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LINEAR OPERATORS AND VECTOR MEASURES

BY

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ABSTRACT. Compact and weakly compact operators on function spaces are studied. Those operators are characterized by properties of finitely additive set functions whose existence is guaranteed by Riesz representation theorems.

1. **Introduction.** In this paper we study operators on function spaces. Criteria for weak and strong compactness of these operators are established in terms of their representing vector measures. Some of these results have been announced by the authors in [10]. The function spaces are as follows. Let E and F be locally convex spaces, and let H be a locally compact Hausdorff space. By $C_0(H,E)$ we denote the space of continuous E-valued functions which vanish at ∞, by $C(H,E)$ we denote the space of continuous E-valued functions equipped with the compact-open topology, and by $U_E(\mathfrak{D})$ we denote the space of totally $\mathfrak{D}$-measurable functions. In §2 a Riesz representation theorem for operators L defined on the continuous function spaces into F is given in terms of a measure m on Σ, the Borel subsets of H, with values in $B(E,F^{**})$, where the latter is the space of operators from E to F^{**} and E and F are locally convex spaces. In symbols, $L(f) = \int f\,dm$, and we write $L \leftrightarrow m$. Our theorem extends and unifies existing representation theorems of this type. Although a number of authors have considered this problem in special cases, none have used the device of embedding isometrically the simple functions in $C(H,E)^{**}$ and thus reducing the problem to utilizing the representation theorem for operators $L\colon U_E(\mathfrak{D}) \to F$, which can be easily established.

In order to study weak compactness of operators, when E and F are Banach spaces, one examines the adjoint which maps F^* into $\mathrm{ca}(\Sigma,E^*)$, the Banach space of E^*-valued measures of bounded variation. Consequently, the problem is reduced to considering weakly compact subsets of $\mathrm{ca}(\Sigma,E^*)$. This enables us to use recent results of Brooks [6] giving necessary and sufficient conditions for sets to be weakly compact in the space of vector measures. This theory is presented in §3 and applied to operators in §4. Essentially the criterion for an operator $L \leftrightarrow m$ to be weakly compact is that $\tilde{m}(E_i) \to 0$ on disjoint sets E_i, where $\tilde{m}$ is the semivariation of m. Other topics in §4 include a discussion concerning the problem of when m takes its values in $B(E,F)$ and criteria for operators to be compact in terms of topologies induced on F_1^* by m. In §5 various topics are

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discussed; a modified example of Lewis [22] is given which settles questions on limits of strongly bounded vector measures.

2. **Representation theorems.** In this section we establish integral representation theorems for various function spaces. Numerous authors have worked on this problem. For example, see Bartle, Dunford, and Schwartz [2], Dinculeanu [11], Foiaş and Singer [15], Singer [31], Swong [32], and Tucker [34]. However, none of these authors has made use of the isometry in Lemma 2.1 and of Theorem 2.0. The approach presented here unifies much of the work in this area and shows the applicability of the representation theorem on the space of totally measurable functions and this isometry.

We next establish some notation and terminology. We denote the classes of all continuous seminorms on the locally convex spaces (lcs) E and F by $\{p\}_E$ and $\{p\}_F$, respectively; if X is an lcs, then X^* will be the continuous dual of X. If X is a Banach space (B-space), then we view X as a closed linear subspace of X^{**}. The term *operator* will always be used to refer to a continuous linear transformation, and the term *measure* will be used to refer to a finitely additive set function. The reader should recall that a bounded regular complex valued finitely additive set function defined on an algebra $\mathfrak{D}$ of subsets of a compact space S is countably additive, e.g. see [13, p. 138]. We denote the characteristic function of a set A by $\mathfrak{X}_A$. The symbol $\mathbf{B}$ will denote $C_0(H,E)$, $\mathbf{B}_c$ will denote those functions in $C_0(H,E)$ with compact support, and $\mathbf{C}$ will denote the space of all continuous E-valued functions on H; the analogous spaces of scalar valued functions will be denoted by $C_0(H)$, $C_c(H)$, and $C(H)$. The topology on $\mathbf{B}$ and $\mathbf{B}_c$ will be given by uniform convergence, and the topology on $\mathbf{C}$ will be the compact-open topology.

Definition 2.1. Let $\mathfrak{D}$ denote a ring of subsets of a universal space S, and let $S_E(\mathfrak{D})$ be the collection of all E-valued simple functions over the ring $\mathfrak{D}$; let $U_E(\mathfrak{D})$ be the collection of all E-valued functions f which vanish outside of some set in $\mathfrak{D}$ such that f is the uniform limit of a net $(f_\alpha) \subset S_E(\mathfrak{D})$. The space $U_E(\mathfrak{D})$ is called the space of *totally measurable functions*; $U_E(\mathfrak{D})$ is equipped with the uniform topology.

Definition 2.2. If L: $U_E(\mathfrak{D}) \to F$ is a linear mapping, $A \in \mathfrak{D}$, $p \in \{p\}_E$, and $q \in \{p\}_F$, then let

$$\|L_A\|_{(p,q)} = \sup\{q(L(f)): f \in U_E(\mathfrak{D}), \mathfrak{X}_A \cdot f = f, p(f) \leq 1\},$$

where $p(f) = \sup\{p(f(t)): t \in H\}$. Let $\mathfrak{E}$ be the set of linear mappings L: $U_E(\mathfrak{D}) \to F$ so that if $q \in \{p\}_F$ then there is a $p \in \{p\}_E$ so that $\|L_A\|_{(p,q)} < \infty$ for each $A \in \mathfrak{D}$.

Definition 2.3. If X and Y are arbitrary lcs with the family of all continuous seminorms $\{p\}_X$ and $\{p\}_Y$ respectively, m: $\mathfrak{D} \to B(X,Y)$ is finitely additive, $q \in \{p\}_Y$, and $A \in \mathfrak{D}$, then define $\tilde{m}_{(p,q)}(A)$ to be

$$\sup_{\pi(A),x_i}\left\{q\left(\sum_{A_i \in \pi(A)} m(A_i)\cdot x_i\right)\right\}$$

where $\pi(A)$ denotes the (disjoint) $\mathfrak{D}$-partitions of A, and $p(x_i) \leq 1$ for each i, $p \in \{p\}_X$. If $B \subset S$, define $\tilde{m}_{(p,q)}(B)$ to be $\sup\{\tilde{m}_{(p,q)}(A): A \subset B, A \in \mathfrak{D}\}$.

In the sequel, $\tilde{m}_{(p,q)}$ will be called the (p, q)-semivariation, where (p, q) is a pairing determined as in Definition 2.2.

We mention that the bilinear integration theory used here is developed in Dinculeanu [11]. In particular, if $m: \mathfrak{D} \to B(E,F)$ has finite (p, q)-semivariation and $f \in U_E(\mathfrak{D})$, then let (f_α) be a net in $S_E(\mathfrak{D})$ which converges uniformly to f, and define $\int f\,dm$ to be $\lim \int f_\alpha\,dm$, where the convergence is in the locally convex completion of F.

We state the following theorem for reference purposes; it will be referred to as the RRT.

Riesz Representation Theorem [29, p. 131]. *To each bounded linear functional L on $C_0(X)$, where X is a locally compact Hausdorff space, there corresponds a unique complex regular Borel measure μ such that*

$$L(f) = \int_X f\,d\mu, \qquad f \in C_0(X). \tag{II.1}$$

Moreover, if L and μ are related as in (II.1), *then*

$$\|L\| = |\mu|(X) \qquad (|\mu| \text{ is the total variation of } \mu). \tag{II.2}$$

Theorem 2.0 [11, p. 145]. *If $L \in \mathfrak{E}$, then there is a unique finitely additive, operator-valued set function $m: \mathfrak{D} \to B(E,F)$ so that*

$$L(f) = \int_H f\,dm, \qquad f \in U_E(\mathfrak{D}).$$

Furthermore, for each $q \in \{p\}_F$ there is a $p \in \{p\}_E$ so that $\tilde{m}_{(p,q)}(A) < \infty$ for each $A \in \mathfrak{D}$, and $\tilde{m}_{(p,q)}(A) = \|L_A\|_{(p,q)}$.

Before proceeding further, it will be necessary to make explicit the various topologies we shall be using. If F is an lcs, then the topology on F^* will be the strong topology and the topology on F^{**} will be the ε^{00}-topology. Therefore the topology on F^{**} will be uniform convergence on the polars of the p-unit balls of F. If A is an lcs and p is a continuous seminorm on A, then $A(p, 1)$ will be the p-unit ball of A; and if $f^{**} \in A^{**}$, then

$$a_p(f^{**}) = \sup\{|f^{**}(f^*)|: f^* \in A^0(p, 1)\},$$

where $A^0(p, 1)$ denotes the polar of $A(p, 1)$.

If $\mathfrak{X}$ is a B-space and $r > 0$, then $\mathfrak{X}_r$ will denote the closed ball of radius r about the origin.

Definition 2.4. Let $\mathfrak{L}$ be the subring of Σ consisting of those sets whose characteristic functions are pointwise limits of sequences in $C_c(H)$. If $A \in \mathfrak{L}$ and (φ_n^A) is a sequence converging pointwise to $\mathfrak{X}_A$ so that $0 \leq \varphi_n^A \leq 1$ for each n, then (φ_n^A) is called a *determining sequence* for A. We write $\varphi \prec A$ to denote that $\mathrm{supp}(\varphi) \subset A$.

Definition 2.5 (see [16] and [13, VI.7.2]). *If $A \in \Sigma$ and $x \in E$, let $\mathfrak{X}_A \cdot x$ be defined on* $\mathbf{B}^*$ *by*

$$\text{(II.3)} \qquad \mathfrak{X}_A \cdot x(f^*) = \mu_{(x,f^*)}(A), \qquad f^* \in \mathbf{B}^*,$$

where $\mu_{(x,f^*)}$ is the unique regular Borel measure in the RRT given by

$$\text{(II.4)} \qquad \langle x,f^*\rangle(f) = f^*(f \cdot x) = \int_H f\, d\mu_{(x,f^*)} = \mu_{(x,f^*)}(f).$$

It is clear that $S_E(\Sigma)$ is a linear manifold in the lcs of bounded E-valued functions defined on H; from Definition 2.5, it follows that $S_E(\Sigma)$ may be viewed as a subspace of $\mathbf{B}^{**}$. However, if E and F are B-spaces, much more can be said, as the following lemma indicates. We give the proof in some detail since we shall have occasion to refer to the construction later.

Lemma 2.1. *If E and F are B-spaces and*

$$\zeta = \sum_{i=1}^{k} \mathfrak{X}_{A_i} \cdot x_i \in S_E(\Sigma),$$

then $\|\zeta\|_{B^{**}} = \|\zeta\|_\infty$, where $\|\cdot\|_\infty$ denotes the uniform norm.

