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Abstract

We prove that the problem of checking whether all linear programming problems whose data range in prescribed intervals have optimal solutions is NP-hard.

Keywords

Linear programming, inexact data, NP-hardness

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1 Introduction

Consider a family of linear programming (LP) problems

$$\min\{c^T x;\ Ax = b,\ x \geq 0\} \tag{1.1}$$

for all data satisfying

$$A \in A^I,\ b \in b^I,\ c \in c^I, \tag{1.2}$$

where

$$A^I = \{A;\ \underline{A} \leq A \leq \overline{A}\}$$

is an $m \times n$ interval matrix, $m \leq n$, and

$$b^I = \{b;\ \underline{b} \leq b \leq \overline{b}\},$$

$$c^I = \{c;\ \underline{c} \leq c \leq \overline{c}\}$$

are interval vectors of dimensions m and n, respectively (the inequalities are understood componentwise). The family (1.1), (1.2) may be interpreted as a linear programming problem with inexact data, or as a fully parametrized parametric linear programming problem.

The problem of existence of optimal solutions of all linear programming problems in the family (1.1), (1.2) was addressed in [6]. There it was proved that each LP problem (1.1) with data satisfying (1.2) has an optimal solution if and only if the LP problem

$$\min\{\underline{c}^T x;\ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0\}$$

has an optimal solution and each of the 2^m systems $Ax = b$ whose each row is either of the form

$$(\underline{A}x)_i = \overline{b}_i$$

or of the form

$$(\overline{A}x)_i = \underline{b}_i$$

($i = 1, \ldots, m$) has a nonnegative solution. Hence, we have a finitely verifiable necessary and sufficient condition, but the number of systems to be checked for nonnegative solvability is exponential in m.

In the main result of this paper we show that the problem in question is NP-hard (cf. Garey and Johnson [1]). Hence, unless the famous conjecture "P≠NP" [1] is false, there does not exist a necessary and sufficient condition for checking existence of optimal solutions of all LP problems (1.1), (1.2) which could be verified in polynomial time. The proof given in section 2 shows that even checking *feasibility* of all LP problems in the family (1.1), (1.2) is NP-hard. Some concluding remarks are given in section 3.

2 Main result

Theorem 1 *The following problem is NP-hard:*

Instance. A^I, b^I, c^I *(with rational bounds).*

Question. *Does each LP problem (1.1) with data (1.2) have an optimal solution?*

Proof. 0) For the purpose of the proof, let us introduce $A_c = \frac{1}{2}(\underline{A} + \overline{A})$, $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$, $b_c = \frac{1}{2}(\underline{b} + \overline{b})$ and $\delta = \frac{1}{2}(\overline{b} - \underline{b})$, so that

$$A^I = [A_c - \Delta, A_c + \Delta]$$

and

$$b^I = [b_c - \delta, b_c + \delta].$$

The proof goes through several steps.

1) First we prove that each system

$$Ax = b,\ x \geq 0 \tag{2.1}$$

with data satisfying

$$A \in A^I,\ b \in b^I \tag{2.2}$$

has a solution if and only if

$$(\forall y)(A_c^T y + \Delta^T |y| \geq 0 \Rightarrow b_c^T y - \delta^T |y| \geq 0) \tag{2.3}$$

holds. "Only if": Let each system (2.1) with data (2.2) have a solution, and let $A_c^T y + \Delta^T |y| \geq 0$ for some $y \in \mathbb{R}^m$. Define a diagonal matrix T by $T_{ii} = 1$ if $y_i \geq 0$, $T_{ii} = -1$ if $y_i < 0$, and $T_{ij} = 0$ if $i \neq j$ $(i, j = 1, \ldots, m)$, then $|y| = Ty$. Consider now the system

$$(A_c + T\Delta)x = b_c - T\delta,\ x \geq 0. \tag{2.4}$$

Since $A_c + T\Delta \in A^I$ and $b_c - T\delta \in b^I$, the system (2.4) has a solution according to the assumption, and $(A_c + T\Delta)^T y = A_c^T y + \Delta^T |y| \geq 0$, hence Farkas lemma [3] applied to (2.4) gives that $b_c^T y - \delta^T |y| = (b_c - T\delta)^T y \geq 0$, which proves (2.3). "If": Assuming that (2.3) holds, consider a system (2.1) with data satisfying (2.2). Let $A^T y \geq 0$ for some y; then $A_c^T y + \Delta^T |y| \geq (A_c + A - A_c)^T y = A^T y \geq 0$, hence (2.3) gives that $b^T y = (b_c + b - b_c)^T y \geq b_c^T y - \delta^T |y| \geq 0$. Thus we have proved that for each y, $A^T y \geq 0$ implies $b^T y \geq 0$, and Farkas lemma proves the existence of a solution to (2.1).

