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Document status and date:
Published: 01/01/1990

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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Download date: 25. Aug. 2022

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by

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November 30, 1990

Abstract

This paper deals with the discrete-time infinite-horizon linear quadratic problem with indefinite cost criterion. Given a discrete-time linear system, an indefinite cost- functional and a linear subspace of the state space, we consider the problem of minimizing the cost-functional over all inputs that force the state trajectory to converge to the given subspace. We give a geometric characterization of the set of all hermitian solutions of the discrete-time algebraic Riccati equation. This characterization forms the discrete-time counterpart of the well-known geometric characterization of the set of all real symmetric solutions of the continuous-time algebraic Riccati equation as developed by Willems [*IEEE Trans. Automat. Control, 16 (1971), pp. 621- 634*] and Coppel [*Bull. Austral. Math. Soc., 10 (1974), pp. 377-401*]. In the set of all hermitian solutions of the Riccati equation we identify the solution that leads to the optimal cost for the above mentioned linear quadratic problem. Finally, we give necessary and sufficient conditions for the existence of optimal controls.

Keywords: Discrete time optimal control, indefinite cost, algebraic Riccati equation, linear endpoint constraints.

AMS subject classification: 93C05, 93C35, 93C55, 93C45.

1 Introduction

This paper has two main goals. Firstly, we want to establish the discrete-time counterpart of the by now 'classical' geometric characterization of the lattice of real symmetric solutions of the continuous-time algebraic Riccati equation as given in [1] and [8]. Subsequently, we want to apply these results to the discrete-time linear quadratic optimization problem with

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linear endpoint-constraints. Given a discrete-time linear system, the latter problem consists of minimizing a general *indefinite* quadratic cost-functional over the class of input functions that force the state trajectory to converge to an *a priori* given subspace of the state-space (or, equivalently, that force a given linear function of the state to converge to zero). A complete treatment of this optimization problem for the continuous-time case was given only very recently in [5] and [6].

With respect to our first goal, it will be shown that like in the continuous- time case, if the algebraic Riccati equation has at least one hermitian solution, then it has a smallest one and a largest one. Furthermore, any hermitian solution of the algebraic Riccati equation can be written as a 'linear combination' of these extremal solutions. In order to derive these results we will make use of the characterization of all hermitian solutions of the discrete-time Riccati equation in terms of certain invariant Lagrangian subspaces, as established in [4] (see also [2]).

With respect to our second goal, we want to note that compared to usual discrete-time linear quadratic optimal control problems, our problem formulation introduces generalizations into two independent directions. Firstly, in contrast to the existing literature on this subject, we do not require the quadratic form in the cost-functional to be positive semi-definite (the 'linear-quadratic regulator problem'). Instead, the quadratic form is allowed to be indefinite. Secondly, our problem formulation includes a fixed, but arbitrary, linear endpoint constraint, in the sense that the optimization is performed over the class of all input functions that force the state trajectory to converge to an *a priori* given subspace. A solution to the usual zero-endpoint problem (in which the optimal state trajectory is required to converge to the origin) can thus be obtained from our results by setting this subspace to be equal to the zero-subspace. On the other hand, a solution to the free-endpoint problem (no constraint on the optimal state-trajectory) can be obtained from our results by taking the subspace to be equal to the entire state space.

The outline of this paper is as follows. In section 2 we shall formulate the optimization problem that we want to consider. This section also contains a statement of the main result of this paper, that is, a characterization of the optimal cost, necessary and sufficient conditions for the existence of optimal controls and an expression for the optimal state feedback control law. In section 3 we shall establish the characterization of the set of all hermitian solutions of the discrete-time algebraic Riccati equation as announced above. Finally, in section 4 we shall give a proof of the main result as stated in section 2.

