Linear Optical Quantum Metrology with Single Photons: Exploiting Spontaneously Generated Entanglement to Beat the Shot-Noise Limit

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Quantum number-path entanglement is a resource for supersensitive quantum metrology and in particular provides for sub-shot-noise or even Heisenberg-limited sensitivity. However, such number-path entanglement has been thought to be resource intensive to create in the first place—typically requiring either very strong nonlinearities, or nondeterministic preparation schemes with feedforward, which are difficult to implement. Very recently, arising from the study of quantum random walks with multiphoton walkers, as well as the study of the computational complexity of passive linear optical interferometers fed with single-photon inputs, it has been shown that such passive linear optical devices generate a superexponentially large amount of number-path entanglement. A logical question to ask is whether this entanglement may be exploited for quantum metrology. We answer that question here in the affirmative by showing that a simple, passive, linear-optical interferometer—fed with only uncorrelated, single-photon inputs, coupled with simple, single-mode, disjoint photodetection—is capable of significantly beating the shot-noise limit. Our result implies a pathway forward to practical quantum metrology with readily available technology.

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Ever since the early work of Yurke and Yuen it has been understood that quantum number-path entanglement is a resource for supersensitive quantum metrology, allowing for sensors that beat the shot-noise limit [1,2]. Such devices would then have applications to supersensitive gyroscopy [3], gravimetry [4], optical coherence tomography [5], ellipsometry [6], magnetometry [7], protein concentration measurements [8], and microscopy [9,10]. This line of work culminated in the analysis of the bosonic NOON state, as defined as $(|N,0\rangle + |0,N\rangle)/\sqrt{2}$, where N is the total number of photons, which was shown to be optimal for local phase estimation with a fixed, finite number of photons, and in fact allows one to hit the Heisenberg limit and the Quantum Cramér-Rao Bound [11–14].

Let us consider the NOON state as an example, where for this state in a two-mode interferometer we have the condition of all N particles in the first mode (and none in the second mode) superimposed with all N particles in the second mode (and none in the first mode). While such a state is known to be optimal for sensing, its generation is also known to be highly problematic and resource intensive. There are two routes to preparing high-NOON states: the first is to deploy very strong optical nonlinearities [15,16], and the second is to prepare them using measurement and feedforward [17–19]. In many ways then NOON-state generators have had much in common with all-optical quantum computers and therefore are just as difficult to build [20]. In addition to the complicated state preparation, typically a complicated measurement scheme, such as parity measurement at each output port, also had to be deployed [21].

Recently two independent lines of research, the study of quantum random walks with multiphoton walkers in passive linear-optical interferometers [22–24], as well as the quantum complexity analysis of the mathematical sampling problem using such devices [25,26], has led to a somewhat startling yet inescapable conclusion—passive, multimode, linear-optical interferometers, fed with only uncorrelated single photon inputs in each mode (Fig. 1), produce quantum mechanical states of the photon field with path-number entanglement that grows superexponentially fast in the two resources of mode and photon number [27]. What is remarkable is that this large degree of number-path entanglement is not generated by strong optical nonlinearities, nor by complicated measurement and feedforward schemes, but by the natural evolution of the single photons in the passive linear optical device. While such devices are often described to have "noninteracting" photons in them, there is a type of photon-photon interaction generated by the demand of bosonic state symmetrization, which gives rise to the superexponentially large number-path entanglement via

