Hjalmarsson, 2005). However, we will not discuss identification of LTI models for control design for nonlinear systems in this chapter, but merely point out that LTI models really can be useful for this purpose and that robust control design can be done also when a bound on the model error is known only for a subset of all possible input signals. Most of the results in this chapter have previously been published in Glad et al. (2005) and some related results can be found also in Glad et al. (2004). In the next section, the robustness result will first be described for general models while the special case of LTI models and saturated inputs will be studied in Section 10.2.

10.1 Robust Control for Constrained Inputs

Consider a general state-space model $\mathcal{M}$ that can be written as

$$\begin{aligned} x_{\mathcal{M}}(t+1) &= f_{\mathcal{M}}(x_{\mathcal{M}}(t), u(t)), \\ y_{\mathcal{M}}(t) &= h_{\mathcal{M}}(x_{\mathcal{M}}(t), u(t)), \\ x_{\mathcal{M}}(0) &= x_{\mathcal{M},0} \end{aligned} \tag{10.2}$$

of a system with input $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^p$. Note that the model (10.2) contains an assumption about the initial state. The output $y(t)$ from the true system can be written

$$y(t) = y_{\mathcal{M}}(t) + \varepsilon(t), \tag{10.3}$$

where the signal $\varepsilon(t)$ contains both the effects of model errors and external disturbances. Assume that the simple model error model

$$u(t) \in U,\ 0 \le t \le N \Rightarrow \|\varepsilon\|_N \le \alpha(N) + \beta\|u\|_N \tag{10.4}$$

holds. Here, U is a subset of $\mathbb{R}^m$, α is a function with $\alpha(N) \ge 0$ for all N, $\beta \ge 0$ is a constant and $\|\cdot\|_N$ denotes the truncated norm

$$\|z\|_N^2 = \sum_{t=0}^{N} z(t)^T z(t).$$

For example, the function $\alpha(N)$ can be used to describe both the effects of erroneous initial conditions in the model (10.2) and bounded external disturbances. By comparing (10.4) with (10.1), it is obvious that (10.4) can be used as an alternative gain definition, valid only for a restricted class of inputs. Hence, β is an upper bound on the gain of the model error for these restricted inputs. It should be mentioned also that, although the model error model (10.4) looks simple, it can be a demanding task to establish it for a particular system.

Assume that a controller is to be designed using the model $\mathcal{M}$ in (10.2) and (10.3) and with the objective to keep a signal $z(t)$ small despite the influence from the error signal $\varepsilon(t)$. Here, the signal $z(t)$ is assumed to be the output of a filter $\mathcal{W}$ with $y(t)$ as input. Furthermore, it is assumed that the filter $\mathcal{W}$ can be written as

$$\begin{aligned} x_{\mathcal{W}}(t+1) &= f_{\mathcal{W}}(x_{\mathcal{W}}(t)) + g_{\mathcal{W}}(x_{\mathcal{W}}(t))y(t), \\ z(t) &= h_{\mathcal{W}}(x_{\mathcal{W}}(t)), \\ x_{\mathcal{W}}(0) &= x_{\mathcal{W},0}. \end{aligned} \tag{10.5}$$

The model $\mathcal{M}$ in (10.2), which can be used when $u(t) \in U$, the filter $\mathcal{W}$ in (10.5) and the output description (10.3) can be combined into a complete state-space model

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) + n(x(t))\varepsilon(t), \\ y(t) &= h(x(t), u(t)) + \varepsilon(t), \\ z(t) &= m(x(t)), \\ x(0) &= x_0, \end{aligned} \tag{10.6}$$

where

$$x(t) = \begin{pmatrix} x_{\mathcal{M}}(t) \\ x_{\mathcal{W}}(t) \end{pmatrix}$$

and

$$\begin{aligned} f(x(t), u(t)) &= \begin{pmatrix} f_{\mathcal{M}}(x_{\mathcal{M}}(t), u(t)) \\ f_{\mathcal{W}}(x_{\mathcal{W}}(t)) + g_{\mathcal{W}}(x_{\mathcal{W}}(t)) h_{\mathcal{M}}(x_{\mathcal{M}}(t), u(t)) \end{pmatrix}, \\ n(x(t)) &= \begin{pmatrix} 0 \\ g_{\mathcal{W}}(x_{\mathcal{W}}(t)) \end{pmatrix}, \\ h(x(t), u(t)) &= h_{\mathcal{M}}(x_{\mathcal{M}}(t), u(t)), \\ m(x(t)) &= h_{\mathcal{W}}(x_{\mathcal{W}}(t)), \\ x_0 &= \begin{pmatrix} x_{\mathcal{M},0} \\ x_{\mathcal{W},0} \end{pmatrix}. \end{aligned}$$

Let

$$J_N = \sum_{t=0}^{N} (m(x(t))^T m(x(t)) + u(t)^T u(t) - \gamma^2 \varepsilon(t)^T \varepsilon(t))$$

and consider the problem of finding $u(t) = k(x(t))$ such that $J_N \leq 0$ for all $N \geq 0$. One approach for solving this problem is described in the following theorem.

Theorem 10.1
Consider the model (10.6) and suppose that there is a positive semidefinite function $V(x(t))$ and a control law $u = k(x)$ such that $u(t) = k(x(t))$ implies that $u(t) \in U$ for all $t \geq 0$ and that

$$m(x)^T m(x) + k(x)^T k(x) - \gamma^2 \varepsilon^T \varepsilon + V(f(x, k(x)) + n(x)\varepsilon) - V(x) \leq 0 \tag{10.7}$$

for all x and for all ε. Here, γ is a nonnegative constant. Then the inequality

$$\|z\|_N^2 + \|u\|_N^2 \leq V(x(0)) + \gamma^2 \|\varepsilon\|_N^2 \tag{10.8}$$

holds for every signal ε and for every $N \geq 0$ when $u(t) = k(x(t))$.

Proof: Since the equality

$$V(x(0)) - V(x(N+1)) + \sum_{k=0}^{N} (V(x(k+1)) - V(x(k))) = 0$$

holds for any function V, we have that

$$\begin{aligned} J_N - V(x(0)) = \sum_{t=0}^{N} \Big(& m(x(t))^T m(x(t)) + k(x(t))^T k(x(t)) - \gamma^2 \varepsilon(t)^T \varepsilon(t) \\ & + V\big(f(x(t), k(x(t))) + n(x(t))\varepsilon(t)\big) - V\big(x(t)\big)\Big) - V(x(N+1)) \leq 0, \end{aligned}$$

where we have used (10.7) and the fact that V is positive semidefinite in the last inequality. The fact that

$$J_N - V(x(0)) \leq 0$$

implies that (10.8) holds for every ε. □

Theorem 10.1 describes a discrete-time control design problem where a function $k(x)$ should be found. For this control design problem to be solved, the inequality (10.7) must hold for all x and all ε. However, if V is assumed to be a quadratic function, $V(x) = x^T Px$, for some choice of a symmetric, positive semidefinite matrix P, it turns out that ε can be eliminated from (10.7). In this case, the robust control design problem involves finding a solution to an inequality for all x, i.e., to solve an easier problem. Using a quadratic V such that $Q = (\gamma^2 I - n^T Pn)$ is positive definite, we have

$$\begin{aligned}
&z^T z + u^T u - \gamma^2 \varepsilon^T \varepsilon + f^T Pf + f^T Pn\varepsilon + \varepsilon^T n^T Pf + \varepsilon^T n^T Pn\varepsilon - x^T Px \\
&= z^T z + u^T u + f^T Pf + f^T PnQ^{-1} n^T Pf - x^T Px \\
&\quad - (\varepsilon - Q^{-1} n^T Pf)^T Q(\varepsilon - Q^{-1} n^T Pf) \\
&\le z^T z + u^T u + f^T Pf + f^T PnQ^{-1} n^T Pf - x^T Px.
\end{aligned}$$

Hence, the condition (10.7) in Theorem 10.1 is satisfied if a positive semidefinite matrix P and a function $k(x)$ can be found such that $Q(x, P) = \gamma^2 I - n(x)^T Pn(x)$ is positive definite and

$$\begin{aligned}
&m(x)^T m(x) + k(x)^T k(x) + f(x, k(x))^T Pf(x, k(x)) \\
&+ f(x, k(x))^T Pn(x)Q(x, P)^{-1} n(x)^T Pf(x, k(x)) - x^T Px \le 0
\end{aligned} \tag{10.9}$$

for all x.

Intuitively, the result (10.8) says that the gain of the closed-loop system from ε to z and u is less than γ whenever Theorem 10.1 can be applied. Since this theorem also implies that $u(t) \in U$ for all $t \ge 0$, it can be used to prove the following result, which can be viewed as a version of the small gain theorem.

Theorem 10.2
Consider a system and a model such that the model error model (10.4) holds and assume that the robust control design problem in Theorem 10.1 has been solved for some $\gamma \ge 0$ such that $\gamma\beta < 1$. If the controller $u = k(x)$ from Theorem 10.1 is used for the true system, the obtained closed-loop system satisfies

$$\|z\|_N^2 \le V(x(0)) + \frac{\gamma^2 \alpha(N)^2}{1 - \gamma^2 \beta^2}. \tag{10.10}$$

Proof: Inserting the model error model (10.4) in (10.8) gives

$$\|z\|_N^2 + \|u\|_N^2 \le V(x(0)) + \gamma^2 (\alpha(N) + \beta \|u\|_N)^2.$$

By completing squares, this expression can be rewritten as

$$\begin{aligned}
\|z\|_N^2 + (1 - \gamma^2\beta^2)(\|u\|_N - \frac{\gamma^2 \alpha(N)\beta}{1 - \gamma^2\beta^2})^2 &\le V(x(0)) + \gamma^2 \alpha(N)^2 + \frac{\gamma^4 \alpha(N)^2 \beta^2}{1 - \gamma^2 \beta^2} \\
&= V(x(0)) + \frac{\gamma^2 \alpha(N)^2}{1 - \gamma^2 \beta^2}
\end{aligned}$$

and since $\gamma\beta < 1$, deleting the positive term on the left hand side gives the result in (10.10). □

Theorem 10.2 shows that $\|z\|_N^2$ has an upper bound that depends on the size of $\alpha(N)$ in the model error model (10.4). If $\alpha(N)$ is bounded, $0 \le \alpha(N) \le \alpha_0$ for all $N \ge 0$, (10.10) implies that the true closed-loop system is stable in the sense that

$$\lim_{N\to\infty} \|z\|_N < \infty.$$

If $\varepsilon(t)$ contains a signal component, which can be both a result of model errors and an external disturbance, with an upper bound at each time instant, the choice $\alpha(N) = \alpha_0\sqrt{N}$ is natural. For such an $\alpha(N)$, Theorem 10.2 can be used to show that the power of $z(t)$ will be bounded.