2   Problem statement and main results

In this paper we will consider the discrete time system

$$x_{k+1} = Ax_k + Bu_k, \qquad (2.1)$$

where the state variable x_k takes its values in C^n and the input variable u_k takes its values in C^m. In 2.1 we have $A \in C^{n \times n}$ and $B \in C^{n \times m}$. As a standing assumption we take (A, B) to be a controllable pair. We will consider optimization problems of the type

$$\inf \sum_{k=0}^{\infty} \begin{pmatrix} x_k \\ u_k \end{pmatrix}^* \begin{pmatrix} Q & C^* \\ C & R \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{2.2}$$

Here, R, Q and C are complex matrices of appropriate dimensions and $R > 0$, $Q = Q^*$. The expression 2.2 of course needs some explanation. For any $x_0 \in C^n$ and any control sequence $u = \{u_k\}_{k=0}^{\infty}$ we define

$$J_T(x_0, u) := \sum_{k=0}^{T} \begin{pmatrix} x_k \\ u_k \end{pmatrix}^* \begin{pmatrix} Q & C^* \\ C & R \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{2.3}$$

Let

$$U(x_0) := \left\{ u \mid \lim_{T \to \infty} J_T(x_0, u) \text{ exists in } \mathcal{R} \cup \{-\infty, +\infty\} \right\}$$

and for any control sequence $u \in U(x_0)$ define the associated cost by

$$J(x_0, u) := \lim_{T \to \infty} J_T(x_0, u). \tag{2.4}$$

The optimization problem of minimizing the cost functional 2.4 over the class of inputs $U(x_0)$ is called the *free-endpoint linear quadratic problem*. The optimal cost associated with this problem is equal to

$$V_f(x_0) := \inf_{u \in U(x_0)} J(x_0, u). \tag{2.5}$$

Compare this problem with the usual *zero-endpoint problem*, where instead of $U(x_0)$ the cost functional is minimized over the class of all inputs that force the corresponding state trajectory to converge to the origin, i.e. over

$$U_s(x_0) := \left\{ u \in U(x_0) \mid \lim_{k \to \infty} x(x_0, u)_k = 0 \right\}. \tag{2.6}$$

The associated optimal cost is given by

$$V_+(x_0) := \inf_{u \in U_s(x_0)} J(x_0, u).$$

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In the present paper we will study a generalization of the above two linear quadratic problems, the *linear quadratic problem with linear endpoint constraints*. Given a linear subspace $\mathcal{L}$ of the state space $\mathcal{C}^n$, the latter problem consists of minimizing the cost functional 2.4 over all inputs u that force the state trajectory to converge to the subspace $\mathcal{L}$:

$$U_{\mathcal{L}}(x_0) := \left\{ u \in U(x_0) \mid \lim_{k \to \infty} d(x(x_0, u)_k, \mathcal{L}) = 0 \right\}. \tag{2.7}$$

In the above, for a given point $x \in \mathcal{C}^n$, $d(x, \mathcal{L})$ denotes the usual distance from the point x to the subspace $\mathcal{L}$. The optimal cost for the latter problem is given by

$$V_{\mathcal{L}}(x_0) := \inf_{u \in U_{\mathcal{L}}(x_0)} J(x_0, u). \tag{2.8}$$

Obviously, both the free-endpoint problem as well as the fixed endpoint problem are special cases of the latter problem formulation: take $\mathcal{L} = \mathcal{C}^n$ and $\mathcal{L} = 0$, respectively.

An important role will be played by the set of hermitian solutions of the discrete time algebraic Riccati equation

$$P = A^* P A + Q - (C + B^* P A)^* (R + B^* P B)^{-1} (C + B^* P A). \tag{2.9}$$

Besides controllability of (A, B) we shall assume throughout that $A - BR^{-1}C$ is nonsingular and that $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$, where

$$\Psi(z) := \begin{pmatrix} B^*(Iz^{-1} - A^*)^{-1} & I \end{pmatrix} \begin{pmatrix} Q & C^* \\ C & R \end{pmatrix} \begin{pmatrix} (Iz - A)^{-1}B \\ I \end{pmatrix}.$$

Here, $\mathcal{T}$ denotes the unit circle. In that case the set of hermitian solutions of 2.9 will turn out to have a maximal element P_+ and a minimal element P_- (see Theorem 3.4 below). Put $\Delta := P_+ - P_-$. Let us denote by A_+ and A_- the matrices

$$A_+ := A - B(R + B^* P_+ B)^{-1}(C + B^* P_+ A),$$

$$A_- := A - B(R + B^* P_- B)^{-1}(C + B^* P_- A).$$

It will be seen that $\sigma(A_+) \subset \bar{\mathcal{D}}$ and $\sigma(A_-) \subset \bar{\mathcal{D}}^e$ (here $\mathcal{D}$ denotes the open unit disk, $\mathcal{D}^e$ denotes the exterior of the closed unit disc). Given a subspace $\mathcal{L}$ of $\mathcal{C}^n$ we introduce the subspace