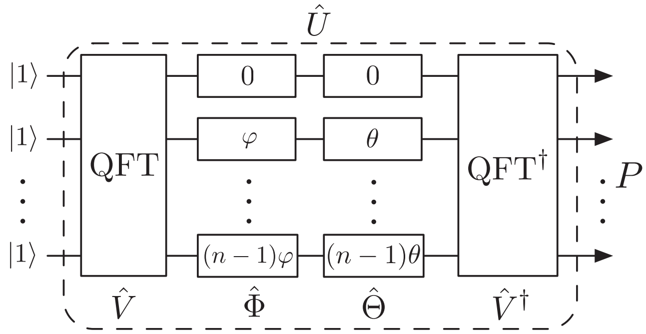

FIG. 1. Architecture of the quantum Fourier transform interferometer (QuFTI) for metrology using single-photon states. The input state comprises n single photons, $|1\rangle^{\otimes n}$. The state evolves via the passive linear optics unitary $\hat{U} = \hat{V} \cdot \hat{\Phi} \cdot \hat{\Theta} \cdot \hat{V}^\dagger$, where $\hat{V}$ is the quantum Fourier transform, $\hat{\Phi}$ is an unknown, linear phase gradient, and $\hat{\Theta}$ is a reference phase gradient used for calibration. At the output we perform a coincidence photodetection projecting on exactly one photon per output mode, measuring the observable $\hat{O} = (|1\rangle\langle 1|)^{\otimes n}$, which, over many measurements, yields the probability distribution $P(\varphi)$ that acts as a witness for the unknown phase φ.

multiple applications of the Hong-Ou-Mandel effect [24]. It is known that linear optical evolution of single photons, followed by projective measurements, can give rise to "effective" strong optical nonlinearities, and we conjecture that there is indeed a hidden Kerr-like nonlinearity at work also in these interferometers [28]. Like boson sampling [25], and unlike universal quantum computing schemes such as that by Knill, Laflamme, and Milburn [29], this protocol is deterministic and does not require any ancillary photons.

The advantage of such a setup for quantum metrology is that resources for generating and detecting single photons have become quite standardized and relatively straightforward to implement in the lab [30–36]. The community then is moving towards single photons, linear interferometers, and single-photon detectors all on a single, integrated, photonic chip, which then facilitates a road map for scaling up devices to large numbers of modes and photons. If all of this work could be put to use for quantum metrology, then a road to scalable metrology with number states would be at hand.

It then becomes a natural question to ask—since number-path entanglement is known to be a resource for quantum metrology—can a passive, multimode interferometer, fed only with easy-to-generate uncorrelated single photons in each mode, followed by uncorrelated single-photon measurements at each output, be constructed to exploit this number-path entanglement for supersensitive (sub-shot-noise) operation? The answer is indeed yes, as we shall now show.

The phase sensitivity, $\Delta\varphi$, of a metrology device can be defined in terms of the standard error propagation formula as,

$$\Delta\varphi = \frac{\sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}}{\left| \frac{\partial \langle \hat{O} \rangle}{\partial \varphi} \right|}, \tag{1}$$

where $\langle \hat{O} \rangle$ is the expectation of the observable being measured and φ is the unknown phase we seek to estimate.

The photons evolve through a unitary network according to $U a_i^\dagger U^\dagger = \sum_j U_{ij} a_j^\dagger$. In our protocol, we construct the n-mode interferometer $\hat{U}$ to be,

$$\hat{U} = \hat{V} \cdot \hat{\Phi} \cdot \hat{\Theta} \cdot \hat{V}^\dagger, \tag{2}$$

which we call the quantum Fourier transform interferometer (QuFTI) because $\hat{V}$ is the n-mode quantum Fourier transform matrix, with matrix elements given by,

$$\mathrm{V}_{j,k}^{(n)} = \frac{1}{\sqrt{n}} \exp\left[\frac{-2ijk\pi}{n}\right]. \tag{3}$$

$\hat{\Phi}$ and $\hat{\Theta}$ are both diagonal matrices with linearly increasing phases along the diagonal represented by

$$\Phi_{j,k} = \delta_{j,k} \exp[i(j-1)\varphi],$$
$$\Theta_{j,k} = \delta_{j,k} \exp[i(j-1)\theta], \tag{4}$$

where φ is the unknown phase one would like to measure and θ is the control phase. $\hat{\Theta}$ is introduced as a reference, which can calibrate the device by tuning θ appropriately. To see this tuning we combine $\hat{\Phi}$ and $\hat{\Theta}$ into a single diagonal matrix with a gradient given by

$$\Phi_{j,k} \cdot \Theta_{j,k} = \delta_{j,k} \exp[i(j-1)(\varphi + \theta)]. \tag{5}$$

The control phase θ can shift this gradient to the optimal measurement regime, which can be found by minimizing $\Delta\varphi$ with respect to n and φ. Since this is a shift according to a known phase, we can for simplicity assume (and without loss of generality) that φ is in the optimal regime for measurements and $\theta = 0$. Thus, $\hat{\Theta} = \hat{I}$ and is left out of our analysis for simplicity.