Proof. Without loss of generality, we may suppose that $\zeta = \sum_{i=1}^k \mathfrak{X}_{A_i} \cdot x_i$ is in the canonical form of a simple function and that $\|x_1\| = \|\zeta\|_\infty$. Let $t \in A_1$ and define $\xi\colon \Sigma \to E^*$ in the following way:

(i) $\xi(A) = 0$ *if $t \notin A$*;

(ii) *if $t \in A$, then $\xi(A) = x^* \in E^*$, where $\|x^*\| = 1$ and $x^*(x_1) = \|x_1\|$.*

Then clearly ξ is countably additive, has finite semivariation, and $|\xi|$ (the total variation function) is regular. Furthermore, $\xi \in \mathbf{B}^*$, and $\|\xi\|_{\mathbf{B}^*} = \tilde{\xi}(H) = 1$. Now

$$\zeta(\xi) = \left(\sum_{i=1}^{k} \mathfrak{X}_{A_i} \cdot x_i\right)(\xi) = \sum_{i=1}^{k} \mathfrak{X}_{A_i} \cdot x_i(\xi)$$

$$= \sum_{i=1}^{k} \mu_{(x_i,\xi)}(A_i).$$

Let $i > 1$, $\varepsilon > 0$, and choose a collection $\{K_i\}_{i=2}^k$ of compact sets so that $K_i \subset A_i$ and $\sum_{i=2}^k |\mu|_{(x_i,\xi)}(A_i - K_i) < \varepsilon$. Then let $\{V_i\}$ be a disjoint collection of conditionally compact open sets so that $t \notin \bigcup V_i$, and $\sum_{i=2}^k |\mu|_{(x_i,\xi)}(V_i - K_i) < \varepsilon$. By the Baire approximation theorem [19, p. 218], and the Lebesgue dominated convergence theorem, we can choose a collection of compact G_δ subsets $\{G_i\}$ and $\{\varphi_i\} \subset C(H)$ so that $K_i \subset G_i \subset V_i$, $0 \le \varphi_i \le 1$, $\mathrm{supp}(\varphi_i) \subset V_i$, and $\sum_{i=2}^k |\mu_{(x_i,\xi)}(G_i) - \mu_{(x_i,\xi)}(\varphi_i)| < \varepsilon$. But $\mu_{(x_i,\xi)}(\varphi) = \int \varphi_i x_i \, d\xi = 0$. Since

$$(\mathfrak{X}_{A_1} \cdot x_1)(\xi) = \xi(A_1) \cdot x_1 = \|x_1\|,$$

it follows that $\|\xi\|_\infty \leq \|\xi\|_{\mathbf{B}^{**}}$.

For the reverse inequality, let ζ be as above, and choose ξ arbitrarily in $\mathbf{B}_1^*$. Let $\{K_i\}$, $\{G_i\}$, $\{V_i\}$, and $\{\varphi_i\}$ be chosen as in the preceding paragraph, except that we now insist that i range from 1 to k. Therefore

$$\left|\sum_{i=1}^{k} (\mu_{(x_i,\xi)}(A_i) - \mu_{(x_i,\xi)}(\varphi_i))\right| \leq \left|\sum_{i=1}^{k} (\mu_{(x_i,\xi)}(A_i) - \mu_{(x_i,\xi)}(G_i))\right| + \left|\sum_{i=1}^{k} (\mu_{(x_i,\xi)}(G_i) - \mu_{(x_i,\xi)}(\varphi_i))\right| < 3\varepsilon,$$

and

$$|\Sigma\, \mu_{(x_i,\xi)}(\varphi_i)| = \left|\xi\left(\sum_{i=1}^{k} \varphi_i \cdot x_i\right)\right| \leq \left\|\sum_{i=1}^{k} \varphi_i \cdot x_i\right\| \leq \|x_1\|.$$

This proves the lemma.

Remark 2.1. If E is an lcs, $q \in \{p\}_E$, and $\xi = \sum_{i=1}^{k} \mathfrak{X}_{D_i} \cdot x_i \in S_E(\Sigma)$, then from the technique of Lemma 2.1 it follows that

$$a_q\left(\sum_{i=1}^{k} \mathfrak{X}_{D_i} \cdot x_i\right) \leq q\left(\sum_{i=1}^{k} \mathfrak{X}_{D_i} \cdot x_i\right).$$

Therefore it follows that the second adjoint $L^{**}\colon S_E(\Sigma) \to F^{**}$ is continuous relative to the uniform topology on $S_E(\Sigma)$.

The following theorem is used in establishing the representation theorem.

Theorem 2.1. *Let $f \in C_0(H, E)$. Then there is a net $(g_\alpha) \subset S_E(\mathfrak{C})$ simultaneously approximating f in the uniform and $\mathbf{B}^{**}$ topologies.*

Proof. For each $p \in \{p\}_E$, let $A_{(p,n)} = \{x\colon p(f(x)) \geq 1/n\}$. Then $A_{(p,n)} \in \mathfrak{C}$ for each p and each n since $A_{(p,n)}$ is a compact G_δ. Let $\mathbf{N}$ denote the natural numbers, and let $\Gamma = \{p\}_E \times \mathbf{N}$. For (p, n), $(q, m) \in \Gamma$, define $(p, n) \geq (q, m)$ if and only $p \geq q$ in the ordering on $\{p\}_E$ and $n \geq m$. Then Γ is a directed set, and for each $\gamma \in \Gamma$, $\gamma = (p, n)$, let $f_\gamma = \mathfrak{X}_{A_{(p,n)}} \cdot f$ be defined as in Definition 2.5. Thus $f_\gamma \in \mathbf{B}^{**}$. It is clear that $f_\gamma \to f$ uniformly. But also $f_\gamma \to f$ in the $\mathbf{B}^{**}$ topology. In fact, if $x^* \in \mathbf{B}^0(p, 1)$ and $\varepsilon > 0$, then choose $k \in \mathbf{N}$ and let $\gamma \in \Gamma$, $\gamma = (q, t)$ so that $1/k \leq \varepsilon/2$ and $q \geq p$, $t \geq k$. Using the fact that there is a determining sequence for $A(q, t)$ which is identically one on $A(q, t)$, we see that $|f_\gamma(x^*) - x^*(f)| \leq 2/t < \varepsilon$.

Now let $\lambda = (p, n)$ be a fixed element of Γ, and note that $f(A(\lambda))$ is a compact subset of E. Therefore for each $m \in \mathbf{N}$ there is a partition $\{A_i^m\}_{i=1}^{k(m)}$ of $A(\lambda)$ into elements of $\mathfrak{C}$ so that p-diam$(f(A_i^m)) < 1/m$ for each i. Then, for each $m > n$, choose one such partition $\{A_i^m\}_{i=1}^{k(m)}$, let $x_i^m \in f(A_i^m)$, and let

$$\zeta(\lambda, m) = \sum_{i=1}^{k(m)} \mathfrak{X}_{A_i^m} \cdot x_i^m.$$

To simplify notation, we shall usually denote $\zeta(\lambda, m)$ by $\sum_{i=1}^{k(m)} \mathfrak{X}_{A_i} \cdot x_i$. Order the set $I = \{(\lambda, m)\colon \lambda = (p, n) \in \Gamma, m \geq n\}$ by $(\lambda, m) \geq (\gamma, j)$ if $\lambda \geq \gamma$ and $m \geq j$. It is easy to see that $\{\zeta(\lambda, m)\}_I \to f$ uniformly.

Finally, we prove that $\{\xi(\lambda, n)\}_I$ is a net which satisfies the conclusion of the theorem. To this end, let $q \in \{p\}_E$, $\varepsilon > 0$, and choose $p \geq q$, $1/n < \varepsilon$, and $\lambda = (d, t) \geq (p, n)$. If $\xi(\lambda, m) = \sum_{i=1}^{k(m)} \mathfrak{X}_{D_i} \cdot x_i$ is chosen as above and $f^* \in \mathbf{B}^0(q, 1)$, then

$$|(\xi(\lambda, m) - f_\lambda)f^*| = \left|\sum_{i=1}^{k(m)} (\mu_{(x_i, f^*)}(D_i) - \mu_{(f, f^*)}(D_i))\right|,$$

where $\mu_{(x_i, f^*)}$ and $\mu_{(f, f^*)}$ are the regular Borel measures of Definition 2.5. Repeat the construction of Lemma 2.1 to select disjoint open sets $\{V_i\}_{i=1}^{k(m)}$ and compact G_δ sets $\{G_i\}_{i=1}^{k(m)}$ so that $G_i \subset V_i$ for each i, d-diam$(F(V_i)) < 2/t$,

$$\sum_{i=1}^{k(m)} |\mu|_{(x_i, f^*)}(G_i \Delta D_i) < \varepsilon,$$

and

$$\sum_{i=1}^{k(m)} |\mu|_{(f, f^*)}(G_i \Delta D_i) < \varepsilon.$$

Therefore

$$|\xi(\lambda, m)(f^*) - f_\lambda(f^*)| < \left|\sum_{i=1}^{k(m)} (\mu_{(x_i, f^*)}(G_i) - \mu_{(f, f^*)}(G_i))\right| + 2\varepsilon.$$

We may apply the Lebesgue dominated convergence theorem since each $G_i \in \mathfrak{L}$. Thus since $d \geq q$ and $f^* \in \mathbf{B}^0(q, 1)$,

$$\left|\sum_{i=1}^{k(m)} (\mu_{(x_i, f^*)}(G_i) - \mu_{(f, f^*)}(G_i))\right| \leq \frac{4}{t};$$

whence $|\zeta(\lambda, m)(f^*) - f_\lambda(f^*)| \leq 6\varepsilon$, and the proof is completed.

Definition 2.6. If $m\colon \Sigma \to B(E, F^{**})$ is finitely additive with finite (p, q)-semivariation, then we say that m is *weakly regular* provided that $x \in E$ and $z \in F^*$ imply that $m_{(x,z)}(\cdot) \equiv \langle m(\cdot)x, z\rangle$ is a finite regular Borel measure.

We should note here that this notion of weak regularity differs from that in [13].