A nice property of the controller $u = k(x)$ is that the control signal is computed only from the states of the model. Hence, no observer has to be designed and the complete controller can be written as

$$\begin{aligned} x(t+1) &= f\big(x(t), k(x(t))\big) + n\big(x(t)\big)\big(y(t) - h(x(t), k(x(t)))\big), \\ u(t) &= k(x(t)), \\ x(0) &= x_0 \end{aligned}$$

or, equivalently, as

$$\begin{aligned} x_{\mathcal{M}}(t+1) &= f_{\mathcal{M}}\big(x_{\mathcal{M}}(t), k\big((x_{\mathcal{M}}(t)^T,\ x_{\mathcal{W}}(t)^T)^T\big)\big), \\ x_{\mathcal{W}}(t+1) &= f_{\mathcal{W}}\big(x_{\mathcal{W}}(t)\big) + g_{\mathcal{W}}\big(x_{\mathcal{W}}(t)\big)y(t), \\ u(t) &= k((x_{\mathcal{M}}(t)^T,\ x_{\mathcal{W}}(t)^T)^T), \\ x_{\mathcal{M}}(0) &= x_{\mathcal{M},0}, \\ x_{\mathcal{W}}(0) &= x_{\mathcal{W},0}. \end{aligned}$$

However, it is important to verify that it is possible to compute the states in this controller with sufficiently high accuracy before using it in practice. For example, it should always be verified that the state-space model used in the controller is stable.

In the next section, robust control design for a nonlinear system based on an LTI model will be discussed.

10.2 Robust Control Using LTI Models

In this section, the special case of LTI models with input saturation will be considered. Assume that (10.6) can be written as

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + N\varepsilon(t), \\ y(t) &= Cx(t) + Du(t) + \varepsilon(t), \\ z(t) &= Mx(t), \\ x(0) &= x_0, \end{aligned} \tag{10.11}$$

where A, B, C, D, M and N are matrices. This implies that the condition (10.9), which can be used instead of (10.7) if

$$Q(P) = \gamma^2 I - N^T PN$$

is positive definite, can be written

$$x^T M^T M x + u^T u + x^T A^T P A x + x^T A^T P B u + u^T B^T P A x + u^T B^T P B u + (Ax + Bu)^T P N Q(P)^{-1} N^T P (Ax + Bu) - x^T P x \leq 0. \tag{10.12}$$

Furthermore, this condition can be rewritten as

$$x^T (A^T P A - P + M^T M + A^T P N Q(P)^{-1} N^T P A - T(P)^T W(P)^{-1} T(P)) x + (u + W(P)^{-1} T(P) x)^T W(P) (u + W(P)^{-1} T(P) x) \leq 0, \tag{10.13}$$

where

$$W(P) = I + B^T P B + B^T P N Q(P)^{-1} N^T P B,$$
$$T(P) = B^T P A + B^T P N Q(P)^{-1} N^T P A.$$

Consider the case of a single input single output system and let

$$u(t) = k(x(t)) \triangleq \begin{cases} -u_0, & W(P)^{-1} T(P) x(t) > u_0, \\ -W(P)^{-1} T(P) x(t), & |W(P)^{-1} T(P) x(t)| \leq u_0, \\ u_0, & W(P)^{-1} T(P) x(t) < -u_0, \end{cases} \tag{10.14}$$

where $u_0 > 0$ is a constant such that $U = \{v \in \mathbb{R} \mid |v| \leq u_0\}$. When

$$W(P)^{-1} T(P) x(t) > u_0,$$

we obtain the condition

$$\begin{aligned} & x^T (A^T P A - P + M^T M + A^T P N Q(P)^{-1} N^T P A - T(P)^T W(P)^{-1} T(P)) x \\ & \quad + (-u_0 + W(P)^{-1} T(P) x)^T W(P) (-u_0 + W(P)^{-1} T(P) x) \\ & = x^T (A^T P A - P + M^T M + A^T P N Q(P)^{-1} N^T P A) x \\ & \quad + u_0^2 W(P) - 2 u_0 T(P) x \leq 0 \end{aligned} \tag{10.15}$$

by inserting the expression for the control signal in (10.14) into (10.13). Since

$$u_0^2 W(P) - 2 u_0 T(P) x = u_0 W(P) (u_0 - 2 W(P)^{-1} T(P) x) \leq 0,$$

condition (10.15) is satisfied if

$$x^T (A^T P A - P + M^T M + A^T P N Q(P)^{-1} N^T P A) x \leq 0.$$

Using, similar calculations, this result can also be obtained when

$$W(P)^{-1} T(P) x(t) < -u_0.$$

If a positive definite matrix P such that

$$A^T P A - P + M^T M + A^T P N Q(P)^{-1} N^T P A \preccurlyeq 0, \tag{10.16a}$$
$$Q(P) = \gamma^2 - N^T P N > 0 \tag{10.16b}$$

can be found, the conditions in Theorem 10.1 are thus satisfied for $V(x) = x^T P x$ and the control law in (10.14). Here, $\preccurlyeq 0$ means negative semidefiniteness of the left hand side. Hence, for an LTI model, a solution to the robust control design problem can be found by solving the Riccati inequality (10.16a) and checking that the solution satisfies (10.16b). This approach is illustrated in the following example.

Example 10.1

Consider the slightly nonlinear system

$$
\begin{aligned}
y(t) &= y_l(t) + \delta y_n(t) + w(t),\\
y_l(t) &= G_L(q)u(t) = \frac{2q-1}{q^2-q+0.2}u(t),\\
y_n(t) &= u(t)^3,
\end{aligned}
$$

where $\delta = -1/15$ is a constant and $w(t)$ is a disturbance. Assume that the linear transfer function $G_L(q)$ is known and that we want to use it for robust control design where the objective is to reduce the effects of low-frequent disturbances $w(t)$ on $y(t)$. This control objective is formulated using the filter

$$
z(t) = W(q)y(t) = \frac{0.2}{q-0.9}y(t).
$$

The model $G_L(q)$ and the filter $W(q)$ give the complete state-space model

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + N\varepsilon(t),\\
y(t) &= Cx(t) + \varepsilon(t),\\
z(t) &= Mx(t),\\
x(0) &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} 1.0 & -0.4 & 0 \\ 0.5 & 0 & 0 \\ 0.2 & -0.2 & 0.9 \end{pmatrix},\\
B &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},\\
N &= \begin{pmatrix} 0 \\ 0 \\ 0.2 \end{pmatrix},\\
C &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix},\\
M &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

For example, with $\gamma = 2.1$, the matrix

$$
P = \begin{pmatrix} 2.11 & -1.67 & 3.64 \\ -1.67 & 1.34 & -2.95 \\ 3.64 & -2.95 & 7.95 \end{pmatrix}
$$

is one solution to the Riccati inequality (10.16a) obtained for this state-space model. Furthermore, this choice of P satisfies (10.16b). This solution has been found using MATLAB and the Riccati equation solver `dare`. With $G_L(q)$ as model, the gain of the nonlinear model error is $\beta = u_0^2/15$. In order to achieve $\gamma\beta < 1$, the input has to be saturated.

For example, the saturation can be at $u_0 = 2.5$ since this value gives the gain

$$\beta = \frac{5}{12} < \frac{10}{21} = \frac{1}{\gamma}$$

of the model error. The resulting control law is

$$u(t) = k(x(t)) = \begin{cases} -2.5, & Lx(t) > 2.5, \\ -Lx(t), & |Lx(t)| \leq 2.5, \\ 2.5, & Lx(t) < -2.5, \end{cases}$$

where

$$L = \begin{pmatrix} 0.429 & -0.337 & 0.709 \end{pmatrix}$$

and the complete controller can be written

$$\begin{aligned} x(t+1) &= Ax(t) + Bk(x(t)) + N(y(t) - Cx(t)), \\ u(t) &= k(x(t)), \\ x(0) &= 0. \end{aligned}$$

The closed-loop system that is obtained when this controller is used for the true nonlinear system has been simulated using MATLAB. In this simulation, the true system was initialized at zero and the disturbance $w(t)$ was a unit step at $t = 5$, i.e.,

$$w(t) = \begin{cases} 1, & t \geq 5, \\ 0, & t < 5. \end{cases}$$

This corresponds to a function

$$\alpha(N) = \begin{cases} \sqrt{N-4}, & N \geq 5, \\ 0, & N < 5 \end{cases}$$

in the model error model. The output signal from the simulation experiment is shown in Figure 10.1. As can be seen in this figure, the feedback controller reduces the stationary effect of the disturbance with more than 60%. Furthermore, from the design of the controller, we know that there is no risk of instability due to the model errors.

Example 10.1 shows that an input saturation might be needed to limit the size of the model error in a nonlinear closed-loop system. In this example, removing the restrictions on the input signal would result in a model error with infinite gain.

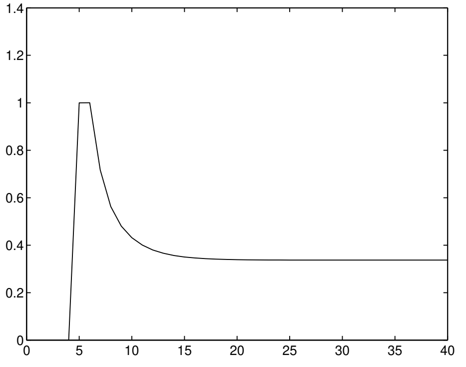


Figure 10.1: *The output signal from the closed-loop system in Example 10.1 when a unit step disturbance is added to the output at* $t = 5$.

10.3 Discussion

In this chapter, some issues concerning robust control design have been discussed. In particular, a gain definition valid only for a restricted set of input signals has been presented. In some cases, this gain definition gives less pessimistic results than the standard gain definition concerning, for example, robust control using LTI models of nonlinear systems. It has been shown that a robust control law can be found by solving (10.7) in Theorem 10.1 for a $\gamma > 0$ that is smaller than the inverse of the gain of the model error for restricted inputs. Hence, the main difference between the approach presented here and an approach based on the small gain theorem is the use of the alternative gain definition.