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$$\mathcal{V}(\mathcal{L}) := < \mathcal{L} \cap \ker P_- \mid A_- > \cap \mathcal{X}_+(A_-). \qquad (2.10)$$

(Here, for a matrix A, $\mathcal{X}_+(A)$ denotes the spectral subspace of A corresponding to its eigenvalues in $\mathcal{D}^e$; likewise we shall denote $\mathcal{X}_0(A)$, $\mathcal{X}_-(A)$ the spectral subspaces of A corresponding to its eigenvalues in $\mathcal{T}$, respectively, $\mathcal{D}$). As usual, the notation $< \mathcal{W} \mid A >$, where $\mathcal{W}$ is a subspace of C^n and A an $n \times n$ matrix, denotes the largest A-invariant subspace in $\mathcal{W}$. If H is a matrix such that $\ker H = \mathcal{L} \cap \ker P_-$ then $\mathcal{V}(\mathcal{L})$ is the undetectable subspace of the pair (H, A_-) with respect to the the stability set $\bar{\mathcal{D}}$. Now, let

$$P_{\mathcal{L}} := P_- \pi_{\mathcal{V}(\mathcal{L})} + P_+(I - \pi_{\mathcal{V}(\mathcal{L})}), \qquad (2.11)$$

where $\pi_{\mathcal{V}(\mathcal{L})}$ is the projection onto $\mathcal{V}(\mathcal{L})$ along $(\Delta \mathcal{V}(\mathcal{L}))^{\perp}$. It will turn out that $P_{\mathcal{L}}$ is a solution of 2.9. Finally, if $\mathcal{L}$ is a subspace of C^n and if P is a hermitian $n \times n$ matrix then we will say that P is *negative semi-definite on* $\mathcal{L}$ if the following conditions hold:

- $\forall x_0 \in \mathcal{L} : \quad x_0^* P x_0 \le 0$,

- $\forall x_0 \in \mathcal{L} : \quad x_0^* P x_0 = 0 \Leftrightarrow P x_0 = 0$.

According to this definition, a hermitian matrix P is negative semi-definite on C^n if and only if it is negative semi-definite in the usual sense. Furthermore, any hermitian matrix is negative semi-definite on the zero-subspace $\{0\}$. The main result of this paper can now be formulated as follows:

Theorem 2.1 *Suppose (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Assume further 2.9 has at least one hermitian solution and assume P_- is negative semi-definite on $\mathcal{L}$. Then we have:*

(i) $V_{\mathcal{L}}(x_0)$ is finite for all x_0, and $V_{\mathcal{L}}(x_0) = x_0^* P_{\mathcal{L}} x_0$,

(ii) for all x_0 there is an input u^+ such that $V_{\mathcal{L}}(x_0) = J(x_0, u^+)$ if and only if $\ker \Delta \subseteq \mathcal{L} \cap \ker P_-$; in that case u^+ is unique and is given by the state feedback control law

$$u_k^+ = -(R + B^* P_{\mathcal{L}} B)^{-1}(C + B^* P_{\mathcal{L}} A) x_k.$$

This result is the discrete time analogue of [6, Theorem 4.1]. We stress that the above theorem also provides information on the free-endpoint problem and on the zero-endpoint problem. Indeed, for the free-endpoint problem we set $\mathcal{L} = C^n$. The corresponding subspace $\mathcal{V}(\mathcal{L})$ is then equal to $\mathcal{V} = < \ker P_- \mid A_- > \cap \mathcal{X}_+(A_-)$. Define

$$P_f := P_- \pi_{\mathcal{V}} + P_+(I - \pi_{\mathcal{V}}),$$

where, again, $\pi_{\mathcal{V}}$ is the projection onto $\mathcal{V}$ along $(\Delta \mathcal{V})^{\perp}$. P_f is a solution of 2.9 and we find:

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Corollary 2.2 *Suppose (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Assume further 2.9 has at least one hermitian solution and assume that $P_- \leq 0$. Then we have:*

(i) $V_f(x_0)$ is finite for all x_0, and $V_f(x_0) = x_0^* P_f x_0$,

(ii) for all x_0 there is an input u^+ such that $V_f(x_0) = J(x_0, u^+)$ if and only if $\ker \Delta \subseteq \ker P_-$; in that case u^+ is unique and is given by the state feedback control law

$$u_k^+ = -(R + B^* P_f B)^{-1}(C + B^* P_f A)x_k.$$

The above corrolary is the discrete time analogue of [5, Theorem 5.1]. In order to get the corresponding result on the zero-endpoint problem we set $\mathcal{L} = \{0\}$. The corresponding subspace $\mathcal{V}(\mathcal{L})$ is then equal to $\mathcal{V} = \{0\}$ and we find that the relevant solution of 2.9 is equal to P_+. Thus we find the following discrete time version of [8, Theorem 7].

