SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853-3801

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# Linear Programming, Complexity Theory and Elementary Functional Analysis[1]

by

James Renegar

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# 1 Introduction

## 1.1

This work is concerned with the analysis of algorithms for linear programming (LP) where the analysis is performed in terms of parameters which are natural to functional analysis rather than in terms of parameters arising from the standard complexity theory frameworks; those frameworks are, in our opinion, best suited to combinatorial and algebraic problems. Of course LP can be viewed as an algebraic problem and special cases of it correspond to combinatorial problems. However, LP can also be developed in a manner consistent with the spirit of functional analysis. We are motivated to consider parameters which are natural to functional analysis because interior-point methods (ipm's), which have had a very pronounced impact on research directions within the LP community, are much more closely tied to functional analysis than to combinatorics or algebra.

## 1.2

To briefly explain the two standard complexity theory frameworks in relation to LP, consider any of the elementary forms in which LP is typically introduced, for example, consider problems of the form

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq \vec{0} \end{array}$$

where $A$ is a matrix and the vector inequalities mean coordinate-wise inequality, assuming vectors to be expressed with respect to the standard bases. When one specifies $A$, $b$ and $c$, one has specified an LP *instance.*

Let us say that an *LP solver* is an algorithm which, given any LP instance, is able to determine if the constraints for the instance are consistent, is able to determine if the instance has an optimal solution, and if the instance has an optimal solution, is able to compute an optimal solution.

Complexity theory relies on the notion of instance *size*, roughly, the amount data needed to encode the instance. How size is measured depends on the complexity theory framework (and largely defines the framework). The two standard complexity theory frameworks relied on for analyzing LP solvers are often referred to as *bit complexity* and *algebraic complexity.* In each of these one speaks of the *data coefficients* for an instance, meaning the coefficients of $A$, $b$ and $c$ when expressed in terms of the standard bases.

In bit complexity as it customarily relates to LP, data coefficients are assumed to be integers specified in binary form. The size of an instance is defined as the total number of binary bits in the data for the instance; here, the size of an instance is often referred to as the *bit-length* of the instance. One considers all computational operations to be bit-wise. Thus, for example, the number of operations required to add two integers depends on the number of bits encoding the integers. Bit complexity is very natural for combinatorial problems where each data coefficient is either 0 or 1. It is much less natural for general LP.

In algebraic complexity as it customarily relates to LP, data coefficients are assumed to be real numbers (possibly irrational) and the size of an instance is defined as the total number of data coefficients for the instance. One considers as operations those defined naturally with respect to the underlying algebraic structure, that is, one considers $+$, $-$, $\cdot$, $\div$ and inequality comparison as basic operations, the latter being used for branching. Here, in contrast to bit complexity, adding two numbers is a single operation. (The algebraic complexity theory framework was formalized by Blum, Shub and Smale[3].)

Regardless of the complexity theory framework, an algorithm is said to require only *polynomial time* if there exists a univariate polynomial $p$ such that for all positive integers $L$, whenever the algorithm is applied to any instance whose size does not exceed $L$, the algorithm terminates within $p(L)$ operations.

Khachiyan[20] was the first to prove that there exists a polynomial time LP solver in the bit complexity framework although it was not until Karmarkar[19] that truly practical polynomial time solvers (i.e., ipm's) began emerging. (Actually, some of the algorithms that "emerged" had in essence been developed years earlier; however, they had not been considered seriously in the context of LP and, in particular, their relation to complexity theory had not been explored.) It is unknown whether there exists a polynomial time LP solver in the algebraic complexity framework; this is the most prominent unresolved problem concerning the complexity of LP.

## 1.3

The literature on ipm's is vast; c.f., den Hertog[7], Goldfarb and Todd[14], Gonzaga[16] and Wright[44]. The most general and extensive development of ipm's is to be found in the work of Nesterov and Nemirovskii[22].

We now discuss a typical ipm result. Our discussion is motivated, in part, by the work of Nesterov and Nemirovskii[22]. Our discussion is rather lengthy as we do not wish to assume, in this introductory section, that the reader is familar with the beautiful and abstract concepts underlying some of the contemporary ipm literature. An understanding of these concepts provides motivation for our results. (See [27] for proofs of the unreferenced assertions that follow; similar generality can be found in Renegar in Shub[28].)

The typical result to be described pertains to a particular ipm (the so-called "barrier method") when applied to solving optimization problems of the form

$$\begin{array}{ll} \inf & \langle c, x\rangle \\ \text{s.t.} & x \in D_f \end{array} \tag{1}$$

where $D_f$ is the domain of a particularly nice functional $f$; the functional is used by the ipm to solve (1), i.e., the functional is used as a means-to-an-end.

The functional $f$ is assumed to satisfy four properties, the first three of which are as follows:

1. The domain of $f$, denoted $D_f$, is an open convex subset of a real Hilbert space, denoted $\mathcal{H}_f$.

2. The functional $f$ is twice continuously Frechet differentiable and the Hessian[2] $H_x$ of $f$ at $x$ is strictly positive-definite[3]. (Hence, $f$ is strictly convex.)

Assuming $f$ to satisfy properties 1 and 2, let $\langle\ ,\ \rangle$ denote the inner product on $\mathcal{H}_f$ and for $x \in D_f$ let $\langle\ ,\ \rangle_x$ denote the inner product defined by

$$\langle u, v\rangle_x := \langle u, H_x v\rangle.$$

The inner product $\langle\ ,\ \rangle_x$ induces a norm,

$$\|\ \|_x := \langle\ ,\ \rangle_x^{1/2}.$$

The third property required of $f$ is as follows:

3. If $x \in D_f$ and $y \in \mathcal{H}_f$ satisfy $\|y-x\|_x < 1$ then $y \in D_f$ and for all $v \in \mathcal{H}_f$,

$$\left|\, \|v\|_y^2 - \|v\|_x^2 \,\right| \leq \|y - x\|_x \|v\|_x^2.$$

Note that the latter bound stipulated in the third property is essentially a bound on the change in the norm $\|\ \|_x$ as $x$ varies; the squares appear for technical convienence.[4]

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[2]Recalling that a Hilbert space $\mathcal{H}_f$ can be identified with its dual space $\mathcal{H}_f^*$ consisting of continuous linear functionals on $\mathcal{H}_f$ (i.e., each continuous linear functional on $\mathcal{H}_f$ is of the form $u \mapsto \langle u, v\rangle$), the *Hessian* of $f$ at $x$ is the linear operator $H_x : \mathcal{H}_f \to \mathcal{H}_f$ for which the second Frechet differential of $f$ at $x$ is given by the map which sends each $y \in \mathcal{H}_f$ to the functional $u \mapsto \langle u, H_x y\rangle$; in the case of the finite dimensional space $I\!R^n$, the Hessian can be thought of as "the matrix of second derivatives." If $H_x$ varies continuously with $x$ (i.e., if $f$ is twice *continuously* Frechet differentiable) then $H_x$ is self-adjoint.

[3]If $v \neq \vec{0}$ then $\langle v, H_x v\rangle > 0$ for all $x \in D_f$.

[4]In applying the latter requirement of the third property, one relies on the identity

$$\sup_{v \neq \vec{0}} \left| \frac{\|v\|_y^2 - \|v\|_x^2}{\|v\|_x^2} \right| = \|I - H_x^{-1} H_y\|_x,$$

the operator norm being that induced by $\|\ \|_x$; see [27]. Conceptually, it is worth mentioning that if the inner product on $\mathcal{H}_f$ is replaced by $\langle\ ,\ \rangle_x$ then the Hessian of $f$ at $y$ becomes $H_x^{-1} H_y$; in particular, the Hessian at $x$ becomes the identity operator and the value $\|I - H_x^{-1} H_y\|_x$ is seen to measure the proximity of the Hessian at $y$ to that at $x$.

We let $\mathcal{F}$ denote the set of functionals satisfying the above three properties. Nesterov and Nemirovskii[22] consider a set of functionals which is essentially similar to $\mathcal{F}$, referring to the functionals as "nondegenerate strongly self-concordant" functionals.

In understanding the role of $f \in \mathcal{F}$ in solving the optimization problem (1), it is useful to consider the "unconstrained" optimization problem

$$\min_x \quad f_x.$$

Given an initial point in $D_f$, the most fundamental iterative solution procedure for such a problem is Newton's method, i.e., if the current iterate is $x \in D_f$ then the next iterate is

$$y := x - H_x^{-1} g_x,$$

$g_x$ denoting the gradient of $f$ at $x$; thus, $y := x + n_x$ where the *Newton step* $n_x$ can be computed by solving the linear equations $H_x n_x = -g_x$. The behavior of Newton's method is characterized by the classical Kantorovich theory:

*Assume that $f \in \mathcal{F}$ has a minimizer $z$. If $x \in D_f$ satisfies*

$$\|x - z\|_z \leq \tfrac{1}{3}$$

*then $y := x - H_x^{-1} g_x$ satisfies*

$$\|y - z\|_z \leq \|x - z\|_z^2.$$

In a sense, the elements of $\mathcal{F}$ are precisely those functionals for which the Kantorovich theory is "cleanest."

## 1.4

The role that $f \in \mathcal{F}$ can play in solving the optimization problem (1) is motivated by observing that if one adds a continuous linear functional to $f$, the resulting functional is also in $\mathcal{F}$ simply because its Hessians are identical to those of $f$; the resulting functional yields the same inner products $\langle \ , \ \rangle_x$ and norms $\| \ \|_x$. In particular, for each $t > 0$, the functional

$$x \mapsto t\langle c, x\rangle + f_x \tag{2}$$

is in $\mathcal{F}$ and hence fits nicely into the Kantorovich theory.

In solving the optimization problem (1), the ipm known as the *barrier method* follows the minimizer of the functional (2) as $t \uparrow \infty$. It follows the minimizer using Newton's method; assuming that $x_t$ is a good approximation to the minimizer $z_t$ of the functional (2), the parameter $t$ is increased to $s$ and one iteration of Newton's method is applied in hopes of obtaining a good approximation to the minimizer $z_s$ of the new functional

$$x \mapsto s\langle c, x\rangle + f_x.$$

Since the gradient at $x_t$ of this new functional is precisely $sc + g_{x_t}$, the iterate computed by Newton's method is

$$x_s := x_t - H_{x_t}^{-1}(sc + g_{x_t}).$$

Observe that the Kantorovich theory implies $x_s$ will be a good approximation to the new minimizer $z_s$ if $x_t$ is a relatively good approximation to $z_s$; more precisely,

$$\text{if } \|x_t - z_s\|_{z_s} \leq 1/3 \text{ then } \|x_s - z_s\|_{z_s} \leq \|x_t - z_s\|_{z_s}^2.$$

However, there is no apriori reason that $x_t$, which is assumed to be a good approximation to $z_t$, will indeed be a relatively good approximation to $z_s$. To guarantee that it will be, a fourth property is assumed of $f$, namely, there exists $K > 0$ such that the following holds:

4. For all $x \in D_f$,
$$\|H_x^{-1} g_x\|_x \leq K.$$

It is not difficult to prove that if $f \in \mathcal{F}$ satisfies property 4 and if the parameter values $t$ and $s$ satisfy

$$K\,|1 - \tfrac{s}{t}| < 1$$

then
$$\|x_t - z_s\|_{z_s} < \left(\|x_t - z_t\|_{z_t} + K\,|1 - \tfrac{s}{t}|\right)\sqrt{1 + K\,|1 - \tfrac{s}{t}|}.$$

Hence, for example, if

$$\|x_t - z_t\|_{z_t} \leq \tfrac{1}{9} \quad \text{and} \quad s = (1 + \tfrac{1}{6K})t$$

then
$$\|x_t - z_s\|_{z_s} \leq \tfrac{1}{3}$$

and thus, by the Kantorovich theory,

$$\|x_s - z_s\|_{z_s} \leq \tfrac{1}{9}.$$

Consequently, induction shows the barrier method to "stay on track" if the parameter $t$ is multiplied by a factor of $1 + 1/6K$ with each iteration, i.e., $t_{i+1} := (1 + \frac{1}{6K})t_i$.[5]

It is not difficult to prove that the objective values of the iterates $x_t$ computed by the barrier method approach the optimal objective value; in fact, for all $x \in \mathcal{H}_f$,

$$\langle c, x\rangle - \inf_{y \in D_f} \langle c, y\rangle \leq \frac{K^2(1 + \|x - z_t\|_{z_t})}{t}.$$

## 1.5

We let $\mathcal{F}(K)$ denote the set of functionals satisfying the four properties listed above. Nesterov and Nemirovskii[22] consider a set of functionals which is essentially similar to the union $\cup_K \mathcal{F}(K)$, referring to the functionals as "nondegenerate self-concordant barrier" functionals. They prove, quite amazingly, that for the finite dimensional space $\mathbb{R}^n$, each open and bounded convex set is the domain of such a functional (in fact, in our notation, a functional in $\mathcal{F}(C\sqrt{n})$ where $C$ is a constant independent of $n$). From a theoretical viewpoint, the convex set $D_f$ appearing in the optimization problem (1) is thus virtually unrestricted for finite dimensional spaces. However, from a computational viewpoint, it is most certainly restricted; at present, computable functionals are only known for a very limited, but very important, family of convex sets. (More on this later.)

We mention that $f \in \mathcal{F}(K)$ has a minimizer if and only if $D_f$ is bounded. Moreover, if $D_f$ is bounded then each functional obtained by adding a continuous linear functional to $f$ also has a minimizer (although such functionals are not necessarily elements of $\mathcal{F}(K)$). Consequently, the minimizers $z_t$ and $z_s$ referred to in the previous discussion do indeed exist if $f \in \mathcal{F}(K)$ and $D_f$ is bounded.

[5] The factor "$1 + 1/6K$" is safe but pessimistic; it yields a "short-step" method. Practical implementations of the barrier method rely on much larger factors. However, when one uses a larger factor, the theory does not guarantee that the point $x_s$ computed will indeed be appropriately close to the minimizer $z_s$. Nonetheless, there are ways to check if it is appropriately close. For example, if

$$\|H_{x_s}^{-1}(sc + g_{x_s})\|_{x_s} \leq \tfrac{1}{2}$$

then $x_s$ is appropriately close to $z_s$.

Theoretically supported versions of the barrier method which are more practical than the short-step method begin by multiplying the parameter $t$ by a large factor to obtain $s$; one or more iterates of Newton's method then yield $x_s$; one checks if $x_s$ is appropriately close to $z_s$; if $x_s$ is not appropriately close to $z_s$ then $x_s$ is discarded and one begins again, multiplying $t$ by a somewhat smaller factor. One knows, from the theory, that the factor will never need to be decreased below, say, $1 + 1/6K$, *even if one does not know the value* $K$. Contrary to the impression one might gain from the literature, it is thus not difficult to design versions of barrier method which are more practical than the short-step method but which possess nearly the same worst-case "complexity" as the short-step method.

## 1.6

In our description of the barrier method we assumed that an initial approximation $x_t$ to a minimizer $z_t$ of the functional (2) was available. For a complete theory, one should instead only assume that an initial point $\breve{x} \in D_f$ is available, this point possibly being nowhere near to a minimizer of any functional of the form (2). Slightly modified versions of the barrier method are appropriate under this weaker hypothesis.[6] These perform best if $D_f$ is relatively symmetric about $\breve{x}$.

To quantify the notion of symmetry of an arbitrary bounded convex set $S$ about $x \in S$ it is natural to rely on the value

$$\mathrm{sym}(x, S) := \sup\{t;\ \forall\, v,\ x + v \in S \Rightarrow x - tv \in S\}.$$

Geometrically, one can think of the quantity $\mathrm{sym}(x, S)$ as follows: Take a line $\ell$ through $x$ for which the interval $\ell \cap S$ has positive length; partition that interval into two intervals, each with an endpoint at $x$, and consider the length of each of these intervals; divide the smaller length by the larger length, thus obtaining a ratio; minimizing this ratio over all lines $\ell$ through $x$ for which the interval $\ell \cap S$ is of positive length, one has the value $\mathrm{sym}(x, S)$. If $\mathrm{sym}(x, S) = 1$ then $S$ is "perfectly symmetric" about $x$, whereas if $\mathrm{sym}(x, S)$ is nearly 0 then $x$ is relatively close to the boundary of $S$.

## 1.7

We are now in position to state a typical ipm result. Let $\nu_{\mathrm{inf}}$ denote the optimal value of the optimization problem (1) to be solved and let $\nu_{\mathrm{sup}}$ denote the optimal value when the objective "inf" is replaced by "sup."

> *Assume that $f \in \mathcal{F}(K)$ and $D_f$ is bounded. Assume that a point $\breve{x} \in D_f$ is available (at which to initiate an algorithm). If $\epsilon > 0$ then using only*
>
> $$O(K \log[K + \frac{1}{\epsilon} + \frac{1}{\mathrm{sym}(\breve{x}, D_f)}])$$

[6]For example, noting that $\breve{x}$ minimizes the functional

$$x \mapsto -\langle g_{\breve{x}}, x\rangle + f_x,$$

one can begin by applying the barrier method "in reverse" to the functionals

$$x \mapsto -t\langle g_{\breve{x}}, x\rangle + f_x,$$

*decreasing* the parameter $t$ rather than increasing it; if $f$ has a minimizer, an approximation point for that minimizer is thus obtained. The approximation point will then also appropriately approximate the minimizer of the functional (2) for small $t > 0$ and hence one is in position to apply the barrier method as described in the text.

*iterations, a (particular) barrier method computes $x \in D_f$ known to satisfy*

$$\frac{\langle c, x\rangle - \nu_{\text{inf}}}{\nu_{\text{sup}} - \nu_{\text{inf}}} \leq \epsilon.$$

(The values $\text{sym}(\check{x}, D_f)$, $\nu_{\text{inf}}$, $\nu_{\text{sup}}$ and $\epsilon$ are not assumed to be known apriori; they naturally appear in the analysis but are not required as input to the algorithm.)

Similar results can be obtained for optimization problems of the form

$$\begin{array}{ll} \inf & \langle c, x\rangle \\ \text{s.t.} & x \in D_f \cap L \end{array}$$

where $f \in \mathcal{F}(K)$ and $L$ is a closed subspace in $\mathcal{H}_f$ specified, say, as the solution set for a given system of linear equations. The crucial point is that the functional obtained by restricting $f$ to $D_f \cap L$ is also an element of $\mathcal{F}(K)$, considering the domain of that functional as lying in the Hilbert space $L$.

## 1.8

The preceding exposition indicates how the mathematics underlying ipm's is rooted in functional analysis rather than in algebra or combinatorics. However, when one pursues the ipm literature, more often than not one finds the emphasis to be on assertions concerning the bit complexity framework; for example, one can use the typical ipm result described above to prove that there exists an LP solver which terminates within $O(\sqrt{m}L)$ iterations when applied to those LP instances of bit-length $L$ which have $m$ linear inequality constraints.[7] [8] It is common, in fact, to find ipm papers in which *all* of the main theorems are stated in terms of bit complexity and the underlying functional analysis is almost totally obscured. Clearly (to us), a large part of the mathematical spirit of ipm's is lost when there is such emphasis on the bit complexity framework. It is unnatural to rely on a complexity theory framework designed primarily for combinatorial problems in order to make assertions concerning the efficiency

[7] The crucial point is that the interior of an bounded LP feasible region

$$\{x;\ \alpha_i^T x \geq b_i \ \text{ for all } \ i = 1, \ldots, m\}$$

is the domain for the functional

$$f_x := -12 \sum_i \ln(\alpha_i^T x - b_i)$$

which is an element of $\mathcal{F}(\sqrt{12\,m})$.

[8] The first such result for an ipm, yet to be improved, was proven in Renegar[23]; Gonzaga[15] established the first such result for a barrier method; Karmarkar[19] proved an $O(mL)$ iteration bound.

of algorithms rooted in analysis. One of our goals is to introduce and explore parameters for analyzing LP algorithms where the parameters are natural to functional analysis. What would a numerical analyst prefer to see in place of the bit-length $L$?

Momentarily we describe the parameters we have been considering and present a few representative theorems concerning them. However, before doing so, we make a few additional remarks in hopes of clarifying the definition of the functional sets $\mathcal{F}$ and $\mathcal{F}(K)$. We also define two closely related functional sets that figure prominently in this work.

## 1.9

When one changes inner products on a Hilbert space, the gradients and Hessians of a functional $f$ also change. For recall that with respect to the inner product $\langle\ ,\ \rangle$, the gradient $g_x$ is the unique vector satisfying $(Df_x)u = \langle g_x, u\rangle$ for all vectors $u$, where $Df_x$ is the first differential of $f$ at $x$. Since $\langle g_x, u\rangle = \langle H_x^{-1} g_x, u\rangle_x$, it follows that the gradient of $f$ at $x$ with respect to the inner product $\langle\ ,\ \rangle_x$ is $H_x^{-1} g_x$ rather than $g_x$. Hence, looking back at the fourth property required of functionals in $\mathcal{F}(K)$, one sees that property simply to be a bound on the norm of the gradient *where the gradient and norm arise from the appropriate inner product.*[9] In a similar vein, it is not difficult to verify that with respect to the inner product $\langle\ ,\ \rangle_x$, the Hessian at $x$ is the identity operator.[10]

Although the definition of what it means for a functional $f$ to be an element of $\mathcal{F}(K)$ (or $\mathcal{F}$) is phrased in terms of the original inner product $\langle\ ,\ \rangle$ on $\mathcal{H}_f$, the definition is in fact largely independent of the inner product. The reason is simply that if the norm for each of two inner products induces the same topology as the other, then the resulting inner product $\langle\ ,\ \rangle_x$ is also the same for each. (Thus, for example, since all norms on the finite dimensional spaces $\mathbb{R}^n$ induce the same topology, the definition of what it means for a functional $f$, with domain in $\mathbb{R}^n$, to be an element of $\mathcal{F}(K)$ (or $\mathcal{F}$) is in fact entirely independent of the particular inner product $\langle\ ,\ \rangle$.) Hence, it is natural, as we do in this paper, to speak of a functional $f$ with domain in a real normed vector space $Y$ as being an element of $\mathcal{F}(K)$ (or $\mathcal{F}$); we simply mean that (i) the norm on $Y$ induces the same topology as the norm given by an inner product $\langle\ ,\ \rangle$ which makes $Y$ into a real Hilbert space, and (ii) with respect to $\langle\ ,\ \rangle$, $f$ satisfies the requirements to be an element of $\mathcal{F}(K)$ (or $\mathcal{F}$).[11]

[9] One rarely sees the appropriate inner products emphasized in the ipm literature which is a shame because they are often the key to understanding the underlying geometry.

[10] Consequently, with respect to the unconstrained optimization problem $\min f_x$ and the inner product $\langle\ ,\ \rangle_x$, the Newton step at $x$ is precisely the negative of the gradient at $x$, i.e., Newton's method coincides with the method of steepest descent.

[11] Alternatively, the definition of $\mathcal{F}(K)$ (and $\mathcal{F}$) can be phrased solely in terms of first and second differentials without reference to an inner product, but one can prove that no generality is thus gained and, more importantly, there seems to be no conceptual advantage in doing so.

## 1.10

We now introduce two additional functional sets. The first, denoted $\mathcal{F}'$, is defined exactly as was $\mathcal{F}$ except the restriction that the Hessians $H_x$ be *strictly* positive definite is replaced by the weaker restriction that the Hessians be positive semi-definite[12]. The resulting definition is somewhat of an abuse of notation; for example,

$$\|v\|_x := \langle v, v \rangle_x^{1/2} := \langle v, H_x v \rangle^{1/2}$$

may not be a norm. Of course $\mathcal{F} \subseteq \mathcal{F}'$. Nesterov and Nemirovskii[22] consider a set of functionals which is essentially similar $\mathcal{F}'$, referring to the functionals as "(possibly degenerate) strongly self-concordant" functionals.

