Linear precision for toric surface patches

Algebraic Geometry and Approximation Theory
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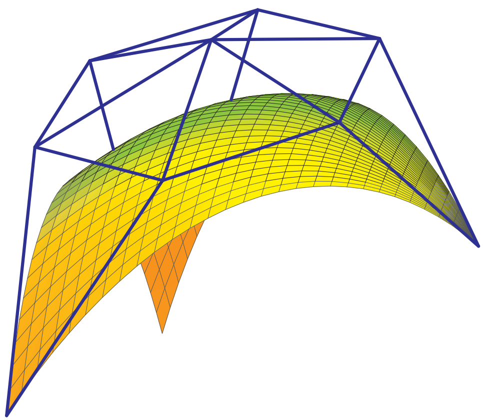
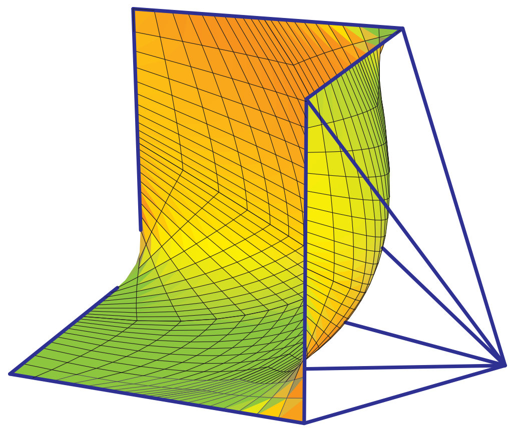




Overview

(With L. Garcia, K. Ranestad, and H.C. Graf v. Bothmer)

Linear precision, the ability of a patch to replicate affine functions, has interesting properties and connections to other areas of mathematics.

– Any patch has a unique reparametrization (possibly non-rational) with linear precision.
– For toric patches, this reparametrization is the maximum likelihood estimator from algebraic statistics, and it is computed by iterative proportional fitting.
– Linear precision by rational functions has an interesting mathematical formulation for toric patches, which leads to a classification, using algebraic geometry, of toric surface patches having linear precision.

(Control-point) patch schemes

Let $\mathcal{A} \subset \mathbb{R}^d$ (e.g. $d = 2$) be a finite set with convex hull Δ, and $\beta := \{\beta_{\mathbf{a}} \colon \Delta \to \mathbb{R}_{\geq 0} | \mathbf{a} \in \mathcal{A}\}$, *basis functions* with $1 = \sum_{\mathbf{a}} \beta_{\mathbf{a}}(x)$.

Given *control points* $\{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$ (e.g. $\ell = 3$), get a map

$$\varphi \;:\; \Delta \longrightarrow \mathbb{R}^\ell \qquad\qquad x \longmapsto \sum \beta_{\mathbf{a}}(x)\, \mathbf{b}_{\mathbf{a}}$$

Image of φ is a patch with shape Δ. Call $(\beta, \mathcal{A})$ a *patch*.

Ex: Bézier curves

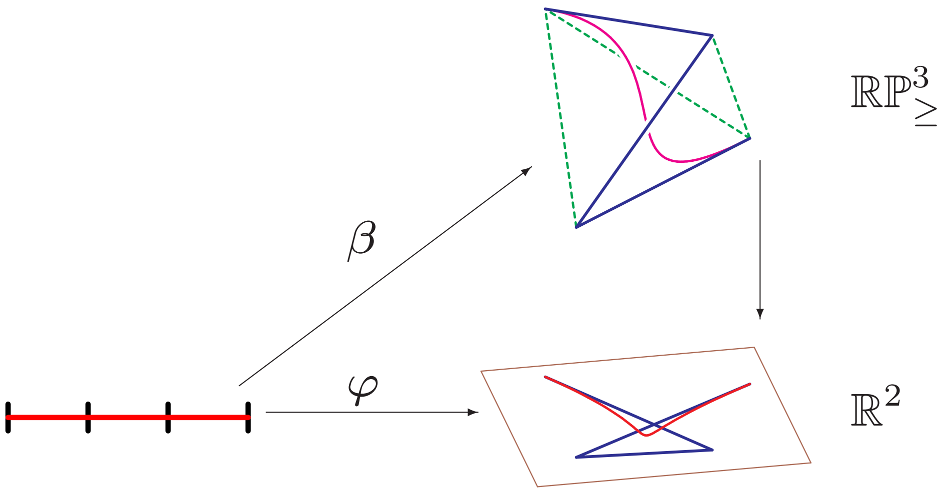
$$\mathcal{A} \;:=\; \{0, 1, \ldots, n\} \subset \mathbb{R}^1\,, \qquad \Delta = [0, n]\,,$$

