Linear Programming with Interval Arithmetic

Herry Suprajitno

Faculty of Science and Technology, University of Airlangga
Kampus C Unair, Jl Mulyorejo Surabaya 60115, Indonesia
herry_sam@yahoo.com

Ismail bin Mohd

Faculty of Science and Technology, Universiti Malaysia Terengganu
Mengabang Telipot, Kuala Terengganu, Terengganu, Malaysia

Abstract

The conventional linear programming model requires the parameters to be known as constants. In the real world, however, the parameters are seldom known exactly and have to be estimated. Interval programming is one of the tools to tackle uncertainty in mathematical programming models. In this paper, it will be presented the interval linear programming problems, where the coefficients and variables are in the form of intervals. The problems will be solved by modification simplex method.

Mathematics Subject Classification: 90C05

Keywords: linear programming, interval programming, simplex method

1 Introduction

Linear programming is among the most widely and successfully used decision tools in the quantitative analysis of practical problems where rational decisions have to be made. The conventional linear programming model requires the parameters to be known as constants. In the real world, however, the parameters are seldom known exactly and have to be estimated. Classical sensitivity analysis allows a study of the effect on the solution of changes to single coefficients or very small groups of coefficients, but only to the extent that the optimal basis is not changed. Therefore, we interest to study interval linear programming where its the coefficients and variables are in the form of interval [14].

In [2], the authors generalize known concepts of the solution of the linear programming problem with interval coefficients in the objective function based

on preference relations between intervals. [11], the authors provide an illustrated overview of the state of the art of interval programming in the context of multiple objective linear programming models. In [10], the author computes the exact upper and lower bounds of optimal values for linear programming problems whose coefficients vary in given intervals. In [4], the author computes exact range of the optimal value for linear programming problem in which input data can vary in some given real compact intervals, and he able to characterize the primal and dual solution sets, the bounds of the objective function resulted from two nonlinear programming problems. In [3], the author propose a new approach in which some or all of the coefficients of the LP are specified as intervals, and find the best optimum and the worst optimum for the model, and the point settings of the interval coefficients that yield these two extremes. In [14], the authors solved linear programming problems with interval in coefficients and variables by transformation interval linear program model into real linear program.

In this paper, we propose a modification of simplex method and solve interval linear programming problems using software Pascal-XSC to implementation the method without transforming interval linear program model into real linear program.

2   Interval Arithmetic

The basic definitions and properties of interval numbers (or interval) and interval arithmetic can be seen in [1], and [8].

Definition 2.1. A closed real interval $[x_I, x_S]$ denoted by $\underline{x}$, is a real interval number which can be defined completely by

$$\underline{x} = [x_I, x_S] = \{x \in R \setminus x_I \leq x \leq x_S; x_I, x_S \in R\}$$

where x_I and x_S are called infimum and supremum, respectively.

Definition 2.2. Two interval numbers $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$ are called equal if and only if $x_I = y_I$ and $x_S = y_S$.

Definition 2.3. An interval number, $\underline{x} = [x_I, x_S]$ is called a point interval number, if $x_I = x_S$.

Definition 2.4. An interval is called unbounded, if the lower bound or the upper bound are infinity

Definition 2.5. A real interval vector $\underline{\mathbf{x}} \in I(R^n)$ is a set of the form $\underline{\mathbf{x}} = (\underline{x}_i)_{nx1}$, where $i = 1, 2, \ldots, n$ and $\underline{x}_i = [x_{iI}, x_{iS}] \in I(R)$.

Definition 2.6. A real interval matrix $\underline{\mathbf{A}} \in I(M(R^n))$ is a set of the form $\underline{\mathbf{A}} = (\underline{a}_{ij})_{nxn}$, where $i = 1, 2, \ldots, n$ and $\underline{a}_{ij} = [a_{ijI}, a_{ijS}] \in I(R)$.
Let $\underline{x} = [x_I, x_S]$ and $\underline{y} = [y_I, y_S]$, then

1. $\underline{x} + \underline{y} = [x_I + y_I, x_S + y_S]$ (addition)

2. $\underline{x} - \underline{y} = [x_I - y_S, x_S + y_I]$ (subraction)

3. $\underline{xy} = [min\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}, max\{x_I y_I, x_I y_S, x_S y_I, x_S y_S\}]$

                                               (multiplication)

4. $\underline{x} \setminus \underline{y} = [x_I, x_S][1 \setminus y_S, 1 \setminus y_I]$ for $0 \notin \underline{y}$.   (division)

    Let $\underline{x}, \underline{y}$ *and* $\underline{z} \in I(R)$

5. $\underline{x} + \underline{y} = \underline{y} + \underline{x}, \underline{xy} = \underline{yx}$ (commutativity)

6. $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z}), (\underline{xy})\underline{z} = \underline{x}(\underline{yz})$ (associativity)

7. $\underline{x}(\underline{y} + \underline{z}) \subseteq \underline{xy} + \underline{xz})$ (subdistributivity)

8. $a(\underline{x} + \underline{y}) = a\underline{x} + a\underline{y}, a \in R$.