Finally, for $K > 0$ let $\mathcal{F}'(K)$ denote those functionals in $\mathcal{F}'$ with the property that for all $x \in D_f$,

$$\limsup_{t \downarrow 0} \langle g_x, (tI + H_x)^{-1} g_x \rangle_x \leq K^2.$$

Comparing this with the fourth property defining $\mathcal{F}(K)$, namely,

$$\|H_x^{-1} g_x\|_x \ (= \langle g_x, H_x^{-1} g_x \rangle_x^{1/2}) \ \leq K,$$

it is apparent that $\mathcal{F}(K) \subseteq \mathcal{F}'(K)$. Nesterov and Nemirovskii[22] consider a set of functionals which is essentially similar to the union $\cup_K \mathcal{F}'(K)$, referring to the functionals as "(possibly degenerate) self-concordant barrier" functionals. They prove that for the finite dimensional space $\mathbb{R}^n$, each open convex set is the domain of such a functional (in fact, in our notation, a functional in $\mathcal{F}'(C\sqrt{n})$ where $C$ is a constant independent of $n$).

As with the definitions of $\mathcal{F}$ and $\mathcal{F}(K)$, the definition of what it means for a functional $f$ to be an element of $\mathcal{F}'$ (or $\mathcal{F}'(K)$) is largely independent of the particular inner product on $\mathcal{H}_f$. Thus one can speak of a functional $f$ with domain in a real normed vector space $Y$ as being an element of $\mathcal{F}'$ (or $\mathcal{F}'(K)$).

Following Nesterov and Nemirovskii[22], one can prove that the following simple and very useful "calculus" is valid:

> If $f_1 \in \mathcal{F}'(K_1)$, $f_2 \in \mathcal{F}'(K_2)$ and $D_{f_1} \cap D_{f_2} \neq \emptyset$ then
>
> $$f_1 + f_2 : D_{f_1} \cap D_{f_2} \to \mathbb{R}$$
>
> is an element of $\mathcal{F}'(\sqrt{K_1^2 + K_2^2})$.
>
> If, in addition, either $f_1 \in \mathcal{F}(K_1)$ or $f_2 \in \mathcal{F}(K_2)$ then $f_1 + f_2$ is an element of $\mathcal{F}(\sqrt{K_1^2 + K_2^2})$.

This concludes our discussion intended to aqquaint the reader with the concepts underlying some of the contemporary ipm literature.

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[12] $\langle v, H_x v \rangle \geq 0$ for all $v$.

## 1.11

We now begin motivating the choice of certain parameters for analyzing LP algorithms. The main parameter can be thought of as the instance size. We have chosen to work with a notion of instance size closely related to condition numbers; this notion applies to general versions of LP that go far beyond the settings of traditional complexity theory.

We motivate the notion of instance size through its relation to condition numbers, first recalling an identity proven in introductory numerical analysis courses.

Assuming $X$ and $Y$ are normed vector spaces, let $\mathcal{L}(X,Y)$ denote the normed vector space of continuous linear operators $A : X \to Y$; the norm on $\mathcal{L}(X,Y)$ is given by $||A|| := \sup_{||x||=1} ||Ax||$.

If $A \in \mathcal{L}(X,X)$ is invertible and $X$ is a Banach space then the (relative) condition number of $A$ is defined to be the quantity

$$\begin{aligned} \text{relcond}(A) &:= \limsup_{||\Delta A|| \downarrow 0} \frac{||(A+\Delta A)^{-1} - A^{-1}||/||A^{-1}||}{||\Delta A||/||A||} \\ &= \limsup_{||\Delta A|| \downarrow 0} \frac{||I - (A+\Delta A)^{-1}A||}{||\Delta A||/||A||} \\ &= ||A||\,||A^{-1}||. \end{aligned}$$

The condition number quantifies the sensitivity of $A^{-1}$ to perturbations in $A$. It indicates a lower bound on the minimal amount of computational accuracy which is sufficient, using floating point arithmetic, to compute a relatively accurate approximation to $A^{-1}$; roughly, for each additional significant digit of accuracy in $A^{-1}$, it is necessary to use $\log(\text{relcond}(A))$ additional significant digits of accuracy in (at least some of) the computations. Hence, recalling that in complexity theory the size of an instance is a measure of the amount of data needed to encode the instance, with respect to the problem of approximating inverses it is natural to think of $\log(\text{relcond}(A))$ as being the size of $A$.

Assuming that $A$ is invertible, a simple and elegant identity proven in introductory numerical analysis courses is

$$\text{relcond}(A) = \frac{1}{\text{reldist}(A, \text{Sing})} \tag{3}$$

where $\text{reldist}(A, \text{Sing})$ is the relative distance from $A$ to the set of non-invertible (i.e., singular) operators, that is,

$$\text{reldist}(A, \text{Sing}) := \inf\{||\Delta A||/||A||;\ A + \Delta A \text{ is not invertible}\}.$$

Thus, it is natural to think of $\log(1/\text{reldist}(A, \text{Sing}))$ as being the size of $A$.

We will define the size of an LP instance to be a quantity analogous to $\log(1/\mathrm{reldist}(A, \mathrm{Sing}))$. The particular quantity depends on the problem. For example, for the problem of determining that the constraints are consistent, and the subsequent problem of computing a feasible point, size will measured in terms of the smallest perturbation needed to obtain an LP instance whose constraints are not consistent. Before we can define size precisely, we must first define what we mean by LP.

## 1.12

The very general definition of LP that we rely on, dating at least to Duffin[8], is as follows.

Let $X$, $Y$ denote normed real vector spaces. Let $X^* := \mathcal{L}(X,\mathbb{R})$, the *dual space* of $X$. Let $C_X$, $C_Y$ be convex cones in $X$, $Y$, each with vertex at the origin, i.e., each is closed under multiplication by positive scalars and under addition.

The cone $C_X$ induces an *ordering*[13] on $X$: Define "$x \geq \bar{x}$" to mean $x - \bar{x} \in C_X$. Similarly, $C_Y$ induces an ordering on $Y$.

Given $A \in \mathcal{L}(X, Y)$, $b \in Y$ and $c^* \in X^*$, the LP instance specified by the *data vector* $d := (A, b, c^*)$ is defined to be the following optimization problem:

$$\begin{array}{ll} \sup & c^* x \\ \text{s.t.} & Ax \leq b \\ & x \geq \vec{0}. \end{array}$$

In considering $d := (A, b, c^*)$ to specify an instance we view $X$, $Y$, $C_X$ and $C_Y$ as fixed.

Although we use the symbols "$\leq$" and "$\geq$" the reader should note that all common forms of linear programming are included in this definition. For example, what one customarily writes as "$Ax = b$" is obtained by letting $C_Y = \{\vec{0}\}$ and what one customarily expresses as "no non-negativity constraints" is obtained by letting $C_X = X$.

We refer to the above very general definition of LP as *analytic LP* in contrast with *elementary LP* where the vector spaces are required to be finite dimensional and the cones are required to be polyhedral. There is a large literature on optimization related to this very general definition of LP (c.f. Andersen and Nash[2], Borwein and Lewis[4], Fiacco and Kortanek[10], Holmes[17], Kallina and Williams[18], Luenberger[21], Rockafellar[29], etc.) including some works which consider ipm's (c.f., Ferris and Philpott[9], Todd[37] and Tuncel[38]), but none which analyze algorithms using parameters similar to the ones that we consider.

[13] The ordering is a partial order iff $C_X$ is pointed; we do not assume pointedness.

## 1.13

We temporarily restrict attention to the problem of determining if the (primal) constraints for an instance $d := (A, b, c^*)$ are consistent and the subsequent problem of computing a so-called feasible point (if the constraints are indeed consistent). Here, the objective functional $c^*$ is irrelevant, so we truncate the data vector $d$, considering instead $d_P := (A, b)$. The subscript "$P$" refers to "primal (constraints)."

The instance $d_P := (A, b)$ is an element of the real vector space

$$\mathbf{D}_P := \mathcal{L}(X, Y) \times Y.$$

Each instance in $\mathbf{D}_P$ specifies a system of (primal) constraints (again emphasizing that we view $X$, $Y$, $C_X$ and $C_Y$ as fixed in doing so); we say that $d_P$ is *consistent* if the system has a solution; if $x$ is a solution for the system then we say that $x$ is *feasible* for $d_P$.

We endow $\mathbf{D}_P$ with a norm: If $d_P := (A, b) \in \mathbf{D}_P$ then let

$$||d_P|| := \max\{||A||, ||b||\}.$$

(It is useful to think of the data for the constraints one wishes to solve as being normalized, i.e., $||A|| \approx 1$, $||b|| \approx 1$ and hence $||d_P|| \approx 1$.)

Let Pri$\emptyset$ denote the set of instances in $\mathbf{D}_P$ which are inconsistent, that is, not consistent; the notation "Pri$\emptyset$" is chosen to make one think, "instances for which the primal feasible region is empty."

If $d_P \in \mathbf{D}_P$ satisfies $d_P \neq \vec{0}$ then define

$$\text{reldist}(d_P, \text{Pri}\emptyset) := \inf\{||\Delta d_P||/||d_P||;\ d_P + \Delta d_P \in \text{Pri}\emptyset\},$$

the relative distance from $d_P$ to the set of inconsistent instances.[14] Observe that if $\text{reldist}(d_P, \text{Pri}\emptyset) > 0$ then, roughly speaking, to determine that $d_P$ is indeed consistent using floating point arithmetic, at least $\log(1/\text{reldist}(d_P, \text{Pri}\emptyset))$ significant digits of accuracy are needed in (at least some of) the computations simply because fewer significant digits do not allow $d_P$ to be distinguished from some inconsistent instance. Once again, recalling that in complexity theory the size of an instance is a measure of the amount of data needed to encode the instance, with respect to the problem of deciding that $d_P$ is consistent (and the subsequent problem of computing a feasible point) it is natural to define the *size* of $d_P$ to be the quantity $\log(1/\text{reldist}(d_P, \text{Pri}\emptyset))$. In light of the identity (3), this notion of size is closely related to condition numbers and essentially reduces to taking the logarithm of the condition number in the case of linear equations, i.e., in the special case $X = Y = C_X$ and $C_Y = \{\vec{0}\}$.

[14] If Pri$\emptyset = \emptyset$ then we define $\text{reldist}(d_P, \text{Pri}\emptyset) = 1$; this allows us to avoid discussion of special cases.

## 1.14

In developing and analyzing LP algorithms, it is necessary to restrict the vector spaces $X$, $Y$ and the cones $C_X$, $C_Y$. However, regardless of the restrictions that one chooses, one can define the size of $d_P$ as $\log(1/\text{reldist}(d_P, \text{Pri}\emptyset))$, i.e., this definition of size is universal. Moreover, certain relations which are useful in complexity theory are virtually always valid with respect to this definition of size, as we now briefly discuss.

In [26] we developed some perturbation theory for analytic LP. Representative results of that theory are as follows: Assume $X$ is reflexive, $C_X$ and $C_Y$ are closed. (Reflexive spaces are common: Hilbert spaces are reflexive as are all normed finite dimensional spaces.) Assume $d_P := (A, b) \in \mathbf{D}_P$ satisfies $\text{reldist}(d_P, \text{Pri}\emptyset) > 0$.

1. There exists $x$ which is feasible for $d_P$ and which satisfies

$$||x|| \leq 1/\text{reldist}(d_P, \text{Pri}\emptyset).$$

2. If $x'$ is feasible for $(A, b + \Delta b)$ then there exists $x$ which is feasible for $d_P$ and which satisfies[15]

$$||x - x'|| \leq \frac{||\Delta b||}{||d_P||} \frac{\max\{1, ||x'||\}}{\text{reldist}(d_P, \text{Pri}\emptyset)}.$$

(There are also results in [26] concerning the size of optimal solutions for an LP instance $d := (A, b, c^*)$, the size of the optimal value, and changes in the optimal value under perturbations. However, besides $\text{reldist}(d_P, \text{Pri}\emptyset)$ those results also involve the relative distance from the LP instance $d$ to the set of instances for which the dual constraints are (strongly) inconsistent. It is shown in [26] that each of the various bounds proven there is the best possible in general.)

Bounds like these are useful in the theory we are attempting to develop. Readers familar with the analysis of ipm's in the bit complexity framework can readily sense why; for example, the above bound on $||x||$ plays a role analogous to the extreme point bound $||x||_\infty \leq 2^L$ which is relied on extensively in bit complexity. However, the above bound on $||x||$ only requires $X$ to be reflexive and the cones to be closed; it applies to LP's far beyond those fitting into the bit complexity framework!

[15] If the cones $C_X$ and $C_Y$ satisfy certain properties as they do, for example, when the linear "inequalities" are actually equations, then the term "$\max\{1, ||x'||\}$" can be replaced simply by "1" as one customarily finds in the linear equation literature. For general cones this replacement is not valid; e.g., it is not valid for some instances $d_P$ if $n \geq 2$, $X = Y = C_X = I\!R^n$ and $C_Y$ is the non-negative orthant.

## 1.15

As mentioned, when developing and analyzing algorithms it is necessary to restrict $X$, $Y$, $C_X$ and $C_Y$. The restrictions we impose allow us to rely on (extensions of) the previously discussed results for the barrier method. In analyzing other algorithms one might impose different restrictions; *the particular restrictions are not so important as the fact that the analysis is performed in terms of parameters like* $\text{reldist}(d_P, \text{Pri}\emptyset)$.

Assume $X$ is a real Hilbert space with inner product $\langle\ ,\ \rangle$. Let $K_X > 0$ and assume that the cone $C_X$ is the *closure* of the set $D_\phi \cap L_X$ where:

1. $D_\phi$ is an open convex cone which is the domain of a functional $\phi \in \mathcal{F}'(K_X)$ (the requirements for $\phi$ to be an element of $\mathcal{F}'(K_X)$ being satisfied with respect to the inner product $\langle\ ,\ \rangle$ on $X$);

2. $L_X$ is a closed subspace of $X$ (possibly $L_X = X$) specified, say, as the solution set for a given system of linear equations.

Make the same assumptions of $Y$, the corresponding entities being $K_Y$, $C_Y$, $D_\psi$ (where $\psi \in \mathcal{F}'(K_Y)$) and $L_Y$.

We assume that the Hilbert space $X$ is endowed with the norm arising from its inner product. By contrast, we only assume that $Y$ is endowed with a norm which generates the same topology as the norm arising from *its* inner product; thus, for example, if $Y$ is finite dimensional then $Y$ can be endowed with any norm. The choice of norm on $Y$ affects parameters appearing in our analysis, parameters like $\text{reldist}(d_P, \text{Pri}\emptyset)$. The need for the stronger restriction on the norm for $X$ is due to the fact that we use the barrier method as a "primal algorithm," i.e., the iterates computed lie in the primal space $X$.

Assume $K$ is a value known to satisfy $K > \sqrt{K_X^2 + K_Y^2} + 1$; the value $K$ is relied on by the barrier method we consider.

## 1.16

Recalling the existence result due to Nesterov and Nemirovskii[22] that each open convex set in $\mathbb{R}^n$ is the domain of a functional in $\mathcal{F}'(C\sqrt{n})$, one sees that from a theoretical viewpoint, our restrictions on the cones $C_X$ and $C_Y$ are virtually nil. However, nearly all of the cones underlying the common forms of LP appearing in practical applications are in fact associated with functionals whose gradients and Hessians are readily computable; hence our restrictions on the cones $C_X$ and $C_Y$ are also virtually nil from a practical viewpoint. Indeed, appropriate functionals $\phi$ and $\psi$ for most of the cones $C_X$ and $C_Y$ appearing in practical applications can be obtained via the previously described "calculus" and the following basic examples for a real Hilbert space $\mathcal{H}$:[16] [17]

1. Any identically constant functional with domain $\mathcal{H}$ is an element of $\mathcal{F}'(0)$.

2. Given $\alpha \in \mathcal{H}$, $\alpha \neq \vec{0}$, the cone

$$\{x;\ \langle \alpha, x\rangle > 0\}$$

is the domain of the functional

$$x \mapsto -12\ \ln\langle \alpha, x\rangle$$

which is an element of $\mathcal{F}'(\sqrt{12})$.

3. If $S : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator whose spectrum contains a single non-negative value, that value being of multiplicity one, then letting $v$ denote an eigenvector for the non-negative value, the cone

$$\{x;\ \langle x, Sx\rangle > 0 \ \text{ and } \ \langle x, v\rangle > 0\}$$

is the domain for the functional

$$x \mapsto -12\ \ln\langle x, Sx\rangle$$

which is an element of $\mathcal{F}(\sqrt{24})$. (Such cones are said to be "elliptical" because when a cross-section is taken orthogonally to $v$, one obtains an ellipsoid.)

4. If $\mathcal{H}$ is the vector space of $n \times n$ symmetric matrices then the cone consisting of the strictly positive definite matrices is the domain of the functional

$$x \to -12\ \ln(\det(x))$$

which is an element of $\mathcal{F}(\sqrt{12\,n})$.[18]

---

[16] The rather strange constant "12" appearing in the examples could have been avoided by incorporating that constant into the defintions of $\mathcal{F}$, $\mathcal{F}(K)$, $\mathcal{F}'$ and $\mathcal{F}'(K)$; we opted for definitions without puzzling constants. Since all of our results are stated in terms of big-O notation, where one puts the constant is irrelevant.

[17] Each of the examples is essentially a special case of the fact that there exists a universal constant $\kappa$ for which the following is true: Assume $\mathcal{H}$ is a real Hilbert space and $F : \mathcal{H} \to IR$ is twice continuously Frechet differentiable. Assume that $m > 0$ and $D$ is an open convex subset of $\mathcal{H}$ with the property that for each $x \in D$ and $y \in \mathcal{H}$, the univariate functional $p(t) := F_{x+ty}$ satisfies the following:

1. $p$ is identically constant or is a polynomial of degree at most $m$ which has only real roots (i.e., no imaginary roots);
2. If $t$ satisfies $p(t) \leq 0$ then $x + ty \notin D$;
3. Each endpoint of the interval $\{t;\ x + ty \in D\}$ is a root of $p$.

Then the functional $f := -\kappa \ln F_x$, with domain $D$, is an element of $\mathcal{F}'(\kappa\sqrt{m})$.

[18] To learn more about this cone see, for example, Alizadeh[1].

Taking finite intersections of these cones and relying on the previously described "calculus," one obtains many cones with associated functionals whose gradients and Hessians are readily computable. For example, for the non-negative orthant in $\mathbb{R}^n$ one has the functional

$$x \mapsto -12 \sum_j \ln x_j$$

which is an element of $\mathcal{F}'(\sqrt{12\,n})$ (in fact, is an element of $\mathcal{F}(\sqrt{12\,n})$).

## 1.17

Recall that in applying a barrier method one needs a point at which to initiate the method. To this end, assume a point $\breve{x}$ in the relative interior of $C_X$ is known; having assumed that $C_X$ is the closure of a set obtained by intersecting a closed subspace and an open convex set, by "relative interior" we mean the interior with respect to the subspace topology. Assume the point $\breve{x}$ satisfies $\|\breve{x}\| < 1$. Letting $C_X(1)$ denote the intersection of $C_X$ with the unit ball, the value $\mathrm{sym}(\breve{x}, C_X(1))$, quantifying the symmetry of $C_X(1)$ about $x$, appears in our theorems.[19]

Similarly, assume that a point $\breve{b}$ in the relative interior of $C_Y$ is available. If $C_Y$ is a subspace then assume $\breve{b} = \vec{0}$. If $C_Y$ is not a subspace then assume $\breve{b} \neq \vec{0}$; in this case, the value $\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|))$ is important, $C_Y(2\|\breve{b}\|)$ denoting the intersection of $C_Y$ with the ball of radius $2\|\breve{b}\|$. Note that $\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|))$ is invariant under positive scaling of $\breve{b}$.[20]

## 1.18

We can now state one of our main theorems. This theorem focuses on determining that the (primal) constraints are consistent and on computing a feasible point.

[19] Those familar with Nesterov and Nemirovskii[22] will know that there exists $\breve{x} \in C_X(1)$ such that

$$\mathrm{sym}(\breve{x}, C_X(1)) \geq \Omega(1/K_X^2).$$

If $\breve{x}$ is indeed such a point then the quantity $\mathrm{sym}(\breve{x}, C_X(1))$ appearing in our theorems can be removed.

[20] In contrast to the footnote preceeding this, there need *not* exist $\breve{b}$ for which

$$\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|)) \geq \Omega(1/K_Y^2),$$

because the norm on $Y$ is so unrestricted.

**Theorem 1.1** *There is a barrier method which upon inputs $d_P$, $\breve{x}$ and $\breve{b}$ terminates only if $d_P$ is consistent, producing a feasible point if it terminates; moreover, if* $\text{reldist}(d_P, \text{Pri}\emptyset) > 0$ *then the method terminates within*

$$O\Bigg(K \log \Bigg[K + \frac{1}{\text{reldist}(d_P, \text{Pri}\emptyset)} + \frac{1}{\text{sym}(\breve{x}, C_X(1))} + \frac{1}{\text{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|\breve{b}\|, \|d_P\|\}}{\min\{\|\breve{b}\|, \|d_P\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving $\breve{b}$ being deleted if $C_Y$ is a subspace.*

**Proof.** See Theorem 3.1. □

Recalling that $\text{sym}(\breve{b}, C_Y(2\|\breve{b}\|))$ is invariant under positive scaling of $\|\breve{b}\|$, observe that by scaling $\breve{b}$ so that $\|\breve{b}\| \approx \|d_P\|$ the final ratio in the iteration bound of the theorem can be removed. However, scaling $\|\breve{b}\|$ in this way requires one to know, at least approximately, the value $\|d_P\|$, an assumption we do not make. We emphasize that none of the parameters appearing in the iteration bound are assumed to be known apriori other than $K$, that is, the algorithm does not rely explicitly on the values for any of the parameters other than $K$.[21]

We also emphasize that the constant hidden by the big-O notation in Theorem 1.1 is universal, being independent $X$, $Y$, $C_X$, $C_Y$, etc.[22] We believe this indicates a certain naturalness in analyzing algorithms in terms of the parameters appearing in the theorem, especially the parameter $\text{reldist}(d_P, \text{Pri}\emptyset)$. Moreover, all of the algorithms discussed in this work are universal in the sense that they can actually be regarded as accepting not only, say, $d_P$, $\breve{x}$ and $\breve{b}$ as input, but also descriptions of $X$, $Y$, $C_X$ and $C_Y$, the latter being primarily in the form of subroutines for evaluating the gradients and Hessians of the functionals $\phi$ and $\psi$.

We chose not to restrict $X$ and $Y$ to be finite dimensional so as to highlight the role of functional analysis; Theorem 1.1 lives in a realm where traditional complexity theory makes no sense. We do not pretend that our theorems are relevant to the practical solution of infinite-dimensional problems, for in

[21] Moreover, in the spirit of the fifth footnote, at the expense of a slightly worse iteration bound, one could remove algorithm's need for the value $K$.

[22] One could elaborate on our proofs to determine specific constants but doing so would likely yield disappointingly large values, at least from a practical viewpoint. This does not concern us for we believe that simply because of the difficulties involved in analyzing algorithms, theorems in computational complexity theory cannot help but fall short of providing complete insight; the theorems must be taken with a grain of salt. Moreover, we prefer the succinct proofs allowed by the use of big-O notation to the computational gynmastics required when specific values are determined unless those values are *especially* nice. We wish that more members of the LP community agreed with us, at least those whose papers we are asked to referee.

infinite-dimensional spaces, each iteration of a barrier method requires solving an infinite-dimensional system of linear equations.