$$\beta_i(x) \;=\; \frac{1}{n^n}\binom{n}{i} x^i (n - x)^{n-i}\,. \qquad \text{(Bernstein polynomials)}$$

(Equivalent to usual definition when $x = ny$ for $y \in [0, 1]$.)

Example: Cubic Bézier Curve

The map $\varphi\colon \Delta \longrightarrow \mathbb{R}^\ell$ factors through a map to projective space and a linear projection. We illustrate this for a cubic Bézier patch.



Properties of Patch Schemes

Patch schemes typically have many useful properties.

Affine invariance and the convex hull property are built into definition.

Linear precision is the ability to replicate linear functions.

We will adopt a precise, but restrictive definition.

Let $\mathcal{A}$ be the control points, $(\mathbf{b}_{\mathbf{a}} = \mathbf{a})$, to get the *tautological map*,

$$\tau \;:\; x \;\longmapsto\; \sum \beta_{\mathbf{a}}(x)\,\mathbf{a} \qquad \tau \;:\; \Delta \to \Delta\,.$$

$(\beta, \mathcal{A})$ has *linear precision* if and only if $\tau =$ identity map.

Characterization of Linear Precision

Theorem (G-S). *If τ is a homeomorphism, the patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has a unique reparametrization with linear precision, $\{\beta_{\mathbf{a}} \circ \tau^{-1} \mid \mathbf{a} \in \mathcal{A}\}$.*

Proof: Immediate from the definitions.

This theoretical result is useless without a method to compute τ^{-1}.

The map τ factors

$$\begin{array}{rccccc} \tau : \Delta & \xrightarrow{\beta} & & \mathbb{RP}^{\mathcal{A}}_{\geq} & & \xrightarrow{\mu} \Delta \\ x & \longmapsto & [1, \beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}] & & e_{\mathbf{a}} = [0, \ldots, 1, \ldots, 0] & \longmapsto \mathbf{a} \end{array}$$

Note $\beta \circ \tau^{-1} = \mu^{-1} \colon \Delta \to X_{\beta}$, where $X_{\beta} :=$ image $\beta(\Delta) \subset \mathbb{RP}^{\mathcal{A}}_{\geq}$.

We see that μ^{-1} is the key to this concept.

Toric patches (After Krasauskas)

A polytope Δ with integer vertices is given by facet inequalities

$$\Delta \;=\; \{x \in \mathbb{R}^d \mid h_i(x) \geq 0 \text{ for } i = 1, \ldots, n\}\,,$$

where h_i is linear with integer coefficients.

For each $\mathbf{a} \in \mathcal{A} := \Delta \cap \mathbb{Z}^d$, define the *toric Bézier function*

$$\beta_{\mathbf{a}}(x) \;:=\; h_1(x)^{h_1(\mathbf{a})} \cdots h_n(x)^{h_n(\mathbf{a})}\,.$$

Let $w = \{w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}_>$ be positive weights. The *toric patch* $(w, \mathcal{A})$ has blending functions $\{w_{\mathbf{a}}\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$. Write $X_{w,\mathcal{A}}$ for its image in $\mathbb{RP}^{\mathcal{A}}$, which is the positive part of a toric variety.