Furthermore, the special case of LTI models of nonlinear systems has been investigated. In this case, the robust control design problem can be solved by finding a solution to the Riccati inequality (10.16a). The method presented here for robust control using linear models of nonlinear systems illustrates one general feature of this type of problems, namely that some restrictions on the input signal typically have to be added in order to reduce the size of the nonlinear model error.

11

Mixed Parametric Nonparametric Identification

The OE-LTI-SOE of a nonlinear system is defined as the LTI model that minimizes the mean-square error and considering only this criterion, there is thus no better LTI approximation. However, the mean-square error is not always a suitable measure of how good or accurate a model is. In the previous chapters, it has been shown that the OE-LTI-SOE of a nonlinear system sometimes can be very sensitive to small nonlinearities. Since this effect is usually undesirable, it is interesting to investigate how it can be reduced. If the input signal to the system can be modified, a careful input design can give an OE-LTI-SOE that is less sensitive to small nonlinearities. For example, as was discussed in Chapter 6, a separable input signal might be useful in some cases. However, if the user is not free to design the input signal, other ways of improving the LTI approximation are necessary.

Of course, one approach for obtaining an LTI approximation of a nonlinear system is to estimate first a nonlinear model. When an accurate nonlinear model of the system has been found, it can be linearized analytically to provide the user with both an LTI model and a detailed mathematical description of the model errors. However, estimating a nonlinear model of a system can be a challenging task where the user has many options. There are various methods for nonlinear system identification and usually a number of nonlinear model structures that can be used with each method (see, for example, Giannakis and Serpedin, 2001).

The use of linear models of complex, nonlinear systems is common in many real-life applications of system identification. For example, tuning of control loops in process industry is often done based on approximate linear models. The reason why nonlinear models are not desired can be that such models are considered too time consuming to estimate. Hence, there are applications where an improved method designed for estimation of LTI models of nonlinear systems might be useful.

In the standard prediction-error method, the obtained estimates of the model parameters are such that the correlation between the model residuals and the past and present input signal components is minimized. Actually, this correlation is zero for the OE-LTI-

SOE (cf. Corollary 4.2). However, if there are small nonlinearities in the system, it might be better not to try to model all correlation between the input and the output but to allow for a small correlated term in the model residuals. This idea was proposed in Mäkilä (2005), but no method was presented there. Here, however, we will discuss one such method for identification of nonlinear autoregressive systems with external inputs (NARX systems) (Sjöberg et al., 1995). This method is described also in Roll et al. (2005a).

NARX systems are a straightforward generalization of linear ARX systems that has been used in many applications. For an NARX system, the mean-square error optimal one step ahead predictor is a nonlinear function of a finite number of past output and past and present input components. Using a version of the prediction-error method (Ljung, 1999), we will here simultaneously estimate both a nonparametric NARX model and a parametric ARX model such that their sum give an as good prediction of the output as possible. Related model structures have been used in semiparametric or partially linear models (see, for example, Heckman, 1988; Chen et al., 2001).

The proposed method can be viewed both as a way to handle insignificant nonlinearities when a linear model is estimated and as a method for nonlinear system identification. A nonparametric nonlinear model will always be estimated but can be ignored if only a linear model is desired. It is interesting to consider nonparametric methods for nonlinear system identification since the assumptions about the true system are usually weaker for such methods than for parametric methods. For a nonlinear system, it can be hard to tell in advance whether a specific assumption about, for example, the shape of the nonlinearities is reasonable or not. Here, the only assumption about the true NARX system is that its nonlinearities are Lipschitz continuous.

This assumption makes it possible to use an approach where the identification problem is formulated as a quadratic programming (QP) problem. By solving this problem, both the parameters of the linear ARX model and the nonparametric NARX model can be estimated at the same time. A version of this idea, without the linear, parametric part, has previously been used for nonparametric regression and for maximum likelihood estimation of unknown parameters in probability density functions (Bertsimas et al., 1999). Other methods for nonparametric regression can be found in, for example, Fan and Gijbels (1996). Lipschitz conditions are a common way to guarantee that a function, or some of its derivatives, will be smooth. For example, functions with a Lipschitz continuous gradient can be identified using local modeling such that the worst-case mean-square error is minimized (Roll et al., 2005b).

The method presented in this chapter can sometimes make the estimate of the linear model more robust against nonlinearities in the system since the nonparametric NARX model can compensate for some of the nonlinear effects. A related concept is the notion of unknown but bounded noise and set membership identification (Garulli et al., 1999), since a bounded nonlinearity might affect the system output in a similar way as such a noise term.

11.1 Identification of NARX and NOE Systems

Consider an NARX system with input $u(t)$ and output $y(t)$ that can be written as

$$y(t) = \theta_0^T \varphi(t) + r_0(\varphi(t)) + e(t), \tag{11.1}$$

where

$$\varphi(t) = \begin{pmatrix} -y(t-1) \\ \vdots \\ -y(t-n_a) \\ u(t-n_k) \\ \vdots \\ u(t-n_k-n_b) \end{pmatrix} \tag{11.2}$$

is a regression vector and $e(t)$ is white noise. The constant vector θ_0 defines a linear ARX part of the system while the function r_0 can be nonlinear. Assume that $(e(t))_{t=-\infty}^{\infty}$ and $(\varphi(t))_{t=-\infty}^{\infty}$ are stationary stochastic processes such that the signal components $e(s)$ and $\varphi(s)$ are independent for every $s \in \mathbb{Z}$ and that r_0 is a Lipschitz continuous function with Lipschitz constant L_0, i.e., that

$$|r_0(\varphi_1) - r_0(\varphi_2)| \leq L_0 |\varphi_1 - \varphi_2|, \ \forall \varphi_1, \varphi_2 \in \mathbb{R}^n, \tag{11.3}$$

where $n = n_a + n_b + 1$. Furthermore, assume that a data set $(\varphi(t), y(t))_{t=1}^{N}$ consisting of N measurements of the regression vector and the system output is available.

Using this data set, estimates $\hat{\theta}_N$ and $\hat{r}_N$ of θ_0 and r_0, respectively, can be obtained by solving the QP problem

$$\begin{array}{ll} \underset{\theta_N, \rho_N}{\text{minimize}} & \frac{1}{N} \sum_{t=1}^{N} (y(t) - \theta_N^T \varphi(t) - \rho_N(t))^2 \\ \text{subject to} & \rho_N(t) - \rho_N(s) \leq L |\varphi(t) - \varphi(s)|, \\ & \qquad \forall s, t \in \{1, 2, \ldots, N\} \\ & \pm \theta_{N,i} \leq m_{\theta,i}, \\ & \qquad \forall i \in \{1, 2, \ldots, n_a + n_b + 1\}. \end{array} \tag{11.4}$$

In this problem, θ_N is a vector with $n_a + n_b + 1$ elements $\theta_{N,i}$ and ρ_N is a vector with N elements $\rho_N(t)$ that can be viewed as estimates of $r_0(\varphi(t))$. The constraints on the variables $\rho_N(t)$ imply that these variables will satisfy

$$|\rho_N(t) - \rho_N(s)| \leq L |\varphi(t) - \varphi(s)|$$

for all $s, t \in \{1, 2, \ldots, N\}$. If the variables $\rho_N(t)$ are viewed as samples from some function, this means that a Lipschitz condition holds for the sample points $(\varphi(t))_{t=1}^{N}$. Note that N of the constraints on ρ_N in (11.4) are trivial ($0 \leq 0$) and present in (11.4) only for notational convenience. These constraints can be removed without changing the solution of the problem. Furthermore, the constraints on θ_N are needed mainly when consistency is proved and can be removed in most applications. In these cases, the QP problem

$$\begin{array}{ll} \underset{\theta_N, \rho_N}{\text{minimize}} & \frac{1}{N} \sum_{t=1}^{N} (y(t) - \theta_N^T \varphi(t) - \rho_N(t))^2 \\ \text{subject to} & \rho_N(t) - \rho_N(s) \leq L |\varphi(t) - \varphi(s)|, \\ & \qquad \forall s, t \in \{1, 2, \ldots, N\}, \end{array} \tag{11.5}$$

which has fewer constraints, can be solved instead of (11.4).

An optimal solution $(\hat{\theta}_N, \hat{\rho}_N)$ to the problem (11.4) or (11.5) can be used to construct one step ahead predictions

$$\hat{y}_N(t) = \hat{\theta}_N^T \varphi(t) + \hat{\rho}_N(t) \tag{11.6}$$

of the system output for the observed regression vectors $(\varphi(t))_{t=1}^N$. In order to obtain a predictor that can be used for an arbitrary regression vector, the nonparametric function estimate $\hat{\rho}_N$ has to be interpolated.

When $\varphi(t)$ is a scalar, linear interpolation is probably the most natural type of interpolation. However, for $\varphi(t) \in \mathbb{R}^n$ with $n > 1$, linear interpolation of the variables $\hat{\rho}_N(t)$ will in general not result in a function that satisfies the Lipschitz condition for the choice of L used in (11.4). Instead, for $n > 1$, an estimate $\hat{r}_N$ of r_0 can be defined as

$$\hat{r}_N(\varphi) = \frac{1}{2} \max_{1 \le t \le N} (\hat{\rho}_N(t) - L|\varphi - \varphi(t)|) + \frac{1}{2} \min_{1 \le t \le N} (\hat{\rho}_N(t) + L|\varphi - \varphi(t)|) \tag{11.7}$$

using a similar construction as in Bertsimas et al. (1999). The function $\hat{r}_N$ is Lipschitz continuous since it is the mean of two Lipschitz continuous functions. Of course, other interpolation methods that can be used instead of (11.7) as long as they guarantee that the resulting function will be Lipschitz continuous with the Lipschitz constant L. Using $\hat{\theta}_N$ and $\hat{r}_N$, a general one step ahead predictor function

$$\hat{f}_N(\varphi) = \hat{\theta}_N^T \varphi + \hat{r}_N(\varphi) \tag{11.8}$$

can be constructed.