Corollary 2.3 *Suppose (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Assume further 2.9 has at least one hermitian solution. Then we have:*

(i) $V_+(x_0)$ is finite for all x_0, and $V_+(x_0) = x_0^* P_+ x_0$,

(ii) for all x_0 there is an input u_+ such that $V_+(x_0) = J(x_0, u_+)$ if and only if $\Delta > 0$; in that case u_+ is unique and is given by the state feedback control law

$$u_k^+ = -(R + B^* P_+ B)^{-1}(C + B^* P_+ A)x_k.$$

In this paper we shall give a proof of 2.1. The proof that we shall give basically follows the line of [6]; the details will be provided in Section 4. In Section 3 we give a description of all solutions of 2.9 in terms of P_- and P_+. The continuous time analogue of this description is due to Coppel [1]. The argument here is somewhat more complicated and uses ideas from [2] and [4].

3 Description of solutions of the algebraic Riccati equation

Consider the discrete time algebraic Riccati equation 2.9. In addition to the controllability of (A, B) and the assumption $R > 0$, we assume throughout that $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some η on the unit circle. We are then in a position to apply [4, Theorem 4.1] (see also [2, Theorem 4.4]), which gives a description of solutions of the algebraic Riccati equation in terms of certain invariant subspaces. To be precise, put

$$T = \begin{pmatrix} A - BR^{-1}C + BR^{-1}B^*(A - BR^{-1}C)^{*-1}(Q - C^*R^{-1}C) & -BR^{-1}B^*(A - BR^{-1}C)^{*-1} \\ -(A - BR^{-1}C)^{*-1}(Q - C^*R^{-1}C) & (A - BR^{-1}C)^{*-1} \end{pmatrix}.$$

$$(3.1)$$

Then we have the following theorem:

Theorem 3.1 *Assume (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Then the following statements are equivalent:*

(i) There exists a hermitian solution of 2.9,

(ii) T has an invariant subspace $\mathcal{M}$ such that

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{M} = \mathcal{M}^\perp, \tag{3.2}$$

(iii) the partial multiplicities of T (i.e., the sizes of the Jordan blocks in the Jordan normal form of T) corresponding to its eigenvalues on the unit circle $\mathcal{T}$ are all even,

(iv) $\Psi(z) \geq 0$ for all z on the unit circle $\mathcal{T}$.

In that case any T-invariant subspace $\mathcal{M}$ for which 3.2 holds is of the form

$$\mathcal{M} = \operatorname{im} \begin{pmatrix} I \\ P \end{pmatrix} \tag{3.3}$$

for some hermitian solution P of 2.9, and, conversely, if $P = P^*$ solves 2.9 then $\mathcal{M}$ given by 3.3 is T-invariant and satisfies 3.2.

Furthermore, in case (i) to (iv) hold, for every T-invariant subspace $\mathcal{N}$ with the property that $\sigma(T \mid \mathcal{N}) \subset \mathcal{D}^e$ there is a unique solution $P = P^*$ of 2.9 with

$$\operatorname{im} \begin{pmatrix} I \\ P \end{pmatrix} \cap \mathcal{X}_+(T) = \mathcal{N}. \tag{3.4}$$

Conversely, for every hermitian solution P of 2.9 the subspace $\mathcal{N}$ given by 3.4 is T-invariant and has the property that $\sigma(T \mid \mathcal{N}) \subset \mathcal{D}^e$. Here $\mathcal{X}_+(T)$ denotes the sum of the generalized eigenspaces of T with respect to its eigenvalues in $\mathcal{D}^e$. $\square$

Now let P be any hermitian solution of 2.9. Then it is a straightforward calculation to see that

$$\Psi(z) = \Delta(\bar{z}^{-1})^*(R + B^* P B)\Delta(z),$$

where

$$\Delta(z) = I + (R + B^* P B)^{-1}(C + B^* P A)(Iz - A)^{-1}B.$$

(see e.g. [2]). So, if (i) - (iv) in the above hold then we have $R + B^*PB > 0$ for *any* solution $P = P^*$ of 2.9 (see also Theorem 2.5 in [2]).