## 1.19

In proving Theorem 1.1, we assume that the system of linear equations arising at each iteration of the barrier method is solved exactly, i.e., infinite precision arithmetic is used in computing the Newton step. This is most obviously contrary to the extensive motivation we gave for the appropriateness of considering $\log(1/\text{reldist}(d_P, \text{Pri}\emptyset))$ as the size of $d_P$.[23] One would hope that if only limited computational precision is used then sufficiently accurate approximations to the Newton steps are obtained; ideally, one would hope that if the computations are performed using only slightly more than $\log(1/\text{reldist}(d_P, \text{Pri}\emptyset))$ significant digits of accuracy then the barrier method will be successful. Indeed, at least in some situations this ideal can be realized as will be shown in future papers which take this paper as their starting point. Due to time (and patience) limitations we chose to be content in this flagship paper with the assumption of infinite precision arithmetic; the assumption certainly makes for more transparent proofs.

Other researchers have made use of quantities analogous to $\text{reldist}(d_P, \text{Pri}\emptyset)$ in their study of algorithms for solving systems of linear and polynomial *equations*; c.f., Smale[33],[34],[35],[36]), Demmel([5],[6]), Shub and Smale([31],[32]), Renegar[24]. We are certainly motivated, in part, by the work of those researchers. Related work pertaining to elementary LP includes Renegar[25], Vera([41],[42],[43]), Filipowski([11],[12]), Freund[13] and Vavasis and Ye([39],[40]).

We consider the problem of determining that the (primal) constraints are inconsistent, rather than consistent, in Section 5. We present three analogues to Theorem 1.1, the main difference being that $\text{reldist}(d_P, \text{Pri}\emptyset)$ is replaced by $\text{reldist}(d_P, \text{Pri})$ where Pri denotes the set of consistent instances. The analogues rely on additional assumptions, for example, pointedness of the cone $C_X$.

## 1.20

In Sections 4 and 6 we present analogues to Theorem 1.1 for the problem of determining that the dual constraints are asymptotically consistent and for the problem of determining that the dual constraints are not asymptotically consistent. We defer defining the dual constraints until Section 2; for now, simply think of the dual constraints as they occur in elementary LP.

[23] Theorem 1.1 is analogous to theorems one often finds in the LP literature where a bound on the number of arithmetic operations is given in terms of the bit-length of the input, i.e., theorems that "mix" complexity theory frameworks.

In this context we truncate the data vectors $d := (A, b, c^*)$ for LP instances, obtaining instances $d_D := (A, c^*)$ in the data space

$$\mathbf{D}_D := \mathcal{L}(X, Y) \times X^*.$$

We rely on the norm

$$||d_D|| := \max\{||A||, ||c^*||\}.$$

Each instance $d_D$ can be thought of as a system of dual constraints. An instance $d_D := (A, c^*)$ is said to be *asymptotically consistent* if it is consistent or can be made consistent by an arbitrarily slight perturbation of $c^*$. As is well known, the relevance of asymptotic consistency is that under very mild assumptions on the vector spaces $X$, $Y$ and cones $C_X$, $C_Y$, a linear programming instance $d := (A, b, c^*)$ has finite optimal objective value if and only if $d_P := (A, b)$ is consistent and $d_D := (A, c^*)$ is asymptotically consistent. (In the context of elementary LP, $d_D$ is asymptotically consistent if and only if it is consistent.)

Instances $d_D$ which are not asymptotically consistent are said to be *strongly inconsistent*; let DualS$\emptyset$ denote the subset of $\mathbf{D}_D$ consisting of these instances.

For the problem of determining that the dual constraints are asymptotically consistent, the principal parameter appearing in our theorems is, not surprisingly,[24]

$$\text{reldist}(d_D, \text{DualS}\emptyset) := \inf\{||\Delta d_D||/||d_D||; d_D + \Delta d_D \in \text{DualS}\emptyset\}.$$

No doubt the reader can guess the principal parameter in our theorems concerning the problem of determining that the dual constraints are strongly inconsistent.

## 1.21

The final section, Section 7, is devoted to the problem of "solving" a linear programming instance in the sense of computing a feasible point at which the objective value nearly equals the optimal objective value. Here, we consider instances $d := (A, b, c^*)$ to be elements of the data space

$$\mathbf{D} := \mathcal{L}(X, Y) \times Y \times X^*$$

and rely on the norm

$$||d|| := \max\{||A||, ||b||, ||c^*||\}.$$

(Again it is useful to think of the data for the problem to be solved as being normalized, i.e., $||A|| \approx 1$, $||b|| \approx 1$, $||c^*|| \approx 1$ and hence $||d|| \approx 1$.) The most important parameter appearing in the analysis is

$$\text{reldist}(d, \text{DualS}\emptyset) := \inf\{||\Delta d_D||/||d||; d_D + \Delta d_D \in \text{DualS}\emptyset\}$$

[24] If DualS$\emptyset = \emptyset$ then we define $\text{reldist}(d_D, \text{DualS}\emptyset) = 1$ to avoid discussing special cases.

where $d := (A, b, c^*)$ and $d_D := (A, c^*)$.

Let $\mathrm{val}(d)$ denote the optimal objective value of $d := (A, b, c^*)$, i.e., the supremum of $c^*x$ over all $x$ which are feasible for $d_P := (A, b)$.

**Theorem 1.2** *There is a barrier method such that for all $\epsilon > 0$ the following is true:*

*Assume $d := (A, b, c^*)$ satisfies $\mathrm{reldist}(d, \mathrm{DualS}\emptyset) > 0$. Assume that the algorithm of Theorem 1.1 terminates when applied to $d_P := (A, b)$, thus producing a feasible point $\bar{x}$.*

*Upon input $d := (A, b, c^*)$, $\bar{x}$ and a user-selected value $\breve{s} > 0$, the barrier method computes an infinite sequence of points $x$, each of those of which is computed after at most*

$$O\left(K \log\left[K + \frac{1}{\epsilon} + \frac{1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)} + \frac{\max\{\breve{s},\ \|d\|(\|\bar{x}\| + 1)\}}{\min\{\breve{s},\ \|d\|(\|\bar{x}\| + 1)\}}\right]\right)$$

*operations being known to satisfy*

$$\frac{\mathrm{val}(d) - c^*x}{\max\{\|d\|, -val(d)\}} \leq \epsilon. \tag{4}$$

**Proof.** See Corollary 7.6. □

Observe that if one chooses the positive parameter $\breve{s}$ to satisfy $\breve{s} \approx \|d\|(\|\bar{x}\| + 1)$, the final ratio in the iteration bound of the theorem can be deleted. Of course choosing $\breve{s}$ in this way requires knowing, at least approximately, the value $\|d\|$, as assumption we do not make. As in Theorem 1.1, none of the parameters appearing in the bound are assumed to be known other than $K$, that is, the algorithm does not rely explicitly on the values of any of the parameters other than $K$.

The ratio (4) is a mixture of a sort of absolute error, when $\|d\| \geq -\mathrm{val}(d)$, and relative error. This mixture allows the iteration bound to be independent of $\mathrm{reldist}(d_P, \mathrm{Pri}\emptyset)$. If one prefers the denominator $\max\{\|d\|, -val(d)\}$ in (4) to be replaced by $\|d\|$ alone then the statement of the theorem remains valid if one adds

$$\frac{1}{\mathrm{reldist}(d_P, \mathrm{Pri}\emptyset)}$$

to the argument of the logarithm. This is proven as Corollary 7.3.

In LP lingo, one can think of the algorithm of Theorem 1.1 as "Phase I" and the algorithm of Theorem 1.2 as "Phase II." In the literature, one often finds these phases combined. We separated Phase I from Phase II to highlight the fact that $\log(\mathrm{reldist}(d_P, \mathrm{Pri}\emptyset))$ is the crucial parameter for Phase I whereas $\log(\mathrm{reldist}(d, \mathrm{DualS}\emptyset))$ is the crucial parameter for Phase II.

The proof of the iteration bound in Theorem 1.2 depends heavily on nice properties of the feasible point $\bar{x}$ produced by the algorithm of Theorem 1.1.

## 1.22

Lastly, we would like to highlight a theorem that characterizes the quantity $\text{reldist}(d_P, \text{Pri}\emptyset)$ for $d_P = (A, b) \in \mathbf{D}_P$. Whereas the quantity $\text{reldist}(d_P, \text{Pri}\emptyset)$ is defined in terms of perturbations for both the operator $A$ and right-hand side vector $b$, the characterization is in terms of perturbations for a right-hand side vector alone.

For $d_P := (A, b)$, consider the constraints in $(x, s) \in X \times \mathbb{R}$ of the form

$$\begin{aligned} Ax - sb &\leq \bar{b} \\ x &\geq \vec{0} \\ s &\geq 0 \\ \|x\| + |s| &\leq 1, \end{aligned} \tag{5}$$

where $\bar{b} \in Y$. Define

$$\rho(d_P) := \inf\{\rho;\ \exists\, \bar{b} \text{ such that } \|\bar{b}\| \leq \rho \text{ and (5) is inconsistent}\}.$$

**Theorem 1.3** *If* $d_P := (A, b)$ *then*

$$\text{reldist}(d_P, \text{Pri}\emptyset) = \frac{\rho(d_P)}{\|d_P\|}.$$

**Proof.** This is a special case of Theorem 3.5. □

Theorem 1.3 generalizes the well-known fact that for an invertible linear operator $A : X \to X$,

$$\frac{1}{\|A^{-1}\|} = \inf\{\rho;\ \exists\, \bar{b} \in X \text{ such that } \|\bar{b}\| \leq \rho \text{ and } Ax = \bar{b},\ \|x\| \leq 1 \text{ is inconsistent}\}.$$

## 1.23

The first part of each of the remaining (sub)sections in this paper is devoted to explaining what is proven in that (sub)section. We have chosen a few of the most interesting theorems for the introduction. The dependence of sections is indicated by the following diagram:

$$\begin{array}{ccccccccc} & & & & & & 4 & \rightarrow & 6 \\ & & & & & \nearrow & & \nearrow & \\ 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 5 & & \\ & & & & & \searrow & & & \\ & & & & & & 7 & & \end{array}$$

On a first reading we suggest the sequence $1 \rightarrow 2 \rightarrow 3 \rightarrow 7$.

When reading theorems keep in mind that, except in Section 2, we implicitly assume that all of the assumptions presented in this introduction hold unless explicitly stated otherwise.

# 2 Preliminaries

In this section we state a typical result concerning the barrier method; the result is an extension of the one discussed in the introduction. The result encapsulates all the reader needs to know about the barrier method in order to verify the correctness of our analysis. After stating the result, we discuss the dual of an LP instance. Finally, we state useful bounds on the magnitude of the optimal value for an LP instance.

Unlike all of the other sections in this paper where the theorems implicitly assume that the assumptions of the introduction hold, in this section the only assumptions made are those explicitly stated.

We rely extensively on order notation (i.e., $O$, $\Omega$ and $\Theta$) throughout this work, supressing constants in hopes of achieving more transparent proofs. Thus, for example, a statement of the form

"If $\alpha = O(\beta)$ then $\gamma = \Theta(\delta)$,"

means that there exist universal positive constants $\kappa_1$, $\kappa_2$ and $\kappa_3$ such that:

If $\alpha$ and $\beta$ satisfy $\alpha \leq \kappa_1\beta$ then $\gamma$ and $\delta$ satisfy $\kappa_2\delta \leq \gamma \leq \kappa_3\delta$.

Similarly,

"If $\alpha = \Theta(\beta)$ then $\gamma = \Omega(\delta)$,"

means that there exist universal positive constants $\kappa_1, \kappa_2$ and $\kappa_3$, where $\kappa_1 < \kappa_2$, such that:

If $\alpha$ and $\beta$ satisfy $\kappa_1\beta \leq \alpha \leq \kappa_2\beta$ then $\gamma$ and $\delta$ satisfy $\kappa_3\delta \leq \gamma$.

And so on.

## 2.1 A Typical IPM Theorem

In this subsection we simply state a theorem concerning the barrier method; for details, see [27].

Recall the set $\mathcal{F}(K)$ consisting of those functionals $f$ satisfying the properties listed at the beginning of Section 1. When we say that "$f \in \mathcal{F}(K)$ is among the inputs" to the barrier method, we mean that the method is provided with subroutines for evaluating the Hessian operator and gradient of $f$ at appropriate $x \in D_f$. When we say that "a closed subspace $L \subseteq \mathcal{H}_f$ is among the inputs," we mean, say, that it is specified as the solution set to a given system of linear equations.

We mention that part 6b of the theorem, perhaps the most technical statement in the theorem, plays a role in our analysis only with respect to the problem of determining that a system of constraints is inconsistent.

**Theorem 2.1.1** *There is an barrier method for which the following is true:*

1. *The method has two modes, an "optimization" mode and an "equation-solving" mode.*

2. *In optimization mode, the method requires five entries as input, which we assume to be*

$$K, \quad f \in \mathcal{F}(K), \quad w \in D_f, \quad c^* \in \mathcal{H}_f^* \ (= \mathcal{H}_f) \quad \text{and} \quad L \subseteq \mathcal{H}_f,$$

   *$L$ being a closed subspace. The goal of optimization mode is to solve*

$$\begin{array}{ll} \inf & c^*x \\ s.t. & x \in D_f \cap (L + w), \end{array}$$

   *$L + w$ denoting the translate of $L$ by $w$. Let $\nu_{\text{inf}}$ denote the optimal value of this optimization problem and let $\nu_{\text{sup}}$ denote the optimal value of the problem obtained by replacing "inf" with "sup".*

3. *In equation-solving mode the method requires six entries as input, the first five of which we assume to be the same as in optimization mode and the last of which we assume to be $\nu \in \mathbb{R}$. The goal of equation solving mode is to compute a point $x$ satisfying*

$$\begin{array}{c} c^*x = \nu \\ x \in D_f \cap (L + w). \end{array}$$

4. *Both modes involve two computational stages, the second stage being initiated if and only if the first stage terminates.*[25]

5. *The first stage terminates in either mode if and only if $D_f$ is bounded. If the first stage terminates it does so within*

$$O\left(K \log\left[K + \frac{1}{\text{sym}(w, D_f \cap (L + w))}\right]\right)$$

   *iterations and it provides a point $x_1 \in D_f \cap (L+w)$ which serves to initiate the second stage.*

6. *If the second stage is initiated in either mode then that stage computes a sequence of iterates $\{x_i\} \subset D_f \cap (L + w)$.*

   (a) *Each iterate $x_i$ satisfies*

$$\text{sym}(x_i, S_i) = \Omega\left(\frac{1}{K^2}\right)$$

   *where*

$$S_i := \{x \in D_f \cap (L + w);\ c^*x = c^*x_i\}.$$

[25] The two stages are like those described in the sixth footnote; a minimizer of $f$ (restricted to $D_f \cap (L + w)$) is approximated in the first stage, if a minimizer exists.

(b) *If $H_{x_i}$ denotes the Hessian of $f$ at $x_i$ and $P$ denotes the operator which projects $\mathcal{H}_f$ orthogonally onto L then the smallest non-zero value $\lambda$ in the spectrum of $PH_{x_i}P$ satisfies*

$$\Omega\left(\frac{1}{\operatorname{diam}(S_i)}\right) = \sqrt{\lambda} = O\left(\frac{K^2}{\operatorname{diam}(S_i)}\right)$$

*where*

$$\operatorname{diam}(S_i) := \sup\{\|x' - x''\|;\ x', x'' \in S_i\}$$

*and $\|\ \|$ is the norm on $\mathcal{H}_f$ induced by the inner product on $\mathcal{H}_f$.*

7. *If the second stage is initiated in optimization mode then the sequence of iterates is infinite. Moreover, if $\epsilon > 0$ and*

$$i = \Omega\left(K \log_2 \left[2 + K + \frac{\nu_{\text{sup}} - \nu_{\text{inf}}}{\epsilon}\right]\right)$$

*then $x_i$ is known to satisfy*

$$c^* x_i - \nu_{inf} \leq \epsilon.$$

8. *If the second stage is initiated in equation-solving mode then the sequence of iterates can be finite or infinite, depending on the input entry $\nu$.*

   (a) *If $\nu_{\text{inf}} < \nu < \nu_{\text{sup}}$ then the sequence consists of*

   $$O\left(K \log_2 \left[2 + K + \frac{\nu_{\text{sup}} - \nu_{\text{inf}}}{\min\{\nu_{\text{sup}} - \nu, \nu - \nu_{\text{inf}}\}}\right]\right)$$

   *iterates, the last of which satisfies $c^* x = \nu$.*

   (b) *If $\nu \leq \nu_{\text{inf}}$ then the sequence is infinite and satisfies the same bounds asserted in item 7. (In fact, the sequence is then identical with that computed in optimization mode.)*

   (c) *If $\nu \geq \nu_{\text{sup}}$ then the sequence is infinite and satisfies the bounds asserted in item 7 if one replaces the difference "$c^* x_i - \nu_{\text{inf}}$" with "$\nu_{\text{sup}} - c^* x_i$." (The sequence is identical with the sequence computed in optimization mode if the objective functional $c^* x$ is replaced with $-c^* x$.)*

## 2.2 The Dual of a Linear Programming Instance

### 2.2.1

Recall that $d := (A, b, c^*)$ represents the optimization problem

$$\begin{array}{ll} \sup & c^*x \\ \text{s.t.} & Ax \leq b \\ & x \geq \vec{0} \end{array} \tag{6}$$

where " $Ax \leq b$ " means $b - Ax \in C_Y$ and where " $x \geq \vec{0}$ " means $x \in C_X$. Now we only assume $X$, $Y$ to be normed spaces, and $C_X$, $C_Y$ to be convex cones each with vertex at the origin and each including the origin.

The instance $d$ is associated with another optimization problem, its so-called "dual." In this problem the domain space is the dual space $Y^*$ rather than $X$, and the range space is the dual space $X^*$ rather than $Y$.

Recall that the norm on $X$, $Y$ induces an operator norm on $X^*$, $Y^*$, respectively. When we consider $X^*$, $Y^*$ as normed spaces, it is with respect to these operator norms.

Associated to the cone $C_X$ is a *dual cone* $C_X^* \subseteq X^*$ defined by

$$C_X^* := \{c^* \in X^*;\ c^*x \geq 0 \text{ for all } x \in C_x\}.$$

Similarly, associated to $C_Y$ is a dual cone $C_Y^* \subseteq Y^*$.

The cones $C_X^*$ and $C_Y^*$ induce orderings on $X^*$ and $Y^*$ just as $C_X$ and $C_Y$ induce orderings on $X$ and $Y$. With this in mind, we define the *dual* of $d := (A, b, c^*)$ as the optimization problem

$$\begin{array}{ll} \inf & y^*b \\ \text{s.t.} & y^*A \geq c^* \\ & y^* \geq \vec{0}, \end{array}$$

" $y^*A \geq c^*$ " meaning the functional $x \mapsto y^*Ax - c^*x$ is an element of $C_X^*$, and " $y^* \geq \vec{0}$ " meaning $y^* \in C_Y^*$.

### 2.2.2

The constraints for the dual problem corresponding to $d := (A, b, c^*)$ are unaffected by $b$. Viewing $X$, $Y$, $C_X$ and $C_Y$ as fixed, we define DualA to be the subset of

$$\mathbf{D}_D := \mathcal{L}(X, Y) \times Y^*$$

consisting of those truncated instances $d_D := (A, c^*)$ with the property that the constraints

$$\begin{array}{l} y^*A \geq c^* \\ y^* \geq \vec{0} \end{array}$$

are either consistent or can be made consistent by an arbitrarily slight perturbation of $c^*$. The elements of DualA are said to be *asymptotically consistent*; the notation "DualA" is meant to suggest "dual constraints are asymptotically consistent."

The set DualS$\emptyset$ is defined to be the complementary subset of DualA in $\mathbf{D}_D$. It is the set of *strongly inconsistent* instances. The notation "DualS$\emptyset$" is meant to suggest "dual feasible region is strongly empty."

The set DualA is relevant to our development through a well-known proposition which has additional hypotheses on $X$, $C_X$ and $C_Y$, but not on $Y$.

The most restrictive additional hypothesis is that $X$ be a "reflexive" space. A reflexive space $X$ is one that can be identified with the second dual space $X^{**}$ (i.e., the dual space of $X^*$), in the following sense.

A normed vector space $X$ can always be considered as a subspace of $X^{**}$. For if $x \in X$ then $x$ induces a continuous linear functional on $X^*$ defined by $c^* \mapsto c^*x$. If under this identification of $X$ with a subspace of $X^{**}$ it happens that $X = X^{**}$ then $X$ is said to be *reflexive.*

Many important spaces are reflexive. For example, if $X$ is a normed space whose norm is compatible with an inner product making $X$ into a Hilbert space then $X$ is reflexive. So, for example, all finite dimensional spaces are reflexive regardless of the norm.

Results concerning the dual problem and extending far beyond the following proposition are well-known.

**Proposition 2.2.1 (Duffin)** *Assume $X$ and $Y$ are normed spaces, $X$ being reflexive. Assume $C_X$ and $C_Y$ are closed in the norm topology.*

*The linear programming instance $d := (A, b, c^*)$ has finite optimal objective value if and only if $d_P \in$ Pri and $d_D \in$ DualA.*

**Proof.** See, for example, [26, Proposition 2.5]. □

## 2.3 Useful Bounds

### 2.3.1

In the introduction we defined the relative distances reldist$(d_P, \text{Pri}\emptyset)$ and reldist$(d_D, \text{DualS}\emptyset)$. Sometimes it is useful to work with the absolute distances instead, that is,

$$\text{dist}(d_P, \text{Pri}\emptyset) := \inf\{||\Delta d_P||;\ \ d_P + \Delta d_P \in \text{Pri}\emptyset\}$$

and

$$\text{dist}(d_D, \text{DualS}\emptyset) := \inf\{||\Delta d_D||;\ \ d_D + \Delta d_D \in \text{DualS}\emptyset\}.$$

In the final section of this paper where we consider the problem of computing a feasible point with nearly optimal objective value for an LP instance $d := (A, b, c^*)$, we rely heavily on the following proposition.

**Proposition 2.3.1** *Assume $X$ and $Y$ are normed spaces, $X$ being reflexive. Assume $C_X$ and $C_Y$ are closed in the norm topology. Assume $d := (A, b, c^*) \in$* $\mathbf{D}$.

1. *If $d_P$ satisfies* $\text{dist}(d_P, \text{Pri}\emptyset) > 0$ *then there exists $x$ which is feasible for $d_P$ and which satisfies*

$$||x|| \leq \frac{||b||}{\text{dist}(d_P, \text{Pri}\emptyset)}.$$

2. *If $d_P$ satisfies* $\text{dist}(d_P, \text{Pri}\emptyset) > 0$ *and $d_D$ satisfies* $\text{dist}(d_D, \text{DualS}\emptyset) > 0$ *then*

$$\frac{-||b||\,||c^*||}{\text{dist}(d_P, \text{Pri}\emptyset)} \leq \text{val}(d) \leq \frac{||b||\,||c^*||}{\text{dist}(d_D, \text{DualS}\emptyset)}$$

*where* $\text{val}(d)$ *denotes the optimal value of $d$.*

3. *If $d_D$ satisfies* $\text{dist}(d_D, \text{DualS}\emptyset) > 0$ *and if $x$ is feasible for $d_P$ then*

$$||x|| \leq \frac{\max\{||b||, -c^*x\}}{\text{dist}(d_D, \text{DualS}\emptyset)}.$$

### 2.3.2

The assertions made in Theorem 2.3.1 would be a subset of the assertions made in [26, Theorem 1.1 and Lemma 3.2] was it not for the fact that in [26] the set DualS$\emptyset$ is replaced by Dual$\emptyset$, the set of inconsistent instances, i.e., instances for which the (dual) feasible region is empty. Thus, to prove Theorem 2.3.1 it suffices to prove that for all instances $d_D$,

$$\text{dist}(d_D, \text{DualS}\emptyset) = \text{dist}(d_D, \text{Dual}\emptyset),$$

that is, it suffices to prove that DualS$\emptyset$ is dense in Dual$\emptyset$ with respect to the norm topology of $\mathbf{D}_D$.