At first sight, it might seem that the $N + n$ variables used in the problem (11.4) and for the construction of the model (11.8) are too many since there are only N measurements. However, thanks to the randomness of the disturbance $e(t)$ in (11.1), the constraints in (11.4) will impose an averaging effect on the nonparametric function estimate. Without these constraints, one optimal solution to (11.4) is $\theta_N = 0$, $\rho_N(t) = y(t)$ for $t = 1, 2, \ldots, N$. Of course, since the measurements of the output are noisy, such a solution does not give a good model of the true system. By adding constraints like in (11.4), two variables $\rho_N(t)$ and $\rho_N(s)$ are allowed to differ only marginally from each other if the distance $|\varphi(t) - \varphi(s)|$ between the corresponding regression vectors is small. In this way, the ρ variables are imposed to have similar properties as samples from the true Lipschitz continuous function r_0. If the set of regression vectors gets more dense when N increases, $\hat{\theta}_N^T \varphi(t) + \hat{\rho}_N(t)$ will approach $\theta_0^T \varphi(t) + r_0(\varphi(t))$. For an intuitive understanding of this convergence, consider a small region in $\mathbb{R}^n$ which contains many regression vectors. The corresponding ρ variables will with a high probability be close to the mean of $y(t) - \hat{\theta}_N^T \varphi(t)$ since the constraints in (11.4) implies that the ρ variables should have values close to each other. The consistency of the predictor function estimator (11.8) will be discussed in Section 11.2.

Several types of extensions can be made to the identification method presented here. For example, if any prior knowledge about the true system can be written as linear constraints on θ_N and ρ_N, this knowledge can easily be incorporated in the QP problem (11.4). Examples of such prior knowledge are:

- Bounds on the function r_0 are known in a subset of its domain.

- The function r_0 is only piecewise Lipschitz continuous (with known breakpoints).
- The function r_0 satisfies different Lipschitz conditions in different parts of its domain.
- The function r_0 is known to be odd or even.
- An expression for the function r_0 is known in a subset of its domain.

Sometimes it could be interesting to estimate only a Lipschitz continuous function. Such an estimate can be obtained by setting all $m_{\theta,i} = 0$ in (11.4). Hence, a nonparametric function estimate can be calculated by solving the QP problem

$$\begin{array}{ll} \underset{\rho_N}{\text{minimize}} & \frac{1}{N}\sum_{t=1}^{N}(y(t) - \rho_N(t))^2 \\ \text{subject to} & \rho_N(t) - \rho_N(s) \leq L|\varphi(t) - \varphi(s)|, \\ & \qquad \forall s,t \in \{1,2,\dots,N\}. \end{array} \tag{11.9}$$

The construction of $\hat{r}_N$ using the interpolation method (11.7) can be used also in this case.

An advantage with the presented method is that the underlying optimization problem (11.4) is convex (Boyd and Vandenberghe, 2004). This convexity follows from the fact that only NARX models are considered. However, OE models have been the standard choice in the earlier chapters of this thesis and thus it would be nice if the mixed parametric and nonparametric method could be generalized to handle this case as well. It turns out that such a generalization is easy to formulate, but that it gives a nonconvex optimization problem. This problem can be formulated as

$$\begin{array}{ll} \underset{\theta_N,\rho_N}{\text{minimize}} & \frac{1}{N}\sum_{t=1}^{N}(y(t) - G(q,\theta_N)u(t) - \rho_N(t))^2 \\ \text{subject to} & \rho_N(t) - \rho_N(s) \leq L|\varphi(t) - \varphi(s)|, \\ & \qquad \forall s,t \in \{1,2,\dots,N\} \\ & \theta_N \in \mathcal{D}_\theta, \end{array} \tag{11.10}$$

where $\varphi(t)$ now only contains input components, $G(q, \theta_N)$ is a parameterized LTI model and $\mathcal{D}_\theta$ is a set of parameters that give stable and causal models. The model in this problem has a particular nonlinear output error (NOE) structure such that the model output can be written as the sum of the outputs from a linear OE model and an NFIR model.

In most cases, the optimization problem (11.10) is nonconvex and thus, the problem of finding its global minimum seems hard. However, it is easy to use a procedure where the problems

A: Keep ρ_N fixed and estimate the parameters θ_N by solving (11.10) for these variables.

B: Keep θ_N fixed and estimate the parameters ρ_N by solving (11.10) for these variables.

are solved iteratively until convergence. Problem A is a standard OE modeling problem while problem B is equivalent to an NFIR version of (11.9). The method can, for example, be initiated with $\rho_N = 0$. Usually, this procedure will give a model that corresponds to a local minimum of (11.10), but we will not analyze the convergence properties of this method here. However, the consistency of the presented nonparametric identification methods based on (11.4) and (11.9) will be shown in the next section.

11.2 Consistency

The identification methods given by (11.4) and (11.9), respectively, turn out to have rather attractive asymptotic properties when the number of measurements tends to infinity. In this section, it will be shown that the prediction function estimator defined by the solution to the optimization problem (11.4) under fairly general conditions is consistent. Since (11.9) is a special case of (11.4), the consistency of the nonparametric function estimator without a linear part follows from the more general case.

A related consistency result for the estimator defined by (11.9) is shown in Bertsimas et al. (1999) using results from Vapnik (1998). However, the result in the following theorem is based on different assumptions and is shown using an alternative proof.

Theorem 11.1
Consider data sets $(\varphi(t), y(t))_{t=1}^{N}$ *generated from the nonlinear system*

$$y(t) = \theta_0^T \varphi(t) + r_0(\varphi(t)) + e(t) \triangleq f_0(\varphi(t)) + e(t), \tag{11.11}$$

where $e(t)$ *is a white stationary stochastic process with zero mean and bounded variance* σ^2 *and where* $\varphi(t)$ *is a stationary stochastic process. For each data set, let* $\hat{\theta}_N$ *and* $\hat{\rho}_N(t)$ *be the optimal solution to (11.4). Furthermore, let* $\hat{f}_N$ *be the predictor function given by this solution, i.e.,*

$$\hat{f}_N(\varphi) = \hat{\theta}_N^T \varphi(t) + \hat{r}_N(\varphi),$$

where $\hat{r}_N$ *is defined in (11.8). Suppose that*

1. $\varphi(t) \in \Phi$, *where* Φ *is a compact set such that the probability density function* $p(\varphi)$ *for* $\varphi(t)$ *is positive for all* $\varphi \in \Phi$ *and that for any* $\varepsilon > 0$, Φ *can be partitioned*

$$\Phi = \bigcup_{i=1}^{d} \Phi_i, \tag{11.12}$$

where $\varphi_1, \varphi_2 \in \Phi_i \Rightarrow |\varphi_1 - \varphi_2| \leq \varepsilon$ *and* $p_i = P(\varphi(t) \in \Phi_i) > 0$ *for all* $i = 1, 2, \ldots, d$,

2. *the stochastic process* $\varphi(t)$ *is such that* $N_i/N \to p_i$ *when* $N \to \infty$ *w.p.1 for all* i *in any* ε*-partitioning* (11.12) *where*

$$N_i = \operatorname{card}(T_i) \text{ and } T_i = \{t \mid \varphi(t) \in \Phi_i,\ t \leq N\}, \tag{11.13}$$

3. $e(t)$ *and* $\varphi(t)$ *are independent, but* $\varphi(t)$ *may depend on past* $e(s)$,
4. $|r_0(\varphi_1) - r_0(\varphi_2)| \leq L|\varphi_1 - \varphi_2|$ *for all* $\varphi_1, \varphi_2 \in \Phi$,
5. $|f_0(\varphi_1) - f_0(\varphi_2)| \leq \tilde{L}|\varphi_1 - \varphi_2|$ *for all* $\varphi_1, \varphi_2 \in \Phi$, *where* $\tilde{L} = L + M_\theta$ *and*

$$M_\theta^2 = \sum_{i=1}^{n_a+n_b+1} m_{\theta,i}^2.$$

Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}(\hat{f}_N(\varphi(t))-f_0(\varphi(t)))^2=0 \quad \textit{w.p.1} \tag{11.14}$$

and

$$\hat{f}_N(\varphi)\to f_0(\varphi) \textit{ uniformly on } \Phi \textit{ as } N\to\infty \quad \textit{w.p.1.} \tag{11.15}$$

Proof: Take an arbitrary $\varepsilon > 0$ and consider an ε-partitioning such that the first assumption is satisfied. Let $I_{\Phi_i}(\varphi)$ be the indicator function for the set Φ_i, i.e.,

$$I_{\Phi_i}(\varphi)=\begin{cases}1, & \varphi\in\Phi_i,\\ 0, & \text{otherwise.}\end{cases}$$

Consider arbitrary realizations of the processes $\varphi(t)$ and $e(t)$. With probability one, these realizations are such that $N_i/N \to p_i$ as $N\to\infty$ and that

$$\lim_{N\to\infty}\frac{1}{p_iN}\sum_{t\in T_i}e(t)=\lim_{N\to\infty}\frac{1}{p_iN}\sum_{t=1}^{N}I_{\Phi_i}(\varphi(t))e(t)=0, \tag{11.16a}$$

$$\lim_{N\to\infty}\frac{1}{p_iN}\sum_{t\in T_i}|e(t)|=\lim_{N\to\infty}\frac{1}{p_iN}\sum_{t=1}^{N}I_{\Phi_i}(\varphi(t))|e(t)|\le C, \tag{11.16b}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}e(t)^2=\sigma^2 \tag{11.16c}$$

for some constant C and for all $i=1,2,\ldots,d$. The limits (11.16) follow from the law of large numbers. Let

$$\tilde{e}(t)=I_{\Phi_i}(\varphi(t))e(t),$$

and consider two arbitrary different time instants s and t. Without loss of generality, we can assume that $s>t$. Since $\varphi(t)$, $\varphi(s)$ and $e(t)$ are all independent of $e(s)$, this implies that

$$\mathrm{E}(\tilde{e}(t)\tilde{e}(s))=\mathrm{E}\big(I_{\Phi_i}(\varphi(t))e(t)I_{\Phi_i}(\varphi(s))\big)\mathrm{E}\big(e(s)\big)=0,$$

i.e., that $\tilde{e}(t)$ and $\tilde{e}(s)$ are uncorrelated. Similarly, with

$$e^*(t)=I_{\Phi_i}(\varphi(t))(|e(t)|-\mathrm{E}(|e(t)|)),$$

$e^*(t)$ and $e^*(s)$ can be shown to be uncorrelated. Furthermore, the variances of $\tilde{e}(t)$ and $e^*(t)$ are finite. Hence, the version of the strong law of large numbers in Theorem 5.1.2 in Chung (1974) imply that (11.16a) and (11.16b) hold. The convergence of the series in (11.16c) follows from the strong law of large numbers for independent variables (Chung, 1974, Theorem 5.4.2).