Theorem 2.2 (Representation Theorem). *If $L\colon \mathbf{B} \to F$ is an operator, then there is a unique weakly regular set function $m\colon \Sigma \to B(E, F^{**})$ so that*

$$L(f) = \int_H f\, dm, \qquad f \in \mathbf{B}.$$

Proof. We know that $S_E(\Sigma) \subset \mathbf{B}^{**}$, and therefore $L^{**}: S_E(\Sigma) \to F^{**}$. From Corollary 2.1.1, it follows that L^{**} has a continuous extension $\hat{L}: U_E(\Sigma) \to C(F^{**})$, where $C(F^{**})$ denotes the locally convex completion of F^{**}. But by Theorem 2.1 and Lemma 2.1 it follows that $\hat{L}(f) = L^{**}(f) = L(f)$. Now we apply Theorem 2.0 and write $L(f) = \int_H f\,dm$. Furthermore, by Theorem 2.0,

$$\hat{L}(\mathfrak{X}_A \cdot x) = L^{**}(\mathfrak{X}_A \cdot x) = \int_H \mathfrak{X}_A \cdot x\,dm = m(A) \cdot x,$$

and $\|\hat{L}_A\|_{(p,q)} = \tilde{m}_{(p,q)}(A) < \infty$ for appropriate pairings (p, q), $p \in \{p\}_E$, $q \in \{p\}_F$. Also, if $z \in F^*$, then

$$\begin{aligned}\langle m(A) \cdot x, z\rangle &= \langle L^{**}(\mathfrak{X}_A \cdot x), z\rangle \\ &= \mathfrak{X}_A \cdot x(L^*(z)) = \mu_{(x, L^*(z))}(A),\end{aligned}$$

where $\mu_{(x,L^*(z))}$ is a finite regular Borel measure. The uniqueness statement follows from the corresponding uniqueness statement in the RRT.

We write $L \leftrightarrow m$ to denote the correspondence established in this theorem.

Next we show that the representation portion of Theorem 2.2 of [14] follows as an immediate corollary to Dinculeanu's representation theorem. To this end, let H be an arbitrary point set, let E and F be normed linear spaces (NLS), and let B be a linear space of bounded E-valued functions.

Theorem 2.3 [14, Theorem 2.2]. *Suppose that B is equipped with a norm $\|\cdot\|'$ which is weaker than the uniform norm, and suppose there is an algebra $\mathfrak{D}$ of subsets of H so that $\|\cdot\|'$ can be extended to $\mathfrak{S} = S_E(\mathfrak{D})$ in such a way that there is a subspace $\mathfrak{S}'$ of $\mathfrak{S}$ satisfying*

(i) *$B \subset U(\mathfrak{S}')$, the uniform closure of $\mathfrak{S}'$, and*

(ii) *there is a linear mapping $\mathfrak{P}: \mathfrak{S} \to B^{**}$ which is continuous relative to $\|\cdot\|'$; and, when we extend $\mathfrak{P}$ to $U_E(\mathfrak{D})$, $\mathfrak{P}$ maps $\eta(f)$ into f for each $f \in B$, where η is the canonical embedding.*

Then, if $T: (B\|\cdot\|') \to F$ is a continuous linear operator, there is a finitely additive set function $m: \mathfrak{D} \to B(E, F^{**})$ with finite semivariation $\tilde{m}$ so that $T(f) = \int_H f\,dm$, $f \in B$. Furthermore $\tilde{m}(A) = \|(T^{**} \circ \mathfrak{P})_A\|$, $A \in \mathfrak{D}$.

(We know that we can extend $\mathfrak{P}$ to $U_E(\mathfrak{D})$ since $\mathfrak{P}$ is continuous on $\mathfrak{S}$ with respect to the uniform norm; we continue to denote the extension by $\mathfrak{P}$.)

Proof. Since $\mathfrak{P}: U_E(\mathfrak{D}) \to B^{**}$ is continuous relative to the uniform norm, then $T^{**} \circ \mathfrak{P}: U_E(\mathfrak{D}) \to F^{**}$ is an operator. By Theorem 2.0 there is a unique finitely additive $m: \mathfrak{D} \to B(E, F^{**})$ with finite semivariation so that

$$T^{**} \circ \mathfrak{P}(g) = \int_H g\,dm, \qquad g \in U_E(\mathfrak{D}),$$

and

$$\tilde{m}(A) = \|(T^{**} \circ \mathfrak{P})_A\|, \qquad A \in \mathfrak{D}.$$

But then if $f \in B$,

$$T^{**} \circ \mathcal{G}(f) = T^{**}(f) = \int f\,dm,$$

and we have the desired conclusion.

Definition 2.7 [20, §1.5]. Let L: $\mathbf{B}_c \to F$ be an operator. The point $t \in H$ will be in the support of L, denoted by supp(L), if and only if for each neighborhood $N(t)$ of t there is an $f \in \mathbf{B}_c$ so that $\operatorname{supp}(f) \subset N(t)$ and $L(f) \neq 0$. If L has $\mathbf{B}$ for its domain, the notion of support is defined similarly.

Theorem 2.4. *Let L: $\mathbf{B}_c \to F$ be an operator so that* supp(L) *is compact. Then L has a unique extension $\hat{L} \in B(C,F)$ so that $\hat{L}(f) = 0$ for each $f \in C$ which vanishes in a neighborhood of* supp(L).

The proof of this theorem very closely parallels the proof of the analogous statement in the scalar case and will be omitted.

Theorem 2.5. *Let F be a B-space. A linear mapping $\hat{L}$: $C \to F$ is continuous if and only if $\hat{L}$ is the unique extension of Theorem* 2.4 *of an operator $L \in B(\mathbf{B}_c, F)$ so that L has compact support.*

Proof. Suppose that $\hat{L} \in B(C,F)$. Then, by the continuity of $\hat{L}$, there is a compact set $K \subset H$, a positive number M, and a continuous seminorm p on E so that

$$\text{(i)} \qquad \|\hat{L}(x)\| \leq M p_K(x), \qquad x \in C,$$

where $p_K(x) = \sup\{p(x(t)) \colon t \in K\}$. Now if $L = \hat{L}\,|_{\mathbf{B}_c}$, then certainly $\operatorname{supp}(L) \subset K$ and $\hat{L}$ is an extension of an operator with compact support. If $x \in C$ vanishes in a neighborhood of supp(L), let $\varphi \in C_c(H)$ so that $0 \leq \varphi \leq 1$ and $\varphi(K) = 1$. Then $x = \varphi \cdot x + (1 - \varphi) \cdot x$, and

$$\begin{aligned} \hat{L}(x) &= \hat{L}(\varphi \cdot x) + \hat{L}((1 - \varphi) \cdot x) \\ &= L(\varphi \cdot x) + \hat{L}((1 - \varphi) \cdot x). \end{aligned}$$

But $\hat{L}((1 - \varphi) \cdot x) = 0$ by (i), and $\varphi \cdot x \in \mathbf{B}_c$ so that $\varphi \cdot x$ vanishes on a neighborhood containing supp(L). By appealing to a partition of unity argument, it follows that $L(\varphi \cdot x) = 0$.

The converse of the theorem follows immediately from Theorem 2.4, and the proof is completed.

We remark that this characterization fails in case we allow F to be an lcs.

Example 2.1. Let H be a locally compact, noncompact, Hausdorff space, let E = normed space, let $\mathbf{C} = C(H,E)$ with the compact-open topology, and let $F = \mathbf{C}$. If I denotes the identity mapping from $\mathbf{C}$ to $\mathbf{C}$, then $I\,|_{\mathbf{B}_c}$ is the identity mapping and obviously does not have compact support.

The following lemma enables us to see the measure theoretic consequences of Theorems 2.4 and 2.5 more clearly.

Lemma 2.2 [23]. *Suppose that H, Σ, E, and F are as in the introduction. If $m\colon \Sigma \to B(E,F)$ is a weakly regular Borel measure, then there is a smallest closed set $D \subset H$ so that if U is an open subset of H, then $\tilde{m}_{(p,q)}(U) = 0$ for each $p \in \{p\}_E$, $q \in \{p\}_F$, if and only if $U \cap D = \varnothing$. Hence m vanishes on $\Sigma \cap (H - D)$.*

This set D, called the support of m, is denoted by $\operatorname{supp}(m)$.

The proof of the following theorem involves primarily the technical details carried out in Lemma 2.1 and Theorem 2.1. Consequently, we omit the proof.

Theorem 2.6. *Let $L\colon \mathbf{B} \to F$ be an operator, with $L \leftrightarrow m$. Then* $\operatorname{supp}(L) = \operatorname{supp}(m)$.

We can now restate Theorem 2.5 as follows.

Corollary 2.6.1. *If F is a B-space, then a linear mapping $L\colon \mathbf{C} \to F$ is an operator if and only if there is a unique weakly regular measure $m\colon \Sigma \to B(E, F^{**})$ so that* $\operatorname{supp}(m)$ *is compact and $L(f) = \int_H f\,dm$, $f \in \mathbf{C}$.*

The following example is interesting in view of Theorem 2.2 of this paper.

Example 2.2. Let H be the natural numbers equipped with the discrete topology, and let $B(H)$ be all the bounded, complex-valued functions on H topologized by the uniform norm. Then Σ is the power class of H. If $A \in \Sigma$, define $m(A)$ to be $\chi_A \in B(H) \cong B(\mathbf{C}, B(H))$. Then $m\colon \Sigma \to B(\mathbf{C}, B(H))$ is finitely additive and has finite semivariation. If $f \in C(H)$, define $L(f)$ to be $\int_H f\,dm$.

Now let βH denote the Stone-Čech compactification of H, and choose $t \in \beta H - H$. For $f \in B(H)$, define $t(f)$ to be $\hat{f}(t)$, where $\hat{f}$ denotes the unique extension of f to all of βH. Certainly $t \in B(H)^*$. We note next that $\hat{\chi}_{\{a_i\}} = \chi_{\{a_i\}}$ for a singleton $a_i \in H$. Therefore

$$\sum_{i=1}^{\infty} m_t\{a_i\} = \sum_{i=1}^{\infty} t(\chi_{\{a_i\}}) = 0.$$

But $m_t(H) = t(\chi_H) = 1$, and m_t is not countably additive. Hence the representing measure of Theorem 2.2 differs considerably from the measure m used to define the operator L.

In the remainder of the paper, E and F will denote B-spaces.

We now turn to more general domain spaces H. As an application of Urysohn's extension theorem in [18] we obtain the following.

Lemma 2.3. *If H is a Hausdorff space, then $C(H,E)\,|_K = C(K,E)$ for each compact $K \subset H$ if and only if the continuous functions on H separate the points of H.*

Such a Hausdorff space H will be called an S-space.

Theorem 2.7. *If H is an S-space and $L\colon \mathbf{C} = C(H,E) \to F$ is a linear transformation continuous with respect to the compact-open topology, then there is a* *unique weakly regular, compactly supported set function* $m: \Sigma \to B(E, F^{**})$ *so that* $L(f) = \int_H f\,dm, f \in \mathbf{C}$.