In addition, one of our central proofs in §3 requires the fact that PriS$\emptyset$ is dense in Pri$\emptyset$ where PriS$\emptyset := \mathbf{D}_P \setminus$ PriA and PriA denotes the set of *asymptotically consistent* instances (i.e., PriA consists of those instance $d_P := (A, b)$ which are consistent or can be made consistent by an arbitrarily small perturbation of $b$).

The author should have taken care of these matters in [26] as they are somewhat out of place here.

**Proposition 2.3.2** *Assume $X$ and $Y$ are normed spaces, $X$ being reflexive. Assume $C_X$ and $C_Y$ are closed in the norm topology.*

*The set* PriS$\emptyset$ *is dense in* Pri$\emptyset$ *with respect to the norm topology of* $\mathbf{D}_P$*. Similarly, the set* DualS$\emptyset$ *is dense in* Dual$\emptyset$ *with respect to the norm topology of* $\mathbf{D}_D$*.*

The remainder of the section is devoted to proving Proposition 2.3.2. The reader is encouraged to now skip to the next section as the proof provides no significant insight into the central problems addressed in this paper.

### 2.3.3

The proof of Proposition 2.3.2 relies on the following lemmas.

**Lemma 2.3.3 (Duffin)** *Assume $X$ and $Y$ are normed vector spaces. If $d_P := (A, b)$ then exactly one of the following two alternatives is true:*

1. $d_P \in$ PriA.
2. *The system*

$$\begin{aligned} y^*A &\geq \vec{0} \\ y^* &\geq \vec{0} \\ y^*b &= -1 \end{aligned}$$

*is consistent.*

**Proof.** See, for example, [26, Proposition 2.1]. □

**Lemma 2.3.4 (Duffin)** *Assume $X$ and $Y$ are normed spaces, $X$ being reflexive. Assume $C_X$ and $C_Y$ are closed in the norm topology. If $d_P := (A, b)$ then exactly one of the following two alternatives is true:*

1. $d_P \in$ Pri.
2. *The system*

$$\begin{aligned} y^*A &\geq \vec{0} \\ y^*b &= -1 \\ y^* &\geq \vec{0} \end{aligned}$$

*is asymptotically consistent, meaning that it can be made consistent by an arbitrarily slight perturbation of $\vec{0}$ in the constraints $y^*A \geq \vec{0}$.*

**Proof.** Follows from, for example, [26, Corollary 2.3], replacing $X$ there with $X \times I\!R$, $C_X$ with $C_X \times I\!R$ and $A$ with the operator $(x, t) \mapsto Ax - tb$. □

**Lemma 2.3.5 (Duffin)** *Assume $X$ and $Y$ are normed spaces, $X$ being reflexive. Assume $C_X$ and $C_Y$ are closed in the norm topology. If $d_D := (A, c^*)$ then exactly one of the following two alternatives is true:*

1. $d_D \in$ DualA.
2. *The system*

$$\begin{aligned} Ax &\leq \vec{0} \\ c^*x &= 1 \\ x &\geq \vec{0} \end{aligned}$$

*is consistent.*

**Proof.** See, for example, [26, Corollary 2.3]. □

**Lemma 2.3.6 (Duffin)** *Assume $X$ and $Y$ are normed spaces. If $d_D := (A, c^*)$ then exactly one of the following two alternatives is true:*

1. *The instance $d_D$ is consistent.*
2. *The system*

$$\begin{aligned} Ax &\leq \vec{0} \\ c^*x &= 1 \\ x &\geq \vec{0} \end{aligned}$$

*is asymptotically consistent, meaning that it can be made consistent by an arbitrarily slight perturbation of $\vec{0}$ in the constraints $Ax \leq \vec{0}$.*

**Proof.** Follows from, for example, [26, Proposition 2.1], replacing $Y$ there with $Y \times I\!R$, $C_Y$ with $C_Y \times I\!R$ and $A$ with the operator $x \mapsto (Ax, c^*x)$. □

**Proof of Proposition 2.3.2.** We first prove that DualS$\emptyset$ is dense in Dual$\emptyset$. Assume $d_D := (A, c^*)$ is an instance with the property that all instances in an open neighborhood of $d_D$ are elements of DualA. It suffices to show that $d_D$ is consistent.

To show $d_D$ is consistent it suffices, by Lemma 2.3.6 to show that the system

$$\begin{aligned} Ax &\leq \vec{0} \\ c^*x &= 1 \\ x &\geq \vec{0} \end{aligned} \tag{7}$$

is not asymptotically consistent.

Assume that the system (7) is asymptotically consistent, that is, assume for each $\delta > 0$ there exists $\Delta b$ such that $||\Delta b|| \leq \delta$ and the system

$$\begin{aligned} Ax &\leq \Delta b \\ c^*x &= 1 \\ x &\geq \vec{0} \end{aligned} \tag{8}$$

is consistent. Consider the instance $\bar{d}_D := (\bar{A}, c^*)$ where

$$\bar{A}x := Ax - (c^*x)\Delta b.$$

Noting that

$$||\bar{d} - d|| = ||\bar{A} - A|| = ||c^*|| \, ||\Delta b|| \leq \delta ||c^*||,$$

by choosing $\delta$ sufficiently small we may assume $\bar{d}_D \in$ DualA. However, since (8) is consistent so is the system

$$\begin{aligned} \bar{A}x &\leq \vec{0} \\ c^*x &= 1 \\ x &\geq \vec{0}. \end{aligned}$$

Hence, by Lemma 2.3.5, $\bar{d}_D \notin$ DualA, a contradiction. Consequently, DualS$\emptyset$ is indeed dense in Dual$\emptyset$.

The proof that PriS$\emptyset$ is dense in Pri$\emptyset$ exactly parallels the preceeding proof, relying on Lemmas 2.3.3 and 2.3.4 in place of Lemmas 2.3.5 and 2.3.6, respectively; we leave the proof to the reader. □

# 3 Primal Consistency

## 3.1

In this section we apply the barrier method to obtain an algorithm for determining that an instance $d_P := (A, b)$ is consistent. If the algorithm terminates upon input $d_P$ then it produces a *strictly* feasible point, that is, a point satisfying

$$\begin{gathered} Ax < b \\ x > \vec{0} \end{gathered}$$

where "$Ax < b$" means that $b - Ax$ is in the relative interior $C_Y^\circ$ of $C_Y$, and where $x > \vec{0}$ means that $x$ is in the relative interior $C_X^\circ$ of $C_X$.

The algorithm is obtained by applying the barrier method behind Theorem 2.1.1 to solving a system of constraints of the form

$$\begin{gathered} (x, s, t) \in D_f \cap L \\ t = 0 \end{gathered} \tag{9}$$

for an appropriate functional $f$ and closed subspace $L$. Each of $f$, $D_f$ and $L$ depends on the instance $d_P = (A, b)$ and the known points $\breve{x} \in C_X^\circ$, $\breve{b} \in C_Y^\circ$.

Recall that $C_X$ is the closure of $D_\phi \cap L_X$ where $\phi \in \mathcal{F}'(K_X)$; hence $C_X^\circ = D_\phi \cap L_X$. Similarly, $C_Y$ is the closure of $D_\psi \cap L_Y$ where $\psi \in \mathcal{F}'(K_Y)$; hence, $C_Y^\circ = D_\psi \cap L_Y$.

Let $\mathcal{H}_f$ denote the Hilbert space

$$\mathcal{H}_f := X \times I\!R \times I\!R.$$

The domain $D_f$ of $f$ is defined to be the set of all points $(x, s, t) \in \mathcal{H}_f$ satisfying the following constraints:

$$\begin{gathered} -Ax + sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b) \in D_\psi \\ x \in D_\phi \\ \|x\| < 1 \\ 0 < s < 1 \\ -1 < t < 2. \end{gathered}$$

Clearly, $D_f$ is bounded.

The functional $f$ is defined by

$$\begin{aligned} f_{x,s,t} \quad \mapsto \quad & \psi_{-Ax+sb+t(A\breve{x}+\breve{b}-\frac{1}{2}b)} \\ & + \phi_x \\ & - 12 \ln(1 - \langle x, x \rangle) \\ & - 12 \ln(s) - 12 \ln(1 - s) \\ & - 12 \ln(t + 1) - 12 \ln(2 - t). \end{aligned}$$

Using the "calculus" described in the introduction, it is not difficult to prove that $f \in \mathcal{F}(K)$ where

$$K := \sqrt{K_X^2 + K_Y^2 + 72}.$$

The closed linear subspace $L$ is defined as

$$L := \{(x, s, t) \in \mathcal{H};\ x \in L_X \text{ and } -Ax + sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b) \in L_Y\}.$$

Observe that $D_f \cap L$ consists precisely of those points $(x, s, t)$ satisfying the following constraints:

$$\begin{gathered} Ax < sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b) \\ x > \vec{0} \\ \|x\| < 1 \\ 0 < s < 1 \\ -1 < t < 2. \end{gathered}$$

Note that $w := (\breve{x}, \frac{1}{2}, 1) \in D_f \cap L$. In attempting to solve the constraints (9), we initiate the barrier method of Theorem 2.1.1 at $w$.

Observe that if $(x, s, t) \in D_f \cap L$ and $t = 0$ then $\frac{1}{s}x$ is strictly feasible for $d_P := (A, b)$. Hence, if the barrier method terminates when it is applied to solving (9) then $d_P$ is consistent and a strictly feasible point is obtained.

If $d_P \in \mathbf{D}_P$ then define

$$\text{Feas}(d_P) := \{x \in X;\ Ax \le b \text{ and } x \ge \vec{0}\},$$

the feasible region for $d_P$, and

$$\text{SFeas}(d_P) := \{x \in X;\ Ax < b \text{ and } x > \vec{0}\},$$

the strictly feasible region for $d_P$.

**Theorem 3.1** *The barrier method applied to solving (9) terminates only if $d_P := (A, b)$ is consistent. If $d_P$ satisfies $\text{dist}(d_P, \text{Pri}\emptyset) > 0$ then the method terminates within*

$$O\Bigg(K \log \Bigg[K + \frac{1}{\text{reldist}(d_P, \text{Pri}\emptyset)} + \frac{1}{\text{sym}(\breve{x}, C_X(1))} + \frac{1}{\text{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|d_P\|, \|\breve{b}\|\}}{\min\{\|d_P\|, \|\breve{b}\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving $\check{b}$ being deleted if $C_Y$ is a subspace. Moreover, if the method terminates then its final iterate $(\widehat{x}, \widehat{s}, 0)$ is such that $\bar{x} := \frac{1}{\widehat{s}}\widehat{x}$ is strictly feasible for $d_P$ and possesses the following properties:*

1. *Either*

$$\|\bar{x}\| = O(K^2)$$

*or*

$$\sup\left\{\frac{\|\bar{x}\|}{\|x\|};\ x \in \mathrm{Feas}(d)\right\} = O(K^2).$$

2.

$$\|\bar{x}\| = O\left(\frac{K^2}{\mathrm{reldist}(d_P, \mathrm{Pri}\emptyset)}\right).$$

3. *There exists*

$$\gamma = \Omega\left(\frac{1}{K^2}\right)$$

*such that if*

$$S_\gamma := \{x \in \mathrm{SFeas}(d_P);\ \|x\| < \|\bar{x}\| + \gamma(\|\bar{x}\| + 1)\}$$

*then*

$$\mathrm{sym}(\bar{x}, S_\gamma) = \Omega\left(\frac{1}{K^2}\right).$$

The properties of $\bar{x}$ stated in Theorem 3.1 are relied on in Section 7 where we consider the problem of computing a feasible point with nearly optimal objective value for an LP instance $d := (A, b, c^*)$; the point $\bar{x}$ will be considered as input to the algorithm we develop there; the properties of $\bar{x}$ are important in analyzing the behavior of that algorithm.

The remainder of this section is devoted to proving Theorem 3.1. We encourage readers uninterested in the proofs to at least look at the statement of Theorem 3.5; that theorem provides an interesting characterization of the quantity $\mathrm{dist}(d_P, \mathrm{Pri}\emptyset)$.

## 3.2

Define $t_{\mathrm{inf}}$ to be the optimal value of the problem

$$\begin{array}{ll} \inf_{x,s,t} & t \\ \text{s.t.} & (x,s,t) \in D_f \cap L, \end{array}$$

that is, the optimal value of the problem

$$\begin{array}{ll} \inf_{x,s,t} & t \\ \text{s.t.} & Ax < sb + t(A\breve{x} + \breve{b} - \frac{1}{2}b) \\ & x > \vec{0} \\ & \|x\| < 1 \\ & 0 < s < 1 \\ & -1 < t < 2. \end{array}$$

Define $t_{\text{sup}}$ to be the optimal value of problem obtained by replacing "inf" with "sup."

Since $r(x,s,t) \in D_f \cap L$ whenever $(x,s,t) \in D_f \cap L$ and $0 < r \leq 1$, we clearly have $t_{\text{inf}} \leq 0$. Also note that $1 \leq t_{\text{sup}} \leq 2$, the upper bound being trivial and the lower bound being a consequence of $w := (\breve{x}, \frac{1}{2}, 1) \in D_f \cap L$.

The proof of Theorem 3.1 depends on the following lemma and two propositions.

**Proposition 3.2**

$$t_{\text{inf}} \leq \frac{-\|d_P\|}{2\|d_P\| + \|\breve{b}\|} \text{reldist}(d_P, \text{Pri}\emptyset).$$

**Proposition 3.3** *If $C_Y$ is not a subspace then the point $w := (\breve{x}, \frac{1}{2}, 1)$ satisfies*

$$\text{sym}(w, D_f \cap L) = \Omega\left( \frac{\|\breve{b}\|}{\|d_P\| + \|\breve{b}\|} \text{sym}(\breve{x}, C_X(1)) \, \text{sym}(\breve{b}, C_Y(2\|\breve{b}\|)) \right).$$

*If $C_Y$ is a subspace then $w$ satisfies*

$$\text{sym}(w, D_f \cap L) = \Omega(\text{sym}(\breve{x}, C_X(1))).$$

**Lemma 3.4** *If $d_P := (A, b)$ satisfies* $\text{SFeas}(d_P) \neq \emptyset$ *then* $\text{SFeas}(d_P)$ *is dense in* $\text{Feas}(d_P)$ *with respect to the norm topology on $X$.*

**Proof.** Follows readily from $C_X$, $C_Y$ being the closure of $C_X^\circ$, $C_Y^\circ$ in the norm topology on $X$, $Y$, respectively. □

Before proving Propositions 3.2 and 3.3 we use them to prove Theorem 3.1.

## 3.3

**Proof of Theorem 3.1.** The operation bound is immediate from Proposition 3.2, Proposition 3.3, the fact that $1 \le t_{\sup} \le 2$ and parts 5 and 8a of Theorem 2.1.1; observe that in the present context $L + w = L$ since $w \in L$.

Assume the method terminates, the final iterate being $\hat{z} := (\hat{x}, \hat{s}, 0) \in D_f \cap L$. Define $\bar{x} := \frac{1}{\hat{s}}\hat{x} \in \mathrm{SFeas}(d_P)$.

Part 6a of Theorem 2.1.1 implies

$$\mathrm{sym}(\hat{z}, T_1) = \Omega\left(\frac{1}{K^2}\right) \tag{10}$$

where

$$T_1 := \{z = (x, s, t) \in D_f \cap L;\ t = 0\}.$$

To prove the first assertion in Theorem 3.1 concerning $\bar{x}$, it suffices to prove the following two implications for all $x \in \mathrm{Feas}(d_P)$:

$$\|x\| < 1 \ \Rightarrow\ \|\bar{x}\| = O(K^2), \tag{11}$$

$$\|x\| \ge 1 \ \Rightarrow\ \|\bar{x}\| = O(K^2\|x\|). \tag{12}$$

Since $\bar{x} \in \mathrm{SFeas}(d_P)$, and hence $\mathrm{SFeas}(d_P) \neq \emptyset$, Lemma 3.4 implies that the implications will be true for all $x \in \mathrm{Feas}(d_P)$ if we prove them for all $x \in \mathrm{SFeas}(d_P)$.

Assume $x \in \mathrm{SFeas}(d_P)$ satisfies

$$\|x\| < 1 \quad (\text{resp. } \|x\| \ge 1).$$

Then

$$z := (x, \tfrac{1}{2}, 0) \in T_1 \quad (\text{resp. } z := (\tfrac{1}{2\|x\|}x, \tfrac{1}{2\|x\|}, 0) \in T_1). \tag{13}$$

Since all elements of $D_f \cap L$ satisfy the constraint $s > 0$ it follows from (10) and (13) that

$$\hat{s} = \Omega\left(\frac{1}{K^2}\right) \quad \left(\text{resp. } \hat{s} = \Omega\left(\frac{1}{K^2\|x\|}\right)\right). \tag{14}$$

Since $\|\hat{x}\| < 1$ (because $\hat{z}$ is an element of $D_f \cap L$), we thus have

$$\|\bar{x}\| = \tfrac{1}{\hat{s}}\|\hat{x}\| = O(K^2) \quad (\text{resp. } \|\bar{x}\| = O(K^2\|x\|))$$

and hence (11) (resp. (12)).

The second assertion concerning $\bar{x}$ in the theorem follows immediately from the first assertion and Proposition 2.3.1.

Towards proving the third assertion concerning $\bar{x}$ note that if we define

$$T_2 := \{z := (x, s, t) \in D_f \cap L;\ s = \hat{s} \text{ and } t = 0\}$$

then (10) implies

$$\text{sym}(\widehat{z}, T_2) = \Omega\left(\frac{1}{K^2}\right). \tag{15}$$

Also note that, by definition of $D_f$ and $L$, for all $x$ we have

$$(x, \widehat{s}, 0) \in D_f \cap L \quad \Leftrightarrow \quad [\tfrac{1}{\widehat{s}}x \in \text{SFeas}(d_P) \;\text{ and }\; ||x|| < 1].$$

Consequently,

$$\text{sym}(\widehat{z}, T_2) = \text{sym}(\bar{x}, T_3) \tag{16}$$

where

$$T_3 := \{x \in \text{SFeas}(d_P);\; ||x|| < \tfrac{1}{\widehat{s}}\}.$$

Define

$$\gamma := \frac{1 - ||\widehat{x}||}{||\widehat{x}|| + \widehat{s}}, \quad S_\gamma := \{x \in \text{SFeas}(d_P);\; ||x|| < ||\bar{x}|| + \gamma(||\bar{x}|| + 1)\}$$

and observe that

$$\frac{1}{\widehat{s}} = ||\bar{x}|| + \gamma(||\bar{x}|| + 1);$$

hence $T_3 = S_\gamma$. It follows from (15) and (16) that

$$\text{sym}(\bar{x}, S_\gamma) = \Omega\left(\frac{1}{K^2}\right).$$

Thus, to prove the third assertion concerning $\bar{x}$ in the theorem it suffices to show

$$\gamma = \Omega\left(\frac{1}{K^2}\right). \tag{17}$$

Observe that

$$[(x, s, 0) \in D_f \cap L \;\text{ and }\; 0 < \alpha \leq 1] \;\Rightarrow\; \alpha(x, s, 0) \in D_f \cap L,$$

and hence there are elements in $T_1$ for which $||x||$ is arbitrarily small. Since all elements in $T_1$ satisfy $||x|| < 1$, it follows from (10) that

$$1 - ||\widehat{x}|| = \Omega\left(\frac{1}{K^2}\right).$$

Since $||\widehat{x}|| < 1$ and $\widehat{s} < 1$, (17) now follows easily from the definition of $\gamma$. □

### 3.4

Proposition 3.2 is a consequence of the following theorem and proposition. The theorem has implications far beyond Proposition 3.2. We set aside our convention of implicitly assuming that all of the conditions described in the Introduction hold; unless explicitly stated otherwise, we only assume that $X$, $Y$ are normed vector spaces and $C_X$, $C_Y$ are convex cones (with vertex at the origin). Moreover, we broaden our definition of the norm on $\mathbf{D}_P$. Letting $\|\ \|$ denote an arbitrary norm on $I\!R^2$, for $d_P := (A, b)$ we define

$$\|d_P\| := \|(\|A\|, \|b\|)\|.$$

(Of course this norm affects the value $\operatorname{dist}(d_P, \mathrm{Pri}\emptyset)$.) We let $\|\ \|^*$ denote the norm on $I\!R^2$ which is dual to the norm $\|\ \|$ on $I\!R^2$.

For $d_P := (A, b)$, consider the constraints in $(x, s) \in X \times I\!R$ of the form

$$\begin{aligned} Ax - sb &\leq \bar{b} \\ x &\geq \vec{0} \\ s &\geq 0 \\ \|(\|x\|, |s|)\|^* &\leq 1, \end{aligned} \tag{18}$$

where $\bar{b} \in Y$. Define

$$\rho(d_P) := \inf\{\rho;\ \exists \bar{b} \text{ such that } \|\bar{b}\| \leq \rho \text{ and (18) is inconsistent}\}.$$

**Theorem 3.5** *If $d_P := (A, b)$ then*

$$\operatorname{dist}(d_P, \mathrm{Pri}\emptyset) \leq \rho(d_P).$$

*If $X$ is reflexive and $C_X$, $C_Y$ are closed then*

$$\operatorname{dist}(d_P, \mathrm{Pri}\emptyset) = \rho(d_P).$$

**Proposition 3.6** *Let $L_X$ denote the minimal subspace containing $C_X$ and assume that the relative interior $C_X^\circ$ of $C_X$ with respect to the subspace topology of $L_X$ is dense in $C_X$; similarly for $L_Y$, $C_Y$ and $C_Y^\circ$.*

*Assume $d_P := (A, b)$ and $\hat{b} \in Y$ are such that the following system of constraints is inconsistent:*

$$\begin{aligned} Ax - sb &< \hat{b} \\ x &> \vec{0} \\ s &> 0 \\ \|(\|x\|, |s|)\|^* &< 1. \end{aligned} \tag{19}$$

*Then $\hat{b}$ can be perturbed by an arbitrarily small amount to obtain a vector $\bar{b}$ for which (18) is inconsistent.*

**Corollary 3.7** *Assume that $X$ is reflexive and $C_X$, $C_Y$ are closed. Assume the conditions of Proposition 3.6 are satisfied. If $d_P := (A, b)$ satisfies* $\mathrm{dist}(d_P, \mathrm{Pri}\emptyset) >$ $0$ *then the set of points $x$ satisfying*

$$\begin{aligned} Ax &< b \\ x &> \vec{0} \end{aligned} \tag{20}$$

*is dense within the set of points $x$ satisfying*

$$\begin{aligned} Ax &\leq b \\ x &\geq \vec{0} \end{aligned} .$$

**Proof:** In light of Lemma 3.4 it suffices to show that if the set of points $x$ satisfying (20) is empty then $\mathrm{dist}(d_P, \mathrm{Pri}\emptyset) = 0$. So assume the set of points $x$ satisfying (20) is indeed empty; hence the system (19) is inconsistent if we define $\bar{b} = \vec{0}$. Given $\epsilon > 0$, Proposition 3.6 implies that there exists $\bar{b}$ such that $||\bar{b}|| \leq \epsilon$ and the system (18) is inconsistent. Consequently, Theorem 3.5 implies $\mathrm{dist}(d_P, \mathrm{Pri}\emptyset) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary we thus have $\mathrm{dist}(d_P, \mathrm{Pri}\emptyset) = 0$. □

Before proving Theorem 3.5 and Proposition 3.6 we use them to prove Proposition 3.2.