For two fixed realizations of $\varphi(t)$ and $e(t)$ for which (11.16) holds, we can thus find an $N'(\varepsilon)$ such that

$$\left| \frac{1}{p_i N} \sum_{t \in T_i} e(t) \right| \leq \varepsilon, \quad \forall i \in \{1, 2, \ldots, d\}, \tag{11.17a}$$

$$\frac{1}{p_i N} \sum_{t \in T_i} |e(t)| \leq 2C, \quad \forall i \in \{1, 2, \ldots, d\}, \tag{11.17b}$$

$$\frac{1}{N} \sum_{t=1}^{N} e(t)^2 \leq 2\sigma^2 \tag{11.17c}$$

for all $N > N'(\varepsilon)$. This result follows since the partitioning is finite for any ε.

The fourth and fifth assumption in the theorem imply that

$$f_N(\varphi(t)) \triangleq \theta_N^T \varphi(t) + \rho_N(t) = f_0(\varphi(t))$$

is a feasible choice of function in the optimization problem (11.4), either with $\rho_N(t) = r_0(\varphi(t))$ and $\theta_N = \theta_0$ or sometimes with some smaller θ_N and larger $\rho_N(t)$:s. Hence, we have

$$\frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{f}_N(\varphi(t)))^2 \leq \frac{1}{N} \sum_{t=1}^{N} (y(t) - f_0(\varphi(t))^2 = \frac{1}{N} \sum_{t=1}^{N} e(t)^2,$$

which means that

$$\frac{1}{N} \sum_{t=1}^{N} (f_0(\varphi(t)) - \hat{f}_N(\varphi(t)))^2 \leq \left| \frac{2}{N} \sum_{t=1}^{N} e(t)(f_0(\varphi(t)) - \hat{f}_N(\varphi(t))) \right|. \tag{11.18}$$

Note first that, by applying Cauchy-Schwarz inequality to the right hand side, we obtain that

$$\frac{1}{N} \sum_{t=1}^{N} (f_0(\varphi(t)) - \hat{f}_N(\varphi(t)))^2 \leq \frac{4}{N} \sum_{t=1}^{N} e(t)^2.$$

Since f_0 is bounded by $C_{f_0} = \sup_{\varphi \in \Phi} |f_0(\varphi)|$ and (11.17c) holds for all $N > N'(\varepsilon)$, $\hat{f}_N(\varphi)$ as defined by (11.7) and (11.8) must be bounded too. Hence, we can choose a constant $C_{\hat{f}}$, such that for $N > N'(\varepsilon)$ we have $C_{\hat{f}} > \sup_{\varphi \in \Phi} |\hat{f}_N(\varphi)|$.

Since $N_i \to \infty$ for all i, there will be many $\varphi(t)$ in every set Φ_i in the ε-partitioning. Choose $t_i^* \in T_i$ and let

$$f_i = f_0(\varphi(t_i^*)),$$
$$\hat{f}_{N,i} = \hat{f}_N(\varphi(t_i^*)).$$

This means that for $t \in T_i$, it holds that

$$|f_0(\varphi(t)) - f_i| \leq \tilde{L}\varepsilon$$

and

$$|\hat{f}_N(\varphi(t)) - \hat{f}_{N,i}| \leq \tilde{L}\varepsilon.$$

Inserting this into the expression (11.18) gives

$$\begin{aligned}
&\frac{1}{N}\sum_{t=1}^{N}(f_0(\varphi(t)) - \hat{f}_N(\varphi(t)))^2 \\
&\leq \left|\frac{2}{N}\sum_{t=1}^{N} e(t)(f_0(\varphi(t)) - \hat{f}_N(\varphi(t)))\right| \\
&= \left|\frac{2}{N}\sum_{i=1}^{d}\sum_{t\in T_i} e(t)(f_0(\varphi(t)) - f_i + f_i - \hat{f}_N(\varphi(t)) + \hat{f}_{N,i} - \hat{f}_{N,i})\right| \\
&= \left|2\sum_{i=1}^{d} p_i\left(\left(\frac{1}{p_i N}\sum_{t\in T_i} e(t)\right)\left(f_i - \hat{f}_{N,i}\right)\right.\right. \\
&\quad \left.\left. + \left(\frac{1}{p_i N}\sum_{t\in T_i} e(t)(f_0(\varphi(t)) - f_i - \hat{f}_N(\varphi(t)) + \hat{f}_{N,i})\right)\right)\right| \\
&\leq 2\sum_{i=1}^{d} p_i\left(\varepsilon \max_i |f_i - \hat{f}_{N,i}| + \frac{1}{p_i N}\sum_{t\in T_i} |e(t)| 2\tilde{L}\varepsilon\right) \\
&\leq C'\varepsilon \qquad \text{for } N > N'(\varepsilon),
\end{aligned}$$

where $C' = 2C_{f_0} + 2C_{\hat{f}} + 8\tilde{L}C$. Since ε and the realizations were arbitrary, (11.14) has been proven.

We will now prove that the result (11.14) implies the uniform convergence in (11.15). First, we will consider pointwise convergence. Consider arbitrary realizations of $\varphi(t)$ and $e(t)$. With probability one these realizations are such that the second assumption in the theorem is satisfied and that the convergence in (11.14) holds. Consider two arbitrary realizations where these limits hold and assume that $\hat{f}_N$ does not converge pointwise to $f_0(\varphi)$ on Φ, i.e., that there exists a φ_0 in Φ, a $\delta > 0$ and an infinite strictly increasing sequence of integers K_j, $j \in \mathbb{Z}_+$, such that

$$|\hat{f}_{K_j}(\varphi_0) - f_0(\varphi_0)| > \delta$$

for all j. Consider a $\delta/4\tilde{L}$-partitioning of Φ such that the two first assumptions are satisfied, i.e., a partitioning where

$$\varphi_1,\ \varphi_2 \in \Phi_i \Rightarrow |\varphi_1 - \varphi_2| < \frac{\delta}{4\tilde{L}}$$

and where $N_i/N \to p_i$ for all $i = 1, 2, \ldots, d$. Consider the set Φ_l that contains φ_0. For every φ in Φ_l, it holds that

$$\begin{aligned}
|\hat{f}_{K_j}(\varphi) - f_0(\varphi)| &\geq |\hat{f}_{K_j}(\varphi_0) - f_0(\varphi_0)| - |\hat{f}_{K_j}(\varphi) - \hat{f}_{K_j}(\varphi_0)| - |f_0(\varphi_0) - f_0(\varphi)| \\
&\geq \delta - \tilde{L}\frac{\delta}{4\tilde{L}} - \tilde{L}\frac{\delta}{4\tilde{L}} = \frac{\delta}{2}.
\end{aligned} \tag{11.19}$$

Furthermore, from the second assumption in the theorem, it holds that $K_{j,l}/K_j \to p_l$ when $j \to \infty$, where $K_{j,l}$ is the number of $\varphi(t)$ in Φ_l when the total number of measurements is K_j. The convergence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (\hat{f}_N(\varphi(t)) - f_0(\varphi(t)))^2 = 0$$

implies that

$$\lim_{j\to\infty} \frac{1}{K_j} \sum_{t=1}^{K_j} (\hat{f}_{K_j}(\varphi(t)) - f_0(\varphi(t)))^2 = 0.$$

However, using (11.19), it follows that

$$\begin{aligned} \frac{1}{K_j} \sum_{t=1}^{K_j} (\hat{f}_{K_j}(\varphi(t)) - f_0(\varphi(t)))^2 &\geq \frac{1}{K_j} \sum_{t\in T_l} (\hat{f}_{K_j}(\varphi(t)) - f_0(\varphi(t)))^2 \\ &\geq \frac{1}{K_j} \sum_{t\in T_l} \frac{\delta^2}{4} = \frac{\delta^2 K_{j,l}}{4K_j}. \end{aligned}$$

Since

$$\frac{\delta^2 K_{j,l}}{4K_j} \to \frac{\delta^2 p_l}{4} > 0, \quad j \to \infty,$$

we have a contradiction. Thus, $\hat{f}$ must converge pointwise to f_0 on Φ.

It turns out that pointwise convergence gives uniform convergence in this case. Take an arbitrary $\tilde{\varepsilon} > 0$ and assume that $\hat{f}_N(\varphi)$ converges pointwise to $f_0(\varphi)$ on Φ. Select a finite number of points $\tilde{\varphi}_k$, $k = 1, 2, \ldots, d_{\tilde{\varepsilon}}$ in Φ such that for every point φ in Φ,

$$|\varphi - \tilde{\varphi}_k| < \frac{\tilde{\varepsilon}}{3\tilde{L}}$$

for some k. Choose an $N_{\tilde{\varepsilon}}$ such that for all k it holds that

$$|\hat{f}_N(\tilde{\varphi}_k) - f_0(\tilde{\varphi}_k))| < \frac{\tilde{\varepsilon}}{3}, \quad \forall N > N_{\tilde{\varepsilon}}.$$

Hence, for an arbitrary point φ in Φ, there is a k such that

$$\begin{aligned} |\hat{f}_N(\varphi) - f_0(\varphi)| &\leq |\hat{f}_N(\varphi) - \hat{f}_N(\tilde{\varphi}_k)| + |f_0(\tilde{\varphi}_k) - f_0(\varphi)| + |\hat{f}_N(\tilde{\varphi}_k) - f_0(\tilde{\varphi}_k)| \\ &< \tilde{L}\frac{\tilde{\varepsilon}}{3\tilde{L}} + \tilde{L}\frac{\tilde{\varepsilon}}{3\tilde{L}} + \frac{\tilde{\varepsilon}}{3} = \tilde{\varepsilon}, \quad \forall N > N_{\tilde{\varepsilon}}, \end{aligned}$$

where we have used that both f_0 and $\hat{f}_N$ satisfy a Lipschitz condition with Lipschitz constant $\tilde{L}$. Since $\tilde{\varepsilon}$ and φ were arbitrary it follows that $\hat{f}_N$ converges uniformly to f_0. □

From the fourth and fifth assumption in Theorem 11.1, it can be seen that the constant L in the QP problem must be an upper bound on the true Lipschitz constant for the nonlinear function r_0 and that $\tilde{L} = L + M_\theta$ must be an upper bound on the Lipschitz constant of the true predictor function f_0 for the consistency result to hold. These conditions are quite intuitive since the estimated function must be allowed to vary at least as much as the true function. However, it is interesting to see that the choice $\theta_N = \theta_0$ does not have to be a feasible point to (11.4) since the linear part of the system (11.1) can be modeled by the nonparametric function $\hat{r}_N$, provided L is large enough. However, although a too hard bound on θ_N might not ruin the consistency, an unnecessarily large L will give a less smooth function estimate with a finite number of measurements. One of the main benefits of including also a linear parametric term in the model structure is that the smoothness of the estimate of the nonlinear part in that case will not depend on how large the linear part of the system is.