3   On Comparing Intervals : A Discussion on Existing Ideas

Let $\underline{x}$ and $\underline{y}$ be two interval numbers. In [1, 2, 5, 6, 7] and [12], the authors present the following ways of comparing them.

Definition 3.1.   1. $\underline{x} \leq \underline{y}$ if and only if $x_S \leq y_I$

  2. $\underline{x} \leq \underline{y}$ if and only if $x_I \geq y_I$ and $x_S \leq y_S$

  3. $\underline{x} \leq \underline{y}$ if and only if $x_I \leq y_I$ and $x_S \leq y_S$

  4. (a) $\underline{x} \leq \underline{y}$ if and only if $x_I \leq y_I$ and $m(\underline{x}) \leq m(\underline{y})$

     (b) $\underline{x} \leq \underline{y}$ if and only if $x_S > y_I$ and $m(\underline{x}) < m(\underline{y})$

     (c) $\underline{x} \leq \underline{y}$ if and only if $m(\underline{x}) \leq m(\underline{y})$ and $w(\underline{x}) \geq w(\underline{y})$
        where $m(\underline{x}) = \frac{x_I + x_S}{2}$, $w(\underline{x}) = \frac{x_S - x_I}{2}$

  5. (a) $\underline{x} \leq \underline{y}$ if and only if $x_I - \varepsilon \leq y_S$

     (b) $\underline{x} \leq \underline{y}$ if and only if $x_S - \varepsilon \leq y_I$

6. $\underline{x} \leq \underline{y}$ if and only if $x_I + x_S \leq y_I + y_S$

In the following, we discuss about comparing two intervals in the Definition 3.1(also see [15].

Definition 3.1.1 only concerns to disjoint intervals, thus the definition cannot be used to compare two intersection intervals.

Definition 3.1.2 only concerns to an interval which is contained in another interval, hence the definition cannot be used to compare two disjoint intervals.

Definition 3.1.3 only concerns to intersection intervals, hence the definition cannot be used to compare two intervals where one of the two intervals is contained in another interval.

Definition 3.1.4.a contradicts with Definition 3.1.2. In Definition 3.1.4.a, $\underline{y} \leq \underline{x}$ if and only if $y_I \leq x_I$ and $m(\underline{y}) \leq m(\underline{x})$. We can determine values of the $m(\underline{y})$ and $m(\underline{x})$ as displayed in the following relation

$$y_I \leq m(\underline{y}) \leq x_I \leq m(\underline{x}) \leq x_S \leq y_S$$

Based on values $m(\underline{y})$ and $m(\underline{x})$, we can write again $\underline{y} \leq \underline{x}$ if and only if $x_I \geq y_I$ and $x_S \leq y_S$. In the other hand, in Definition 3.1.2, $\underline{x} \leq \underline{y}$ if and only if $x_I \geq y_I$ and $x_S \leq y_S$. So that there are contradict among the two definitions.

Definition 3.1.4.b concerns to intersection interval, hence the definition cannot be used to compare two disjoint intervals.

Definition 3.1.4.b contradicts with Definition 3.1.2. Analog Definition 3.1.4.a. There is some contradiction in Definition 3.1.5.a. Let we have relations $x_I < y_I < y_S < x_S$ then $x_I - \varepsilon \leq y_S$, hence $\underline{x} \leq \underline{y}$. But based on the relations to, we have $y_I - \varepsilon \leq x_S$, that mean $\underline{y} \leq \underline{x}$.

Definition 3.1.5.b cannot be used to compare two intervals where one of the two intervals is contained in another interval.

Definition 3.1.6 contradicts with Definition 3.1.2. Let we have relations $y_I \leq x_I \leq x_S \leq y_S$ and $m(\underline{y}) \leq x_I$ then $2m(\underline{y}) \leq 2x_I$, hence $y_I + y_S \leq 2x_I \leq x_I + x_S$. We can write again $y_I + y_S \leq x_I + x_S$, that mean $\underline{y} \leq \underline{x}$ (Definition 3.1.6). But in Definition 3.1.2, $y_I \leq x_I \leq x_S \leq y_S$ if and only if $\underline{x} \leq \underline{y}$.

In order to make the validity of Definition 3.1.6, we introduce a new definition as follows.