### 3.5

**Proof of Proposition 3.2.** In proving the proposition we may assume that $t_{\mathrm{inf}} > -1$. Thus, for sufficiently small $\epsilon > 0$, if we define

$$\bar{b} := (t_{\mathrm{inf}} - \epsilon)(A\check{x} + \check{b} - \tfrac{1}{2}b),$$

then it is easily verified that the following system of constraints is inconsistent:

$$\begin{aligned} Ax &< sb + \bar{b} \\ x &> \vec{0} \\ ||x|| &< 1 \\ 0 < s &< 1. \end{aligned}$$

Consequently, the following system of constraints is inconsistent

$$\begin{aligned} Ax &< sb + \bar{b} \\ x &> \vec{0} \\ s &> 0 \\ ||x|| + |s| &< 1. \end{aligned}$$

Hence, Theorem 3.5 and Proposition 3.6 imply

$$\mathrm{dist}(d_P, \mathrm{Pri}\emptyset) \leq ||\bar{b}||.$$

Since $\epsilon > 0$ can be arbitrarily small and $t_{\inf} \leq 0$, the proof is completed by observing that $||\check{x}|| < 1$ implies

$$\begin{aligned} ||\bar{b}|| &\leq (|t_{\inf}| + \epsilon)(||A||\,||\check{x}|| + ||\check{b}|| + \tfrac{1}{2}||b||) \\ &\leq (|t_{\inf}| + \epsilon)(2||d_P|| + ||\check{b}||). \end{aligned}$$

□

### 3.6

Theorem 3.5 is an immediate consequence of the following two propositions. Other than that $X$, $Y$ are normed vector spaces and $C_X$, $C_Y$ are convex cones (with vertex at the origin), the only assumptions made in the propositions are those explicitly stated.

**Proposition 3.8** *If* $d_P := (A, b)$ *then*

$$\text{dist}(d_P, \text{Pri}\emptyset) \leq \rho(d_P).$$

**Proof:** Assume that $\bar{b} \in Y$ is such that the system (18) is inconsistent. It suffices to show that

$$\text{dist}(d_P, \text{Pri}\emptyset) \leq ||\bar{b}||. \tag{21}$$

Introduce a variable $t \in I\!R$ and let $S$ denote the set of points $(x, s, t)$ satisfying the following system of constraints:

$$\begin{aligned} Ax - sb &\leq t\bar{b} \\ x &\geq \vec{0} \\ s &\geq 0. \end{aligned}$$

Observe that $S$ is a convex cone.

Let $T$ denote the open and convex set consisting of those points $(x, s, t)$ satisfying the following system of constraints:

$$\begin{aligned} ||(||x||, |s|)||^* &< 1 \\ t &> 1. \end{aligned}$$

Note that $S \cap T = \emptyset$; for if $(x, s, t) \in S \cap T$ then $\frac{1}{t}(x, s, 1)$ satisfies the constraints (18) contradicting our assumption that those constraints are inconsistent.

Since $S \cap T = \emptyset$ and $T$ is open, the Hahn-Banach Theorem (c.f., [30, Theorem 3.4a]) implies that there exists $(\alpha^*, \sigma, \tau) \in X^* \times I\!R \times I\!R$ such that if

$$(x_1, s_1, t_1) \in S \text{ and } (x_2, s_2, t_2) \in T$$

then

$$\alpha^* x_1 + \sigma s_1 + \tau t_1 < \alpha^* x_2 + \sigma s_2 + \tau t_2.$$

Since $S$ is a cone and $T$ is open, it follows that

$$(x, s, t) \in S \;\Rightarrow\; \alpha^* x + \sigma s + \tau t \leq 0 \tag{22}$$

and

$$(x, s, t) \in T \;\Rightarrow\; \alpha^* x + \sigma s + \tau t > 0. \tag{23}$$

Since $(\vec{0}, 0, t) \in T$ whenever $t > 1$, (23) implies $\tau > 0$; scaling $(\alpha^*, \sigma, \tau)$ if necessary, we may thus assume $\tau = 1$.

Let $\epsilon > 0$ and consider the instance $d_P + \Delta d_P$ where $\Delta d_P := (\Delta A, \Delta b)$,

$$\Delta b := (-\sigma + \epsilon)\bar{b}$$

and $\Delta A : X \to Y$ is the linear operator defined by

$$(\Delta A)x := (\alpha^* x)\bar{b}.$$

Since $\epsilon > 0$ arbitrary, to prove the proposition it suffices to show that

$$d_P + \Delta d_P \in \text{Pri}\emptyset \;\text{ and }\; ||\Delta d_P|| \leq ||\bar{b}||(1 + O(\epsilon)).$$

Assume that $d_P + \Delta d_P \notin \text{Pri}\emptyset$ and let $x'$ denote a feasible point for $d_P + \Delta d_P$. It is easily verified that

$$(x', 1, -\alpha^* x' - \sigma + \epsilon) \in S.$$

Hence, (22) and $\tau = 1$ imply

$$\alpha^* x' + \sigma + (-\alpha^* x' - \sigma + \epsilon) \leq 0 \tag{24}$$

which is a contradition since the quantity on the left of (24) equals $\epsilon$ which is positive. Hence, $d_P + \Delta d_P \in \text{Pri}\emptyset$.

Finally, note that

$$\begin{aligned}
||\Delta d_P|| &= ||(||\Delta A||, ||\Delta b||)|| \\
&= ||(||\alpha^*||\,||\bar{b}||, |-\sigma + \epsilon|\,||\bar{b}||)|| \\
&= ||\bar{b}||\,||(||\alpha^*||, |-\sigma + \epsilon|)|| \\
&\leq ||\bar{b}||(||(||\alpha^*||, |-\sigma|)|| + O(\epsilon)).
\end{aligned}$$

However, (23) and $\tau = 1$ give

$$||(||x||, |s|)||^* < 1 \;\Rightarrow\; \alpha^* x + \sigma s < 1,$$

that is,

$$||(||\alpha^*||, |s|)|| \leq 1.$$

Hence,

$$||\Delta d_P|| \leq ||\bar{b}||(1 + O(\epsilon))$$

concluding the proof. □

**Proposition 3.9** *If $X$ is reflexive and $C_X$, $C_Y$ are closed then*

$$\text{dist}(d_P, \text{Pri}\emptyset) \geq \rho(d_P).$$

**Proof:** Since $X$ is reflexive and $C_X$, $C_Y$ are closed, Proposition 2.3.2 implies

$$\text{dist}(d_P, \text{Pri}\emptyset) = \text{dist}(d_P, \text{PriS}\emptyset),$$

where PriS$\emptyset$ is the set of instances $(A', b')$ which are strongly inconsistent, i.e., cannot be made consistent by an arbitrarily slight perturbation of $b'$. Consequently, assuming that $\Delta d_P := (\Delta A, \Delta b)$ satisfies $d_P + \Delta d_P \in \text{PriS}\emptyset$, it suffices to show

$$||\Delta d_P|| \geq \rho(d_P). \tag{25}$$

Assuming $d_P := (A, b)$ let $S$ denote the convex set defined by

$$S := \{b' \in Y;\ b' = (b + \Delta b) - b'' - (A + \Delta A)x \text{ for some } x \geq 0 \text{ and } b'' \geq \vec{0}\}.$$

Let $T$ denote the convex cone defined by

$$T := \{b' \in Y;\ b' = s(b+\Delta b) - b'' - (A+\Delta A)x \text{ for some } s > 0,\ x \geq 0 \text{ and } b'' \geq \vec{0}\}.$$

Note that

$$T = \{tb';\ t > 0 \text{ and } b' \in S\}.$$

Since $d_P + \Delta d_P$ is strongly inconsistent we have

$$\vec{0} \notin \bar{S}$$

where $\bar{S}$ denotes the closure of $S$. Hence, the Hahn-Banach Theorem implies that there exists $y^* \in Y^*$ such that $||y^*|| = 1$ and

$$b' \in \bar{S} \ \Rightarrow\ y^*b' > 0.$$

Hence,

$$b' \in T \ \Rightarrow\ y^*b' > 0$$

and thus

$$b' \in \bar{T} \ \Rightarrow\ y^*b' \geq 0 \tag{26}$$

where $\bar{T}$ denotes the closure of $T$.

Let $\epsilon > 0$. Since $||y^*|| = 1$ there exists $\bar{b} \in Y$ satisfying

$$y^*\bar{b} \geq ||\Delta d_P|| + \epsilon \text{ and } ||\bar{b}|| \leq ||\Delta d_P|| + 2\epsilon.$$

Since $\epsilon$ is arbitrary, to prove (25) it suffices to show that for this vector $\bar{b}$, the system (18) is inconsistent.

Assume to the contrary that the system (18) is consistent, letting $(\bar{x}, \bar{s})$ denote a feasible point. Observe that there then exists $b'' \geq \vec{0}$ such that

$$\bar{s}b - b'' - A\bar{x} = -\bar{b}.$$

Hence

$$y^*(\bar{s}b - b'' - A\bar{x}) = -y^*(\bar{b}) \leq -||\Delta d_P|| - \epsilon.$$

Thus,

$$\begin{aligned} & y^*(\bar{s}(b + \Delta b) - b'' - (A + \Delta A)\bar{x}) \\ \leq \; & -||\Delta d_P|| - \epsilon + y^*(\bar{s}(\Delta b) - (\Delta A)\bar{x}) \\ \leq \; & -||\Delta d_P|| - \epsilon + ||y^*|| \, ||\Delta d_P|| \, ||(||\bar{x}||, ||\bar{s}||)||^* \\ \leq \; & -\epsilon, \end{aligned}$$

in contradiction to (26) and the fact that

$$\bar{s}(b + \Delta b) - b'' - (A + \Delta A)\bar{x}) \in \bar{T}.$$

Hence the system is (18) is inconsistent and the proof is complete. □

**Proof of Theorem 3.5:** Immediate from Propositions 3.8 and 3.9. □

## 3.7

**Proof of Proposition 3.6:** Assume $\widehat{b}$ is such that the system (19) is inconsistent.

Define the following subsets of $X \times \mathbb{R} \times Y$:

$$L := \{(x, s, \tilde{b});\ x \in L_X,\ \tilde{b} \in L_Y\},$$

$$S := \{(x, s, \tilde{b});\ \tilde{b} = \widehat{b} + sb - Ax\},$$

$$T := \{(x, s, \tilde{b});\ x \geq \vec{0},\ s \geq 0,\ \tilde{b} \geq \vec{0} \text{ and } ||(||x||, |s|)||^* \leq 1\},$$

and

$$T^\circ := \{(x, s, \tilde{b});\ x > \vec{0},\ s > 0,\ \tilde{b} > 0 \text{ and } ||(||x||, |s|)|| < 1\}.$$

Note that $T^\circ \subseteq T \subseteq L$ and $T^\circ$ is dense in $T$.

Since the system (19) is inconsistent we have

$$S \cap T^\circ = \emptyset.$$

Since $T^\circ$ is open in the subspace topology of $L$, the Hahn-Banach Theorem thus implies that there exists $z^* \in L^*$ such that,

$$\left[(x_1, s_1, \tilde{b}_1) \in S \cap L \text{ and } (x_2, s_2, \tilde{b}_2) \in T^\circ\right] \;\Rightarrow\; z^*(x_1, s_1, \tilde{b}_1) < z^*(x_2, s_2, \tilde{b}_2).$$

Since $T^\circ$ is dense in $T$,

$$\left[(x_1, s_1, \tilde{b}_1) \in S \cap L \text{ and } (x_2, s_2, \tilde{b}_2) \in T\right] \Rightarrow z^*(x_1, s_1, \tilde{b}_1) \leq z^*(x_2, s_2, \tilde{b}_2). \tag{27}$$

Let $v' := (x', s', b') \in L$ be such that

$$z^* v' < 0. \tag{28}$$

Let $\epsilon > 0$ and define

$$\bar{b} := \widehat{b} + \epsilon(b' + Ax' - s'b).$$

Since $\epsilon > 0$ is arbitrary, to prove the proposition it suffices to show that the system (18) is inconsistent for this choice of $\bar{b}$.

Note that (27) and (28) imply

$$[(S \cap L) + \epsilon v'] \cap T = \emptyset.$$

Since $v' \in L$ and $T \subseteq L$ we thus have

$$(S + \epsilon v') \cap T = \emptyset. \tag{29}$$

Noting that

$$\begin{aligned} S + \epsilon v' &= \{(x + \epsilon x', s + \epsilon s', \tilde{b} + \epsilon b');\ \tilde{b} = \widehat{b} + sb - Ax\} \\ &= \{(x'', s'', b'');\ b'' - \epsilon b' = \widehat{b} + (s'' - \epsilon s')b - A(x'' - \epsilon x')\} \\ &= \{(x'', s'', b'');\ b'' = \bar{b} + s''b - Ax''\}, \end{aligned}$$

we see that (29) is equivalent to the system (18) being inconsistent, thus completing the proof. □

## 3.8

The proof of Proposition 3.3 depends on the next five lemmas and the proposition following them.

**Lemma 3.10** *If $x \in S_1 \cap S_2$ where $S_1$ and $S_2$ are convex subsets of a real vector space then*

$$\text{sym}(x, S_1 \cap S_2) \geq \min\{\text{sym}(x, S_1), \text{sym}(x, S_2)\}.$$

**Proof.** Simple. □

**Lemma 3.11** *Assume $V_1$, $V_2$ are real vector spaces and $M : V_1 \to V_2$ is a linear operator. Assume $S_2$ is a convex subset of $V_2$ and let*

$$S_1 := \{x \in V_1;\ Mx \in S_2\}.$$

*If $x \in S_1$ then*

$$\mathrm{sym}(x, S_1) \geq \mathrm{sym}(Mx, S_2).$$

**Proof.** Simple. □

**Lemma 3.12** *Assume $S, T \subseteq V$ where $V$ is a real vector space and $S$ is convex. If*

$$x, y \in S, \quad r \geq 0, \quad y + (1+r)(x-y) \in S \ \text{ and } \ y + T \subseteq S$$

*then*

$$x + \tfrac{r}{1+r} T \subseteq S. \tag{30}$$

**Proof.** The set on the left of (30) is the image of $y + T$ under the contraction

$$z \mapsto [y + (1+r)(x-y)] + \tfrac{r}{1+r}(z - [y + (1+r)(x-y)]).$$

Since $y + (1+r)(x-y) \in S$ and $S$ is convex, the lemma follows. □

**Lemma 3.13** *Assume $S, T \subseteq V$ where $V$ is a real vector space and $S$ is convex. If*

$$x, y \in S, \quad y + T \subseteq S \ \text{ and } \ 0 \leq r < \mathrm{sym}(x, S)$$

*then*

$$x + \tfrac{1}{2} r T \subseteq S.$$

**Proof.** Immediate from Lemma 3.12, the definition of $\mathrm{sym}(x, S)$ and the fact that $\mathrm{sym}(x, S) \leq 1$. □

**Lemma 3.14** *Assume $C$ is a convex cone with vertex $\vec{0}$ in a normed vector space $V$. For $r \in I\!R$ define*

$$C(r) := \{y \in C;\ \|y\| < r\}.$$

*If $x \in C(r_1)$ then for all $r_2 \geq r_1$,*

$$\mathrm{sym}(x, C(r_2)) \geq \tfrac{r_1}{2r_2}\mathrm{sym}(x, C(r_1)).$$

**Proof.** Assuming $r_2 \geq r_1$, $y \in C(r_2)$ and $0 \leq r < \text{sym}(x, C(r_1))$ we must show

$$x + r\tfrac{r_1}{2r_2}(x-y) \in C(r_2)$$

for which it suffices to show

$$x + r\tfrac{r_1}{2r_2}(x-y) \in C(r_1). \tag{31}$$

Define

$$T := \{v \in V;\ v = t(x-y) \text{ for some } 0 \leq t \leq 1\}.$$

Since $y \in C(r_2)$, $y + T \subseteq C(r_2)$ and $C$ is a convex cone with vertex $\vec{0}$,

$$\tfrac{r_1}{r_2}y \in C(r_1) \quad \text{and} \quad \tfrac{r_1}{r_2}y + \tfrac{r_1}{r_2}T \subseteq C(r_1).$$

Hence, by Lemma 3.13,

$$x + r\tfrac{r_1}{2r_2}T \subseteq C(r_1)$$

from which (31) is immediate. □

**Proposition 3.15** *Assume $V_1$, $V_2$ are real vector spaces and $M : V_1 \to V_2$ is a linear operator. Assume $x \in S \subseteq V_1$ where $S$ is convex and let*

$$S_1 := \{x_1 \in V_1;\ Mx_1 = Mx\},$$

$$S_2 := \{x_2 \in V_2;\ x_2 = Mx_1 \text{ for some } x_1 \in S\}.$$

*The following inequality is valid:*

$$\text{sym}(x, S) \geq \tfrac{1}{4}\text{sym}(x, S_1)\,\text{sym}(Mx, S_2).$$

**Proof.** We may assume $\text{sym}(x, S_1), \text{sym}(Mx, S_2) > 0$.

To prove the proposition it suffices to show that if

$$y \in S \quad \text{and} \quad 0 \leq r < \tfrac{1}{4}\text{sym}(x, S_1)\,\text{sym}(Mx, S_2) \tag{32}$$

then

$$x + r(x-y) \in S. \tag{33}$$

Fix $y$ and $r$ satisfying (32) and let $r_1$, $r_2$ be values satisfying

$$r_1 < \text{sym}(x, S_1), \quad r_2 < \text{sym}(Mx, S_2)$$

and

$$r < \tfrac{1}{4}r_1 r_2. \tag{34}$$

Define

$$T := \{x_1 \in V_1;\ x_1 = t(x-y) \text{ for some } 0 \leq t \leq 1.\}.$$

By definition of $\text{sym}(Mx, S_2)$ there exists $y' \in S$ satisfying

$$My' = Mx + r_2 M(x - y).$$

Define

$$y'' := y + \tfrac{1}{1+r_2}(y' - y),$$

an element of $S$ since $y, y' \in S$ and $S$ is convex. Since

$$y + (1 + r_2)(y'' - y) = y' \in S \text{ and } y + T \subseteq S$$

Lemma 3.12 implies

$$y'' + \tfrac{r_2}{1+r_2} T \subseteq S. \tag{35}$$

Noting that $y'' \in S_1$, the assumption $0 \leq r_1 < \text{sym}(x, S)$ implies

$$y'' + (1 + r_1)(x - y'') \in S. \tag{36}$$

Together, (35), (36) and Lemma 3.12 imply

$$x + \tfrac{r_1}{1+r_1} \tfrac{r_2}{1+r_2} T \subseteq S$$

and thus by definition of $T$,

$$x + \tfrac{r_1}{1+r_1} \tfrac{r_2}{1+r_2}(x - y) \in S.$$

Since $r_1$ and $r_2$ satisfy $r_1 \leq 1,\ r_2 \leq 1$ and (34), (33) follows. □

## 3.9

**Proof of Proposition 3.3.** We first prove the proposition assuming $C_Y$ is not a subspace. Afterwards, we record the minor alterations needed to prove the proposition when $C_Y$ is a subspace.

Letting $d := (A, b, c)$, define $S_1$ to be the convex set consisting of all points $(x, s, t) \in \mathcal{H}$ satisfying the following constraints:

$$\begin{gathered} Ax < sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b) \\ x > \vec{0} \\ ||x|| < 1 \\ ||sb + t(A\breve{x} + \breve{b} - b)|| < 4||d_P|| + 2||\breve{b}||. \end{gathered}$$

Define $S_2$ to be the set of all points $(x, s, t) \in \mathcal{H}$ satisfying the following constraints:

$$\begin{gathered} 0 < s < 1 \\ -1 < t < 2. \end{gathered}$$

We first note that

$$D_f \cap L = S_1 \cap S_2. \tag{37}$$

This identity would be immediate from the definitions was it not for the constraint

$$||sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b)|| < 4||d_P|| + 2||\breve{b}|| \tag{38}$$

occurring in the definition of $S_1$. However, we claim that if this constraint is removed then $S_1 \cap S_2$ is not enlarged. For if $(x, s, t)$ satisfies all constraints defining $S_1 \cap S_2$ except perhaps (38) then since $||\breve{x}|| < 1$,

$$\begin{aligned} ||sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b)|| &< 2(||A|| + ||b|| + ||\breve{b}||) \\ &\leq 4||d_P|| + 2||\breve{b}||. \end{aligned}$$

Hence (37).

Trivially, $\text{sym}(w, S_2) = \Omega(1)$. Hence, Lemma 3.10 and (37) imply that to prove the theorem it suffices to show

$$\text{sym}(w, S_1) = \Omega\left( \frac{||\breve{b}||}{||d_P|| + ||\breve{b}||} \text{sym}(\breve{x}, C_X(1))\, \text{sym}(\breve{b}, C_Y(2||\breve{b}||)) \right). \tag{39}$$

Define $S'$ to be the convex set consisting of all points $(x, b') \in X \times Y$ satisfying the following constraints:

$$\begin{gathered} Ax < b' \\ x > \vec{0} \\ ||x|| < 1 \\ ||b'|| < 4||d_P|| + 2||\breve{b}||. \end{gathered}$$

Letting $M : X \times I\!R \times I\!R \to X \times Y$ denote the linear operator defined by

$$M(x, s, t) := (x, sb + t(A\breve{x} + \breve{b} - \tfrac{1}{2}b))$$

we have

$$S_1 = \{(x, s, t);\ M(x, s, t) \in S'\}$$

and hence, by Lemma 3.11, to prove (39) it suffices to show

$$\text{sym}(z, S') = \Omega\left( \frac{||\breve{b}||}{||d_P|| + ||\breve{b}||} \text{sym}(\breve{x}, C_X(1))\, \text{sym}(\breve{b}, C_Y(2||\breve{b}||)) \right) \tag{40}$$

where

$$z := Mw = (\breve{x}, A\breve{x} + \breve{b}).$$

Define

$$S'_1 := \{(x, b') \in S';\ x = \breve{x}\},$$
$$S'_2 := \{x \in X;\ (x, b') \in S' \text{ for some } b' \in Y\}.$$

With reference to the operator projecting $X \times Y$ onto $X$, Proposition 3.15 implies

$$\mathrm{sym}(z, S') \geq \tfrac{1}{4}\mathrm{sym}(z, S_1')\,\mathrm{sym}(\breve{x}, S_2').$$

Hence, to prove (40) it suffices to show

$$\mathrm{sym}(z, S_1') = \Omega\left(\frac{\|\breve{b}\|}{\|d_P\| + \|\breve{b}\|}\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|))\right) \tag{41}$$

and

$$\mathrm{sym}(\breve{x}, S_2') = \mathrm{sym}(\breve{x}, C_X(1)). \tag{42}$$

Observe that $S_1'$ consists precisely of points of the form

$$z + (\vec{0}, b'') \text{ where } \breve{b} + b'' \in C_Y(4\|d_P\| + 2\|\breve{b}\|)$$

and

$$C_Y(4\|d_P\| + 2\|\breve{b}\|) := \{b' \in C_Y;\ \|b'\| < 4\|d_P\| + 2\|\breve{b}\|\}.$$

Hence,

$$\mathrm{sym}(z, S_1') = \mathrm{sym}(\breve{b}, C_Y(4\|d_P\| + 2\|\breve{b}\|)) \tag{43}$$

which, by Lemma 3.14, implies

$$\mathrm{sym}(z, S_1') \geq \tfrac{\|\breve{b}\|}{8\|d_P\| + 4\|\breve{b}\|}\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|))$$

establishing (41).

Finally, observe that if $x \in C_X(1)$ then

$$(x, b') \in S' \text{ where } b' := Ax + \breve{b}.$$

Since $C_X(1) \subseteq S_2'$, it follows that

$$S_2' = C_X(1)$$

and hence (42). This completes the proof when $C_Y$ is not a subspace.

To prove the proposition when $C_Y$ is a subspace, one simply makes the following substitutions:

1. Relation (39) is replaced with

$$\mathrm{sym}(w, S_1) = \Omega(\mathrm{sym}(\breve{x}, C_X(1)).$$

2. Relation (40) is replaced with

$$\mathrm{sym}(z, S') = \Omega(\mathrm{sym}(\breve{x}, C_X(1)).$$

3. Relation (41) is replaced with

$$\mathrm{sym}(z, S_1') = \Omega(1).$$

4. The sentence containing relation (43) is replaced with the following sentence:

   Hence,

   $$\mathrm{sym}(z, S_1') = \mathrm{sym}(\check{b}, C_Y(4\|d_P\| + 2\|\check{b}\|))$$

   which, since $\check{b} = \vec{0}$ and $C_Y$ is a subspace, gives

   $$\mathrm{sym}(z, S_1') = 1.$$

□

# 4 Strong Dual Inconsistency

## 4.1

By employing a "theorem of the alternatives," it is easy to obtain an algorithm for determining dual inconsistency from the algorithm of the previous section for determining primal consistency.