Proof. Since L is continuous, there is a constant $M > 0$ and a compact set $K \subset H$ so that $\|L(f)\| \leq M\|f\|_K$, where $\|f\|_K = \sup\{\|f(t)\|: t \in K\}$. Therefore, by Theorem 2.5, L induces an operator $T: C(K, E) \to F$. By Theorem 2.2, we may write $T(f) = \int_K f\,dm'$, where $m': \Sigma(K) \to B(E, F^{**})$ is weakly regular. Since $\Sigma(H) \cap K = \Sigma(K)$, we may define $m: \Sigma(H) \to B(E, F^{**})$ by $m(A) = m'(A \cap K)$. Then clearly m is weakly regular and $L(f) = \int_H f\,dm, f \in C$. The uniqueness follows from the uniqueness statement in Theorem 2.2.

In the future, we shall identify m and m'.

As an application of the following theorem, we can say more than simply that m is weakly regular. In fact, it follows that if $m_z: \Sigma \to E^*$ is defined by $m_z(A) \cdot x = \langle m(A) \cdot x, z \rangle$, $x \in E$, then $|m_z| \in \text{rca}(\Sigma, \mathbf{C})$.

Definition 2.8. If H is a locally compact Hausdorff space, then an operator $L: \mathbf{B} \to F$ is said to be dominated provided there is a positive linear functional $P \in C_0(H)^*$ so that $\|L(f)\| \leq P(\|f\|), f \in \mathbf{B}$.

The following theorem is due to N. Dinculeanu.

Theorem 2.8. *If* $L: \mathbf{B} \to F$ *is an operator with* $L \leftrightarrow m$, *then* L *is dominated if and only if* $|m| \in \text{rca}(\Sigma, \mathbf{C})$.

Proof. If m has finite total variation, which we denote by $|m|$, then certainly $\|\int f\,dm\| \leq \int \|f\|\,d|m|$. But $\int_H (\cdot)\,d|m|$ is a positive operator, and therefore L is dominated.

Conversely, suppose that L is dominated by the positive operator $P = \int (\cdot)\,d\mu$ (μ is the positive regular Borel measure given by the RRT). Then, using the weak regularity of m and repeating the construction in Lemma 2.1, it follows that $|m|(A) \leq \mu(A)$, $A \in \Sigma$. But, from this inequality, it follows that $|m| \in \text{rca}(\Sigma, \mathbf{C})$, and m is countably additive.

We remark that if the set function given by Theorem 2.2 has finite total variation, then it must be countably additive.

Thus, if $L \leftrightarrow m$ as in Theorem 2.2 or Theorem 2.7, then $(z \circ L)(f) = z(L(f)) = \int_H f\,dm_z$, $\|z \circ L\| = \tilde{m}_z(H) = |m_z|(H)$, and it follows that $|m_z| \in \text{rca}(\Sigma, \mathbf{C})$ for each $z \in F^*$. We should also point out that $L^*(z) = m_z$. Therefore we have the following corollary.

Corollary 2.8.1. *If* $m: \Sigma \to B(E, F^{**})$ *is finitely additive with finite semivariation and* $m_{(x,z)}$ *is a regular Borel measure for each* $(x, z) \in E \times F^*$, *then* $|m_z|$ *is a nonnegative regular Borel measure.*

This prompts us to make the following definition.

Definition 2.9. By a representing measure $m: \Sigma \to B(E, F^{**})$ we shall mean a finitely additive set function with finite semivariation so that $|m_z|$ is a regular Borel measure for each $z \in F^*$.

Remark. It is clear that an S-space H is the most general domain space which will yield unique compactly supported representing measures. In the $C_0(H,E)$ setting, the support of a representing measure need only be closed and not necessarily compact.

3. **Weak compactness of vector measures.** In this section we establish criteria for weak compactness in the B-space of vector measures. This will enable us to determine the compactness or weak compactness of operators on function spaces by examining the weak compactness of the set of vector measures formed by the image of the unit sphere under the adjoint operator. The various results concerning weak compactness of vector measures are due to Brooks and are essentially taken from [6], where the proofs are briefly outlined (for the reflexive case). We are grateful to C. Swartz for providing us with Lemma 3.3 which enabled us to eliminate a weak sequential completeness condition that was imposed earlier.

The following definition is fundamental in characterizing representing measures of weakly compact operators.

Definition 3.1. Let $\mathfrak{D}$ be a ring of subsets of a set S. A finitely additive set function $m\colon \mathfrak{D} \to B(E,F^{**})$ is *strongly bounded* (*s-bounded*) if $\tilde{m}(A_i) \to 0$, whenever (A_i) is a disjoint sequence of sets from $\mathfrak{D}$. An operator L is said to be s-bounded provided that its representing measure is s-bounded.

A similar concept was introduced by Lewis [22] under the name variational semiregularity (vsr), where the requirement took the form $\tilde{m}(A_i) \to 0$, where A_i are Borel sets satisfying $A_i \searrow \varnothing$. We remark that countable additivity, even on a σ-algebra, need not imply s-boundedness (Lewis [22]), as Example 5.1 *infra* will show. Rickart [28] also introduced a continuity condition in a different setting.

Let X be a S-space. By $\mathrm{fa}(\mathfrak{D},X)$ we denote the Banach space of all finitely additive set functions $\mu\colon \mathfrak{D} \to X$ with bounded total variation; the norm of μ is given by $|\mu|(S)$. The subspace consisting of countably additive measures is denoted by $\mathrm{ca}(\mathfrak{D},X)$.

A set is said to be conditionally compact with respect to a topology τ if its τ-closure is τ-compact. In the sequel, we shall use the following equality [11]:

$$\text{(III.1)} \qquad \tilde{m}(A) = \sup\{|m_z|(A)\colon z \in F_1^*\}, \qquad A \in \mathfrak{D}.$$

Lemma 3.1. *Suppose that D is an S-space and $m\colon \Sigma(D) \to B(E,F)$ is a representing measure. Then the following are equivalent*:

(i) *m is s-bounded*;

(ii) *if $A_n \searrow \varnothing$, then $\tilde{m}(A_n) \to 0$*;

(iii) *$\{|m_z|\colon z \in F_1^*\}$ is conditionally weakly compact in* $\mathrm{ca}(\Sigma, F^*)$;

(iv) *$\sum m(A_i)x_i$ converges in F for each disjoint sequence (A_i) and $(x_i) \subset E_1$.*

Proof. (ii) $\Rightarrow$ (i). Let (A_i) be a disjoint sequence. Then $B_k \searrow \varnothing$, where $B_k = \bigcup_{n \geq k} A_n$. Hence $\tilde{m}(A_k) \leq \tilde{m}(B_k) \to 0$ and m is s-bounded.

(ii) $\Rightarrow$ (iii). In view of equality (III.1), m satisfies (ii) if and only if $\{|m_z|: z \in F_1^*\}$ is uniformly countably additive. The result follows by Theorem IV.9.1 in [13].

(i) $\Rightarrow$ (iv). Let (A_i) be a disjoint sequence, $(x_i) \subset E_1$ and suppose that $\sum m(A_i)x_i$ does not converge. Then there is a subsequence (i_k) and an $\varepsilon > 0$ such that

$$\left\| \sum_{n=i_k+1}^{i_{k+1}} m(A_n)x_n \right\| > \varepsilon, \qquad k = 1, 2, \ldots.$$

Let $D_k = \bigcup_{n=i_k+1}^{i_{k+1}} A_n$. Then (D_k) is a disjoint sequence and $\tilde{m}(D_k) > \varepsilon$ for each k. Hence m is not s-bounded.

(iv) $\Rightarrow$ (i). If m is not s-bounded, there exists an $\varepsilon > 0$ and a disjoint sequence (A_i) such that $\tilde{m}(A_i) > \varepsilon$ for each i. From the definition of $\tilde{m}$, there exist partitions $(B_k^i)_{k=1}^{n(i)}$ of A_i and $(x_k^i)_{k=1}^{n(i)} \subset E$ so that $\|\sum_k m(B_k^i)x_k^i\| > \varepsilon$ for each i. Consequently (iv) does not hold.

(i) $\Rightarrow$ (ii). If (ii) does not hold, there exists an $\varepsilon > 0$ and $A_i \searrow \varnothing$ such that $\tilde{m}(A_i) > \varepsilon$. There is a $z_1 \in F_1^*$ such that $|m_{z_1}|(A_1) > \varepsilon$. Choose $N_2 > N_1 = 1$ so that $|m_{z_1}|(A_{N_1} - A_{N_2}) > \varepsilon$. Proceeding inductively, we obtain an increasing sequence (N_i) so that $\tilde{m}(A_{N_i} - A_{N_{i+1}}) > \varepsilon$, $i = 1, 2, \ldots$. Thus m is not s-bounded. This completes the proof of the lemma.

Definition 3.2. A B-space X has the *Radon-Nikodym property* (*property* R-N) if every countably additive X-valued measure m of bounded variation defined on a σ-algebra, which is absolutely continuous with respect to a scalar measure ν, can be expressed as the indefinite integral of a Bochner integrable function f; in symbols, $dm/d\nu = f \in L^1(\nu, X)$.

Remark 3.1. The following spaces are known to have property R-N: reflexive spaces [27] and separable dual spaces [12].

The next result is a key lemma in obtaining criteria for weak compactness in the countably additive case. We denote weak convergence by $\to^w$ and uniform convergence by $\to^u$.

Lemma 3.2. *Let $\mathcal{Q}$ be a σ-algebra and let X be a B-space such that X and X^* have property R-N. Suppose that (μ_n) is a sequence in* $\mathrm{ca}(\mathcal{Q}, X)$ *such that* $\sup_n |\mu_n|(S) = M < \infty$. *Assume that $\{|\mu_n|: n = 1, 2, \ldots\}$ is uniformly countably additive and $\mu_n(A) \to^w \mu(A)$ in X for each $A \in \mathcal{Q}$. Then $\mu: \mathcal{Q} \to X$ belongs to* $\mathrm{ca}(\mathcal{Q}, X)$ *and $\mu_n \to^w \mu$ in* $\mathrm{ca}(\mathcal{Q}, X)$.