Classical patches are toric.

Example: Bézier triangles

Bézier triangles are toric surface patches.

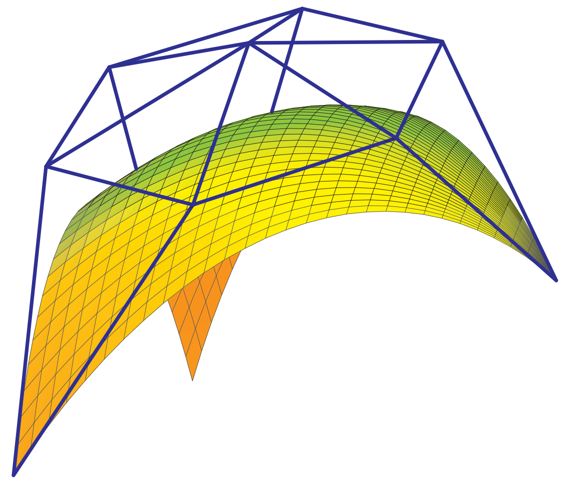
Set $\mathcal{A} := \{(i,j) \in \mathbb{N}^2 \mid i \geq 0,\ j \geq 0, n-i-j \geq 0\}$, then
$w_{(i,j)}\beta_{(i,j)} := \frac{n!}{i!j!(n-i-j)!}x^i y^j (n-x-y)^{n-i-j}$.

These are essentially the Bernstein polynomials, which have linear precision.

The corresponding toric variety is the Veronese surface of degree n.

Choosing control points, get Bézier triangle of degree n.

This picture is a cubic Bézier triangle.



Algebraic moment map

Let $X_{w,\mathcal{A}} \subset \mathbb{RP}^{\mathcal{A}}_{\geq}$ be the image of Δ under the toric blending functions $\{w_{\mathbf{a}}\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$.

Recall that the tautological map $\tau\colon \Delta \to \Delta$ factors

$$\Delta \xrightarrow{\ \beta\ } X_{w,\mathcal{A}} \subset \mathbb{RP}^{\mathcal{A}}_{\geq} \xrightarrow{\ \mu\ } \Delta\,,$$

where μ is the linear projection defined by $e_{\mathbf{a}} \mapsto \mathbf{a}$ for $\mathbf{a} \in \mathcal{A}$.

We call $\mu\colon X_{w,\mathcal{A}} \to \Delta$ the *algebraic moment map*.

Its inverse, μ^{-1} gives the blending functions with linear precision.

(The moment map of symplectic geometry factors through μ.)

Digression: algebraic statitics

In algebraic staitstics, the probability simplex is $\mathbb{RP}^n_>$, the positive part of $\mathbb{RP}^n$, and its subvarieties $X_{w,\mathcal{A}}$ are called *toric statistical models*.

For example, Bézier triangles correspond to *trinomial distributions*.

The algebraic moment map $\mu\colon \mathbb{RP}^n_> \to \Delta$ is called the *expectation map*, and, for $p \in \mathbb{RP}^n_>$, the point $\mu^{-1}(\mu(p)) \in X_{w,\mathcal{A}}$ is the *maximum likelihood estimator*; the distribution in $X_{w,\mathcal{A}}$ which 'best' explains p.

Iterative proportional fitting (IPF) is a fast numerical algorithm to compute μ^{-1}. IPF may be useful in modeling.

Linear precision means maximum likelihood degree 1.

Many statistical models have MLD 1.

Properties of toric patches

Toric patches are very appealing mathematically, but it is not clear if they have enough good properties to be useful or interesting for modeling.

The unique map with linear precision, μ^{-1}, may be computed with iterative proportional fitting.

Krasauskas asked: for which toric patches $(w, \mathcal{A})$ is this map μ^{-1} rational. When this happens, we say that the patch has (rational) linear precision.

This property has an appealing mathematical reformulation.