**Proposition 4.1 (Duffin)** *Assume $X$ and $Y$ are normed spaces, $X$ being reflexive. Assume $C_X$ and $C_Y$ are closed in the norm topology.*

*If $d_D := (A, c^*) \in \mathbf{D}_D$ then exactly one of the following two alternatives is true:*

1. *$d_D \in$ DualA, meaning that $d_D$ is consistent or can be made consistent by an arbitrarily slight perturbation of $c^*$.*
2. *The system*

$$\begin{aligned} Ax &\leq \vec{0} \\ c^*x &> 0 \\ x &\geq \vec{0} \end{aligned} \tag{44}$$

*is consistent.*

**Proof.** See, for example, [26, Corollary 2.3]. □

Assuming $\check{t}$ is a positive real number, an algorithm for determining dual inconsistency is obtained simply by applying the algorithm of the previous section to determining the (primal) consistency of the following constraints:

$$\begin{aligned} Ax &\leq \vec{0} \\ c^*x &= \check{t} \\ x &\geq \vec{0}. \end{aligned} \tag{45}$$

Proposition 4.1 implies that the algorithm will terminate only if

$$d_D \in \text{DualS}\emptyset := \mathbf{D}_D \setminus \text{DualA},$$

that is, only if $d_D$ is strongly inconsistent.

The value $\check{t}$ enters into the iteration bound for the algorithm as does $||\check{b}||$ (when $C_Y$ is not a subspace); ideally, $\check{t} \approx ||d_D|| \approx ||\check{b}||$.

In applying the algorithm for determining the consistency of (45), we view (45) as an instance $\widehat{d}_P := (\widehat{A}, \widehat{b})$ where $\widehat{A} : X \to Y \times I\!R$ is defined by

$$\widehat{A}x := (Ax, c^*x)$$

and where $\widehat{b} := (\vec{0}, \check{t})$. The system (45) can then be expressed as

$$\begin{aligned} \widehat{A}x &\,\widehat{\leq}\, \widehat{b} \\ x &\geq \vec{0} \end{aligned} \tag{46}$$

where " $\widehat{A}x \,\widehat{\leq}\, \widehat{b}$ " is defined as meaning $\widehat{b} - \widehat{A}x \in C_Y \times \{0\}$. The cone $C_Y \times \{0\}$ defining non-negativity in $Y \times I\!R$ is the closure of

$$(D_\psi \times I\!R) \cap (L_Y \times \{0\}),$$

$L_Y \times \{0\}$ being a closed linear subspace of $Y \times I\!R$. For the corresponding functional in $\mathcal{F}(K_Y)$, we use

$$(y,t) \mapsto \psi_y,$$

the domain of which is $D_\psi \times I\!R$. For the known point in the relative interior of $C_Y \times \{0\}$ we use $(\check{b}, 0)$. Note that

$$\text{sym}((\check{b},0), C_Y(2\|\check{b}\|) \times \{0\}) = \text{sym}(\check{b}, C_Y(2\|\check{b}\|)). \tag{47}$$

**Theorem 4.2** *The algorithm terminates only if $d_D$ is strongly inconsistent. Moreover, if $d_D := (A, c^*)$ satisfies $\text{dist}(d_D, \text{DualA}) > 0$ then the algorithm terminates within*

$$O\Bigg(K \log\Bigg[K + \frac{1}{\text{reldist}(d_D, \text{DualA})} + \frac{1}{\text{sym}(\check{x}, C_X(1))} + \frac{\max\{\|d_D\|, \check{t}\}}{\min\{\|d_D\|, \check{t}\}} + \frac{1}{\text{sym}(\check{b}, C_Y(2\|\check{b}\|))} + \frac{\max\{\|d_D\|, \|\check{b}\|\}}{\min\{\|d_D\|, \|\check{b}\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving $\check{b}$ being deleted if $C_Y$ is a subspace.*

The remainder of this section is devoted to proving Theorem 4.2.

## 4.2

In proving the theorem we consider (45) as an instance $\widehat{d}_P := (\widehat{A}, \widehat{b})$ as discussed above. We endow $Y \times I\!R$ with the norm

$$\|(y,t)\| := \max\{\|y\|, |t|\}.$$

This and the original norm on $X$ induce a norm on the data space of instances

$$\mathcal{L}(X, Y \times I\!R) \times (Y \times I\!R)$$

containing $\widehat{d}_P$, thus giving meaning to the quantity $\text{dist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset})$ where $\widehat{\text{Pri}\emptyset}$ is the set of instances $\bar{d}_P := (\bar{A}, \bar{b})$ for which

$$\begin{aligned} \bar{A}x &\widehat{\leq} \bar{b} \\ x &\geq \vec{0} \end{aligned} \tag{48}$$

is inconsistent, " $\widehat{\geq}$ " being defined as in (46).

The proof of Theorem 4.2 depends on the following Proposition.

**Proposition 4.3** *Let* $\widehat{d}_P$ *and* $\widehat{\text{Pri}\emptyset}$ *be as above. If*

$$\text{dist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset}) \leq \tfrac{1}{2}\check{t}$$

*then*

$$\text{dist}(d_D, \text{DualA}) = O\left(\text{dist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset})\left[1 + \frac{\|d_D\|}{\check{t}}\right]\right).$$

Before proving Proposition 4.3 we use it to prove Theorem 4.2.

**Proof of Theorem 4.2.** Theorem 3.1 and (47) imply that the method terminates within

$$O\left(K\log\left[K + \frac{1}{\text{reldist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset})} + \frac{1}{\text{sym}(\check{x}, C_X(1))} + \frac{1}{\text{sym}(\check{b}, C_Y(2\|\check{b}\|))} + \frac{\max\{\|\widehat{d}_P\|, \|\check{b}\|\}}{\min\{\|\widehat{d}_P\|, \|\check{b}\|\}}\right]\right)$$

operations, the ratios involving $\check{b}$ being deleted if $C_Y$ is a subspace. Noting that $\|\widehat{d}_P\| = \max\{\|d_D\|, \check{t}\}$, to prove the proposition it thus suffices to show

$$\log\frac{1}{\text{reldist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset})} \leq \log\frac{1}{\text{reldist}(d_D, \text{DualA})} + \log\frac{\max\{\|d_D\|, \check{t}\}}{\min\{\|d_D\|, \check{t}\}} + O(1). \tag{49}$$

In showing (49) we may assume

$$\text{dist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset}) \leq \tfrac{1}{2}\check{t} \tag{50}$$

since otherwise

$$\frac{1}{\text{reldist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset})} = \frac{\max\{\|d_D\|, \check{t}\}}{\text{dist}(\widehat{d}_P, \widehat{\text{Pri}\emptyset})} \leq 2\max\left\{\frac{\|d_D\|}{\check{t}}, 1\right\}$$

from which (49) follows easily.

Assuming (50), we claim that to prove (49) it suffices to show

$$\text{dist}(d_D, \text{DualA}) = O\Big(\text{dist}(\hat{d}_P, \widehat{\text{Pri}\emptyset})\left[1 + \frac{\|d_D\|}{\check{t}}\right]\Big). \tag{51}$$

Recalling that $\|\hat{d}_P\| = \max\{\|d_P\|, \check{t}\}$, the claim follows easily by separate consideration of the two cases $\|\hat{d}_P\| = \|d_P\|$ and $\|\hat{d}_P\| = \check{t}$.

Finally, Proposition 4.3 establishes (51) under the assumption (50). □

## 4.3

**Proof of Proposition 4.3.** To establish the proposition it suffices to show

$$\begin{gathered} \bar{d}_P \in \widehat{\text{Pri}\emptyset} \text{ and } \|\bar{d}_P - \hat{d}_P\| \le \tfrac{1}{2}\check{t} \\ \Rightarrow \\ \exists\, \tilde{d}_D \in \text{DualA} \text{ such that } \|\tilde{d}_D - d_D\| = O\Big(\|\bar{d}_P - \hat{d}_P\|\left[1 + \frac{\|d_D\|}{\check{t}}\right]\Big). \end{gathered} \tag{52}$$

In proving (52) assume $\bar{d}_P$ in $\widehat{\text{Pri}\emptyset}$ satisfies $\|\bar{d}_P - \hat{d}_P\| \le \frac{1}{2}\check{t}$. If the constraints (48) corresponding to $\bar{d}_P$ are rewritten as

$$\begin{aligned} (A + \Delta A)x &\le \Delta\bar{b}_1 \\ (c^* + \Delta c^*)x &= \check{t} + \Delta\check{t} \\ x &\ge \vec{0} \end{aligned} \tag{53}$$

then

$$\|\Delta A\|, \|\Delta\bar{b}_1\|, \|\Delta c^*\|, \|\Delta\check{t}\| \le \|\bar{d}_P - \hat{d}_P\|. \tag{54}$$

Consider the instance $\tilde{d}_D := (\tilde{A}, \tilde{c}^*)$ where

$$\tilde{A}x := (A + \Delta A)x - \frac{(c^* + \Delta c^*)x}{\check{t} + \Delta\check{t}}\Delta\bar{b}_1$$

and

$$c^* := c^* + \Delta c^*.$$

We claim that $\tilde{d}_D \in$ DualA. For otherwise, by Proposition 4.1, the following system is consistent:

$$\begin{aligned} (A + \Delta A)x - \frac{(c^* + \Delta c^*)x}{\check{t} + \Delta\check{t}}\Delta\bar{b}_1 &\le \vec{0} \\ (c^* + \Delta c^*)x &= \check{t} + \Delta\check{t} \\ x &\ge \vec{0}. \end{aligned}$$

Hence (53) is consistent, a contradiction.

Finally, note that (54) and $||\bar{d}_P - \widehat{d}_P|| \leq \frac{1}{2}\check{t}$ (hence $\check{t} + \Delta\check{t} \geq \frac{1}{2}\check{t}$) imply

$$\begin{aligned} ||\tilde{d} - d|| &\leq \max\{||\Delta A|| + \frac{||c^* + \Delta c^*||}{\check{t} + \Delta\check{t}}||\Delta\bar{b}_1||, \, ||\Delta c^*||\} \\ &\leq \max\{||\bar{d}_P - \widehat{d}_P|| + 2\frac{||d_D|| + ||\bar{d}_P - \widehat{d}_P||}{\check{t}}||\bar{d}_P - \widehat{d}_P||, \, ||\bar{d}_P - \widehat{d}_P||\} \\ &= O\left(||\bar{d}_P - \widehat{d}_P|| \left[1 + \frac{||d_D||}{\check{t}}\right]\right). \end{aligned}$$

The implication (52) follows. □

# 5 Primal Inconsistency

We consider three algorithms for determining primal inconsistency. The assumptions relied on differ markedly between the algorithms.

The first algorithm requires the cone $C_X$ to be pointed and a bound on its pointedness to be known. The method easily extends to the situation that $C_X$, although perhaps not pointed itself, is a finite union of pointed cones.

The second algorithm requires, roughly speaking, that one be able to approximate the smallest non-zero element in the spectrum of an operator of the form $PHP$ for particular positive definite operators $H$ and orthogonal projection operators $P$.

The third algorithm is obtained simply by applying the algorithm for determining dual inconsistency (i.e., the algorithm in Section 4), to the dual rather than the primal, assuming the dual cones satisfy appropriate requirements and assuming $Y$ is a Hilbert space (rather than $X$). The algorithm then determinines strong inconsistency in the dual of the dual which, assuming $X$ is reflexive and $C_X$ and $C_Y$ are closed, is the same as strong primal inconsistency.

## 5.1 The First Algorithm

### 5.1.1

The first two algorithms rely on exactly the same framework as the algorithm for determining primal consistency considered in Section 3. We suggest the reader again browse the introductory paragraphs in Section 3 through the statement of Proposition 3.3. In particular, recall the dependency of $f$ and $D_f$ on the instance $d_P := (A, b)$ whose consistency is in question.

For the first algorithm we assume the cone $C_X$ is pointed and a bound on its pointedness is known. More precisely, we assume $C_X \neq \{\vec{0}\}$ and we assume known a value $\alpha > 0$ for which there exists $\bar{c}^* \in X^* \ (= X)$, $\|\bar{c}^*\| = 1$, such that

$$x \in C_X \;\Rightarrow\; \bar{c}^* x \geq \alpha \|x\|.$$

( We do not assume $\bar{c}^*$ is known.) To succinctly express these assumptions we say that " $C_X$ has pointedness at least $\alpha > 0$ ".

Recall that the algorithm for determining primal consistency was obtained by applying the barrier method to solving the constraints

$$\begin{array}{c} (x, s, t) \in D_f \cap L \\ t = 0. \end{array} \tag{55}$$

Our first algorithm for determining primal inconsistency revolves around simple comparisons involving the magnitudes of iterates arising in the second stage of the barrier method. Specifically, if $(x, s, t)$ is an iterate computed in the second stage then the comparisons are

$$\|x\| \leq \frac{C\alpha}{K^2}, \quad s \leq \frac{C}{K^2} \quad \text{and} \quad 0 < t \leq 1 \tag{56}$$

where $C > 0$ is an appropriately small universal constant[26]. As we prove, if the iterate satisfies all of the the inequalites (56) then $d_P$ is inconsistent.

The first algorithm begins by simply applying the barrier method to solving (55), initiated at $w := (\breve{x}, 1, 1)$, continuing until the second stage is reached. For each iterate in the second stage the comparisons (56) are made. If those inequalities are all satisfied then the algorithm terminates. Otherwise the next iterate is computed and so on. Since for simplicity and uniformity we are separating the issues of determining inconsistency from those of determining consistency, we assume that if the barrier method determines a solution of (55) when being used for the task of determining inconsistency then an infinite loop is activated; in other words, the algorithm terminates only if the input instance is inconsistent.

Recall that Pri denotes the subset of $\mathbf{D}_P$ consisting of consistent instances.

**Theorem 5.1.1** *Assume $C_X$ has pointedness at least $\alpha > 0$ ,where $\alpha$ is known, and assume the positive constant $C$ in (56) satisfies $C = \Theta(1)$, i.e., a universal constant.*

*The first algorithm terminates only if the input $d_P := (A, b)$ is inconsistent. Moreover, if $\text{dist}(d_P, \text{Pri}) > 0$ then the algorithm terminates within*

$$O\left(K \log \left[K + \frac{1}{\alpha} + \frac{1}{\text{reldist}(d_P, \text{Pri})} + \frac{1}{\text{sym}(\breve{x}, C_X(1))} + \frac{1}{\text{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|d_P\|, \|\breve{b}\|\}}{\min\{\|d_P\|, \|\breve{b}\|\}}\right]\right)$$

*iterations, the ratios involving $\breve{b}$ being deleted if $C_Y$ is a subspace.*

Before proving Theorem 5.1.1 we note that it has implications for the situation that $C_X$, although perhaps not pointed itself, is a finite union of pointed cones $C_X^j$ assuming:

1. $C_X^j$ has an underlying functional in $\mathcal{F}'(K_j)$ where $K_j$ is known;
2. $C_X^j$ has pointedness at least $\alpha_j > 0$ where $\alpha_j$ is known;

[26] We do not specify a value for $C$ although one could deduce an appropriate value by elaborating on the proofs.

3. a point $\breve{x}_j$ satisfying $\|\breve{x}_j\| < 1$, and in the relative interior of $C_X^j$, is known.

For observe that the instance $d_P$ is inconsistent if and only if it is inconsistent when the cone $C_X^j$ is substituted for $C_X$ for every index $j$. Hence, to determine inconsistency of $d_P$ it is (necessary and) sufficient to determine inconsistency under all of the substitutions. Moreover, observing that if $C_X^j$ is substituted for $C_X$ the resulting value analogous to $\text{dist}(d_P, \text{Pri})$ is not less than $\text{dist}(d_P, \text{Pri})$ since $C_X^j \subseteq C_X$, it follows by Theorem 5.1.1 that we have at hand an algorithm requiring at most

$$O\left(K \sum_j \log\left[K + \frac{1}{\alpha_j} + \frac{1}{\text{reldist}(d_P, \text{Pri})} + \frac{1}{\text{sym}(\breve{x}_j, C_X^j(1))} + \frac{1}{\text{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|d_P\|, \|\breve{b}\|\}}{\min\{\|d_P\|, \|\breve{b}\|\}}\right]\right)$$

iterations assuming $\text{dist}(d_P, \text{Pri}) > 0$, where the ratios involving $\breve{b}$ are deleted if $C_Y$ is a subspace.

The remainder of this subsection is devoted to proving Theorem 5.1.1.

### 5.1.2

The proof of theorem relies on the following two propositions, the first of which does not require $C_X$ to be pointed.

**Proposition 5.1.2** *Assume* $d_P := (A, b)$ *satisfies* $\text{dist}(d_P, \text{Pri}) > 0$. *If* $(x, s, t) \in D_f \cap L$ *then*

$$\|x\|, s = O\left(t\left[\frac{\|d_P\| + \|\breve{b}\|}{\text{dist}(d_P, \text{Pri})}\right]\right).$$

**Proposition 5.1.3** *Assume* $C_X$ *has pointedness at least* $\alpha$ *where* $\alpha > 0$. *Let* $(x, s, t)$, $0 < t \le 1$, *denote an iterate computed in the second stage of the barrier method applied to (55). If* $d_P := (A, b) \in \text{Pri}$ *then either*

$$\|x\| = \Omega\left(\frac{\alpha}{K^2}\right) \quad \text{or} \quad s = \Omega\left(\frac{1}{K^2}\right). \tag{57}$$

Before proving the propositions we use them to prove Theorem 5.1.1.

**Proof of Theorem 5.1.1.** Assume the positive constant $C$ in (56) is chosen sufficiently small so that if $x$, $s$ and $t$ satisfy (56) then (57) is not satisfied. Proposition 5.1.3 then implies that the algorithm will not terminate if the input instance $d_P$ is consistent.

If $d_P$ is inconsistent then $t_{\text{inf}} = 0$. Consequently, if $\text{dist}(d_P, \text{Pri}) > 0$ then parts 5 and 8b of Theorem 2.1.1, Proposition 3.3 and Proposition 5.1.2 easily imply the method will terminate within the desired number of operations. □

### 5.1.3

The proof of Proposition 5.1.2 depends on the following lemma.

**Lemma 5.1.4** *If $(x, s, t) \in D_f \cap L$ and $x \neq \vec{0}$ then $d_P := (A, b)$ satisfies*

$$\text{dist}(d_P, \text{Pri}) = O\left( t[\|d_P\| + \|\breve{b}\|] \min\left\{ \frac{1}{\|x\|}, \frac{1}{s} \right\} \right).$$

**Proof.** Assume $(x, s, t) \in D_f \cap L$ and $x \neq \vec{0}$. Consider the instances

$$d'_P := (A, b + \tfrac{t}{2s}(A\breve{x} + \breve{b} - b))$$

and $d''_P := (A'', b)$ where $A''$ is the operator defined by

$$A''\bar{x} := A\bar{x} - \tfrac{t}{\|x\|^2}\langle x, \bar{x}\rangle(A\breve{x} + \breve{b} - b).$$

Since $(x, s, t) \in D_f \cap L$ it is easily verified that x is feasible for $d'_P$ and $\frac{1}{2s}x$ is feasible for $d''_P$. Hence,

$$\text{dist}(d_P, \text{Pri}) \leq \min\{\|d_P - d'_P\|, \|d_P - d''_P\|\}.$$

Observing that since $\|\breve{x}\| < 1$ ,

$$\|d_P - d'_P\| = O\left( \frac{t[\|d_P\| + \|\breve{b}\|]}{s} \right) \quad \text{and} \quad \|d_P - d''_P\| = O\left( \frac{t[\|d_P\| + \|\breve{b}\|]}{\|x\|} \right),$$

the proposition follows. □

**Proof of Proposition 5.1.2.** Immediate by Lemma 5.1.4. □

**5.1.4**

The proof of Proposition 5.1.3 depends on the following two lemmas, the proof of the second of which depends on the lemma following it.

**Lemma 5.1.5** *Assume $C_X$ has pointedness at least $\alpha$ where $\alpha > 0$. Assume $t \in I\!R$ satisfies*

$$S_t := \{(x, s, \bar{t}) \in D_f;\ \bar{t} = t\} \neq \emptyset.$$

*Define*

$$\rho_t := \sup\{||x||;\ \exists\, s \text{ such that } (x, s, t) \in D_f\}$$

*and*

$$\sigma_t := \sup\{s;\ \exists\, x \text{ such that } (x, s, t) \in D_f\}.$$

*If $z := (\bar{x}, \bar{s}, t) \in D_f$ then*

$$\operatorname{sym}(z, S_t) \leq \min\left\{\frac{||\bar{x}||}{\alpha\rho_t}, \frac{\bar{s}}{\sigma_t}\right\}.$$

**Proof.** The inequality

$$\operatorname{sym}(z, S_t) \leq \frac{\bar{s}}{\sigma_t}$$

is a simple consequence of the fact that all elements $(x, s, t) \in D_f$ satisfy $s > 0$. Similarly, the inequality

$$\operatorname{sym}(z, S_t) \leq \frac{||\bar{x}||}{\alpha\rho_t}$$

is a simple consequence of the fact that all elements $(x, s, t) \in D_f$ satisfy $x \in C_X$. □

**Lemma 5.1.6** *If $d_P := (A, b) \in \text{Pri}$ and $0 < t \leq 1$ then there exist $x$ and $s$ satisfying*

$$(x, s, t) \in D_f \cap L \text{ and } \max\{||x||, s\} = \Omega(1).$$

*If, in addition, $0 < t \leq \frac{1}{2}$ then*

$$\operatorname{diam}(S_t) = \Omega(1).$$

The proof of Lemma 5.1.6 depends on the following lemma.

**Lemma 5.1.7** *Assume $x \in S \subseteq U \subseteq V$ where $V$ is a normed vector space, $U$ is a subspace and $S$ is a convex set which is open in the relative topology of $U$. If $y \in U$ is in the closure of $S$ then*

$$y + t(x - y) \in S \text{ for all } 0 < t \leq 1.$$

**Proof.** Elementary. □

**Proof of Lemma 5.1.6.** Assume $x'$ is feasible for $d_P$ and let

$$z' := \frac{1}{\max\{2\|x'\|,\, 2\}}(x', 1, 0).$$

Let $M : \mathcal{H} \to X \times Y$ denote the linear operator defined by

$$M(x, s, t) := (x, -Ax + 2sb + t(A\breve{x} + \breve{b} - b)).$$

Noting that

$$Mw \in C_X^{\circ} \times C_Y^{\circ} \;\text{ and }\; Mz' \in C_X \times C_Y$$

where $w := (\breve{x}, \frac{1}{2}, 1)$ , Lemma 5.1.7 implies

$$M[rz' + t(w - rz')] \in C_X^{\circ} \times C_Y^{\circ} \;\text{ for all }\; 0 < t \leq 1 \;\text{ and }\; 0 < r \leq 1.$$

Consequently, relying on the definitions of $M$, $w$ and $z'$ it is easily proven that

$$tw + (1 - t)rz' \in S_t \;\text{ for all }\; 0 < t \leq 1 \;\text{ and }\; 0 < r \leq 1.$$

The lemma follows easily. □

**Proof of Proposition 5.1.3.** Assume $d_P \in$ Pri and assume $z := (x, s, t)$ is an iterate computed in the second stage of the barrier method applied to (55). Assume $0 < t \leq 1$. Define $S_t$, $\rho_t$ and $\sigma_t$ as in Lemma 5.1.5. Observe that Lemma 5.1.6 implies

$$\rho_t = \Omega(1) \;\text{ or }\; \sigma_t = \Omega(1).$$

Hence, by Lemma 5.1.5,

$$\|x\| = \Omega(\alpha\, \mathrm{sym}(z, S_t)) \;\text{ or }\; s = \Omega(\mathrm{sym}(z, S_t)).$$

However, part 6a of Theorem 2.1.1 implies

$$\mathrm{sym}(z, S_t) = \Omega\left(\frac{1}{K^2}\right)$$

and hence the proof is complete. □

## 5.2 The Second Algorithm

### 5.2.1

The second algorithm is much like the first, but does not require $C_X$ to be pointed. In place of the comparison (56) made for each iterate $v := (x, s, t)$ computed during the second stage of the barrier method we now make a comparison based on an approximation to the smallest non-zero value in the spectrum of $PH_vP$ where $H_v$ denotes the Hessian of $f$ at $v$ and $P$ denotes the operator which projects $\mathcal{H}_f := X \times I\!R \times I\!R$ orthogonally onto the subspace

$$\{(x, s, t) \in L;\ t = 0\}$$

recalling that

$$L := \{(x, s, t) \in \mathcal{H}_f;\ x \in L_X \text{ and } -Ax + 2sb + t(A\breve{x} + \breve{b} - b) \in L_Y\}.$$

Since $H_v$ is positive definite and the spectrum is compact there is indeed such a smallest non-zero element.