The fact that the linear part of the nonlinear system can be described by the nonparametric nonlinear part of the model explains why Theorem 11.1 does not discuss consistency for the individual linear and nonlinear estimators. In general, these estimators are not consistent since the separation of the system into a linear and a nonlinear part is not unique if a too large Lipschitz constant L is used. Some further properties of the proposed mixed parametric and nonparametric method will be discussed in the examples in the next section.

11.3 Examples

The previously presented method for combined parametric and nonparametric estimation of NARX or NOE systems has been used in a couple of numerical examples.

11.3.1 NARX Models

The first example concerns identification of a static nonlinearity. Although this example is very simple, it illustrates a number of general properties of the proposed method.

Example 11.1

Consider the system

$$y(t) = 0.4u(t) + r_0(u(t)) + e(t), \tag{11.20}$$

where both $u(t)$ and $e(t)$ are white noise processes and independent of each other. The input $u(t)$ has uniform distribution on the interval $[-10, 10]$ while the noise $e(t)$ is normally distributed such that its mean is zero and its variance is 25. The nonlinearity in this system is

$$r_0(u(t)) = \frac{40}{5+|u(t)|}\left(\frac{u(t)}{1+|u(t)|} - \frac{u(t)-3}{1+|u(t)-3|} - \frac{u(t)+6}{1+|u(t)+6|} + \frac{3}{28}\right).$$

This function is Lipschitz continuous with $L_0 = 7.4$ and bounded since $|r_0'(x)| < 7.4$ and $|r_0(x)| < 3.1$, for all $x \in \mathbb{R}$.

A small data set consisting of 40 measurements of the input and output in (11.20) has been generated and is shown in Figure 11.1. Note that the shape of the nonlinear function is not obvious in this figure. The method (11.5) with $L = 7.4$ has been used with this data set and linear interpolation has been used to construct $\hat{r}_N$. The resulting predictor function $\hat{f}_N(\varphi)$ is shown in Figure 11.2. From this figure, it seems that the function estimate has

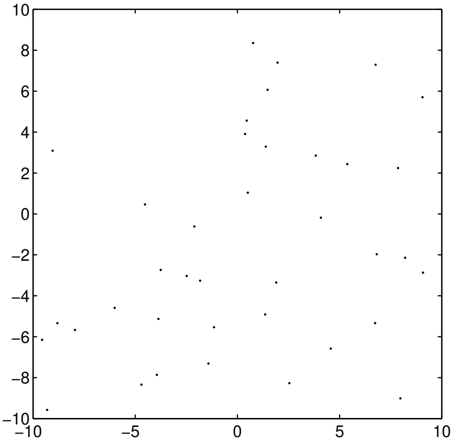


Figure 11.1: *The values of $y(t)$ plotted against $u(t)$ for the data set with 40 measurements used in Example 11.1.*

managed to pick up some key features of the true function, despite the small number of measurements.

In this case, the L value used in the method is equal to L_0. In a more realistic example, the true Lipschitz constant would typically be unknown. An alternative would then be to divide the data set into estimation data and validation data and try different values of L. By evaluating the predictor (11.8) on the validation data for different choices of Lipschitz constant, it would be possible to find a good choice of L.

If this approach would be used for the identification of the system (11.20), the resulting Lipschitz constant L would probably be smaller than L_0. This results follows since the variations in $\hat{r}_N(\varphi)$ in regions where $|r_0'(\varphi)|$ is rather small can be reduced if a bias is accepted in regions where $|r_0'(\varphi)|$ is close to L_0.

A larger data set consisting of 500 measurements of the input and output in (11.20) has also been generated and a couple of models have been estimated using an extended version of (11.5) where bounds $\pm\rho_N(t) \leq 4$ have been added. One model was estimated using $L = 15$ and the resulting predictor function is shown in Figure 11.3a. The choices $L = 7.4$ and $L = 4$ gave the results shown in Figure 11.3b and 11.3c, respectively. From these figures, it seems that the function estimates contain no significant systematic errors and that a larger value of L gives more variations. Note that for $L = 4$, the true function r_0 is not a feasible solution to the identification problem. However, the obtained function estimate gives a rather good approximation of r_0 anyway.

In the case with $L = 15$, the obtained estimate of the linear regression parameter $\theta_0 = 0.4$ was $\hat{\theta}_N = 0.31$ while $L = 7.4$ gave $\hat{\theta}_N = 0.33$ and $L = 4$ gave $\hat{\theta}_N = 0.39$. Using the same data set but with a completely linear model, the least-squares method gave an estimate $\hat{\theta}_{\mathrm{LS}} = 0.23$. Hence, it seems that including a bounded nonlinear Lipschitz continuous term in the model sometimes can improve the estimate of the linear part.

Although the estimated nonlinear predictor functions in the previous example are rela-

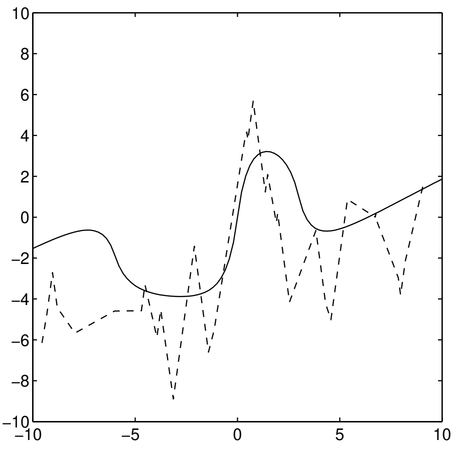


Figure 11.2: *The predictor function estimated from 40 measurements (dashed) and the true predictor function (solid) from Example 11.1.*

tively good approximations of the true nonlinear function, they are not very smooth. The fact that the function estimate $\hat{r}_N$ will not be differentiable everywhere in its domain might be a problem in some applications. However, the nonparametric estimate presented here can still be useful as an initial estimate of the nonlinearities.

The method (11.5) combined with the interpolation (11.7) has also been used on a NARX system where the regression vector consists of two past output components and one input component. The results of this numerical experiment are described in Roll et al. (2005a) and indicates that the proposed method gives useful estimates of the predictor function also when $\varphi(t)$ is a vector.

11.3.2 NOE Models

The iterative method described in Section 11.1 for finding a local minimum to the NOE optimization problem (11.10) has been used on data from the nonlinear system in Example 8.1. The results of this experiment are described in the following example.

Example 11.2

Consider again the nonlinear system

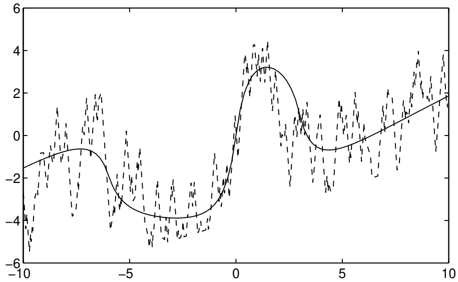
$$\begin{aligned} y(t) &= y_l(t) + \alpha y_n(t), \\ y_l(t) &= u(t), \\ y_n(t) &= u(t)^3, \end{aligned}$$

which was studied in Examples 8.1, 8.2 and 8.3. Just like in Example 8.1, assume that the input signal is

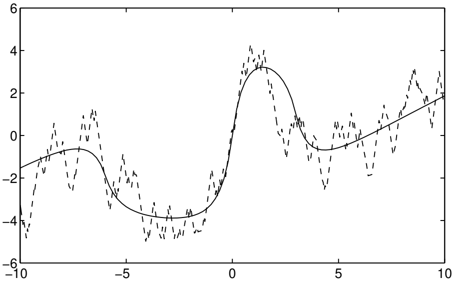
$$u(t) = L_m(q, c)e(t),$$

where

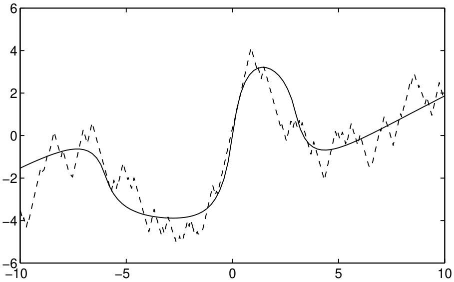
$$L_m(q, c) = \left(1 - cq^{-1}\right)^2 = 1 - 2cq^{-1} + c^2q^{-2}, \quad 0 < c < 1,$$



(a) $L = 15$



(b) $L = 7.4$



(c) $L = 4$

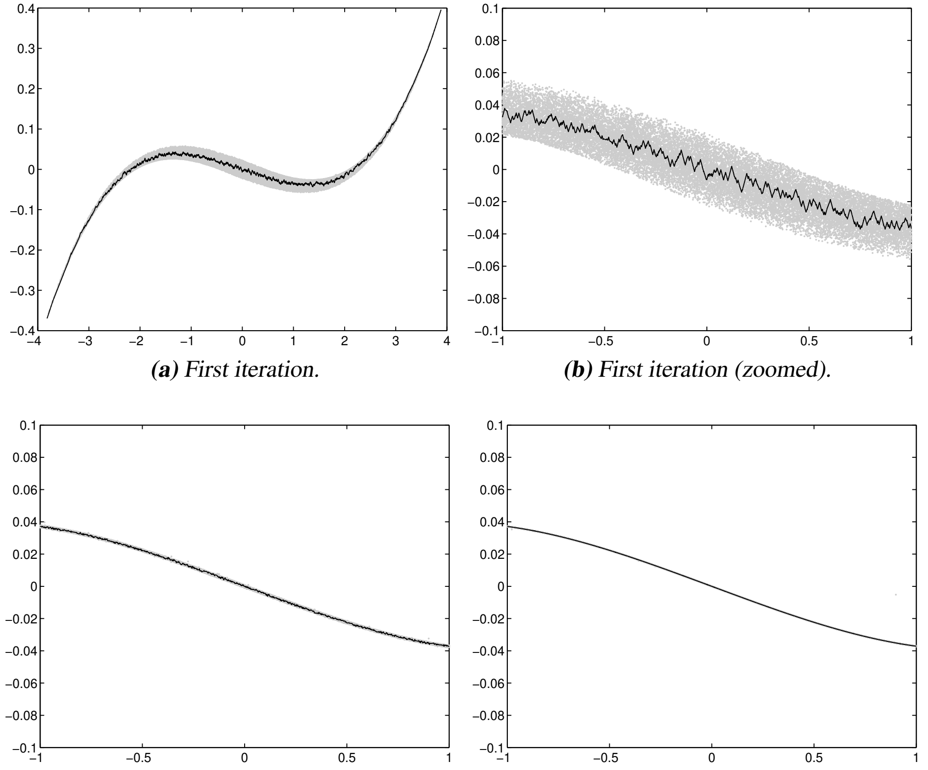
Figure 11.3: *The predictor function estimated from 500 measurements for three choices of L (dashed) and the true predictor function (solid) from Example 11.1.*

$e(t)$ is a white noise process with uniform distribution over the interval $[-1, 1]$ and where $c = 0.99$ and $\alpha = 0.01$.