Proof. Let $x^* \in X^*$. Since $x^*\mu_n(A) \to x^*\mu(A)$ for each A, $x^*\mu$ is countably additive by the Nikodym theorem. It then follows from the Orlicz-Pettis theorem that μ is countably additive. Let $(A_i)_{i=1}^m$ be a partition of S. For suitable $x_i^* \in X_1^*$,

$$\sum_{i=1}^m \|\mu(A_i)\| = \sum_{i=1}^m |x_i^*\mu(A_i)| = \lim_k \sum_{i=1}^m |x_i^*\mu_k(A_i)|$$
$$\leq \varliminf_k \sum_{i=1}^k \|\mu_k(A_i)\| \leq M.$$

Since $|\mu|(S) \leq M$, $\mu \in \text{ca}(\mathcal{Q},X)$. Let $\mu_0 = \mu$ and define $\nu(A) = \sum_{i=0}^{\infty} |\mu_i|(A)/2^i$, $A \in \mathcal{Q}$. Note that ν is finite, countably additive and $\mu \ll \nu$ for each i. Since X has property R-N, for each i there exists an $f_i \in L^1(\nu,X)$ such that $d\mu_i/d\nu = f_i$. To show that $\mu_i \to^w \mu_0$, it suffices to prove that $f_i \to^w f_0$ in $L^1(\nu,X)$. Since X^* has property R-N, one can show that $L^1(\nu,X)^* = L^\infty(\nu,X^*)$. Consequently, we have to show that $\int_s \varphi f_i \to \int_s \varphi f_0$ for each $\varphi \in L^\infty(\nu,X^*)$. To establish this, choose a sequence (ψ_n) of $\mathcal{Q}$-simple X^*-valued functions satisfying $\psi_n \to \varphi$ a.e. ν and $\|\psi_n(t)\| < 2\|\varphi(t)\|$ for each n. Since $|\mu_i| \ll \nu$ uniformly in i (see [8, Theorem 1]), given an $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\mu_i|(A) < \varepsilon$, $i = 1, 2, \ldots,$ if $\nu(A) < \delta$. By Egoroff's theorem, there is an $A \in \mathcal{Q}$ such that $\nu(A) < \delta$, and $\Psi_i \to^u \varphi$ on $S - A$. By considering the inequality

$$\left|\int \varphi f_i - \int \varphi f_0\right| \leq \left|\int \Psi_n f_i - \int \varphi f_i\right| + \left|\int \Psi_n f_i - \int \Psi_n f_0\right| + \left|\int \Psi_n f_0 - \int \varphi f_0\right|,$$

one can show that $|\int \varphi f_i - \int \varphi f_0| < K\varepsilon$ for $i \geq I_\varepsilon$ and a suitable constant K. The lemma then follows.

Proposition 3.1. (i) *Assume that X and X^* have property R-N. Let* $\mathcal{K} \subset \text{ca}(\mathcal{Q},X)$ *satisfy the following conditions*:

(a) $\mathcal{K}$ *is bounded*;

(b) $\{|\mu|: \mu \in \mathcal{K}\}$ *is uniformly countably additive*;

(c) $\{\mu(A): \mu \in \mathcal{K}\}$ *is conditionally weakly compact in X for each* $A \in \mathcal{Q}$.

Then $\mathcal{K}$ *is conditionally weakly compact in* $\text{ca}(\mathcal{Q},X)$.

(ii) *Conversely, if X is any B-space and* $\mathcal{K} \subset \text{ca}(\mathcal{Q},X)$ *is conditionally weakly compact, then* (a), (b), *and* (c) *hold.*

Proof. To prove (i), it suffices in view of the Eberlein-Šmulian theorem to prove that if (μ_n) is a sequence of elements from $\mathcal{K}$, there is a weakly convergent subsequence. Construct a countable subalgebra $\mathcal{Q}_0 = \{A_1, A_2, \ldots\}$ of $\mathcal{Q}$ such that $|\mu_n|(S) = |\mu_n|_{\mathcal{Q}_0}|(S)$, $n = 1, 2, \ldots$. Let $\mathcal{Q}'$ be the σ-algebra generated by $\mathcal{Q}_0$. Thus the mapping π: $\overline{\text{span}(\mu_n)} \to \text{ca}(\mathcal{Q}',X)$ is an isometry, where $\pi(\mu) = \mu|_{\mathcal{Q}'}$. Using (c) we can choose a subsequence (β_k) of (μ_k) such that $\beta_k(A_n) \to^w_k$, $n = 1, 2, \ldots$. By Theorem 4 in [5], $x^*\beta_k(A) \to$ for every $A \in \mathcal{Q}'$ and $x^* \in X^*$. Using (c) we conclude that $\beta_k(A) \to^w$ for every $A \in \mathcal{Q}'$. If we define $\beta_0(A)$ to be this weak limit, then $\pi\beta_k \to^w \beta_0$ in $\text{ca}(\mathcal{Q}',X)$ by Proposition 3.1. Hence $\beta_k \to^w \pi^{-1}(\beta_0)$ in $\text{ca}(\mathcal{Q},X)$.

To prove the converse, assume that $\mathcal{K}$ is conditionally weakly compact and $\{|\mu|: \mu \in \mathcal{K}\}$ is not uniformly countably additive. There exists an $\varepsilon > 0$, $(\mu_i) \subset \mathcal{K}$ and a disjoint sequence (A_i) such that $|\mu_i|(A_i) > \varepsilon$, $i = 1, 2, \ldots$. Obtain partitions $\{B_{ij}\}_{j=1}^{n(i)}$ of A_i and $x_{ij}^* \in X_1^*$ such that

$$(\#) \qquad \sum_{j=1}^{n(i)} |x_{ij}^* \mu_i(B_{ij})| > \varepsilon, \qquad i = 1, 2, \ldots.$$

The map $\Upsilon: \mathcal{K} \to l_1$ defined by $\Upsilon(\mu) = \{x_{ij}^* \mu(B_{ij})\}_{j=1;i=1}^{n(i);\infty} \in l_1$ is continuous with respect to weak topologies on $\mathcal{K}$ and l_1. Thus $\Upsilon(\mathcal{K})$ is a conditionally weakly compact subset of l_1; by the Schur theorem, $\Upsilon(\mathcal{K})$ is conditionally strongly compact. One can show that in view of (#), this leads to a contradiction. This completes the proof of the proposition.

We now turn our attention to the finitely additive case. A family $\mathcal{K}$ of finitely additive set functions $\mu: \mathfrak{D} \to X$ is *uniformly strongly additive* if $\mu(A_i) \to 0$ uniformly with respect to $\mu \in \mathcal{K}$ whenever (A_i) is a disjoint sequence. When $\mathcal{K}$ consists of countably additive measures and $\mathfrak{D}$ is a σ-algebra, this concept coincides with uniform countable additivity.

First we need the following lemma.

Lemma 3.3. *Let I be a nonempty set. Suppose that there exists a sequence of functions $f_n: I \to X$ such that $f_n(I)$ is a conditionally weakly compact set in X for each n. Assume that $f: I \to X$ and* $\lim_n \|f_n(\alpha) - f(\alpha)\| = 0$ *uniformly for $\alpha \in I$. Then $f(I)$ is conditionally weakly compact.*

Proof. Let $B(I)$ denote the B-space of all bounded functions with norm defined by the sup norm. Let $f \equiv f_0$ and for each $n \geq 0$ define $T_n: X^* \to B(I)$ by $T_n(x^*) = x^* f_n(\cdot)$. By Theorem 2 in [33], each T_n is a weakly compact operator. Since $T_n \to T$ in the uniform operator topology, T is weakly compact. Again by the above-mentioned theorem, $f(I)$ is conditionally weakly compact in X.

We can now state the main theorem on weak compactness in $\text{fa}(\mathfrak{D}, X)$.

Theorem 3.1. (i) *Assume that X and X^* have property R-N. Let $\mathcal{K} \subset$* $\text{fa}(\mathfrak{D}, X)$ *satisfy the following conditions*:

(a) *$\mathcal{K}$ is bounded*;

(b) *$\{|\mu|: \mu \in \mathcal{K}\}$ is uniformly strongly additive*;

(c) *$\{\mu(A): \mu \in \mathcal{K}\}$ is conditionally weakly compact in X for each $A \in \mathfrak{D}$.*

Then $\mathcal{K}$ is conditionally weakly compact in $\text{fa}(\mathfrak{D}, X)$.

(ii) *Conversely, if X is any B-space and $\mathcal{K} \subset$* $\text{fa}(\mathfrak{D}, X)$ *is conditionally weakly compact, then* (a), (b) *and* (c) *hold.*

Remark 3.2. In view of Remark 3.1, if X is reflexive or $X = Y^*$ and X^* is separable, then X and X^* have property R-N and thus satisfy the hypotheses of (i).

Proof. Assume $\mathfrak{D} = \mathfrak{R}$ is an algebra of sets–that is, $S \in \mathfrak{R}$. Let $\mathfrak{R}_1$ be the Stone algebra [13, p. 312] of all open-closed subsets of the compact Hausdorff space S_1. The σ-algebra generated by $\mathfrak{R}_1$ is denoted by $\mathfrak{R}_2$. We shall use the fact that $\text{fa}(\mathfrak{R}, X)$ and $\text{ca}(\mathfrak{R}_2, X)$ are isometrically isomorphic (see [11]). Let $\mathcal{K}'$ denote the usual image of $\mathcal{K}$ in $\text{ca}(\mathfrak{R}_1, X)$; $\mathcal{K}''$ denotes the extensions of $\mathcal{K}'$ to R_2. It suffices to show that $\mathcal{K}''$ satisfies (a), (b) and (c) of Proposition 3.1 if $\mathcal{K}$ satisfies the hypotheses of (i). Clearly (a) is satisfied. Next, (b) follows from Theorem 3 in [5], which implies that since $\{|\mu'|: \mu' \in \mathcal{K}\}$ is uniformly strongly additive on $\mathfrak{R}_1$,

$\{|\mu''|: \mu'' \in \mathcal{K}''\}$ is uniformly countably additive on $\mathfrak{R}_2$. Note that we use the fact that the $|\mu''|$ is the extension of $|\mu'|$. To prove (c), let $A \in \mathfrak{R}_2$ and let $(\sigma_n) \equiv (\mu''_n) \subset \mathcal{K}''$. Set $\lambda = \sum |\sigma_n|/2^n$. Then λ is a bounded measure such that $|\sigma_n| \ll \lambda$ for each n. By Theorem 1 in [8], the σ_n are uniformly absolutely continuous with respect to λ. Choose a sequence of sets $B_n \in \mathfrak{R}_1$ such that $\lambda(B_n \Delta A) \to 0$. Hence $\sigma_i(B_n) \to_i \sigma_i(A)$ uniformly in i. Let I be the set of positive integers. Define $f, f_n: I \to \mathfrak{X}$ by $f_n(i) = \sigma_i(B_n), f(i) = \sigma_i(A)$. Since each $f_n(I)$ is conditionally weakly compact and $f_n \to f$ uniformly on I, we conclude by Lemma 3.3 that $f(I) = \{\sigma_i(A): i = 1,2,\dots\}$ is conditionally weakly compact. This establishes (c).

In the general case when $\mathfrak{D}$ is a ring, we consider $\mathfrak{R}$ the algebra generated by $\mathfrak{D}$. Then $\mathcal{K}$ is extended to $\mathcal{K}'$ on $\mathfrak{R}$ in such a way that (a), (b) and (c) hold for $\mathcal{K}'$ on $\mathfrak{R}$ (see 3.V in [6]). The problem is then reduced to the above case. This completes the proof of the theorem.