In order to understand how such a linearly increasing array of unknown phase shifts may be arranged in a practical device, it is useful to consider a specific example. Let us suppose that we are to use the QuFTI as an optical magnetometer. We consider an interferometric magnetometer of the type discussed in Ref. [37], where each of the sensing modes of the QuFTI contains a gas cell of Rubidium prepared in a state of electromagnetically induced transparency manually designed to implement the linear phase gradient. In this state, a photon passing through the cell at the point of zero absorption in the electromagnetically induced transparency spectrum acquires a phase shift that is proportional to the product of an applied uniform (but unknown) magnetic field and the length of the cell. We assume that the field is uniform across the QuFTI, as would be the case if the entire interferometer was constructed on an all optical chip and the field gradient across the chip were negligible. Since we are carrying out local phase measurements (not global) we are not interested in the magnitude of the magnetic field but wish to know if the field changes and if so by how much. (Often we are interested in if the field is oscillating and with what frequency.) Neglecting other sources of noise then in an ordinary Mach-Zehnder interferometer, this limit would be set by the photon shot-noise limit. To construct a QuFTI with the linear cascade of phase shifters, as shown in Fig. 1, we simply increase the length of the cell by integer amounts in each mode. The first cell has length L, the second length $2L$, and so forth. This will then give us the linearly increasing configuration of unknown phase shifts required for the QuFTI to beat the shot-noise limit.

One might question why one would employ a phase gradient rather than just a single phase. Investigation into using a single phase in $\hat{\Phi}$ indicates that this yields no benefit. We conjecture that this is because the number of paths interrogating a phase in a single mode is not superexponential as is the case when a phase gradient is employed.

The interferometer may always be constructed efficiently following the protocol of Reck *et al.* [38], who showed that an $n \times n$ linear optics interferometer may be constructed from $O(n^2)$ linear optical elements (beam splitters and phase shifters), and the algorithm for determining the circuit has run time polynomial in n. Thus, an experimental implementation of our protocol may always be efficiently realized.

The input state to the device is $|1\rangle^{\otimes n}$, i.e., single photons inputted in each mode. If $\varphi = 0$ then $\hat{\Phi} = \hat{I}$ and thus $\hat{U} = \hat{V} \cdot \hat{I} \cdot \hat{V}^{\dagger} = \hat{I}$. In this instance, the output state is exactly equal to the input state, $|1\rangle^{\otimes n}$. Thus, if we define P as the coincidence probability of measuring one photon in each mode at the output, then $P = 1$ when $\varphi = 0$. When $\varphi \neq 0$, in general $P < 1$. Thus, intuitively, we anticipate that $P(\varphi)$ will act as a witness for φ.

In the protocol, assuming a lossless device, no measurement events are discarded. Upon repeating the protocol many times, let x be the number of measurement outcomes with exactly one photon per mode, and y be the number of measurement outcomes without exactly one photon per mode. Then P is calculated as $P = x/(x+y)$. Thus, all measurement outcomes contribute to the signal and none are discarded. Note that, due to preservation of the photon number and the fact that we are considering the antibunched outcome, $P(\varphi)$ may be experimentally determined using non-number-resolving detectors if the device is lossless. If the device is assumed to be lossy, then number-resolving detectors would be necessary to distinguish between an error outcome and one in which more than one photon exits the same mode. The circuit for the architecture is shown in Fig. 1.

