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LINEAR OPERATORS COMMUTING WITH TRANSLATIONS ON $\mathscr{D}(\mathbf{R})$ ARE CONTINUOUS

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Dedicated to the Memory of Dr. Lilian Louise Colombe Asam (née Graue)
Born 18 July 1953; died 12 September 1987

Lilian was an early, enthusiastic and dedicated TILFer. (A TILFer is one who has investigated Translation Invariant Linear Functionals.) Her joyfulness, good-nature, intelligence and energy made a unique, important and permanent impact on the lives of all who knew her. Lilian was tragically drowned in a boating accident on the Starnbergersee near Munich on September 12, 1987. We cherish her memory and we miss her presence among us on Earth.

ABSTRACT. Let $\mathscr{D}(\mathbf{R})$ denote the Schwartz space of all C^∞-functions $f\colon \mathbf{R} \to \mathbf{C}$ with compact supports in the real line $\mathbf{R}$. An earlier result of the author on the automatic continuity of translation-invariant linear functionals on $\mathscr{D}(\mathbf{R})$ is combined with a general version of the Closed-Graph Theorem due to A. P. Robertson and W. J. Robertson in order to prove that every linear mapping S of $\mathscr{D}(\mathbf{R})$ into itself, which commutes with translations, is automatically continuous.

1. INTRODUCTION

As usual, $\mathscr{D}(\mathbf{R})$ denotes the Schwartz space of all C^∞ complex-valued test functions with compact supports on the real line $\mathbf{R}$. There have been a number of papers written on the automatic continuity of translation-invariant linear functionals (TILFs) on $\mathscr{D}(\mathbf{R})$ and other spaces of functions and Schwartz distributions. These include [11, 12] by the author and, more recently, [1, 3, 8, 9, 16, 18, 20] by several other TILFers. See the author's *Math Review* of Willis [20] for a brief survey of results and some further references.

The purpose of this note is to prove a new result of this general type, except for operators rather than functionals, which was stated without proof in [13, §6, p. 442].

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Barry Johnson [6, 7] was one of the first to study the automatic continuity of linear operators commuting with translations or other continuous linear operators. See Sinclair [19], Dales [4], and Bachar [2] for surveys of the entire area of automatic continuity and many other references. See also Loy [10].

It is known that every *continuous* linear operator $S: \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})$ which commutes with translations

$$S\tau_a = \tau_a S \tag{1}$$

can be represented as convolution with a unique distribution T of compact support:

$$Sf = T * f, \qquad \text{for all } f \text{ in } \mathscr{D}(\mathbf{R}). \tag{2}$$

See, for example, Donoghue [5, pp. 121–122] or Rudin [17, Theorem 6.33].

It is the purpose of this note to show that the hypothesis of continuity in this statement is superfluous. Specifically we prove the following

Theorem. *A linear mapping $S: \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})$ satisfying (1) for all a in $\mathbf{R}$ is necessarily continuous.*

The translation operator $\tau_a: \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})$ is defined, as usual, by the formula

$$(\tau_a f)(t) \equiv f(t-a), \qquad t, a \in \mathbf{R}.$$

We shall use the following notation for the Fourier Transform $\hat{f}$ of f in $\mathscr{D}(\mathbf{R})$:

$$\hat{f}(z) \equiv \int_{\mathbf{R}} e^{-2\pi i z t} f(t)\,dt, \qquad z \in \mathbf{C}.$$

The proof of the above theorem is based on two lemmas (given in the next section) and the following general form of the Closed-Graph Theorem proved by A. P. Robertson and W. J. Robertson in [14] and stated in [15, p. 124]:

Closed Graph Theorem. *If E is a separated (= Hausdorff) inductive limit of convex Baire spaces and if F is a separated inductive limit of a sequence of fully complete spaces, then any linear mapping, with a closed graph, of E into F is continuous.*

We shall apply this Closed-Graph Theorem to the space $E = F = \mathscr{D}(\mathbf{R})$ which is known to be a separated locally convex topological vector space which is an inductive limit of a sequence of locally convex Fréchet spaces. Since Fréchet spaces are both fully complete and Baire spaces, the hypotheses of this Closed-Graph Theorem are satisfied for $E = F = \mathscr{D}(\mathbf{R})$.

2. Two lemmas

Lemma 1. *If $\varphi: \mathscr{D}(\mathbf{R}) \to \mathbf{C}$ is a (not assumed continuous) translation-invariant linear functional on $\mathscr{D}(\mathbf{R})$, then there is a complex constant c such that*

$$\varphi(f) = c\hat{f}(0) \equiv c\int_{\mathbf{R}} f(t)\,dt \tag{3}$$

for all f in $\mathscr{D}(\mathbf{R})$. That is, translation-invariant linear functionals on $\mathscr{D}(\mathbf{R})$ are automatically continuous.

Proof. See [11], or [12, Theorem 2 page 182], or [13, Corollary of Theorem 1, p. 427].