2) For a given *square* $m \times m$ interval matrix $A_0^I = [A_c^0 - \Delta^0, A_c^0 + \Delta^0]$, construct an $m \times 2m$ interval matrix

$$A^I = [A_c - \Delta, A_c + \Delta] \tag{2.5}$$

with

$$A_c = (A_c^{0T}, -A_c^{0T}), \tag{2.6}$$

$$\Delta = (\Delta^{0T}, \Delta^{0T}), \tag{2.7}$$

and an interval m-vector

$$b^I = [-e, e], \tag{2.8}$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^m$. We shall prove that A_0^I is regular (i.e., each $A \in A_0^I$ is nonsingular) if and only if each system

$$Ax = b,\ x \geq 0 \tag{2.9}$$

with data satisfying

$$A \in A^I,\ b \in b^I \tag{2.10}$$

(A^I, b^I given by (2.5)–(2.8)) has a solution. In fact, according to part 1), Eq. (2.3), some system (2.9) with data (2.10) does *not* have a solution if and only if there exists a vector y satisfying

$$\begin{pmatrix} A_c^0 \\ -A_c^0 \end{pmatrix} y + \begin{pmatrix} \Delta^0 \\ \Delta^0 \end{pmatrix} |y| \geq 0$$

and

$$e^T|y| > 0,$$

which is equivalent to

$$|A_c^0 y| \leq \Delta^0 |y| \tag{2.11}$$

and

$$y \neq 0. \tag{2.12}$$

Then the Oettli–Prager theorem [4] (see the reformulation in [7], Lemma 2.1) gives that (2.11), (2.12) is equivalent to existence of a singular matrix in $A_0^I = [A_c^0 - \Delta^0, A_c^0 + \Delta^0]$. This proves the assertion.

3) Given a square $m \times m$ interval matrix A_0^I, construct an $m \times 2m$ interval matrix A^I and an interval vector b^I by (2.5)–(2.8). This can be done in polynomial time. According to part 2), checking regularity of A_0^I can be reduced in polynomial time to checking solvability of all systems (2.9), (2.10). But since the problem of checking regularity of interval matrices is NP-hard (Poljak and Rohn [5], Theorem 2.8), the problem of checking whether each system (2.9) with data satisfying (2.10) has a solution is NP-hard as well.

4) For an $m \times n$ interval matrix A^I and an interval m-vector b^I, consider the family of LP problems

$$\min\{c^T x;\ Ax = b,\ x \geq 0\} \tag{2.13}$$

for

$$A \in A^I,\ b \in b^I,\ c \in [e, e]. \tag{2.14}$$

Since the objective $e^T x$ is bounded from below, a problem (2.13) has an optimal solution if and only if it is feasible. Hence each system

$$Ax = b,\ x \geq 0$$

with data satisfying

$$A \in A^I,\ b \in b^I$$

has a solution if and only if each LP problem (2.13) with data (2.14) has an optimal solution. Since the former problem was proved to be NP-hard in 3), the latter one is NP-hard as well. This concludes the proof. ∎

3 Concluding remarks

Khachiyan [2] proved that an LP problem (1.1) can be solved in polynomial time. The above result shows that this nice property is lost when inexact data are present. Even more, since existence of an optimal solution to an LP problem (1.1) is not affected by premultiplying the equation $Ax = b$ by a nonzero number, and in view of part 4) of the proof, Eq. (2.14), we can see that the problem remains NP-hard if we consider only instances A^I, b^I, c^I satisfying

$$\max_{i,j} |\overline{A}_{ij} - \underline{A}_{ij}| \leq \varepsilon,$$

$$\max_{i} |\overline{b}_i - \underline{b}_i| \leq \varepsilon,$$

$$\underline{c} = \overline{c},$$

where ε is any prescribed positive rational number. Thus the NP-hardness of our problem has nothing to do with the amount of uncertainty in the data (1.2); it is caused by the exponential number of vertices of an interval matrix A^I (Poljak and Rohn [5]).

Nevertheless, the worst–case–type result of Theorem 1 does not preclude efficient solvability of many practical examples. The criterion from [6] quoted in the Introduction requires solving one LP problem and checking nonnegative solvability of all systems of linear equations whose ith row is either of the form $(\underline{A}x)_i = \overline{b}_i$ or of the form $(\overline{A}x)_i = \underline{b}_i$ $(i = 1, \ldots, m)$. The number of mutually different such systems is 2^p, where p is the number of rows i having at least one inexact coefficient (i.e., either $\underline{b}_i < \overline{b}_i$, or $\underline{A}_{ij} < \overline{A}_{ij}$ for some j). Thus the criterion can be efficiently applied to practical examples with small values of p.

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