The state at the output to the device is a highly path-entangled superposition of $\binom{2n-1}{n}$ terms, which grows superexponentially with n. This corresponds to the number of ways to add n non-negative integers whose sum is n, or equivalently, the number of ways to put n indistinguishable balls into n distinguishable boxes. We conjecture that this superexponential path-entanglement yields improved phase sensitivity as the paths query the phases a superexponential number of times.

The observable being measured is the projection onto the state with exactly one photon per output mode, $\hat{O} = (|1\rangle\langle 1|)^{\otimes n}$. Thus, $\langle \hat{O} \rangle = \langle \hat{O}^2 \rangle = P$. And, the phase-sensitivity estimator reduces to,

$$\Delta\varphi = \frac{\sqrt{P - P^2}}{|\frac{\partial P}{\partial \varphi}|}. \tag{6}$$

Following the result of Ref. [39], P is related to the permanent of $\hat{U}$ as,

$$P = |\mathrm{Per}(U)|^2. \tag{7}$$

Here the permanent of the full $n \times n$ matrix is computed, since exactly one photon is going into and out of every mode. This is unlike the boson-sampling protocol [25] where permanents of submatrices are computed.

We will now examine the structure of this permanent. The matrix form for the n-mode unitary $\hat{U}^{(n)}$ is given by

$$U_{j,k}^{(n)} = \frac{1 - e^{in\varphi}}{n(e^{2i\pi(j-k)/n} - e^{i\varphi})}, \tag{8}$$

as derived in Ref. [40]. Taking the permanent of this matrix is challenging as calculating permanents are in general **P**-hard. However, based on calculating $\mathrm{Per}(\hat{U}^{(n)})$ for small n, we observe the empirical pattern,

$$\mathrm{Per}(\hat{U}^{(n)}) = \frac{1}{n^{n-1}} \prod_{j=1}^{n-1} [j e^{in\varphi} + n - j], \tag{9}$$

as conjectured in Ref. [40]. This analytic pattern we observe is not a proof of the permanent, but an empirical pattern—a conjecture—that has been verified by brute force to be correct up to $n = 25$. Although we don't have a proof beyond that point, $n = 25$ is well beyond what will be experimentally viable in the near future, and thus the pattern we observe is sufficient for experimentally enabling supersensitive metrology with technology available in the foreseeable future.

Following as a corollary to the previous conjecture, the coincidence probability of measuring one photon in each mode is

$$\begin{aligned} P &= |\mathrm{Per}(\hat{U}^{(n)})|^2 \\ &= \frac{1}{n^{2n-2}} \prod_{j=1}^{n-1} [a_n(j)\cos(n\varphi) + b_n(j)], \end{aligned} \tag{10}$$

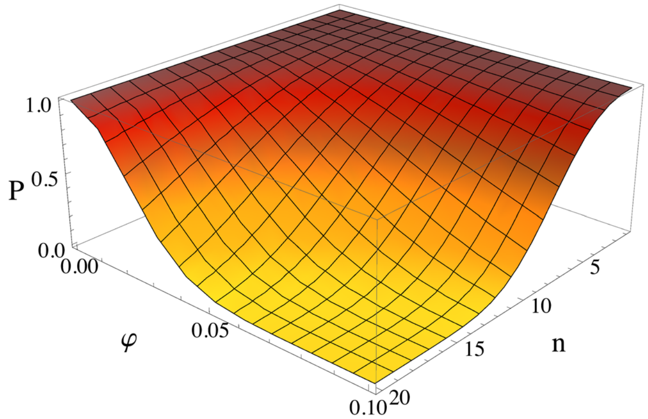

FIG. 2 (color online). Coincidence photodetection probability P against the unknown phase φ and the number of photons and modes n. As n increases, the dependence of P on φ increases, resulting in improved phase sensitivity.

as shown in Ref. [40], where

$$a_n(j) = 2j(n-j),$$
$$b_n(j) = n^2 - 2jn + 2j^2. \tag{11}$$

The dependence of P on n and φ is shown in Fig. 2.