A realization of this input has been generated and a data set with 50 000 input and output measurements has been collected. The iterative method described after (11.10) in Section 11.1 for finding a local minimum to (11.10) has been used with this data set with Lipschitz constant $L = 0.48$ and additional bounds $|\rho_N(t)| \leq 0.64$. These values correspond to the true bounds on the nonlinearity for inputs in the interval $[-4, 4]$. The orders of the estimated OE models have been $n_b = 3$, $n_f = 2$ and $n_k = 0$, i.e., equal to the orders of the OE-LTI-SOE of this system (cf. Example 8.1).

In order to minimize the computation time, only 2 000 measurements have been used to calculate each $\hat{\rho}_N$. However, different measurements have been used for each estimation of the nonlinearity, i.e., the measurements for $t = 1, 2, \ldots, 2000$ were used to calculate the first $\hat{\rho}_N$, the measurements for $t = 2001, 2002, \ldots, 4000$ were used to calculate the next one, etc. Furthermore, the largest and the smallest input components and the corresponding output components have been added to each of these small data sets, if they were not already included.

Some of the resulting nonparametric function estimates are shown in Figure 11.4. As can be seen from Figure 11.4a, the residuals from the first LTI model contain a lot of information about the unmodeled nonlinearity. However, the difference between the estimated LTI model and the linear part of the system implies that the residuals only can



(a) First iteration. *(b) First iteration (zoomed).*

(c) Second iteration (zoomed). *(d) Ninth iteration (zoomed).*

Figure 11.4: *The residuals from the LTI models (grey) and the corresponding nonparametric function estimates (black) from some of the iteration steps in Example 11.2. In this case, the output from the true system contains no measurement noise.*

give a blurred picture of the nonlinearity. In Figures 11.4b to 11.4d, it can be seen that the residual plots get sharper after each iteration. After nine iterations, there is a close fit between the estimated and the true nonlinearity.

The frequency responses of some of the estimated LTI models are shown in Figure 11.5. After nine iterations, the estimated LTI model is essentially a constant. Hence, it seems that the nonlinear term in the model is able to describe a significant part of the true nonlinearity and thus to reduce its effect on the estimated LTI model.

The iterative method has also been tested on a data set with the same realization of the input but where a realization of Gaussian measurement noise with zero mean and variance 0.0025 has been added to the output. In this case, the same orders of the OE model and bounds on the ρ variables as for the noise-free case have been used. The residuals from the LTI model and the estimated nonlinearity from the first iteration are shown in Figure 11.6. The estimate of the nonlinearity seems to be able to describe the true nonlinearity rather well also in this case. The frequency responses of some of the estimated LTI models are

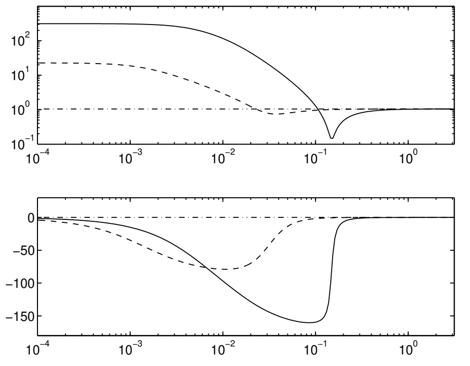


Figure 11.5: *The estimated LTI models of the noise-free system in Example 11.2 from the first (solid), second (dashed) and ninth (dash-dotted) iteration.*

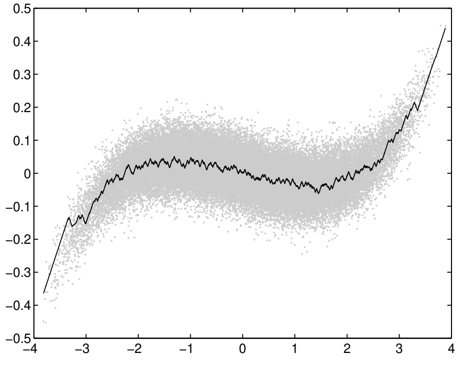


Figure 11.6: *The residuals from the initial LTI model (grey) and the corresponding nonparametric function estimate (black) for the system with measurement noise in Example 11.2.*

shown in Figure 11.7. Again, it seems that the sensitivity of the LTI models to the small nonlinearity has been reduced significantly.

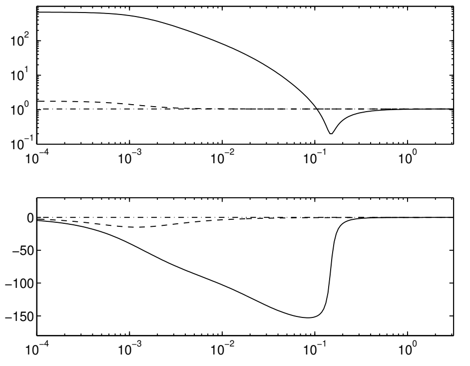


Figure 11.7: *The estimated LTI models of the system with measurement noise in Example 11.2 from the first (solid), second (dashed) and ninth (dash-dotted) iteration.*

11.4 A General Perspective

As was mentioned previously in Section 11.1, it is easy to incorporate various kinds of prior knowledge into the identification problem. In fact, we can regard the presented approach as a special instance of the more general identification problem

$$\begin{aligned} \underset{\theta_N,\rho_N}{\text{minimize}} \quad & \tfrac{1}{N}\textstyle\sum_{t=1}^{N}(y(t) - \theta_N^T\varphi(t) - \rho_N(t))^2 \\ \text{subject to} \quad & A\begin{pmatrix}\rho_N \\ \theta\end{pmatrix} \preccurlyeq b, \end{aligned} \tag{11.21}$$

where $\preccurlyeq$ denotes component-wise inequality. This is still a convex QP problem. An interesting special case of (11.21) is

$$\begin{aligned} \underset{\theta_N,\rho_N}{\text{minimize}} \quad & \tfrac{1}{N}\textstyle\sum_{t=1}^{N}(y(t) - \theta_N^T\varphi(t) - \rho_N(t))^2 \\ \text{subject to} \quad & |\rho_N(t)| \le M, \\ & \qquad \forall t \in \{1, 2, \ldots, N\}. \end{aligned} \tag{11.22}$$

It turns out that minimizing (11.22) gives exactly the same linear part as using an ε-*insensitive norm* for identification of ARX models, i.e.,

$$\underset{\theta_N}{\text{minimize}} \ \frac{1}{N}\sum_{t=1}^{N}\left|y(t) - \theta_N^T\varphi(t)\right|_{\varepsilon}^{k}, \tag{11.23}$$

with

$$|x|_\varepsilon = \begin{cases} 0, & |x| \le \varepsilon, \\ |x| - \varepsilon, & |x| > \varepsilon \end{cases}$$

and with $k = 2$ and $\varepsilon = M$. This norm (or the corresponding norm with $k = 1$) is often used in support vector machines (Vapnik, 1998), and similar approaches are also used in robust adaptive control (Peterson and Narendra, 1982). To see the equivalence between (11.22) and (11.23), define

$$\bar{r}(t,\theta) = \begin{cases} M, & y(t) - \theta^T\varphi(t) > M, \\ y(t) - \theta^T\varphi(t), & -M \leq y(t) - \theta^T\varphi(t) \leq M, \\ -M, & y(t) - \theta^T\varphi(t) < -M. \end{cases}$$

Then we can write (11.23) as

$$\underset{\theta_N}{\text{minimize}} \ \frac{1}{N} \sum_{t=1}^{N} \left| y(t) - (\theta_N^T\varphi(t) + \bar{r}(t,\theta_N)) \right|^k .$$

On the other hand it is easy to see that, for a given θ_N, the minimum of (11.22) is obtained precisely when $\rho_N(t) = \bar{r}(t,\theta_N)$. Since $\bar{r}(t,\theta_N)$ automatically has a magnitude not greater than M, the desired equivalence follows.

The advantage with using the formulation (11.22) instead of (11.23) is that the explicit representation of ρ_N again makes it possible to combine different types of requirements on the nonlinearity.

11.5 Discussion

In this chapter, a method has been proposed for identification of NARX systems using a model structure with a parametric linear part and a nonparametric nonlinear part. The model estimate is computed by solving a convex optimization problem where the only assumption on the nonlinearities in the system is that they are Lipschitz continuous. Theorem 11.1 gives a direct proof of the consistency of this method and its usefulness has also been illustrated in examples.

An iterative method for identification of a class of NOE systems has also been presented. The model structure that can be used with this method contains a linear OE part and a nonparametric NFIR part. This NOE identification method can sometimes reduce the sensitivity of OE models to small nonlinearities (see Example 11.2).

A nice property of both presented methods is that it is easy to include prior system knowledge in them if this knowledge can be written as linear constraints. Furthermore, since a nonparametric model of the nonlinearities is used, the methods might be particularly useful for computing initial estimates of unknown nonlinear systems where the shapes of the nonlinearities are completely unknown.

12

Conclusions

In this thesis, system identification using LTI models of nonlinear systems with random inputs has been studied. The main focus has been on analysis of LTI models that are optimal approximations in the sense that they minimize a mean-square error criterion. This is a very rich research field with many theoretically and practically interesting problems to consider.

Here, properties of the optimal LTI approximations for different types of input signals have been studied. For example, several useful properties of the LTI models have been shown for classes of minimum phase filtered white noise inputs, Gaussian inputs and separable inputs. Furthermore, LTI approximations of nonlinear systems have been discussed from a robust control point of view and a method for mixed parametric and nonparametric identification has been proposed.