Linear precision for toric patches

Given the data $(w, \mathcal{A})$ of a toric patch, define a polynomial

$$F_{w,\mathcal{A}} := \sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} x^{\mathbf{a}},$$

where $x^{\mathbf{a}}$ is the multivariate monomial.

Theorem (G-S). *A toric patch* $(w, \mathcal{A})$ *has linear precision if and only if*

$$\mathbb{C}^d \ni x \longmapsto \left(x_1\frac{\partial F_{w,\mathcal{A}}}{\partial x_1},\ x_2\frac{\partial F_{w,\mathcal{A}}}{\partial x_2},\ \ldots,\ x_d\frac{\partial F_{w,\mathcal{A}}}{\partial x_d}\right) \qquad (*)$$

defines a birational isomorphism $\mathbb{C}^d \longrightarrow \mathbb{C}^d$.

We say that F defines *a toric polar Cremona transformation*, when its toric derivatives $(*)$ define a birational map.

Algebraic relaxation

The algebraic relaxation of linear precision for toric patches is

Question: Classify (up to equivalence) the polynomials F that define a toric polar Cremona transformation.

Those F with positive coefficients correspond to toric patches.

Thus Question is an algebraic relaxation of the classification of toric patches with linear precision, because it makes no reference to the field or sign of the coefficients.

This relaxation is amenable to tools from algebraic geometry, specifically the study of birational maps $\mathbb{C}^d - \rightarrow \mathbb{C}^d$.

Linear precision for toric surface patches

Theorem (vB-R-S). *A polynomial $F \in \mathbb{C}[x, y]$ defines a toric polar Cremona transformation if and only if it is equivalent to one of the following forms*

- $(x + y + 1)^n$ *($\Longleftrightarrow$ Bézier triangle).*
- $(x + 1)^m(y + 1)^n$ *($\Longleftrightarrow$ tensor-product patch).*
- $(x + 1)^m\big((x + 1)^d + y\big)^n$ *($\Longleftrightarrow$ trapezoidal patch).*
- $x^2 + y^2 + z^2 - 2(xy + xz + yz)$. *(no analog in modeling).*

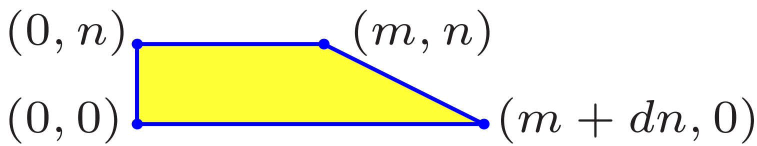
In particular, this classifies toric surface patches with linear precision.

Trapezoidal patch

Let $n, d \geq 1$ and $m \geq 0$ be integers, and set

$$\mathcal{A} \ := \ \{(i,j) \ : \ 0 \leq j \leq n \ \text{ and } \ 0 \leq i \leq m + dn - dj\}\,,$$

which are the integer points inside the trapezoid below.



Choose weights $w_{i,j} := \binom{n}{j}\binom{m+dn-dj}{i}$.
Then the toric Bézier functions are

$$\binom{n}{j}\binom{m+dn-dj}{i} s^i (m+dn-s-dt)^{m+dn-dj-i} t^j (n-t)^{n-j}\,.$$

Outline of Proof

We use the classification and structure of plane Cremona transformations, in an essential way.

The polynomial F either factors or it is irreducible.

When F factors, the structure of plane Cremona transformations allows us to show that F has two factors and identify them.

When F is irreducible, $F = 0$ defines a rational curve. Parametrizing it and using the structure of plane Cremona transformations leads to the proof of the Theorem.

Future work?

- When is it possible to tune a patch (move the points $\mathcal{A}$) to acheive linear precision?
- Linear precision for 3- and higher-dimensional patches.
- Algebraic statistics furnishes many higher dimensional toric patches with linear precision.
- Can iterative proportional fitting be useful to compute patches? Recent work of Dustin Cartwright suggests the answer is yes.

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