Let P_+ and P_- be the unique solutions for which

$$\text{im} \begin{pmatrix} I \\ P_+ \end{pmatrix} \cap \mathcal{X}_+(T) = \{0\}, \quad \text{im} \begin{pmatrix} I \\ P_- \end{pmatrix} \cap \mathcal{X}_+(T) = \mathcal{X}_+(T). \tag{3.5}$$

respectively. We shall show that P_+ is the maximal solution and P_- the minimal solution of the equation 2.9. First we prove a lemma.

Lemma 3.2 *Let P_- be the solution introduced above, and suppose P is an arbitrary hermitian solution. Introduce*

$$A_- = A - B(R + B^*P_-B)^{-1}(C + B^*P_-A),$$

$$S_- = R + B^*P_-B.$$

Then $X := P - P_-$ satisfies the algebraic Riccati equation

$$X = A_-^*XA_- - A_-^*XB(S_- + B^*XB)^{-1}B^*XA_-. \tag{3.6}$$

Conversely, any hermitian solution X of 3.6 gives a solution of 2.9 via $P = X + P_-$.

Proof: Introduce the following matrices

$$\begin{aligned} S_- &:= R + B^*P_-B, & S &:= R + B^*PB, \\ E_- &:= C + B^*P_-A, & E &:= C + B^*PA, \\ L_- &:= S_-^{-1}E_-, & L &:= S^{-1}E. \end{aligned}$$

To prove 3.6, compute

$$\begin{aligned} X - A_-^*XA_- &= X - (A - BL_-)^*X(A - BL_-) \\ &= (P - P_-) - A^*(P - P_-)A + A^*(P - P_-)BL_- \\ &\quad + L_-^*B^*(P - P_-)A - L_-^*B^*(P - P_-)BL_-. \end{aligned} \tag{3.7}$$

Since P and P_- solve 2.9 we have

$$X - A^*XA = E_-^*S_-^{-1}E_- - E^*S^{-1}E. \tag{3.8}$$

Furthermore,

$$\begin{aligned}
A^*(P - P_-)BL_- &= (A^*PB - A^*P_-B)L_- \\
&= (E^* - E_-^*)L_- \\
&= (E^* - E_-^*)S_-^{-1}E_-
\end{aligned}$$

and

$$L_-^*B^*(P - P_-)BL_- = E_-^*S_-^{-1}B^*(P - P_-)BS_-^{-1}E_-.$$

Using these equalities and 3.8 in 3.7 we obtain

$$\begin{aligned}
X - A_-^*XA_- &= E_-^*S_-^{-1}E_- - E^*S^{-1}E + (E^* - E_-^*)S_-^{-1}E_- \\
&\quad + E_-^*S_-^{-1}(E - E_-) - E_-^*S_-^{-1}B^*(P - P_-)BS_-^{-1}E_- \\
&= -E_-^*S_-^{-1}E_- - E^*S^{-1}E + E^*S_-^{-1}E_- + E_-^*S_-^{-1}E \\
&\quad - E_-^*S_-^{-1}B^*(P - P_-)BS_-^{-1}E_- \\
&= -E_-^*S_-^{-1}(S_- + B^*(P - P_-)B)S_-^{-1}E_- - E^*S^{-1}E \\
&\quad + E^*S_-^{-1}E_- + E_-^*S_-^{-1}E \\
&= -E_-^*S_-^{-1}SS_-^{-1}E_- - E^*S^{-1}E + E^*S_-^{-1}E_- + E_-^*S_-^{-1}E \\
&= -(E^* - E_-^*S_-^{-1}S)S^{-1}(E - SS_-^{-1}E_-). \tag{3.9}
\end{aligned}$$

Here we used that $S_- + B^*XB = S$. Moreover,

$$SS_-^{-1} = (R + B^*PB)(R + B^*P_-B)^{-1} = I + B^*XBS_-^{-1},$$

so

$$\begin{aligned}
E - SS_-^{-1}E_- &= E - E_- - B^*XBS_-^{-1}E_- \\
&= B^*XA - B^*XBS_-^{-1}E_- \\
&= B^*X(A - BS_-^{-1}E_-) \\
&= B^*XA_-.
\end{aligned}$$

Hence, from 3.9 we see

$$X - A_-^* X A_- = -A_-^* X B (S_- + B^* X B)^{-1} B^* X A_-,$$

which proves 3.6. The converse follows by a similar computation. □

According to Theorem 3.1 the subspaces im $\begin{pmatrix} I \\ P_+ \end{pmatrix}$ and im $\begin{pmatrix} I \\ P_- \end{pmatrix}$ are T-invariant. The following lemma states that the restrictions of T to these subspaces are semi-stable and semi-antistable, respectively.