4. Weakly compact and compact operators. In this section we establish criteria for operators to be (weakly) compact on certain function spaces.

Definition 4.1. Let T: $C(H,E) \to F$ be an operator. We say that T is (weakly) compact if there exists a compact set K such that $T(\{f: \|f\|_K \leq 1\})$ is conditionally (weakly) compact in F.

Theorem 4.1. *Let L be an operator defined on either $C(H,E)$ or $C_0(H,E)$ into F, with $L \leftrightarrow m$. If L is weakly compact then m is s-bounded, and $m(A)$: $E \to F$ is weakly compact for each A.*

Conversely, if E^* and E^{**} have property R-N (e.g. if E is reflexive or E^{**} is separable), m is s-bounded and $m(A)$ is a weakly compact operator in $B(E,F)$ for each A, then L is weakly compact.

Proof. We shall only consider the case where L is defined on $C(H,E)$. Suppose L is weakly compact. One can find a compact set $K \subset H$ and a constant $M > 0$ such that $\|L(f)\| \leq M\|f\|_K, f \in C(H,E)$ and $L(\{f: \|f\|_K \leq 1, f \in C(H,E)\})$ is conditionally weakly compact in F. Thus we may consider L as an operator on $C(K,E)$ into F. By Gantmacher's theorem,

$$L^*: F^* \to C^*(K,E) \subset \mathrm{ca}(\Sigma(H),E^*)$$

(see Corollary 2.9.1) is weakly compact. Consequently, $\{m_z: z \in F_1^*\} = \{L^*(z): z \in F_1^*\}$ is conditionally weakly compact in $\mathrm{ca}(\Sigma, E^*)$. By Proposition 3.1, $\{|m_z|: z \in F_1^*\}$ is uniformly countably additive, hence conditionally weakly compact in $\mathrm{ca}(\Sigma, C)$. By Lemma 3.1, m is s-bounded.

Conversely, suppose the conditions of the second part of the theorem are fulfilled. As before we regard L as an operator on $C(K,E)$ for some compact set K. Consider the set $\mathcal{K} = \{m_z: z \in F_1^*\} = L^*(F_1^*)$. It suffices to show that $\mathcal{K}$ is conditionally weakly compact in $\mathrm{ca}(\Sigma(K),E^*)$. But $\mathcal{K}$ is bounded since $\tilde{m}(K) < \infty$, and $\{|m_z|: z \in f_1^*\}$ is uniformly countably additive since m is s-bounded. Let $A \in \Sigma(K)$. Since $m(A)$: $E \to F$ (see Corollary 4.4.1 *infra*) is weakly compact, $\{m(A)^*z: z \in F_1^*\} = \{m_z(A): z \in F_1^*\}$ is conditionally weakly compact in E^*. The conditional weak compactness of $\mathcal{K}$ now follows from Proposition 3.1.

Remark 4.1. It follows from the above theorem that if L is weakly compact, then m is countably additive. If L is a dominated operator, then $|m|$ is finite and countably additive, hence s-bounded. Thus in this case L is weakly compact if E^* and E^{**} have property R-N and $m(A)$ is a weakly compact operator for each A. The above theorem strengthens Theorem VI.7.3 of [13] as follows: L: $C_0(H) \to F$ is weakly compact if and only if m is countably additive. The hypothesis that m maps into F is unnecessary. We mention that if the hypotheses on E^* and E^{**} are omitted in the above theorem, then the result is false. In fact if E is not reflexive, one can exhibit nonweakly compact operators which correspond to s-bounded measures. Special cases of the above result have been obtained by Batt and Berg [3] and C. Swartz (private communication).

Next we consider operators on the space of totally measurable functions $U_E(\mathfrak{D})$ (Definition 2.1) into F. First we state a lemma whose proof will be left to the reader.

Lemma 4.1. *Let* L: $U_E(\mathfrak{D}) \to F$ *be an operator. Then there exists a unique finitely additive set function* m: $\mathfrak{D} \to B(E,F)$ *such that* $\tilde{m}(S) = \|L\| < \infty$ *and* $L(f) = \int f\,dm, f \in U_E(\mathfrak{D})$.

If L and m are related as above, we write $L \leftrightarrow m$. The following theorem characterizes weakly compact operators on $U_E(\mathfrak{D})$.

Theorem 4.2. *Let* L: $U_E(\mathfrak{D}) \to F$ *be an operator, with* $L \leftrightarrow m$. *If* L *is weakly compact, then* m *is s-bounded.*

Conversely, if E^* *and* E^{**} *have property R-N (e.g. if* E *is reflexive or* E^{**} *is separable),* m *is s-bounded and* $m(A)$: $E \to F$ *is weakly compact for each* $A \in \mathfrak{D}$, *then* L *is weakly compact.*

Proof. We first note that $U_E^*(\mathfrak{D}) \subset \mathrm{fa}(\mathfrak{D}, E^*)$. Since $L^*(F_1^*) = \{m_z : z \in F_1^*\}$, the theorem follows in view of Theorem 3.1.

Using Theorem 4.2 we establish a result concerning the pointwise limit of weakly compact operators on the space of totally measurable functions.

Theorem 4.3. (i) *If* $c_0 \not\subset F$ *and* L: $U_E(\mathfrak{D}) \to F$ *is an operator, then* L *is s-bounded.* (ii) *Let* E^* *and* E^{**} *have property R-N, let* L: $U_E(\mathfrak{D}) \to F$ *be an operator, let* L_n *be a sequence of weakly compact operators, and suppose that* $c_0 \not\subset F$. *If* $L_n(\mathfrak{X}_A \cdot x) \to L(\mathfrak{X}_A \cdot x)$ *for each* A *and uniformly for* $x \in E_1$, *then* L *is weakly compact.*

Proof. (i) Suppose that L is not s-bounded. Then there is a disjoint sequence (A_i), a subsequence (n_k) of positive integers, a sequence $(x_i) \subset E_1$, and an $\varepsilon > 0$ so that

$$(\#) \qquad \left\| \sum_{i=n_k+1}^{n_{k+1}} m(A_i) \cdot x_i \right\| > \varepsilon.$$

However, if we select $z \in F_1^*$, then $\sum_{i=1}^{\infty} |\langle m(A_i)x_i, z\rangle| \leq \sum_{i=1}^{\infty} |m_z|(A_i) \leq \sup\{|m_z|(A): A \in \mathfrak{D}\} \leq \sup\{\tilde{m}(A): A \in \mathfrak{D}\} < \infty$, and (#) is a weakly unconditionally convergent series which is not unconditionally convergent. Thus by Bessaga and Pełczyński [4, p. 160], F contains c_0, and (i) is proved.

(ii) Let $m_n \leftrightarrow L_n$ and $m \leftrightarrow L$. Since $m_n(A)x = L_n(\mathfrak{X}_A \cdot x) \to m(A)x = L(\mathfrak{X}_A \cdot x)$ for each A and uniformly for $x \in E_1$, then $m(A)$ is a weakly compact operator for each A. And, since m is s-bounded by (i), the result follows from the preceding theorem.

Next we turn to the problem posed by Dinculeanu [11, p. 416] of characterizing those operators $L: \mathbf{B} \to F$ whose representing measures actually map into $B(E,F)$. This question has also been studied in [3] and [16]. We give a characterization analogous to the theory developed by Bartle, Dunford and Schwartz [2].

If $L: C_0(H,E) \to F$ is an operator and $x \in E$, let $L_x: C_0(H) \to F$ be the usual induced operator. Note that $L^{**}(X_A \cdot x) = L_x^{**}(X_A)$, $A \in \Sigma(H)$. For $x \in E$ and $y \in F^{***}$, m_x and m_y are defined in the obvious manner.

Theorem 4.4. *Let $L: \mathbf{B} \to F$ be an operator with $L \leftrightarrow m$. Then the following are equivalent*:

(i) $m: \Sigma \to B(E,F)$;

(ii) m_x *is countably additive for each* $x \in E$;

(iii) $L_x: C_0(H) \to F$ *is weakly compact for each* $x \in E$;

(iv) m_y *is countably additive for each* $y \in F^{***}$.

Proof. (iii) $\Rightarrow$ (i). Since $L_x \leftrightarrow m_x$ and $L_x^{**}: U_{\mathbf{C}}(\Sigma) \to F$, we have $m_x(A) \in F$ for each $x \in E$. Therefore $m: \Sigma \to B(E,F)$.

(i) $\Rightarrow$ (ii). Since $m_x: \Sigma \to F$ and m_x is weakly countably additive, by the Orlicz-Pettis theorem m_x is countably additive.

(ii) $\Rightarrow$ (iii). If m_x is countably additive for each x, it follows that m_x is s-bounded, and thus L_x is weakly compact by Theorem 4.1.

(iv) $\Rightarrow$ (ii). Since m_y is countably additive for each $y \in F^{***}$, it follows that m_x is weakly countably additive, and by the Orlicz-Pettis theorem m_x is countably additive.

(i) $\Rightarrow$ (iv). Since $m: \Sigma \to B(E,F)$, it follows that if $y \in F^{***}$ and $z = y|_F$, then $m_y = m_z$. Therefore m_y is countably additive since $|m_z| \in \mathrm{rca}(\Sigma, \mathbf{C})$.

The next corollary follows from the above theorem.

Corollary 4.4.1. *Let m be a representing measure. Then m takes its values in $B(E,F)$ in any of the following cases*: (a) *m is countably additive*; (b) *m is s-bounded*; (c) *m corresponds to a weakly compact operator.*

We remark here that the weak extension $\hat{L}: U_E(\Sigma) \to F$ of L which is given in Theorem 1(b) of Batt and Berg [3] is simply L^{**}. Therefore if $L \leftrightarrow m$, then it is immediate that L is (weakly) compact if and only if $m: \Sigma \to B(E,F)$ and

$A = \{\sum_{\pi(H)} m(A_i)x_i \colon x_i \in E_1\}$ is (weakly) conditionally compact. This is the principal content of Theorem 6 of [3].

We mention that if $L \leftrightarrow m$ and m is s-bounded, then m is variationally regular. That is, if $\varepsilon > 0$ and $A \in \Sigma$, then there is a compact set $K \subset A$ and an open set $U \supset A$ so that $\tilde{m}(U - K) < \varepsilon$. This follows since there is a positive $\lambda \in \mathrm{rca}(\Sigma, \mathbf{C})$ so that $|m_z|(A) \to 0$ as $\lambda(A) \to 0$ uniformly for $z \in F_1^*$.

We next turn our attention to compact operators on function spaces.

