



# ***Linear Optimization and Extensions: Theory and Algorithms***

*Shu-Cherng Fang*

*North Carolina State University*

*Sarat Puthenpura*

*AT&T Bell Labs*



Prentice Hall, Englewood Cliffs, New Jersey 07632

# Contents

<b>PREFACE</b>	<b>xiii</b>
<b>1 INTRODUCTION</b>	<b>1</b>
1.1 History of Linear Programming	1
1.2 The Linear Programming Problem	2
1.2.1 Standard-Form Linear Program,	3
1.2.2 Embedded Assumptions,	3
1.2.3 Converting to Standard Form,	4
1.3 Examples of Linear Programming Problems	5
1.4 Mastering Linear Programming	9
References for Further Reading	10
Exercises	11
<b>2 GEOMETRY OF LINEAR PROGRAMMING</b>	<b>14</b>
2.1 Basic Terminologies of Linear Programming	14
2.2 Hyperplanes, Halfspaces, and Polyhedral Sets	15
2.3 Affine Sets, Convex Sets, and Cones	17

2.4	Extreme Points and Basic Feasible Solutions	19
2.5	Nondegeneracy and Adjacency	21
2.6	Resolution Theorem for Convex Polyhedrons	23
2.7	Fundamental Theorem of Linear Programming	24
2.8	Concluding Remarks: Motivations of Different Approaches	25
	References for Further Reading	26
	Exercises	26

### **3 THE REVISED SIMPLEX METHOD**

**29**

3.1	Elements of an Iterative Scheme	29
3.2	Basics of the Simplex Method	30
3.3	Algebra of the Simplex Method	31
3.3.1	<i>Stopping the Simplex Method—Checking for Optimality,</i>	<i>33</i>
3.3.2	<i>Iterations of the Simplex Method—Moving for Improvement,</i>	<i>33</i>
3.4	Starting the Simplex Method	39
3.4.1	<i>Two-Phase Method,</i>	<i>39</i>
3.4.2	<i>Big-M Method,</i>	<i>41</i>
3.5	Degeneracy and Cycling	42
3.6	Preventing Cycling	44
3.6.1	<i>Lexicographic Rule,</i>	<i>44</i>
3.6.2	<i>Bland's Rule,</i>	<i>44</i>
3.7	The Revised Simplex Method	45
3.8	Concluding Remarks	50
	References for Further Reading	50
	Exercises	51

### **4 DUALITY THEORY AND SENSITIVITY ANALYSIS**

**55**

4.1	Dual Linear Program	56
4.2	Duality Theory	57

4.3	Complementary Slackness and Optimality Conditions	61
4.4	An Economic Interpretation of the Dual Problem	63
4.4.1	<i>Dual Variables and Shadow Prices</i> ,	63
4.4.2	<i>Interpretation of the Dual Problem</i> ,	64
4.5	The Dual Simplex Method	65
4.5.1	<i>Basic Idea of the Dual Simplex Method</i> ,	65
4.5.2	<i>Sherman-Morrison-Woodbury Formula</i> ,	66
4.5.3	<i>Computer Implementation of the Dual Simplex Method</i> ,	70
4.5.4	<i>Find an Initial Dual Basic Feasible Solution</i> ,	72
4.6	The Primal Dual Method	73
4.6.1	<i>Step-by-Step Procedure for the Primal-Dual Simplex Method</i> ,	75
4.7	Sensitivity Analysis	78
4.7.1	<i>Change in the Cost Vector</i> ,	78
4.7.2	<i>Change in the Right-Hand-Side Vector</i> ,	80
4.7.3	<i>Change in the Constraint Matrix</i> ,	82
4.8	Concluding Remarks	86
	References for Further Reading	87
	Exercises	87

## **5 COMPLEXITY ANALYSIS AND THE ELLIPSOID METHOD**

**92**

5.1	Concepts of Computational Complexity	93
5.2	Complexity of the Simplex Method	94
5.3	Basic Ideas of the Ellipsoid Method	96
5.4	Ellipsoid Method for Linear Programming	100
5.5	Performance of the Ellipsoid Method for LP	103
5.6	Modifications of the Basic Algorithm	104
5.6.1	<i>Deep Cuts</i> ,	104
5.6.2	<i>Surrogate Cuts</i> ,	106
5.6.3	<i>Parallel Cuts</i> ,	106
5.6.4	<i>Replacing Ellipsoid by Simplex</i> ,	107
5.7	Concluding Remarks	108
	References for Further Reading	108
	Exercises	109

**6 KARMARKAR'S PROJECTIVE SCALING ALGORITHM****112**

- 6.1 Basic Ideas of Karmarkar's Algorithm 112
- 6.2 Karmarkar's Standard Form 114
  - 6.2.1 *The Simplex Structure, 115*
  - 6.2.2 *Projective Transformation on the Simplex, 116*
- 6.3 Karmarkar's Projective Scaling Algorithm 117
- 6.4 Polynomial-Time Solvability 120
- 6.5 Converting to Karmarkar's Standard Form 126
- 6.6 Handling Problems with Unknown Optimal Objective Values 128
- 6.7 Unconstrained Convex Dual Approach 135
  - 6.7.1  *$\epsilon$ -Optimal Solution, 136*
  - 6.7.2 *Extension, 139*
- 6.8 Concluding Remarks 141
  - References for Further Reading 141
  - Exercises 142