It then follows that

$$\left|\frac{\partial P}{\partial \varphi}\right| = nP|\sin(n\varphi)|\sum_{j=1}^{n-1}\frac{a_n(j)}{a_n(j)\cos(n\varphi)+b_n(j)}, \tag{12}$$

as shown in Ref. [40].

Finally, we wish to establish the scaling of $\Delta\varphi$. With a small φ approximation $[\sin(\varphi)\approx\varphi,\ \cos(\varphi)\approx 1-\frac{1}{2}\varphi^2]$ we find

$$\Delta\varphi = \sqrt{\frac{3}{2n(n+1)(n-1)}} = \frac{1}{2\sqrt{\binom{n+1}{3}}}, \tag{13}$$

as shown in Ref. [40]. Thus, the phase sensitivity scales as $\Delta\varphi = O(1/n^{3/2})$ as shown in Fig. 3.

We would like to compare the performance of our QuFTI to an equivalent multimode interferometer baseline for which we will construct the shot-noise limit (SNL) and Heisenberg limit (HL). This is a subtle comparison, due to the linearly increasing unknown phase shifts, $\{0, \varphi, \ldots, (n-1)\varphi\}$, that the QuFTI requires to operate. The mathematical relation is shown in Fig. 3, where we have converted the number of resources, N, to the number of photons, n. There is disagreement on how such resources should be counted. This is the method that we feel most

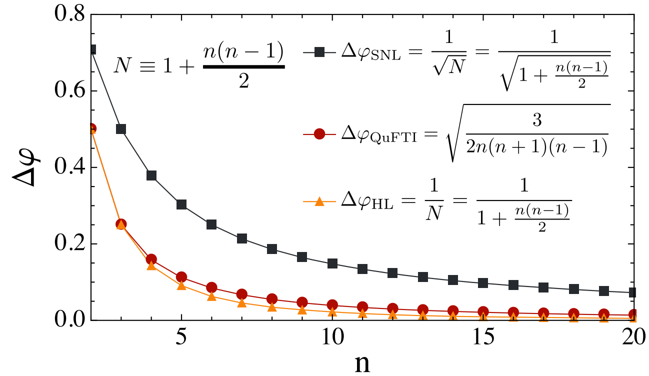

FIG. 3 (color online). Phase sensitivity $\Delta\varphi$ against the number of photons n (red circles). The shot-noise limit of $1/\sqrt{N}$ (black squares) and Heisenberg limit of $1/N$ (orange triangles) are shown for comparison. The QuFTI exhibits phase sensitivity significantly better than the shot-noise limit, and only slightly worse than the Heisenberg limit.

fairly counts our resources. A more detailed supporting discussion can be found in Ref. [40].

We have shown that a passive linear optics network fed with single-photon Fock states may implement quantum metrology with phase sensitivity that beats the shot-noise limit. Unlike other schemes that employ exotic states such as NOON states, which are notoriously difficult to prepare, single-photon states may be readily prepared in the laboratory using present-day technology. Furthermore, we show in Ref. [40] that this network is far more robust against dephasing than the NOON state. This new approach to metrology via easy-to-prepare single-photon states and disjoint photodetection provides a road towards improved quantum metrology with frugal physical resources.

While computing the sensitivity using the standard error propagation formula of Eq. (1) provides clear evidence that our scheme does indeed beat the SNL, it would be instructive to carry out a calculation of the quantum Fisher information and thereby provide the quantum Cramér-Rao bound, which would be a true measure of the best performance of this scheme possible, according to the laws of quantum theory. However, due to the need to compute the permanent of large matrices with complex entries, this calculation currently remains intractable. We will continue to investigate such a computation for a future work. In general, analytic solutions to matrix permanents are not possible. In this instance, the analytic result is facilitated by the specific structure of the QuFTI unitary. Other inhomogeneous phase gradients may yield analytic results, but we leave this for future work.

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