We assume that for each iterate $v := (x, s, t)$ which is computed during the second stage of the barrier method the smallest value $\lambda_v$ in the spectrum of $PH_vP$ can be bounded in a way that involves a quantity $\beta$ where $0 < \beta \leq 1$. Specifically, we assume that one can compute a quantity $\widehat{\lambda}_v$ satisfying

$$\beta\lambda_v \leq \widehat{\lambda}_v \leq \lambda_v.$$

We call this operation "$\beta$-bounding the spectrum". We do not assume the value $\beta$ is known, that is, it is not input for the algorithm.

In place of the comparisons (56) relied on by the first method we now make the comparisons

$$\widehat{\lambda}_v \geq CK^2 \text{ and } 0 < t \leq \tfrac{1}{2} \tag{58}$$

where $C$ is an appropriately large universal constant. Except for this change the second algorithm is identical to the first.

**Theorem 5.2.1** *Assume the positive constant $C$ in (58) satisfies $C = \Theta(1)$, i.e., a universal constant.*

*The second algorithm terminates only if the input $d_P := (A, b)$ is inconsistent. Moreover, if $\operatorname{dist}(d_P, \text{Pri}) > 0$ then the algorithm terminates within*

$$O\Bigg(K \log \Bigg[K + \frac{1}{\beta} + \frac{1}{\operatorname{reldist}(d_P, \text{Pri})} + \frac{1}{\operatorname{sym}(\breve{x}, C_X(1))} + \frac{1}{\operatorname{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|d_P\|, \|\breve{b}\|\}}{\min\{\|d_P\|, \|\breve{b}\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving $\breve{b}$ being deleted if $C_Y$ is a subspace.*

The remainder of this subsection is devoted to proving Theorem 5.2.1.

### 5.2.2

The proof of Theorem 5.2.1 depends on the following two propositions.

**Proposition 5.2.2** *Assume* $d_P := (A, b)$ *satisfies* $\text{dist}(d_P, \text{Pri}) > 0$. *If* $v := (x, s, t)$, $0 < t \le 1$, *is an iterate computed during the second stage of the interior point method applied to (55) then the smallest non-zero value* $\lambda_v$ *in the spectrum of* $PH_vP$ *satisfies*

$$\sqrt{\lambda_v} = \Omega\left(\frac{1}{t}\left[\frac{\text{dist}(d_P, \text{Pri})}{||d_P|| + ||\breve{b}||}\right]\right).$$

**Proof.** Assume $v := (x, s, t)$ is as in the statement of the proposition. Define

$$S_t := \{(\bar{x}, \bar{s}, \bar{t}) \in D_f;\ \bar{t} = t\}.$$

Part 6b of Theorem 2.1.1 implies

$$\sqrt{\lambda_v} = \Omega\left(\frac{1}{\text{diam}(S_t)}\right) \tag{59}$$

where $\text{diam}(S_t)$ denotes the diameter of $S_t$. However, Proposition 5.1.2 implies

$$\text{diam}(S_t) = O\left(t\left[\frac{||d_P|| + ||\breve{b}||}{\text{dist}(d_P, \text{Pri})}\right]\right). \tag{60}$$

Substituting (60) into (59) completes the proof. □

**Proposition 5.2.3** *Assume* $d_P := (A, b)$ *is consistent. If* $v := (x, s, t)$, $0 < t \le \frac{1}{2}$, *is an iterate computed during the second stage of the barrier method applied to (55) then the smallest non-zero element* $\lambda_v$ *in the spectrum of* $PH_vP$ *satisfies*

$$\sqrt{\lambda_v} = O(K^2). \tag{61}$$

**Proof.** Proceeds exactly as does the proof of Proposition 5.2.2 except that the lower bound on $\sqrt{\lambda_v}$ given by (59) is replaced by the upper bound

$$\sqrt{\lambda_v} = O\left(\frac{K^2}{\text{diam}(S_t)}\right),$$

given by part 6b of Theorem 2.1.1 , whereas the upper bound on $\text{diam}(S_t)$ given by (60) is replaced by the lower bound

$$\text{diam}(S_t) = \Omega(1)$$

which is implied by Lemma 5.1.6. □

**Proof of Theorem 5.2.1.** Assume the constant $C$ in (58) is chosen sufficiently large so that if $\widehat{\lambda}_v$ ($\leq \lambda_v$) satisfies (58) then (61) is not satisfied. Proposition 5.2.3 then implies that the algorithm will not terminate if the input instance $d_P$ is consistent.

Recall that if $d_P$ is inconsistent then $t_{\text{inf}} = 0$. Also recall that $\widehat{\lambda}_v \geq \beta\lambda_v$. Consequently, if $\text{dist}(d_P, \text{Pri}) > 0$ then parts 5 and 8b of Theorem 2.1.1, Proposition 3.3 and Proposition 5.2.2 easily imply that the method will terminate within the desired number of operations. □

## 5.3 The Third Algorithm

The third algorithm is obtained simply by applying the algorithm for determining dual inconsistency (presented in Section 4) to the dual rather than the primal. The algorithm then determines strong inconsistency in the dual of the dual which, assuming $X$ is reflexive and $C_X$ and $C_Y$ are closed, is the same as strong primal inconsistency.

More formally, assume $X$ and $Y$ are real vector spaces, $X$ being reflexive and $Y$ being a Hilbert space ( hence $Y$ is also reflexive). Assume $C_X$ and $C_Y$ are closed convex cones each with vertex at the origin. Assume $C_X^*$, $C_Y^*$ fulfill the conditions we generally assume for $C_X$ and $C_Y$, have underlying functionals $\phi \in \mathcal{F}'(K_X)$, $\psi \in \mathcal{F}'(K_Y)$. Assume that a value $K$ satisfying $K \geq (K_X^2 + K_Y^2 + 1)^{1/2}$ is known. Assume $\breve{y}^*$ is a known point in the relative interior of $C_Y^*$ (relative to the subspace topology of $L_Y^*$) satisfying $\|\breve{y}^*\| < 1$ and assume $\breve{c}^*$ is a known point in the relative interior of $C_X^*$; if $C_X^*$ is a subspace then we assume $\breve{c}^* = \vec{0}$.

Assuming $d_P := (A, b)$ is the system whose primal inconsistency is in question we apply the algorithm for determining dual inconsistency to the instance $d_D^* := (A^*, b^{**})$, $A^*$ denoting the dual operator of $A$ and $b^{**}$ denoting the continuous linear functional on $Y^*$ defined by $y^* \mapsto y^* b$. The instance $d_D^*$ is an element of the data space

$$\mathcal{L}(Y^*, X^*) \times X^{**}.$$

Since $X$ and $Y$ are reflexive, each instance in this data space is of the form $\bar{d}_D^* := (\bar{A}^*, \bar{b}^{**})$ for some

$$\bar{d}_P := (\bar{A}, \bar{b}) \in \mathcal{L}(X, Y) \times Y.$$

Moreover, as is easily verified,

$$\|\bar{d}_D^*\| := \max\{\|\bar{A}^*\|, \|b^{**}\|\} = \|\bar{d}_P\| := \max\{\|\bar{A}\|, \|b\|\}. \tag{62}$$

To say that "$\bar{d}_D^* := (\bar{A}^*, \bar{b}^{**})$ is consistent" is to say that the following system is consistent:

$$\begin{array}{c} \bar{b}^{**} - x^{**}\bar{A}^* \in C_Y^{**} \\ x^{**} \in C_X^{**} \\ x^{**} \in X^{**}. \end{array} \tag{63}$$

As is well-known (c.f., [3], Lemma 2.2), reflexivity of $X$ and closedness of $C_X$ and $C_Y$ imply the system (63) to have precisely the same solutions as the following system:

$$\begin{array}{c} b - Ax \in C_Y \\ x \in C_X \\ x \in X. \end{array}$$

Hence $\bar{d}_D^*$ is consistent if and only if $\bar{d}_P$ is consistent. In particular, $d_P$ is inconsistent if and only if $d_D^*$ is inconsistent. Consequently, applying the algorithm for determining dual inconsistency to $d_D^*$ is appropriate for determining the primal consistency of $d_D$ under the above assumptions.

It follows from the preceding discussion that

$$\text{dist}(d_P, \text{Pri}) = \text{dist}(d_D^*, \text{Dual}^*) \tag{64}$$

where Dual$^*$ denotes the set of instances $\bar{d}_D^*$ which are inconsistent. Thus, although when we apply the algorithm for determining dual inconsistency to $d_D^*$, the resulting operation bounds implied by Theorem 4.2 involve the quantity $\text{dist}(d_D^*, \text{Dual}^*)$ we may substitute $\text{dist}(d_P, \text{Pri})$ in its place. Similarly, (62) implies we may substitute $\|d_P\|$ for $\|d_D^*\|$.

In light of the preceding discussion the following theorem is now immediate from Theorem 4.2.

**Theorem 5.3.1** *The third algorithm terminates only if the input $d_P := (A, b)$ is strongly inconsistent. Moreover, if* $\text{dist}(d_P, \text{Pri}) > 0$ *then the algorithm terminates within*

$$O\left(K \log \left[K + \frac{1}{\text{reldist}(d_P, \text{Pri})} + \frac{1}{\text{sym}(\breve{y}^*, C_Y^*(1))} + \frac{1}{\text{sym}(\breve{c}^*, C_X^*(2\|\breve{c}^*\|))} + \frac{\max\{\|d_P\|, \|\breve{c}^*\|\}}{\min\{\|d_P\|, \|\breve{c}^*\|\}}\right]\right)$$

*iterations, the ratios involving $\breve{c}^*$ being deleted if $C_X^*$ is a subspace.*

# 6 Dual Asymptotic Consistency

We relied on Proposition 4.1, a so-called "theorem of the alternatives", to easily obtain an algorithm for determining dual strong inconsistency from an algorithm for determining primal consistency. Now we apply that proposition to easily obtain an algorithm for determining dual asymptotic consistency from the the first two algorithms for determining primal inconsistency; although the same can be done for the third algorithm, the assumptions relied on for that algorithm support unaltered application of the algorithm for determining primal consistency to the dual rather than the primal, as is briefly discussed in Section 6.2.

In Section 7, where we consider optimization, we obtain as a byproduct yet another dual asymptotic consistency recognition method, that one requiring an initial strictly feasible point be known.

## 6.1 The First and Second Algorithms

The development and analysis of these algorithms proceeds almost identically to the development in Section 4, the main differences being that some substitutions of terms are required, and reliance on the theorem in Section 3 is replaced with reliance on the theorems in Section 5. In the following we only highlight the differences, assuming the reader has read Sections 4 and 5.

The algorithms are obtained simply by respectively applying the first two algorithms of Section 5 to the system (45), termination occurring if and only if (45) is found to be inconsistent. The correctness of this approach is immediate from Proposition 4.1.

For the second algorithm, " $\beta$-bounding the spectrum " is now with regards to smallest non-zero element in the spectrum of the Hessian operators of the functional associated with the system (45) rather than that associated with the system $Ax \leq b$, $x \geq \vec{0}$.

In analyzing the algorithms one proceeds exactly as in Section 5 except for the following changes:

1. The set DualA is replaced by DualS$\emptyset$ consisting of strongly inconsistent instances $(A, c^*)$.
2. The set $\widehat{\text{Pri}\emptyset}$ is replaced by its complement $\widehat{\text{Pri}}$.
3. The terms "asymptotically consistent" and "strongly inconsistent" are interchanged as are the terms "consistent" and "inconsistent".
4. Reference to Theorem 3.1 is replaced by reference to Theorem 5.1.1 or Theorem 5.2.1 depending on which algorithm for determining primal inconsistency is being used.

With these changes, all proofs go through verbatim.

**Theorem 6.1.1** *Assume $C_X$ has pointedness at least $\alpha > 0$, where $\alpha$ is known, and assume the positive constant $C$ in (56) satisfies $C = \Theta(1)$.*

*The first algorithm terminates only if the input $d_D := (A, c^*)$ is asymptotically consistent. Moreover, if* $\mathrm{dist}(d_D, \mathrm{DualS}\emptyset) > 0$ *then the algorithm terminates within*

$$O\Bigg(K \log \Bigg[K + \frac{1}{\alpha} + \frac{1}{\mathrm{reldist}(d_D, \mathrm{DualS}\emptyset)} + \frac{1}{\mathrm{sym}(\breve{x}, C_X(1))} + \frac{1}{\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|d_D\|, \|\breve{b}\|\}}{\min\{\|d_D\|, \|\breve{b}\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving $\breve{b}$ being deleted if $C_Y$ is a subspace.*

**Theorem 6.1.2** *Assume the positive constant $C$ in (58) satisfies $C = \Theta(1)$.*

*The second algorithm terminates only if the input $d_D := (A, c^*)$ is asymptotically consistent. Moreover, if* $\mathrm{dist}(d_D, \mathrm{DualS}\emptyset) > 0$ *then the algorithm terminates within*

$$O\Bigg(K \log \Bigg[K + \frac{1}{\beta} + \frac{1}{\mathrm{reldist}(d_D, \mathrm{DualS}\emptyset)} + \frac{1}{\mathrm{sym}(\breve{x}, C_X(1))} + \frac{1}{\mathrm{sym}(\breve{b}, C_Y(2\|\breve{b}\|))} + \frac{\max\{\|d_D\|, \|\breve{b}\|\}}{\min\{\|d_D\|, \|\breve{b}\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving $\breve{b}$ being deleted if $C_Y$ is a subspace.*

## 6.2 The Third Algorithm

The third algorithm, which determines consistency rather than asymptotic consistency, relies on the assumptions of Section 5.3. Assuming $d_D := (A, c^*)$ is the instance whose consistency is in question, we simply apply the algorithm for determining primal consistency directly to $d_D$. Relying on elementary observations analogous to those used in establishing (62) and (64), the resulting iteration bound given by Theorem 5.3.1 can be expressed in terms of $\|d_D\|$ and $\mathrm{dist}(d_D, \mathrm{Dual}\emptyset)$, where $\mathrm{Dual}\emptyset$ is the set of dual inconsistent instance. We thus obtain the following theorem.

**Theorem 6.2.1** *The third algorithm terminates only if the input* $d_D := (A, c^*)$ *is consistent. Moreover, if* $\text{dist}(d_D, \text{Dual}\emptyset) > 0$ *then the algorithm terminates within*

$$O\Bigg(K \log \Bigg[K + \frac{1}{\text{reldist}(d_D, \text{Dual}\emptyset)} + \frac{1}{\text{sym}(\breve{y}^*, C_Y^*(1))} + \frac{1}{\text{sym}(\breve{c}^*, C_X^*(2\|\breve{c}^*\|))} + \frac{\max\{\|d_D\|, \|\breve{c}^*\|\}}{\min\{\|d_D\|, \|\breve{c}^*\|\}}\Bigg]\Bigg)$$

*iterations, the ratios involving* $\breve{c}^*$ *being deleted if* $C_X^*$ *is a subspace.*

# 7 Optimization

## 7.1

We now consider the efficiency of the barrier method in computing feasible $x$ for which $c^*x$ is close to the optimal value of

$$\begin{array}{ll} \sup & c^*x \\ \text{s.t.} & Ax \leq b \\ & x \geq \vec{0}. \end{array} \tag{65}$$

We consider the method as applied to

$$\begin{array}{ll} \sup & c^*x \\ \text{s.t.} & Ax < b \\ & c^*x > c^*\bar{x} - \breve{s} \\ & x > \vec{0}. \end{array} \tag{66}$$

where $\bar{x}$ is assumed to be a known point which is strictly feasible for (65) and where $\breve{s}$ is an arbitrary positive constant. We assume the method is initiated at $\bar{x}$. We consider both arbitrary $\bar{x}$ and $\bar{x}$ as computed by the algorithm for determining primal consistency, i.e., the algorithm in Section 3.

Let $d := (A, b, c^*)$ denote the data vector for (65) and define val$(d)$ to be the optimal value of (65). Let

$$\text{Feas}(d) := \{x \in X;\ Ax \leq b \ \text{ and } \ x \geq \vec{0}\},$$

the set of feasible points, and let

$$\text{SFeas}(d) := \{x \in X;\ Ax < b \ \text{ and } \ x > \vec{0}\},$$

the set of strictly feasible points. We continue to use the notation $d_P := (A, b)$ and $d_D := (A, c^*)$.

Recasting (66) into the framework of Theorem 2.1.1, we define

$$\mathcal{H}_f := X,$$

$$D_f := \{x \in \text{SFeas}(d); c^*x > c^*\bar{x} - \breve{s}\},$$

$$f_x := \phi_x + \psi_{b-Ax} - 12\ln(c^*x - [c^*\bar{x} - \breve{s}])$$

and

$$L := \{x \in X; x \in L_X \ \text{ and } \ Ax \in L_y\}.$$

If we let $w := \bar{x}$ then these definitions allow (66) to be rewritten as

$$\begin{array}{ll} \sup & c^*x \\ \text{s.t.} & x \in D_f \cap (L + w), \end{array}$$

conforming to the notation for the optimization mode of the barrier method of Theorem 2.1.1.

The barrier method applied to (66) may not be well-defined in the sense that the method involves solving linear operator equations for which the operator may not be invertible. Although our results pertain to situations where this does not occur, for definiteness one might assume that if a non-invertible operator equation is encountered then the method stalls and makes no further computations. (In earlier sections, the barrier method was always applied in situations for which the domain $D_f$ was bounded (due to additional constraints like $||x|| < 1$, etc.); consequently non-invertibility was never as issue.)

If the interior point method applied to (66) is well-defined then, by part 5 of Theorem 2.1.1, stage one of the method terminates if and only if the feasible region for (66) is bounded. If the feasible region is bounded then $\text{val}(d)$ is finite, hence $d_D$ is asymptotically consistent. Thus, stage one of the interior point method can be considered as an algorithm for determining dual asymptotic consistency; if stage one terminates then the instance is dual asymptotically consistent. (Of course this algorithm requires that appropriate primal feasible $\bar{x}$ be available whereas those of Section 6 do not.)

Since $\bar{x}$ is optimal if $c^* = \vec{0}$ it is natural to assume, as we do, that $c^* \neq \vec{0}$.

The relative distance of $d$ to the set of dual strongly inconsistent instances is especially important in our analysis:

$$\text{reldist}(d, \text{DualS}\emptyset) := \frac{\text{dist}(d_D, \text{DualS}\emptyset)}{||d||}.$$

(If $\text{DualS}\emptyset = \emptyset$ then we define $\text{dist}(d_D, \text{DualS}\emptyset) = ||d_D||$ to avoid discussing special cases.) The analysis is correct for either of two definitions of $||d||$:

$$||d|| := \max\{||A||, ||b||, ||c^*||\} \quad \text{or} \quad ||d|| := \max\{||b||, ||c^*||\}.$$

The first definition is more in line with tradition but the second provides stonger results in some cases. (In the latter definition we are slightly abusing norm notation.)

The initial point $\bar{x}$ enters into our operation bounds in various ways. One way it enters is in terms of its proximity to the boundary of $\text{Feas}(d)$ relative to other feasible points. Since $\text{Feas}(d)$ may well be unbounded we do not measure the proximity of $\bar{x}$ to the boundary relative to all other feasible points, but rather only relative to feasible points within some radius of the origin, assuming the radius is greater than $||\bar{x}||$. Thus, letting $\gamma > 0$, our operation bounds depend on the quantity $\text{sym}(\bar{x}, S_\gamma)$ where

$$S_\gamma := \{x \in \text{SFeas}(d);\ ||x|| < ||\bar{x}|| + \gamma(||\bar{x}|| + 1)\}.$$

If $\bar{x}$ is as computed by the algorithm (of Section 3) for determining primal consistency then Theorem 3.1 shows that there exists $\gamma$ satisfying

$$\gamma = \Omega\left(\frac{1}{K^2}\right) \quad \text{and} \quad \mathrm{sym}(\bar{x}, S_\gamma) = \Omega\left(\frac{1}{K^2}\right).$$

Another way the intial point $\bar{x}$ enters into our bounds is in relation to the positive constant $\breve{s}$ appearing in (66). The choice of $\breve{s}$ is unrestricted but its choice does affect the operation bounds. In particular, its size relative to $||\bar{x}||$ and $||d||$ is important.

In the following theorem we weaken our customary assumption that the norm on the Hilbert space $X$ be the norm defined by the inner product, requiring instead that the norm only induce the same topology as that defined by an inner product which makes $X$ into a Hilbert space, i.e., be compatible with such an inner product.

**Theorem 7.1** *Assume the norm on $X$ is compatible with an inner product making $X$ into a Hilbert space. Assume $\bar{x}$ is strictly feasible for $d := (A, b, c^*)$ and* $\mathrm{dist}(d_D, \mathrm{DualS}\emptyset) > 0$. *Assume $\gamma, \epsilon > 0$.*

*If the barrier method is initiated at $\bar{x}$ then the method computes a strictly feasible point $x$ known to satisfy*

$$\frac{\mathrm{val}(d) - c^* x}{\mathrm{val}(d) - (c^*\bar{x} - \breve{s})} \le \epsilon$$

*within*

$$O\Bigg(K \log\Bigg[K + \frac{1}{\epsilon} + \frac{1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)} + \frac{1}{\mathrm{sym}(\bar{x}, S_\gamma)} + \max\left\{1,\ \frac{||d||(||\bar{x}|| + 1)}{\breve{s}},\ \frac{1}{\gamma} + \frac{\breve{s}}{\gamma||d||(||\bar{x}|| + 1)}\right\}\Bigg]\Bigg)$$

*iterations.*

The proof of Theorem 7.1 is deferred until later in the section.

We return to assuming that the norm on $X$ is that defined by an inner product making $X$ into a Hilbert space.

**Corollary 7.2** *Assume $\epsilon > 0$ and* $\mathrm{dist}(d_D, \mathrm{DualS}\emptyset) > 0$ *where $d := (A, b, c^*)$ . Assume that when the algorithm (of Section 3) for determing primal consistency is applied to $d_P$, it terminates, thus providing a strictly feasible point $\bar{x}$.*

*If the barrier method is initiated at $\bar{x}$ then the method computes a strictly feasible point $x$ known to satisfy*

$$\frac{\mathrm{val}(d) - c^* x}{\mathrm{val}(d) - (c^* \bar{x} - \breve{s})} \leq \epsilon$$

*within*

$$O\left(K \log\left[K + \frac{1}{\epsilon} + \frac{1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)} + \frac{\max\{\breve{s},\ \|d\|(\|\bar{x}\| + 1)\}}{\min\{\breve{s},\ \|d\|(\|\bar{x}\| + 1)\}}\right]\right)$$

*iterations.*

**Proof.** Follows immediately from Theorem 7.1 since, as noted in the preceding discussion, Theorem 3.1 implies there exists $\gamma$ satisfying

$$\gamma = \Omega\left(\frac{1}{K^2}\right) \text{ and } \mathrm{sym}(\bar{x}, S_\gamma) = \Omega\left(\frac{1}{K^2}\right).$$

□

The next corollary, like the previous one, considers $\bar{x}$ as computed by the algorithm of Section 3. However, unlike the previous corollary, $\bar{x}$ does not enter explicitly into the bounds.