Lemma 3.3

$$\sigma(T \mid \text{im } \begin{pmatrix} I \\ P_+ \end{pmatrix}) \subset \bar{\mathcal{D}}, \quad \sigma(T \mid \text{im } \begin{pmatrix} I \\ P_- \end{pmatrix}) \subset \bar{\mathcal{D}}^e.$$

Proof: The first statement is an immediate consequence of 3.5. To prove the second statement, define

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Note that T is J-unitary, i.e., $T^* J T = J$. This implies that T is invertible and that $T^{-1} = -J T^* J$. Now, assume that $x \in \text{im } \begin{pmatrix} I \\ P_- \end{pmatrix}$, $T x = \lambda x$ and $\lambda \in \mathcal{D}$. Then $T^{-1} x = \lambda^{-1} x$ whence $T^* J x = \lambda^{-1} J x$. This implies that $J x \in \mathcal{X}_+(T^*)$, since $\lambda^{-1} \in \mathcal{D}^e$. We also have

$$J x \in J \text{ im } \begin{pmatrix} I \\ P_- \end{pmatrix} = (\text{im } \begin{pmatrix} I \\ P_- \end{pmatrix})^\perp.$$

Using 3.5 the latter implies $J x \in \mathcal{X}_+(T)^\perp$. Since $\mathcal{X}_+(T)^\perp = \mathcal{X}_0(T^*) \oplus \mathcal{X}_-(T^*)$ we obtain $J x = 0$ so $x = 0$. □

It is a straightforward, but tedious, calculation to show that

$$T \mid \text{im } \begin{pmatrix} I \\ P_+ \end{pmatrix}, \quad T \mid \text{im } \begin{pmatrix} I \\ P_- \end{pmatrix}$$

are similar to A_+ and A_- respectively. Consequently, $\sigma(A_+) \subset \bar{\mathcal{D}}$ and $\sigma(A_-) \subset \bar{\mathcal{D}}^e$. Introduce

$$A_P = A - B(R + B^* P B)^{-1}(C + B^* P A). \tag{3.10}$$

The following theorem states that P_- is the smallest hermitian solution of 2.9. Likewise, P_+ is the largest hermitian solution of 2.9. Furthermore, $P = P_-$ is the only hermitian solution with the property that $\sigma(A_P) \subset \bar{\mathcal{D}}^e$ and $P = P_+$ is the only hermitian solution with the property that $\sigma(A_P) \subset \bar{\mathcal{D}}$.

Theorem 3.4 *Assume (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Assume that 2.9 has at least one hermitian solution. Let P_- and P_+ be the solutions determined by 3.5. Then $\sigma(A_+) \subset \bar{\mathcal{D}}$ and $\sigma(A_-) \subset \bar{\mathcal{D}}^e$. Furhermore, for any hermitian solution of 2.9 we have*

$$P_- \leq P \leq P_+.$$

In addition, if P is a hermitian solution with the property that $\sigma(A_P) \subset \bar{\mathcal{D}}^e$, then $P = P_-$. If P is a hermitian solution with the property that $\sigma(A_P) \subset \bar{\mathcal{D}}$, then $P = P_+$.