Translation-invariance for linear functionals $\varphi: \mathscr{D}(\mathbf{R}) \to \mathbf{C}$ means that for all a in $\mathbf{R}$ and for all f in $\mathscr{D}(\mathbf{R})$,

$$\varphi(\tau_a f) = \varphi(f).$$

Lemma 2. *Let φ be a (not assumed continuous) linear functional on $\mathscr{D}(\mathbf{R})$ such that, for some z in $\mathbf{C}$,*

$$\varphi(\tau_a f) = e^{-2\pi i z a}\varphi(f) \tag{4}$$

for all f in $\mathscr{D}(\mathbf{R})$ and for all a in $\mathbf{R}$. Then there is a complex constant c such that, for all f in $\mathscr{D}(\mathbf{R})$,

$$\varphi(f) = c\hat{f}(z) \equiv c\int_{\mathbf{R}} e^{-2\pi i z t} f(t)dt. \tag{5}$$

Proof. Define $\psi: \mathscr{D}(\mathbf{R}) \to \mathbf{C}$ by

$$\psi(f) = \varphi_t(e^{2\pi i z t} f(t)) \tag{6}$$

for all f in $\mathscr{D}(\mathbf{R})$, where the subscript notation φ_t indicates the variable of the function upon which the linear functional φ is acting: Thus

$$\varphi_t(g(t)) \equiv \varphi(g).$$

Note that, for all f in $\mathscr{D}(\mathbf{R})$, $e^{2\pi i z t} f(t)$ is also in $\mathscr{D}(\mathbf{R})$. Then for each a in $\mathbf{R}$,

$$\begin{aligned}
\psi(\tau_a f) &= \varphi_t(e^{2\pi i z t}(\tau_a f)(t)) && \text{by (6)} \\
&= e^{2\pi i z a}\varphi_t(\tau_a e^{2\pi i z t} f(t)) && \text{by linearity of } \varphi \\
&= e^{2\pi i z a} e^{-2\pi i z a}\varphi_t(e^{2\pi i z t} f(t)) && \text{by (4)} \\
&= \psi(f) && \text{by (6).}
\end{aligned}$$

In other words, ψ is a translation-invariant linear functional on $\mathscr{D}(\mathbf{R})$. It follows from Lemma 1 that, for some constant c in $\mathbf{C}$, and for all f in $\mathscr{D}(\mathbf{R})$,

$$\psi(f) = c\hat{f}(0). \tag{7}$$

We may now compute as follows.

$$\begin{aligned}
\varphi(f) &= \varphi_t(e^{2\pi i z t} e^{-2\pi i z t} f(t)) \\
&= \psi_t(e^{-2\pi i z t} f(t)) && \text{by (6)} \\
&= c[e^{-2\pi i z t} f(t)]\hat{}\,(0) && \text{by (7)} \\
&= c\int_{\mathbf{R}} e^{-2\pi i z t} f(t)\,dt \\
&= c\hat{f}(z). && \text{Q.E.D.}
\end{aligned}$$

3. Proof of the theorem

We may write

$$(S\tau_a f)^\frown(z) = (\tau_a Sf)^\frown(z) = e^{-2\pi iza}(Sf)^\frown(z)$$

for all f in $\mathscr{D}(\mathbf{R})$ and for all z in $\mathbf{C}$. Therefore, the linear functional φ on $\mathscr{D}(\mathbf{R})$ defined by

$$\varphi(f) \equiv (Sf)^\frown(z)$$

has the property

$$\varphi(\tau_a f) = e^{-2\pi iza}\varphi(f)$$

which is the hypothesis (2) of Lemma 2. It follows from Lemma 2 that for each z in $\mathbf{C}$ there is a constant C_z in $\mathbf{C}$ such that

$$(Sf)^\frown(z) = C_z\hat{f}(z)$$

for all f in $\mathscr{D}(\mathbf{R})$. We now apply the Closed-Graph Theorem (stated in the Introduction) to show that S is continuous:

Suppose that f_α is a net converging to zero in $\mathscr{D}(\mathbf{R})$ and that Sf_α converges to an element h in $\mathscr{D}(\mathbf{R})$. The linear functionals φ^z (one for each z in $\mathbf{C}$) defined by

$$\varphi^z(f) = \hat{f}(z), \qquad f \in \mathscr{D}(\mathbf{R}),$$

are continuous and so, for each z in $\mathbf{C}$,

$$\begin{aligned}\hat{h}(z) &= \varphi^z(h)\\ &= \lim_\alpha \varphi^z(Sf_\alpha)\\ &= \lim_\alpha (Sf_\alpha)^\frown(z)\\ &= \lim_\alpha C_z\hat{f}_\alpha(z)\\ &= C_z\lim_\alpha \int_{\mathbf{R}} e^{-2\pi izt}f_\alpha(t)\,dt\\ &= C_z\lim_\alpha \varphi^z(f_\alpha)\\ &= C_z\varphi^z(0) = 0.\end{aligned}$$

Since the Fourier transform on $\mathscr{D}(\mathbf{R})$ is one-to-one, $h = 0$. Thus the graph of S is closed. It now follows from the above-stated Closed-Graph Theorem that S is continuous. Q.E.D.

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