**7 AFFINE SCALING ALGORITHMS****144**

- 7.1 Primal Affine Scaling Algorithm 145
  - 7.1.1 *Basic Ideas of Primal Affine Scaling, 145*
  - 7.1.2 *Implementing the Primal Affine Scaling Algorithm, 155*
  - 7.1.3 *Computational Complexity, 160*
- 7.2 Dual Affine Scaling Algorithm 165
  - 7.2.1 *Basic Ideas of Dual Affine Scaling, 165*
  - 7.2.2 *Dual Affine Scaling Algorithm, 167*
  - 7.2.3 *Implementing the Dual Affine Scaling Algorithm, 169*
  - 7.2.4 *Improving Computational Complexity, 172*
- 7.3 The Primal-Dual Algorithm 177
  - 7.3.1 *Basic Ideas of the Primal-Dual Algorithm, 178*
  - 7.3.2 *Direction and Step-Length of Movement, 180*
  - 7.3.3 *Primal-Dual Algorithm, 184*
  - 7.3.4 *Polynomial-Time Termination, 184*
  - 7.3.5 *Starting the Primal-Dual Algorithm, 188*
  - 7.3.6 *Practical Implementation, 189*

7.3.7	<i>Accelerating via Power-Series Method</i>	193
7.4	Concluding Remarks	194
	References for Further Reading	195
	Exercises	197
<b>8</b>	<b>INSIGHTS INTO THE INTERIOR-POINT METHODS</b>	<b>201</b>
8.1	Moving Along Different Algebraic Paths	201
8.1.1	<i>Primal Affine Scaling with Logarithmic Barrier Function</i>	203
8.1.2	<i>Dual Affine Scaling with Logarithmic Barrier Function</i>	204
8.1.3	<i>The Primal-Dual Algorithm</i>	205
8.2	Missing Information	207
8.2.1	<i>Dual Information in the Primal Approach</i>	207
8.2.2	<i>Primal Information in the Dual Approach</i>	207
8.3	Extensions of Algebraic Paths	208
8.4	Geometric Interpretation of the Moving Directions	209
8.4.1	<i>Primal Affine Scaling with Logarithmic Barrier Function</i>	211
8.4.2	<i>Dual Affine Scaling with Logarithmic Barrier Function</i>	212
8.4.3	<i>The Primal-Dual Algorithm</i>	213
8.5	General Theory	217
8.5.1	<i>General Primal Affine Scaling</i>	217
8.5.2	<i>General Dual Affine Scaling</i>	219
8.6	Concluding Remarks	220
	References for Further Reading	221
	Exercises	221
<b>9</b>	<b>AFFINE SCALING FOR CONVEX QUADRATIC PROGRAMMING</b>	<b>224</b>
9.1	Convex Quadratic Program with Linear Constraints	225
9.1.1	<i>Primal Quadratic Program</i>	225
9.1.2	<i>Dual Quadratic Program</i>	225
9.2	Affine Scaling for Quadratic Programs	227
9.2.1	<i>Primal Affine Scaling for Quadratic Programming</i>	227

9.2.2	<i>Improving Primal Affine Scaling for Quadratic Programming, 237</i>	
9.3	Primal-Dual Algorithm for Quadratic Programming	241
9.3.1	<i>Basic Concepts, 241</i>	
9.3.2	<i>A Step-By-Step Implementation Procedure, 243</i>	
9.3.3	<i>Convergence Properties of the Primal-Dual Algorithm, 245</i>	
9.4	Convex Programming with Linear Constraints	246
9.4.1	<i>Basic Concepts, 246</i>	
9.4.2	<i>A Step-by-Step Implementation Procedure, 248</i>	
9.5	Concluding Remarks	249
	References for Further Reading	249
	Exercises	250
<b>10</b>	<b>IMPLEMENTATION OF INTERIOR-POINT ALGORITHMS</b>	<b>253</b>
10.1	The Computational Bottleneck	253
10.2	The Cholesky Factorization Method	254
10.2.1	<i>Computing Cholesky Factor, 255</i>	
10.2.2	<i>Block Cholesky Factorization, 257</i>	
10.2.3	<i>Sparse Cholesky Factorization, 259</i>	
10.2.4	<i>Symbolic Cholesky Factorization, 263</i>	
10.2.5	<i>Solving Triangular Systems, 263</i>	
10.3	The Conjugate Gradient Method	265
10.4	The LQ Factorization Method	268
10.5	Concluding Remarks	275
	References for Further Reading	276
	Exercises	277
	<b>BIBLIOGRAPHY</b>	<b>280</b>
	<b>INDEX</b>	<b>295</b>