Definition 4.2. Suppose that $\mathfrak{D}$ is an arbitrary ring of subsets of S and $m \colon \mathfrak{D} \to B(E,F)$ is finitely additive with finite semivariation. If $A \in \mathfrak{D}$ and $z \in F^*$, then let $p_{(m,A)}(z) = |m_z|(A)$. The topology induced on F^* by the seminorms $\{p_{(m,A)} \colon A \in \mathfrak{D}\}$ will be denoted by $\delta(m)$. We note that if $S \in \mathfrak{D}$, then $(F^*, \delta(m))$ is a pseudometric space whose topology is determimed by $p_{(m,S)}$.

It is known that if H is compact and $L \colon C(H) \to F$, with $L \leftrightarrow m$, where L is an operator, then L is compact if and only if m takes its values in a compact subset of F [13]. However, if L has $C(H,E)$ as its domain, then the range of m need not be compact if L is compact [3]. Conversely, Example 5.1 *infra* shows that if the range of m is compact and for each A, $m(A)$ is compact (even nuclear), then m may not be s-bounded; hence L is not even weakly compact.

Theorem 4.5. *Let* $L \colon C(H,E) \to F$ *be an operator, with* $L \leftrightarrow m$. *Then*

(i) L *is compact if and only if* $(F_1^*, \delta(m))$ *is a compact pseudometric space;*

(ii) L *is compact with dense range if and only if* $\delta(m)$ *induces the* w^**-topology on* F_1^*.

Proof. Suppose $L \colon C(H,E) \to F$ is a compact operator. As we have seen before, we may consider $L \colon C(K,E) \to F$, where K is a compact set. To show $(F_1^*, \delta(m))$ is a compact space, let (z_α) be a net in F_1^*. By the Banach-Alaoglu theorem, there is a subnet $(z_{\alpha'})$ of (z_α) converging to $z \in F_1^*$ in the w^*-topology. By Theorem VI.5.6 in [13], $L^*(z_{\alpha'}) \to L^*(z)$ in $\mathrm{ca}(\Sigma, E^*)$. This means that $z_{\alpha'} \to z$ in $(F_1^*, \delta(m))$.

Conversely, suppose that $(F_1^*, \delta(m))$ is a compact space. Let (z_n) be a sequence in F_1^*. There exists a subsequence (z_{n_i}) such that $|m_{(z_{n_i} - z)}|(K) \to 0$. This implies that $L^*(z_{n_i}) \to L^*(z)$ in $\mathrm{ca}(\Sigma, E^*)$. Hence L^* is compact. This proves (i).

To establish (ii) we make the following observations. The operator L has dense range if and only if L^* is a one to one mapping; hence $\delta(m)$ is a metric if and only if L has dense range. Thus by (i), L is compact with dense range if and only if $(F_1^*, \delta(m))$ is a compact metric space; and by (i) the w^*-topology is stronger than the $\delta(m)$-topology if L is compact. Consequently, the theorem is proved.

5. **Properties of s-bounded measures.** In this section we continue our investigation of strongly bounded representing measures. An example is given which illustrates the pathological behavior of operator valued representing measures. Following this example, three notions of absolute continuity are discussed and additional miscellaneous results are given.

Theorem 5.1. *The following conditions are equivalent*:

(a) *the B-space F does not contain a topological isomorph of* c_0;

(b) *for each E and each H, the limit of every pointwise convergent uniformly bounded* (*in semivariation*) *sequence of s-bounded representing measures* $\mu^{(i)}$: $\Sigma \to B(E,F)$ *is an s-bounded representing measure*;

(c) *for each E and each H, a representing measure m is s-bounded if and only if* $m: \Sigma \to B(E,F)$.

Proof. Since Example 5.1 *infra* and the remarks following it show that if $F \supset c_0$ then there is a representing measure m which is countably additive, not s-bounded, and the pointwise limit of a uniformly bounded sequence of s-bounded representing measures, it suffices to show that (a) ⇒ (b) and (a) ⇒ (c).

(a) ⇒ (c). Suppose that F does not contain c_0 and $m: \Sigma \to B(E,F)$. Choose (A_i) disjoint, $(x_i) \subset E_1$, and $z \in F^*$. Since $\sum |\langle m(A_i)x_i, z\rangle| \leq \sum |m_z|(A_i) < \infty$, $\sum m(A_i) \cdot x_i$ is weakly unconditionally convergent, and by a result of Bessaga and Pełczyński [4, p. 160], $\sum m(A_i)x_i$ is unconditionally convergent. Thus m is s-bounded by Lemma 3.1.

Conversely, if m is s-bounded, then m is countably additive and $m: \Sigma \to B(E,F)$ by Corollary 4.4.1.

(a) ⇒ (b). Suppose that $c_0 \not\subset F$ and (μ^i) is a sequence satisfying the hypothesis of (b). Set $\mu(A) = \lim \mu^i(A)$, $A \in \Sigma$. Since $\tilde{\mu}^i(H)$ is uniformly bounded, it is clear that $\tilde{\mu}(H) < \infty$. Therefore to complete the argument it is enough to show that μ is countably additive and $|\mu_z| \in \mathrm{rca}(\Sigma, \mathbf{C})$ for each $z \in F^*$. For we know that $\mu: \Sigma \to B(E,F)$, and, by the proof of the preceding implication, to see that μ is s-bounded it suffices to prove that μ is a representing measure. Since each μ^i is s-bounded, there is a $\lambda_i \geq 0$, $\lambda_i \in \mathrm{rca}(\Sigma, \mathbf{C})$, so that $\tilde{\mu}^i(A) \to 0$ as $\lambda_i(A) \to 0$ (see Dunford and Schwartz, [13, IV.13.20(iii)]). Let $\lambda = \sum_{i=1}^{\infty} \lambda_i/2^i(|\lambda_i|(H) + 1)$; note that $\mu^i \ll \lambda$. By the Nikodym theorem, μ is countably additive. Furthermore, by the Vitali-Hahn-Saks theorem, $\mu_z^i \ll \lambda$ uniformly in i for each $z \in F^*$. Therefore $\mu_z: \Sigma \to E^*$ is regular for each z, and, by Corollary 2.8.1, $|\mu_z| \in \mathrm{rca}(\Sigma)$. This completes the proof of the theorem.

Recall that an operator Y from an arbitrary B-space X into a B-space Y is *unconditionally convergent* provided it maps weakly unconditionally convergent series in X to unconditionally convergent series in Y. The Orlicz-Pettis theorem implies that every weakly compact operator $T: X \to Y$ is unconditionally convergent.

Theorem 5.2. *If* $c_0 \not\subset E$, *then every s-bounded operator* $m \leftrightarrow L: C_0(H,E) \to F$ *is unconditionally convergent.*

Proof. Let $\sum f_i$ be weakly unconditionally convergent in $C_0(H,E)$. Since $H \otimes E^* \subset B^*$, it follows that $\sum f_i(t)$ is weakly unconditionally convergent in E for each $t \in H$. But, $E \not\supset c_0$, and $\sum f_i(t)$ is unconditionally convergent in E. Set $f_0(t) = \lim_n \sum_{i=1}^n f_i(t)$, $t \in H$, and let $\mathbb{S}_n(t) = \sum_{i=1}^n f_i(t)$, $n = 1, 2, \ldots$. By the uniform boundedness principle, $\sup\{\|\mathbb{S}_n\|_\infty\} < \infty$.

Now since m is s-bounded, $K = \{|m_z|: z \in F_1^*\}$ is conditionally weakly compact in $\mathrm{ca}(\Sigma, C)$; thus by [13, IV.9.1] there is a positive $\lambda \in \mathrm{ca}(\Sigma, \mathbf{C})$ so that $K \ll \lambda$ uniformly and $\lambda(A) \leq \sup\{|m_z|(A): z \in F_1^*\}$. Therefore $\lambda(A) \to 0$ if and only if $\tilde{m}(A) \to 0$, and λ is a control measure for m in the sense of Bartle [1]. By the bilinear dominated convergence theorem in [1] it follows that $\int f_0\, dm = \lim_n \int \mathbb{S}_n\, dm = \lim_n \sum_{i=1}^n L(f_i)$.

Finally, if (n_k) is any permutation of the natural numbers, then $\sum_k f_{n_k}(t) \to f_0(t)$, $t \in H$, since $\sum f_i(t)$ is unconditionally convergent. Therefore $\lim_p \sum_{k=1}^p L(f_{n_k}) = \int f_0\, dm$, and the theorem is proved.

The following example was mentioned in Theorem 5.1.

Example 5.1. Let E_n be Euclidean $2^n + 1$ dimensional space with the l^1-norm, and let $E = (\prod_{n=1}^\infty E_n)$, i.e. E is the collection of all functions f defined on the natural numbers so that $f(i) \in E_i$ and $\sup\{\|f(i)\|: i = 1,2,\ldots\} < \infty$. Note that elements of E may act on E and transform it into l^∞, i.e. if $(x_n), (y_n) \in E$, then $\langle (x_n),(y_n)\rangle = (z_n) \in l^\infty$, where z_n is the inner product of x_n and y_n. Let H be a countably infinite discrete space whose elements are written in the form $\{\mathscr{Q}_{i,j(i)}: i = 1,2,\ldots, 1 \leq j(i) \leq 2^i + 1\}$, and let $A_i = \{\mathscr{Q}_{i,1}, \ldots, \mathscr{Q}_{i,2^i+1}\}$. Now define $m(\{\mathscr{Q}_{i,j}\})$ to be that point in E with $1/2^i$ in jth component of the ith coordinate and 0 elsewhere; define $m(\varnothing)$ to be 0. Therefore $m(\{\mathscr{Q}_{i,j}\}) \in E$ for each i and j; in addition $m(\{\mathscr{Q}_{i,j}\}): E \to c_0$ is an operator. If $A \in \Sigma$, define $m(A)$ to be the point in E whose jth component of the ith coordinate is $m(A \cap \{\mathscr{Q}_{i,j}\})$. Then $m(A): E \to c_0$ is an operator for each $A \in \Sigma$. In fact, if $y = (y_n) \in E_1$, then $m(A_i)y = (z_n) \in c_0$ so that $t_n = 0$ if $n \neq i$ and $|z_i| = |\langle (1/2^i, \ldots, 1/2^i), y_i\rangle| \leq 1/2^i$. From this inequality, it follows easily that m is countably additive. But, since $\tilde{m}(A_i) = (2^i + 1)/2^i$, m is not s-bounded.

Remark 5.2. As we indicated in Theorem 5.1, this example answers two other questions dealing with vector measures. In [9], using Rickart's notion of s-boundedness, Brooks and Jewett showed that the pointwise limit of a sequence of s-bounded set functions was s-bounded. This result fails for representing measures. In fact, if (A_i) and m are as in Example 5.1, then defining $m_n(A)$ to be $m(A \cap (\bigcup_{i=1}^n A_i))$ and using the countable additivity of m, it follows that $m_n(A) \to m(A)$. In addition, it is not difficult to see that each m_n is s-bounded.