**Corollary 7.3** *Assume $\epsilon > 0$, $\mathrm{dist}(d_D, \mathrm{DualS}\emptyset) > 0$ and $\mathrm{dist}(d_P, \mathrm{Pri}\emptyset) > 0$ where $d := (A, b, c^*)$. Assume that when the algorithm (of Section 3) for determing primal consistency is applied to $d_P$, it terminates, thus providing a strictly feasible point $\bar{x}$.*

*If the barrier method is initiated at $\bar{x}$ then the method computes a strictly feasible point $x$ known to satisfy*

$$\frac{\mathrm{val}(d) - c^* x}{\|d\|} \leq \epsilon \tag{67}$$

*within*

$$O\left(K \log\left[K + \frac{1}{\epsilon} + \frac{1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)} + \frac{1}{\mathrm{reldist}(d_P, \mathrm{Pri}\emptyset)} + \frac{\max\{\breve{s},\ \|d\|\}}{\min\{\breve{s},\ \|d\|\}}\right]\right)$$

*iterations.*

**Proof.** Letting

$$\epsilon' := \epsilon / \left(\frac{1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)} + \frac{1}{\mathrm{reldist}(d_P, \mathrm{Pri}\emptyset)} + \frac{\breve{s}}{\|d\|}\right)$$

consider the iteration bound provided by Corollary 7.2 for computing a feasible point $x$ satisfying

$$\frac{\text{val}(d) - c^* x}{\text{val}(d) - (c^* \bar{x} - \breve{s})} \leq \epsilon'. \tag{68}$$

Since, by Proposition 2.3.1,

$$\|\bar{x}\| \leq \frac{\|b\|}{\text{reldist}(d_P, \text{Pri}\emptyset)} \tag{69}$$

those bounds are easily verified not to exceed the stated operation bounds of the present corollary. Thus, to prove the present corollary it suffices to show that (68) implies (67). However, this implication is a simple consequence of the fact that

$$\begin{aligned}
\text{val}(d) - (c^* \bar{x} - \breve{s}) &\leq \text{val}(d) + \|c^*\| \, \|\bar{x}\| + \breve{s} \\
&\leq \frac{\|b\| \, \|c^*\|}{\text{dist}(d_D, \text{DualS}\emptyset)} + \frac{\|b\| \, \|c^*\|}{\text{dist}(d_P, \text{Pri}\emptyset)} + \breve{s}
\end{aligned}$$

as implied by Proposition 2.3.1 and (69). □

**Corollary 7.4** *Assume $\epsilon > 0$ and let $d := (A, b, c^*)$. If* $\text{dist}(d_D, \text{DualS}\emptyset) > 0$ *and* $\text{dist}(d_P, \text{Pri}\emptyset) > 0$ *then the primal consistency of $d$ can be determined, and a strictly feasible point $x$ known to satisfy*

$$\frac{\text{val}(d) - c^* x}{\|d\|} \leq \epsilon$$

*can be computed, all within*

$$O\left(K \log \left[K + \frac{1}{\epsilon} + \frac{1}{\text{reldist}(d, \text{DualS}\emptyset)} + \frac{1}{\text{reldist}(d_P, \text{Pri}\emptyset)} + \frac{\max\{\breve{s}, \ \|d\|\}}{\min\{\breve{s}, \ \|d\|\}}\right]\right)$$

*iterations, where $\breve{s}$ is an arbitrary positive constant chosen as part of the input for an algorithm.*

**Proof.** Simply combine Theorem 3.1 and Corollary 7.3. □

In Corollary 7.3 the ratio (67) which measures the closeness of $c^*x$ to $\text{val}(d)$ does not depend on $\bar{x}$ whereas the corresponding ratio in Theorem 7.1 and Corollary 7.2 does depend on $\bar{x}$. The price we paid for removing $\bar{x}$ was dependence of the iteration bound on $\text{reldist}(d, \text{Pri}\emptyset)$ . If one measures the proximity of $c^*x$ to $\text{val}(d)$ in a slightly different manner than (67), but in a way that is still independent of $\bar{x}$, then dependence of the iteration bound on $\text{dist}(d, \text{Pri}\emptyset)$ can be avoided, as we now show. The resulting iteration bound does depend on $\bar{x}$ but only in relation to $\breve{s}$; by appropriate choice of $\breve{s}$ the dependence of the iteration bound on $\bar{x}$ vanishes.

To measure how close the objective value $c^*x$ of a feasible point $x$ is to the optimal value val($d$) we now consider the following ratio:

$$\frac{\text{val}(d) - c^*x}{\max\{||d||, -\text{val}(d)\}}. \tag{70}$$

This ratio, which is appropriately invariant under positive scaling of $d$, can be viewed as a mixture of a sort of "absolute error" (when $||d|| \geq -\text{val}(d)$), and relative error (when $-\text{val}(d) \geq ||d||$).

In considering the ratio (70) we introduce a parameter, denoted $\rho$, related to $||\bar{x}||$. The value $\rho$ is assumed to satisfy $\rho \geq 1$ as well as at least one of the following alternatives:

$$||\bar{x}|| \leq \rho \quad \text{or} \quad \{x \in \text{Feas}(d);\ \rho||x|| < ||\bar{x}||\} = \emptyset. \tag{71}$$

If $\bar{x}$ is as computed by the algorithm for determining primal consistency then Theorem 3.1 shows $\rho = O(K^2)$ is appropriate.

**Theorem 7.5** *Assume the norm on $X$ is compatible with an inner product making $X$ into a Hilbert space. Assume $\bar{x}$ is strictly feasible for $d := (A, b, c^*)$ and* $\text{dist}(d_D, \text{DualS}\emptyset) > 0$. *Assume $\gamma, \epsilon > 0$ and assume $\rho \geq 1$ satisfies (71).*

*If the barrier method is initiated at $\bar{x}$ then the method computes a strictly feasible point $x$ known to satisfy*

$$\frac{\text{val}(d) - c^*x}{\max\{||d||, -\text{val}(d)\}} \leq \epsilon$$

*within*

$$O\Bigg(K \log\Bigg[K + \rho + \frac{1}{\epsilon} + \frac{1}{\text{reldist}(d, \text{DualS}\emptyset)} + \frac{1}{\text{sym}(\bar{x}, S_\gamma)} + \max\left\{1, \frac{||d||(||\bar{x}|| + 1)}{\breve{s}}, \frac{1}{\gamma} + \frac{\breve{s}}{\gamma||d||(||\bar{x}|| + 1)}\right\}\Bigg]\Bigg)$$

*iterations.*

The proof of Theorem 7.5 is deferred until later in the section.

**Corollary 7.6** *Assume $\epsilon > 0$ and* $\text{dist}(d_D, \text{DualS}\emptyset) > 0$ *where $d := (A, b, c^*)$. Assume that when the algorithm (of Section 3) for determining primal consistency is applied to $d_P$, it terminates, thus providing a strictly feasible point $\bar{x}$.*

*If the barrier method is initiated at $\bar{x}$ then the method computes a feasible point $x$ known to satisfy*

$$\frac{\text{val}(d) - c^*x}{\max\{\|d\|, -\text{val}(d)\}} \leq \epsilon$$

*within*

$$O\left(K \log\left[K + \frac{1}{\epsilon} + \frac{1}{\text{reldist}(d, \text{DualS}\emptyset)} + \frac{\max\{\check{s},\ \|d\|(\|\bar{x}\|+1)\}}{\min\{\check{s},\ \|d\|(\|\bar{x}\|+1)\}}\right]\right)$$

*iterations.*

**Proof.** Follows immediately from Theorem 7.5 since, as noted in the preceding discussion, Theorem 3.1 implies there exists $\gamma$ such that

$$\gamma = \Omega\left(\frac{1}{K^2}\right) \text{ and } \text{sym}(\bar{x}, S_\gamma) = \Omega\left(\frac{1}{K^2}\right),$$

and also implies that $\rho = O(K^2)$ satisfies the appropriate requirements. $\square$

The remainder of this section is devoted to proving Theorems 7.1 and 7.5.

## 7.2

The proof of Theorem 7.1 depends on the following proposition.

**Proposition 7.7** *Assume $C_X$, $C_Y$ are closed and $X$ is reflexive. Assume $s, \gamma > 0$ and $\bar{x} \in \text{SFeas}(d)$ where $d := (A, b, c^*)$. If*

$$S_\gamma := \{x \in \text{SFeas}(d);\ \|x\| < \|\bar{x}\| + \gamma(\|\bar{x}\| + 1)\}$$

*and*

$$T := \{x \in \text{SFeas}(d);\ c^*x > c^*\bar{x} - s\}$$

*then*

$$\text{sym}(\bar{x}, T) =$$

$$\Omega\left(\text{sym}(\bar{x}, S)\,\text{reldist}(d, \text{DualS}\emptyset)\,\min\left\{1,\ \frac{s}{\|d\|(\|x\|+1)},\ \frac{\gamma}{1 + \frac{s}{\|d\|(\|x\|+1)}}\right\}\right).$$

Before proving Proposition 7.7 we use it to prove Theorem 7.1.

**Proof of Theorem 7.1.** The proof is a simple application of parts 5 and 7 of Theorem 2.1.1 and Proposition 7.7 noting that in the present context the domain

$D_f$ of Theorem 2.1.1 is precisely the the set $T$ as defined in Proposition 7.7, and the quantity $t_{\sup} - t_{\inf}$ of Theorem 2.1.1 equals $\mathrm{val}(d) - (c^*\bar{x} - \check{s})$. □

The proof of Proposition 7.7 depends on the following lemma.

**Lemma 7.8** *Assume $C_X$, $C_Y$ are closed and $X$ is reflexive. Assume $s \geq 0$ and $\bar{x} \in \mathrm{Feas}(d)$ where $d := (A, b, c^*)$. If $v \in X$ satisfies*

$$\bar{x} + v \in \mathrm{Feas}(d) \quad \textit{and} \quad c^*v \geq -s$$

*then*

$$\|v\| = O\left(\frac{\|\bar{x}\| + 1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)}\left[1 + \frac{s}{\|d\|(\|\bar{x}\| + 1)}\right]\right).$$

**Proof.** If $\bar{x} + v \in \mathrm{Feas}(d)$ then part c of Proposition 2.3.1 implies

$$\|\bar{x} + v\| \leq \frac{\max\{\|b\|, -c^*(\bar{x} + v)\}}{\mathrm{dist}(d, \mathrm{DualS}\emptyset)}. \tag{72}$$

If $c^*v \geq -s$, and hence $-c^*(\bar{x} + v) \leq \|c^*\|\,\|\bar{x}\| + s$, it easily follows from (72) that

$$\|\bar{x} + v\| \leq \frac{\|\bar{x}\| + 1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)}\left(1 + \frac{s}{\|d\|(\|\bar{x}\| + 1)}\right).$$

Since $\mathrm{reldist}(d, \mathrm{DualS}\emptyset) \leq 1$ the lemma is now immediate. □

**Proof of Proposition 7.7.** To prove the proposition it suffices to show that if $r$ satisfies

$$0 \leq r < \mathrm{sym}(\bar{x}, S) \tag{73}$$

and if $t$ satisfies bounds of the form

$$0 \leq t = O\left(\mathrm{reldist}(d, \mathrm{DualS}\emptyset)\min\left\{1, \frac{s}{\|d\|(\|x\| + 1)}, \frac{\gamma}{1 + \frac{s}{\|d\|(\|x\|+1)}}\right\}\right) \tag{74}$$

then

$$v \in X \text{ and } \bar{x} + v \in T \quad \Rightarrow \quad \bar{x} - rtv \in S \text{ and } rtc^*v < s. \tag{75}$$

Assume $r$ satisfies (73) and assume $v \in X$ satisfies $\bar{x} + v \in T$.

If $t$ satisfies

$$0 \leq t = O\left(\frac{\gamma\,\mathrm{reldist}(d, \mathrm{DualS}\emptyset)}{1 + \frac{s}{\|d\|(\|x\|+1)}}\right),$$

Lemma 7.8 implies $\bar{x} + tv \in S$ and hence, since $r$ satisfies (73), the definition of $\mathrm{sym}(\bar{x}, S)$ implies

$$\bar{x} - rtv \in S.$$

In proving

$$rtc^*v < s \tag{76}$$

we may assume $c^*v > 0$. Consequently, Lemma 7.8 applied as $s \downarrow 0$, implies

$$||v|| = O\left(\frac{||x|| + 1}{\text{reldist}(d, \text{DualS}\emptyset)}\right)$$

and thus

$$c^*v = O\left(\frac{||d||(||x|| + 1)}{\text{reldist}(d, \text{DualS}\emptyset)}\right).$$

If $t$ satisfies

$$0 \leq t = O\left(\frac{s\,\text{reldist}(d, \text{DualS}\emptyset)}{||d||(||x|| + 1)}\right)$$

it follows that $tc^*v < s$ . Since $r < 1$, (76) is now immediate. □

## 7.3

The proof of Theorem 7.5 depends on the following proposition.

**Proposition 7.9** *Assume $C_X$, $C_Y$ are closed and $X$ is reflexive. Assume $\rho \geq 1$ , $s \geq 0$ and $\bar{x} \in \text{Feas}(d)$ where $d := (A, b, c^*)$. If $\bar{x}$ satisfies*

$$||\bar{x}|| \leq \rho \quad or \quad \{x \in \text{Feas}(d);\ \rho||x|| < ||\bar{x}||\} = \emptyset \tag{77}$$

*then*

$$\frac{\text{val}(d) - (c^*\bar{x} - s)}{\max\{||d||, -\text{val}(d)\}} = O\left(\frac{\rho}{[\text{reldist}(d, \text{DualS}\emptyset)]^2}\left[1 + \frac{s}{||d||(||\bar{x}|| + 1)}\right]\right). \tag{78}$$

Before proving Proposition 7.9 we use it to prove Theorem 7.5.

**Proof of Theorem 7.5.** Proposition 7.9 implies that if $d$, $\bar{x}$ and $\breve{s}$ satisfy the assumptions stated in the theorem then for all $x \in \text{Feas}(d)$ ,

$$\frac{\text{val}(d) - c^*x}{\max\{||d||, -\text{val}(d)\}} \leq C\frac{\text{val}(d) - c^*x}{\text{val}(d) - (c^*\bar{x} - \breve{s})}$$

where

$$C = O\left(\frac{\rho}{[\text{reldist}(d, \text{DualS}\emptyset)]^2}\left[1 + \frac{\breve{s}}{||d||(||\bar{x}|| + 1)}\right]\right).$$

Consequently, the theorem follows immediately from Theorem 7.1. □

The proof of Proposition 7.9 depends on the following proposition.

**Proposition 7.10** *Assume $C_X$, $C_Y$ are closed and $X$ is reflexive. Assume $d := (A, b, c^*)$ is an instance satisfying $c^* \neq \vec{0}$. If $x \in \text{Feas}(d)$ satisfies $||x|| \geq 1$ then*

$$\text{reldist}(d, \text{DualS}\emptyset) = O\left(\max\left\{\frac{-c^*x}{||d||\,||x||}, \frac{1}{\sqrt{||x||}}\right\}\right).$$

**Proof.** It suffices to prove that under the stated assumptions if

$$\text{reldist}(d, \text{DualS}\emptyset) \geq 3\max\left\{\frac{-c^*x}{||d||\,||x||}, \frac{1}{\sqrt{||x||}}\right\} \tag{79}$$

then

$$\text{reldist}(d, \text{DualS}\emptyset) = O\left(\frac{1}{\sqrt{||x||}}\right). \tag{80}$$

Assuming $x$ satisfies the assumptions of the proposition as well as (79) consider the instance $\bar{d} := (A, b, \bar{c}^*)$ where

$$\bar{c}^* := c^* + 2\max\left\{\frac{-c^*x}{||x||^2}, \frac{||c^*||}{||x||^{3/2}}\right\} x^*,$$

$x^*$ denoting a functional satisfying $||x^*|| = ||x||$ and $x^*x = ||x||^2$; the Hahn-Banach Theorem implies $x^*$ to exist. Observe that,

$$||\bar{d} - d|| = 2\max\left\{\frac{-c^*x}{||x||}, \frac{||c^*||}{\sqrt{||x||}}\right\}. \tag{81}$$

Moreover, (79) and (81) imply

$$\begin{aligned}
\text{dist}(\bar{d}, \text{DualS}\emptyset) &\geq \text{dist}(d, \text{DualS}\emptyset) - ||\bar{d} - d|| \\
&= \text{dist}(d, \text{DualS}\emptyset) - 2\max\left\{\frac{-c^*x}{||x||}, \frac{||c^*||}{\sqrt{||x||}}\right\} \\
&\geq ||d||\left[\text{reldist}(d, \text{DualS}\emptyset) - 2\max\left\{\frac{-c^*x}{||d||\,||x||}, \frac{1}{\sqrt{||x||}}\right\}\right] \\
&\geq \tfrac{1}{3}||d||\,\text{reldist}(d, \text{DualS}\emptyset) \\
&= \tfrac{1}{3}\text{dist}(d, \text{DualS}\emptyset)
\end{aligned} \tag{82}$$

Also note that $x \in \text{Feas}(\bar{d})$ and hence

$$\begin{aligned}
\text{val}(\bar{d}) &\geq \bar{c}^*x \\
&= c^*x + 2\max\{-c^*x, ||c^*||\sqrt{||x||}\} \\
&\geq ||c^*||\sqrt{||x||}.
\end{aligned} \tag{83}$$

Proposition 2.3.1 shows

$$\mathrm{val}(\bar{d}) \leq \frac{||b|| \, ||\bar{c}^*||}{\mathrm{dist}(\bar{d}, \mathrm{DualS}\emptyset)}.$$

Substituting by means of (81), (82) and (83) then rearranging and simplifying yields

$$\mathrm{dist}(d, \mathrm{DualS}\emptyset) \leq \frac{6||b||}{\sqrt{||x||}}$$

and hence (80). □

**Proof of Proposition 7.9.** Since

$$0 < \mathrm{reldist}(d, \mathrm{DualS}\emptyset) \leq 1 \leq \rho \tag{84}$$

to prove the proposition it suffices to show

$$\frac{\mathrm{val}(d)}{\max\{||d||, -\mathrm{val}(d)\}} \leq \frac{1}{\mathrm{reldist}(d, \mathrm{DualS}\emptyset)}, \tag{85}$$

$$\frac{-c^* \bar{x}}{\max\{||d||, -\mathrm{val}(d)\}} = O\left(\frac{\rho}{[\mathrm{reldist}(d, \mathrm{DualS}\emptyset)]^2}\right) \tag{86}$$

and

$$\frac{s}{\max\{||d||, -\mathrm{val}(d)\}} = O\left(\frac{\rho s}{||d||(||\bar{x}|| + 1)[\mathrm{reldist}(d, \mathrm{DualS}\emptyset)]^2}\right). \tag{87}$$

The relation (85) is implied by Proposition 2.3.1 so we need only prove (86) and (87).

If $||\bar{x}|| \leq \rho$ then (86) and (87) are easy consequences of (84). Henceforth we assume $||\bar{x}|| > \rho$ and thus, by (77),

$$x \in \mathrm{Feas}(d) \quad \Rightarrow \quad ||x|| \geq \frac{||\bar{x}||}{\rho} \geq 1. \tag{88}$$

The proof depends on two cases corresponding to the following alternatives:

$$\forall\, x \in \mathrm{Feas}(d), \quad -c^* x \geq ||d|| \sqrt{||x||}; \tag{89}$$

$$\exists\, x \in \mathrm{Feas}(d), \quad -c^* x < ||d|| \sqrt{||x||}.$$

Assume (89). Observe that (88), (89) and Proposition 7.10 imply

$$\forall\, x \in \mathrm{Feas}(d) \quad -c^* x = \Omega(||d|| \, ||x|| \mathrm{reldist}(d, \mathrm{DualS}\emptyset)). \tag{90}$$

Let $\{x_i\}$ denote a sequence satisfying

$$\{x_i\} \subseteq \mathrm{Feas}(d) \quad \text{and} \quad c^* x_i \to \mathrm{val}(d). \tag{91}$$

Note that (88),(89) and (91) imply

$$-\operatorname{val}(d) \geq \|d\|. \tag{92}$$

Relying on (88), (90) and (92) we have for all $i$,

$$\begin{aligned}
\frac{-c^*\bar{x}}{\max\{\|d\|, -\operatorname{val}(d)\}} &= \frac{c^*\bar{x}}{\operatorname{val}(d)} \\
&= \frac{c^*x_i}{\operatorname{val}(d)}\left(\frac{-c^*\bar{x}}{-c^*x_i}\right) \\
&= \frac{c^*x_i}{\operatorname{val}(d)}\, O\!\left(\frac{\|c^*\|\,\|\bar{x}\|}{\|d\|\,\|x_i\|\operatorname{reldist}(d, \mathrm{DualS}\emptyset)}\right) \\
&= \frac{c^*x_i}{\operatorname{val}(d)}\, O\!\left(\frac{\rho}{\operatorname{reldist}(d, \mathrm{DualS}\emptyset)}\right).
\end{aligned}$$

Together with (91) and the relation $\operatorname{reldist}(d, \mathrm{DualS}\emptyset) \leq 1$ this implies (86).

Similarly,

$$\begin{aligned}
\frac{s}{\max\{\|d\|, -\operatorname{val}(d)\}} &= \frac{s}{-\operatorname{val}(d)} \\
&= \frac{c^*x_i}{\operatorname{val}(d)}\left(\frac{s}{-c^*x_i}\right) \\
&= \frac{c^*x_i}{\operatorname{val}(d)}\, O\!\left(\frac{s}{\|d\|\,\|x_i\|\operatorname{reldist}(d, \mathrm{DualS}\emptyset)}\right) \\
&= \frac{c^*x_i}{\operatorname{val}(d)}\, O\!\left(\frac{\rho s}{\|d\|\,\|\bar{x}\|\operatorname{reldist}(d, \mathrm{DualS}\emptyset)}\right).
\end{aligned}$$

Together with (91) and the relation $\operatorname{reldist}(d, \mathrm{DualS}\emptyset) \leq 1$ this implies (87).

Now we consider the other case, assuming $x \in \operatorname{Feas}(d)$ satisfies

$$-c^*x < \|d\|\,\sqrt{\|x\|}.$$

Proposition 7.10 then implies

$$\|x\| = O\!\left(\frac{1}{[\operatorname{reldist}(d, \mathrm{DualS}\emptyset)]^2}\right). \tag{93}$$

We have by (88) and (93),

$$\begin{aligned}
\frac{-c^*\bar{x}}{\max\{\|d\|, -\operatorname{val}(d)\}} &\leq \|\bar{x}\| \\
&\leq \rho\|x\| \\
&\leq \frac{\rho}{[\operatorname{reldist}(d, \mathrm{DualS}\emptyset)]^2}
\end{aligned}$$

and hence (86). Similarly,

$$\begin{aligned}\frac{s}{\max\{||d||, -\mathrm{val}(d)\}} &\le \frac{s}{||d||}\\ &\le \frac{\rho s||x||}{||d||\,||\bar{x}||}\\ &\le \frac{\rho s}{||d||\,||\bar{x}||[\mathrm{reldist}(d, \mathrm{DualS}\emptyset)]^2}\end{aligned}$$

and hence (87). □

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