Definition 3.2. Let $\underline{x}$ and $\underline{y}$ be two real interval numbers. The statement defined as $\underline{x} \leq \underline{y}$ if and only if $x_I + x_S \leq y_I + y_S$ is valid when one of the following conditions satisfied.

1. $x_S \leq y_I$

2. $x_I \leq y_I \leq x_S \leq y_S$

3. $y_I \leq x_I \leq x_S \leq m(\underline{y}) \leq y_S$

4. $y_I < x_I < m(\underline{y})$ and $\mathrm{m}(\underline{y}) \le x_S < m(\underline{y}) + \varepsilon, (\varepsilon = \mathrm{m}(\underline{y}) - x_I)$

Therefore, based on Definition 3.2, we can build a theorem about the comparison between two intervals.

Theorem 3.1. *Let $\underline{x}$ and $\underline{y}$ be two real intervals. $\underline{x} \le \underline{y}$ if and only if $ux_I + vx_S \le uy_I + vy_S$ where $u, v \in (0, 1]$ and $u \le v$.*

Proof. By Definition 3.2 and inequality $u \le v(u, v > 0)$, we have

$$v(x_I - y_I) \le v(y_S - x_S)$$

and

$$u(x_I - y_I) \le v(x_I - y_I) \le v(y_S - x_S).$$

Therefore if

$$u(x_I - y_I) \le v(y_S - x_S),$$

then we can write

$$ux_I - uy_I \le vy_S - vx_S$$

and also

$$ux_I + vx_S \le uy_I + vy_S.$$

$\square$

4 Linear programming

Linear programming is a mathematical tool developed to handle the optimization of a linear function subject to a set of linear constraints. The subject of linear programming is a very important area in applied mathematics and it has a large number of uses and applications in many industries. Some of the current applications include mix production, allocation of resources, transportation problems, and scheduling of operations.

A linear programming problem (LP) can be represented as follows. Maximize (objective function)

$$Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, (i = 1, 2, \dots, m)$$

$$x_j \in R(j = 1, 2, \dots, n)$$

where $x_1, x_2, \dots, x_n$ are the problem variables, $c_1, c_2, \dots, c_n$ are constants so-called the cost coefficients, $b_1, b_2, \dots, b_m$ are the resource value coefficients, and $a_{ij}, (i = 1, 2, \dots, m)(j = 1, 2, \dots, n)$ are the constraint coefficients.

5 Linear Programming with Interval Coefficients

Linear program with interval coefficients is an extension of linear programming, to anticipate when unknown data value correctly, but can be estimated by its upper and lower bounds. The problem can be expressed as maximizing

$$Z = \sum_{j=1}^{n} \underline{c}_j x_j$$

subject to

$$\sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \underline{b}_i, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

where $x_j \in R, \underline{c}_j, \underline{a}_{ij}, \underline{b}_i \in I(R)$ is the set of all interval number in R

6 Interval Linear Programming

Interval linear programming (ILP) is an extension of linear programming with interval coefficients. In the interval linear programming, all of coefficients and variables are intervals. Interval linear programming problem can be formulated as follows. Maximize (objective function)

$$Z = \sum_{j=1}^{n} \underline{c}_j \ \underline{x}_j$$

subject to

$$\sum_{j=1}^{n} \underline{a}_{ij}\underline{x}_j \leq \underline{b}_i, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

where $x_j \in R, \underline{c}_j, \underline{a}_{ij}, \underline{b}_i, \underline{x}_j \in I(R)$ is the set of all interval number in R

7 Modified Simplex method

In [9, 13] and [14], the authors implement simplex method for variable with real value. In this paper we will use simplex method to solve interval linear programming. We need to modify the classical simplex method in order that the method can be used for variable with real interval value. The modified simplex method can be written as in the following algorithm

MS Interval Simplex Algorithm. Data : $\underline{A}, \underline{c}$ and $\underline{b}$ in the ILP model $max\{\underline{c}\ \underline{x} \setminus \underline{A}\ \underline{x} \leq \underline{b}, \underline{x} \geq 0\}$

1. **case true of**

   - no optimal solution :
   - 1.1.**stop!** write message "no optimal solution"
   - unbounded :
   - 1.2.**stop!** write message "unbounded"
   - optimal solution :
   - 1.3.**stop!** write message "optimal solution"
   - **default** :
   - 1.4.{ } ! do nothing ! Check an initial feasible solution.

2. **if** we cannot find any initial basic feasible solution.