However, it must be admitted that this thesis mainly concerns theoretical approximation aspects and that no particular application has been considered. Hence, it would be interesting to investigate LTI models for some real-life applications in order to see which are the dominating effects of unmodeled nonlinearities. Hopefully, such a study would show that at least some of the theoretical results in this thesis are nice illustrations of the following classic saying, which has been attributed to many famous scientists.

There is nothing more practical than a good theory.

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A

Calculations for Example 4.3

Since $\mathrm{E}(e(t)^3) = 0$ and $\mathrm{E}(e(t)^4) = 3$, we get

$$\begin{aligned}
R_u(0) &= \mathrm{E}(u(t)^2) = \mathrm{E}((e(t) + e(t-1)^2 - 1)^2) \\
&= \mathrm{E}(e(t)^2) + \mathrm{E}(e(t-1)^4) + 1 - 2\mathrm{E}(e(t-1)^2) = 1 + 3 + 1 - 2 = 3, \\
R_u(\pm 1) &= \mathrm{E}(u(t)u(t-1)) = 0, \\
R_u(\tau) &= 0, \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}.
\end{aligned}$$

Furthermore, since $\mathrm{E}(e(t)^5) = \mathrm{E}(e(t)^7) = 0$, $\mathrm{E}(e(t)^6) = 15$ and $\mathrm{E}(e(t)^8) = 105$, we get

$$\begin{aligned}
R_{yu}(0) &= \mathrm{E}(y(t)u(t)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t) + e(t-1)^2 - 1\big)\big) \\
&= 2\mathrm{E}(e(t)^2 e(t-1)^2) - 2\mathrm{E}(e(t)^2) + \mathrm{E}(e(t-1)^6) - \mathrm{E}(e(t-1)^4) \\
&\quad - 2\mathrm{E}(e(t-1)^4) + 2\mathrm{E}(e(t-1)^2) \\
&= 2 - 2 + 15 - 3 - 2 \cdot 3 + 2 = 8, \\
R_{yu}(1) &= \mathrm{E}(y(t)u(t-1)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t-1) + e(t-2)^2 - 1\big)\big) = 0, \\
R_{yu}(-1) &= \mathrm{E}(y(t)u(t+1)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t+1) + e(t)^2 - 1\big)\big) = \mathrm{E}(e(t)^4) - \mathrm{E}(e(t)^2) = 3 - 1 = 2, \\
R_{yu}(\tau) &= 0, \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}
\end{aligned}$$

and

$$\begin{aligned}
R_y(0) &= \mathrm{E}(y(t)^2) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big)\big) \\
&= \mathrm{E}(e(t)^4) + 2\mathrm{E}(e(t)^2 e(t-1)^4) - 4\mathrm{E}(e(t)^2 e(t-1)^2) - 4\mathrm{E}(e(t)^2) \\
&\quad + \mathrm{E}(e(t-1)^8) - 4\mathrm{E}(e(t-1)^6) - 4\mathrm{E}(e(t-1)^4) + 4\mathrm{E}(e(t)^2 e(t-1)^4) \\
&\quad - 8\mathrm{E}(e(t)^2 e(t-1)^2) + 4\mathrm{E}(e(t)^2) + 4\mathrm{E}(e(t-1)^4) + 8\mathrm{E}(e(t-1)^2) + 4 \\
&= 3 + 2\cdot 3 - 4 - 4 + 105 - 4\cdot 15 - 4\cdot 3 + 4\cdot 3 - 8 + 4 + 4\cdot 3 + 8 + 4 \\
&= 66, \\
R_y(\pm 1) &= \mathrm{E}(y(t)y(t-1)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t-1)^2 + e(t-2)^4 + 2e(t-1)e(t-2)^2 - 2e(t-1) \\
&\quad -2e(t-2)^2 - 2\big)\big) \\
&= \mathrm{E}(e(t)^2 e(t-1)^2) + \mathrm{E}(e(t-1)^6) - 2\mathrm{E}(e(t-1)^4) \\
&\quad - 2\mathrm{E}(e(t-1)^2) + \mathrm{E}(e(t)^2 e(t-2)^4) + \mathrm{E}(e(t-1)^4 e(t-2)^4) \\
&\quad - 2\mathrm{E}(e(t-1)^2 e(t-2)^4) - 2\mathrm{E}(e(t-2)^4) - 2\mathrm{E}(e(t)^2 e(t-2)^2) \\
&\quad - 2\mathrm{E}(e(t-1)^4 e(t-2)^2) + 4\mathrm{E}(e(t-1)^2 e(t-2)^2) + 4\mathrm{E}(e(t-2)^2) \\
&\quad - 2\mathrm{E}(e(t)^2) - 2\mathrm{E}(e(t-1)^4) + 4\mathrm{E}(e(t-1)^2) + 4 \\
&= 1 + 15 - 2\cdot 3 - 2 + 3 + 9 - 2\cdot 3 - 2\cdot 3 - 2 - 2\cdot 3 + 4 + 4 - 2 \\
&\quad - 2\cdot 3 + 4 + 4 = 8, \\
R_y(\tau) &= 0, \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}.
\end{aligned}$$

The z-spectra are thus

$$\begin{aligned}
\Phi_u(z) &= 3, \\
\Phi_{yu}(z) &= 2z + 8, \\
\Phi_y(z) &= 8z + 66 + 8z^{-1}.
\end{aligned}$$

Inserted in (4.2b) this gives

$$\Phi_\zeta(z) = \begin{pmatrix} 3 & 2 + 8z \\ 2 + 8z^{-1} & 8z + 66 + 8z^{-1} \end{pmatrix}.$$

In order to compute the canonical spectral factorization $\Phi_\zeta(z) = T(z) Q_\zeta T^T(z^{-1})$ we first pre- and postmultiply $\Phi_\zeta(z)$ with matrices $T_1(z)$ and $T_1^T(z^{-1})$, respectively. If $T_1(z)$ is chosen as

$$T_1(z) = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} - \frac{8}{3}z^{-1} & 1 \end{pmatrix},$$

the result is the following diagonal matrix

$$D(z) = T_1(z)\Phi_\zeta(z)T_1^T(z^{-1}) = \begin{pmatrix} 3 & 0 \\ 0 & \frac{2}{3}\left(4z^{-1} + 65 + 4z\right) \end{pmatrix}.$$

The matrix element $D_{22}(z) = \frac{2}{3} \cdot \left(4z^{-1} + 65 + 4z\right)$ can be factorized as

$$D_{22}(z) = \frac{8}{3}\left(z - z_0\right)\left(1 - \frac{1}{z_0 z}\right) = -\frac{8}{3z_0}\left(z - z_0\right)\left(z^{-1} - z_0\right),$$

where $z_0 = (-65 + \sqrt{4161})/8$. Hence, the matrix $D(z)$ can be factorized as $D(z) = T_2(z)T_2^T(z^{-1})$, where

$$T_2(z) = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \kappa\left(z - z_0\right) \end{pmatrix},$$

and where $\kappa = \sqrt{-\frac{8}{3z_0}}$. The two matrices $T_1(z)$ and $T_2(z)$ defines a spectral factorization, $\Phi_\zeta(z) = T_p(z)T_p^T(z^{-1})$ with

$$T_p(z) = T_1^{-1}(z)T_2(z) = \begin{pmatrix} \sqrt{3} & 0 \\ \frac{2+8z^{-1}}{\sqrt{3}} & \kappa\left(z - z_0\right) \end{pmatrix}.$$

However, this is not the *canonical* spectral factorization as $T_p(\infty) \neq I$. Let

$$T_3(z) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ -\frac{2}{3\kappa z} & \frac{1}{\kappa z} \end{pmatrix}$$

and let

$$\begin{aligned} T(z) = T_p(z)T_3(z) &= \begin{pmatrix} 1 & 0 \\ \left(\frac{8+2z_0}{3}\right) z^{-1} & 1 - z_0 z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{4161}-33}{12} z^{-1} & 1 + \frac{65-\sqrt{4161}}{8} z^{-1} \end{pmatrix}, \\ Q_\zeta = T_3^{-1}(z)T_3^{-T}(z^{-1}) &= \begin{pmatrix} \sqrt{3} & 0 \\ \frac{2}{\sqrt{3}} & \kappa z \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & \kappa z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 \\ 2 & \frac{4}{3} + \kappa^2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 23 + \frac{\sqrt{4161}}{3} \end{pmatrix}. \end{aligned}$$

This gives $\Phi_\zeta(z) = T(z)Q_\zeta T^T(z^{-1})$ and both $T(z)$ and $T^{-1}(z)$ are analytical on and outside the unit circle, $T(\infty) = I$ and $Q_\zeta \succ 0$. Hence we have found the canonical spectral factorization of $\Phi_\zeta(z)$.

B

MATLAB Code

B.1 Example 5.2

MATLAB commands for Example 5.2:

```
N=10000;
e=2*rand(N,1)-1;

% ----------------------
% Minimum phase case
% ----------------------

ump=filter([1 .5],1,e);
ymp=ump.^3;
zmp=iddata(ymp,ump,1);

G0mpspa=spa(zmp,30);
G0mp=tf([.85 .575],[1 .5],1)

figure(1)
clf
bode(G0mpspa,G0mp)

% ----------------------
% Nonminimum phase case
% ----------------------

unmp=filter([.5 1],1,e);
```

```
ynmp=unmp.^3;
znmp=iddata(ynmp,unmp,1);

G0nmpspa=spa(znmp,30);
G0nmp=tf([.925 .425],[1 .5],1)

figure(2)
clf
bode(G0nmpspa,G0nmp)
```

B.2 Example 7.2

MATLAB commands for Example 7.2:

```
N=100001;

e=randn(N,1);
u=filter([1 -.8 .1],[1 -.2],e);
v=u(1:end-1).^2.*atan(u(2:end));
w=randn(N-1,1);
y=filter(1,[1 .6 .1],v)+w;

z=iddata(y,u(2:end),1);
G=oe(z,[2 2 0],'lim',0,'cov','none')
```

B.3 Example 7.3

MATLAB commands for Example 7.3:

```
N=100001;

e=randn(N,1);
u=filter([1 -.8 .1],[1 -.2],e);
n=filter(1,[1 .6 .1],u);
w=randn(N-1,1);
y=n(1:end-1).^2.*atan(n(2:end))+w;

z=iddata(y,u(2:end),1);
G=oe(z,[2 2 0],'lim',0,'cov','none')
```

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