Secondly, Batt and Berg [3] have given an example of a compact operator $L: \mathbf{B} \to F$ whose representing measure does not have conditionally compact range. Example 5.1 shows that the conditional compactness of range (m) and the compactness (even nuclearity) of each $m(A)$ does not imply that the represented operator is even weakly compact. In fact, if $A \in \Sigma$, then $f_n = m(A \cap A_n)$ may be naturally interpreted as an element of E^*. As such, $\|f_n\| \leq 1/2^n$. Now let e_n denote the nth unit vector in l^∞, and note that $f_n \otimes e_n$ is an operator from E to c_0, where $\otimes$ denotes tensor product, e.g., see Schaefer [30, pp. 97–100]. The countable additivity of m then implies that

$$m(A) = \sum_{n=1}^{\infty} \frac{1}{2^n}(2^n f_n) \otimes e_n,$$

where $\|2^n f_n\| \leq 1$ and $\|e_n\| = 1$. But this is Grothendieck's characterization of nuclear operators [17].

Now let $K = \{m(A): A \in \Sigma\}$, and let $\varepsilon > 0$. There is an N so that if $A \in \Sigma(H)$, $A \subset B = \cup A_n$, $n \geq N$, then $\|m(A)\| < \varepsilon$. But there are only finitely many sets in Σ which do not meet B. Thus K is totally bounded; consequently, K is conditionally compact.

Next we define three types of absolute continuity for representing measures and briefly study permanence of compact and weakly compact operators with respect to these concepts of absolute continuity.

Definition 5.1. Let each of n, m: $\Sigma(H) \to B(E, F^{**})$ be a representing measure.

(i) We say that n is weakly absolutely continuous with respect to m ($n < m$) if for each $A \in \Sigma(H)$ and $x \in E$

$$n(A) \cdot x \in \overline{\left\{ \bigcup_{\pi(A)} \sum m(A_i) \cdot x_i : x \in E \right\}}.$$

(ii) We say that n is absolutely continuous with respect to m ($n \ll m$) if for each $\varepsilon > 0$ there is a $\delta > 0$ so that if $\tilde{m}(A) < \delta$, then $\tilde{n}(A) < \varepsilon$.

(iii) We say that n is strongly absolutely continuous with respect to m ($n \lll m$) if for each $A \in \Sigma$ and $x \in E_1$ then

$$n(A) \cdot x \in \overline{\left\{ \bigcup_{\pi(A)} \sum m(A_i) \cdot x_i : x_i \in E_1 \right\}}.$$

Remarks 5.3. In Lewis [24] it was shown that $n < m$ if and only if $|n_z| \ll |m_z|$; in [23] it was shown that if $n \lll m$, then $|n_z| \leq |m_z|$ for each $z \in F^*$.

If $n \leftrightarrow T$ and $m \leftrightarrow L$, then we write $T < (\ll)(\lll) L$ to indicate $n < (\ll)(\lll) m$. We are concerned with the following questions. If $T < (\ll)(\lll) L$, and L is (weakly) compact, then must T be (weakly) compact? We obtain positive answers in the following cases.

Theorem 5.3. (i) *If L is a compact operator and $T \lll L$ then T is a compact operator.*

(ii) *If E^* and E^{**} have property R-N, L is weakly compact, and $T \lll L$, then T is weakly compact.*

(iii) *If E is reflexive, L is weakly compact, and $\tilde{m}(A) = 0 \Rightarrow \tilde{n}(A) = 0$, then T is weakly compact if and only if $n \ll m$.*

Proof. (i) Since L is compact, $(F_1^*, \delta(m))$ is a compact pseudometric space. By Remark 5.3 above, $|n_z|(A) \leq |m_z|(A)$, $z \in F^*$, $A \in \Sigma$. Therefore $\delta(n)(z) \leq \delta(m)(z)$, $(F_1^*, \delta(n))$ is a compact space, and T is a compact operator.

(ii) Let $\Gamma(A) = \{\cup_{\pi(A)} \sum m(A_i) x_i : x_i \in E_1\}$. Then, since $\Gamma(A) \subset L^{**}(S_E(\Sigma)_1)$ and L is weakly compact, $\Gamma(A)$ is conditionally weakly compact. Thus $\overline{\Gamma(A)}$ is conditionally weakly compact, and since $\{m(A)x : x \in E_1\} \subset \overline{\Gamma(A)}$, $m(A)$ is a weakly compact operator for each A. Therefore, by Remark 5.3, n is s-bounded, and, by Theorem 4.1, T is weakly compact.

(iii) Since E is reflexive, we immediately see that $n(A)$ is weakly compact for each $A \in \Sigma$. And in Lewis [25, Theorem 3.1] it was shown that if m is s-bounded and $\tilde{m}(A) = 0 \Rightarrow \tilde{n}(A) = 0$, then n is s-bounded if and only if $n \ll m$. Therefore, by Theorem 4.1, (iii) follows, and the theorem is proved.

Our next theorem is an analog of Theorem 7, p. 158 of Dinculeanu [11], i.e. if E is a B-space we obtain a measure theoretic identification of the extreme points of $\mathbf{B}_1^*$. Denote this set by $\operatorname{ext}(\mathbf{B}_1^*)$. Clearly we cannot conclude that an extreme point will be multiplicative in our setting.

Theorem 5.4. *If* $L \in \operatorname{ext}(\mathbf{B}_1^*)$ *and* $L \leftrightarrow m^*$ *then* $\operatorname{supp}(m)$ *is a singleton.*

Proof. If A_1 and A_2 are disjoint members of Σ, we first show that

(i) $$m(A_1) = 0 \quad \text{or} \quad m(A_2) = 0.$$

For, if not, let $m_1(A) = m(A \cap A_1)$ and $m_2(A) = m(A - A_1)$, $A \in \Sigma$. Then

$$\begin{aligned} \tilde{m}(H) &= |m|(H) = \tilde{m}_1(A) + \tilde{m}_2(H - A_1) \\ &= |m_1|(A_1) + |m_2|(H - A) = 1, \end{aligned}$$

since the semivariation and total variation are the same in this case. Now let $n_1 = m_1/(|m_1|(H))$, and let $n_2 = m_2/(|m_2|(H))$. Then $n_1 \neq m$, $n_2 \neq m$, and $|m_1|(H)n_1 + |m_2|(H)n_2 = m$. Therefore L cannot be extreme, and we have a contradiction. Thus if $A_1 \cap A_2 = \varnothing$, then $m(A_1) = 0$ or $m(A_2) = 0$, and the only values assumed by m are 0 and $m(H)$. Therefore, in particular, $m(A) = 0$ if and only if $\tilde{m}(A) = 0$.

Now if x and y are distinct points in $\operatorname{supp}(m)$, let U_x and U_y be disjoint open sets containing x and y respectively. But then $m(U_x) = 0$ or $m(U_y) = 0$, and $\tilde{m}(U_x) = 0$ or $\tilde{m}(U_y) = 0$. Consequently, either x or y does not belong to $\operatorname{supp}(m)$, a contradiction, and the proof is finished.

We remark that establishing (i) is the key to the proof of Theorem 7 in [11]. Furthermore, the converse of this theorem is false in this setting.

Example 5.2. Suppose that H is a nontrivial compact Hausdorff space $E = l^1$, and $\mathbf{B} = C(H, E)$. Let $t \in H$, and, if $(x_n) \in l^\infty$, then define

$$((x_n), t)(f) \equiv \langle (x_n), f(t) \rangle, \qquad f \in \mathbf{B}.$$

Then certainly $((x_n), t) \in \mathbf{B}^*$, and $\|((x_n), t)\| \leq \|(x_n)\|$. Then let $m(A) = (1, 0, \ldots, 0, \ldots)$ if $t \in A$ and 0 otherwise. Therefore $\tilde{m}(H) = |m|(H) = 1$. But $m = 1/2m_1 + 1/2m_2$, where $m_1(A) = (1, 1, 0, \ldots, 0, \ldots)$ if $t \in A$ and 0 otherwise and $m_2(A) = (1, -1, 0, \ldots, 0, \ldots)$ if $t \in A$ and 0 otherwise. Since $|m_1|(H) = |m_2|(H) = 1$, it follows that $L(\cdot) = \int_H (\cdot)\, dm = (m(t), t)$ is not extreme.

We next state two theorems obtained in Lewis [26]—one providing us with a generalization of a result due to Grothendieck [17] and Bartle, Dunford, and

Schwartz [2]—the other providing us with a partial result which raises an interesting question.

Theorem 5.5. *Suppose E satisfies the Schur condition (weak and norm convergence of sequences are the same), and let L: $\mathbf{B} \to F$ be an s-bounded operator, with representing measure m. Then*

(i) m: $\Sigma \to B(E,F)$;

(ii) *if* (f_n) *is* $\mathbf{B}^*$*-Cauchy in* $U_E(\Sigma)$, *then* $(L(f_n))$ *converges in* F;

(iii) *if* $f_n \to^{\mathbf{B}^*} f$ *in* $U_E(\Sigma)$, *then* $L(f_n) \to L(f)$.

Conversely, if E is any B-space and m is a representing measure which satisfies (i), (ii), *and* (iii), *then m is s-bounded.*

Theorem 5.6. *If m is a representing measure, $m \leftrightarrow L$, then m is s-bounded if and only if for each sequence of sets $A_i \searrow \varnothing$, there is a nested sequence (U_n) of open sets so that $A_n \subset U_n$ and $L(f_n) \to 0$ uniformly for each sequence (f_n) so that* $\operatorname{supp}(f_n) \subset U_n$, $\|f_n\| \leq 1$.

Theorem 5.7. *If E satisfies the Schur condition, then no infinite dimensional reflexive subspace of $C_0(H,E)$ is complemented in $C_0(H,E)$.*

Proof. Suppose X is an infinite dimensional reflexive subspace of $C_0(H,E)$ and P: $C_0(H,E) \to X$ is a continuous projection. Since X is reflexive, P is weakly compact, and, by Theorem 5.5, $P^2 = P$ is compact. But then X_1 must be compact in the norm topology, and this is a contradiction since X is infinite dimensional.

In [2] and [17] this result was established for E being the scalar field.

We conclude with the following two problems.

Problem 1. Using Theorem 5.6, it is not difficult to see that if $F = \mathbf{B}$, then $\mathfrak{B}$ (= set of s-bounded operators) is a closed left ideal. Is $\mathfrak{B}$ also a right ideal?

Problem 2. Give a measure theoretic characterization of the extreme points (provided any exist) of the closed unit ball of $B(\mathbf{B}, F)$.

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