3. **then**

   - 3.1.**stop!** the model has no optimal solution.
   - **else!** Selection of the entering variable.
   - 3.2.Determine values for the coefficients objective function for Test optimality.
   - 3.3.**If** all coefficient's values positive or zero.
     - **then**
     - 3.3.1.**stop!** the current basic feasible solution is optimal.
     - **else**
     - 3.3.2.Choose the entering variable with the most negative coefficient's values.
     - 3.3.3.Selection of the leaving variable.
     - 3.3.4.Compute vector right hand side.
     - 3.3.5.Compute value vector column pivot.
     - 3.3.6.**If** all vector's values negative or zero.
       * **then**
       * 3.3.6.1.**stop!** the model has no bounded solution.
       * **else**
       * 3.3.6.2.Choose the leaving variable with the minimum ratio test.
       * 3.3.6.3.Updating the inverse basis matrix.

∗ 3.3.6.4.Determine the value of basic variable.

∗ 3.3.6.5.Determine new basic feasible solution.

∗ 3.3.6.6.Test the improvement solution.

∗ 3.3.6.7.**If** the new solution better than the current solution

· then

· 3.3.6.7.1.set solution with the new solution.

· 3.3.6.7.2.go to 3.2.

· else

· 3.3.6.7.3.go to 3.3.2

8   Numerical Example

Let us consider the following example of ILP [2] and [14]. Maximize

$$[-20, 50]\underline{x} + [0, 10]\underline{y},$$

subject to

$$10\underline{x} + 60\underline{y} \leq 1080$$

$$10\underline{x} + 20\underline{y} \leq 400$$

$$10\underline{x} + 10\underline{y} \leq 240$$

$$30\underline{x} + 10\underline{y} \leq 420$$

$$40\underline{x} + 10\underline{y} \leq 520$$

$$\underline{x}, \underline{y} \geq 0$$

In the implementation of the algorithm using **Pascal-XSC**, we have results for (u = v = 1; u = 0.25 and v = 0.75) as follows.

a. for $u = v = 1$

$$\underline{x} = [9.99999999999997, 10.0000000000001]$$
$$\underline{y} = [11.9999999999999, 12.0000000000001]$$

b. for $u = 0.25$ and $v = 0.75$

$$\underline{x} = [12.99999999999999, 13.0000000000001]$$
$$\underline{y} = [0.000000000000000, 0.00000000000000]$$

We obtain the same solution with [2] for second last and [14] for example 2. In [2] author used lower and upper bounds (real number) of interval to solve the problem. In [14], author used classical simplex method to solve the problem. In this paper we execute interval directly.

9 Conclusions

We have modified simplex method so that can be used in real interval. Interval linear programming problems can be solved with execute real interval directly.

Acknowledgment

References

[1] Alefeld G. and Herzberger J., *Introduction to interval computations*, Academic Press, New York, 1983.

[2] Chanas S. and Kuchta D., *Multiobjective programming in optimization of interval objective functions - a generalized approach*, EJOR, 94(1996) 594-598.

[3] Chinneck J.W. and Ramadan K., *Linear programming with interval coefficients*, JORS, 51(2000), 209-220

[4] Hladik M., *Optimal value range in interval linear programming*, Faculty of Mathematics and Physics, Charles University, Prague, 2007.

[5] Klatte R., Kulisch U., Neaga M., Ratz D. and Ullrich Ch., *PASCAL-XSC Language Reference with Examples*, Springer-Verlag, Berlin, 1999.

[6] Kuchta D., *A modification of a solution concept of the linear programming problem with interval coefficients in the constraints*, CEJOR, 16(2008), 307-316.

[7] Malek H.R., Tata M. and Mashinchi M., *Linear Programming with Fuzzy Variables*, Fuzzy Sets and Systems, 109(2000) 21-33.

[8] Mohd I.B., *A global optimization using interval arithmetic*, Journal of Fundamental Sciences, 2(2006), 76-88.

[9] Mohd I.B., *Teori dan Penggunaan Pengaturcaraan Linear*, Dewan Bahasa dan Pustaka, Kuala Lumpur, 1991.

[10] Mraz F., *Calculating the exact bounds of optimal values in LP with interval coefficients*, Annals of Operations Research, 81(1998), 51-62.

[11] Oliveira D. and Antunes C.H., *Multiple objective linear programming models with interval coefficients - an illustrated overview*, EJOR, 181(2007), 1434 - 1463

[12] Sengupta A., Pal T.K. and Chakraborty D., *Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming*, Fuzzy Sets and Systems 119(2001) 129-138.

[13] Sierksma G., *Linear and Integer Programming*, Marcel Dekker, New York, 1996

[14] Suprajitno H. and Mohd I.B., *Interval Linear Programming*, presented in ICoMS-3, 2008, Bogor, Indonesia.

[15] Suprajitno H., Mohd I.B., Mamat M. and Ismail F., *Linear Programming with Interval Variables*, presented in 2nd ICOWOBAS, 2009, Johor Bahru, Malaysia.