Proof: Let P be a hermitian solution. Then $X = P - P_-$ solves 3.6 and $S_- + B^* X B = R + B^* P B > 0$. We shall prove that $X \geq 0$. First we shall prove that $\mathcal{X}_0(A_-) \subseteq \ker X$. In order to prove this, choose a basis of $\mathcal{X}_0(A_-)$ consisting of eigenvectors and generalized eigenvectors. Such a basis consists of chains of vectors $x_1, \ldots, x_k$ with the property that

$$
\begin{aligned}
A_- x_1 &= \lambda x_1 \\
A_- x_2 &= \lambda x_2 + x_1 \\
&\vdots \\
A_- x_k &= \lambda x_k + x_{k-1},
\end{aligned}
$$

where $|\lambda| = 1$. We will show by induction that $X x_1 = \ldots = X x_k = 0$. Assume $A_- x_1 = \lambda x_1$. Using 3.6 we obtain

$$x_1^* X x_1 = |\lambda|^2 x_1^* X x_1 - |\lambda|^2 x_1^* X B (S_- + B^* X B)^{-1} B^* X x_1.$$

This yields $x_1^* X B (S_- + B^* X B)^{-1} B^* X x_1 = 0$, from which we obtain $x_1^* X B = 0$. Again using 3.6 this implies $x_1^* X = \bar{\lambda} x_1^* X A_-$ whence

$$x_1^* X (A_- - \bar{\lambda}^{-1} I \quad B) = 0$$

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By controllability of (A,B) the latter implies that $Xx_1 = 0$. Now, assume that $A_-x_r = x_{r-1} + \lambda x_r$ and $Xx_{r-1} = 0$. Using 3.6 we find

$$
\begin{aligned}
x_r^* X x_r &= (x_{r-1}^* + \bar{\lambda} x_r^*) X (x_{r-1} + \lambda x_r) \\
&\quad - (x_{r-1}^* + \bar{\lambda} x_r^*) X B (S_- + B^* X B)^{-1} B^* X (x_{r-1} + \lambda x_r)
\end{aligned}
$$

The latter implies that $(x_{r-1}^* + \bar{\lambda} x_r^*) X B = 0$ and hence $x_r^* X B = 0$. Also, $x_r^* X = \bar{\lambda} x_r^* X A_-$. Again, by controllability of (A,B) this yields $Xx_r = 0$. This proves our claim that $\mathcal{X}_0(A_-) \subseteq \ker X$.

To proceed, let U be a unitary matrix such that

$$
U^* A_- U = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},
$$

with $\sigma(A_{11}) \subset \mathcal{T}$ and $\sigma(A_{22}) \subset \mathcal{D}^e$. Let $Y = U^* X U$. Obviously, since $\mathcal{X}_0(A_-) \subseteq \ker X$, we have

$$
Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_{22} \end{pmatrix}.
$$

Furthermore if follows from 3.6 that $Y_{22} - A_{22}^* Y_{22} A_{22} \leq 0$. Since the eigenvalues of A_{22} lie *strictly* outside the unit disc the latter can be shown to imply $Y_{22} \geq 0$. Thus we have proven $X \geq 0$.

A completely similar argument can be used to show that any hermitian solution P satisfies $P \leq P_+$. Finally, note that in the above proof we only used the facts that $\sigma(A_-) \subset \bar{\mathcal{D}}^e$ and $\sigma(A_+) \subset \bar{\mathcal{D}}$. Hence, if P is a hermitian solution with the property that $\sigma(A_P) \subset \bar{\mathcal{D}}^e$, then we must have $P \leq Q$ for any hermitian solution Q, in particular for $Q = P_-$. This shows that $P = P_-$. The statement on P_+ is proven similarly. $\qquad \square$

Next we prove an analogue for the discrete time case of a theorem first proved by Coppel [1] for the continuous time case.

Theorem 3.5 *Assume that (A,B) is controllable, that $A - BR^{-1}C$ is nonsingular and that $\Psi(\eta) > 0$ for some η on the unit circle. Let P_- and P_+ be the minimal and maximal hermitian solution of 2.9, respectively. Put $\Delta := P_+ - P_-$ and*

$$
A_- := A - B(R + B^* P_- B)^{-1}(C + B^* P_- A).
$$

Then for every A_--invariant subspace $\mathcal{V}$ of $\mathcal{X}_+(A_-)$ we have

$$\mathcal{C}^n = \mathcal{V} \oplus (\Delta \mathcal{V})^\perp. \tag{3.11}$$

Let $\pi_\mathcal{V}$ denote the projection onto $\mathcal{V}$ along $(\Delta \mathcal{V})^\perp$. For every A_--invariant subspace $\mathcal{V} \subseteq \mathcal{X}_+(A_-)$ the matrix P defined by

$$P = P_- \pi_\mathcal{V} + P_+(I - \pi_\mathcal{V}) \tag{3.12}$$

is a hermitian solution of 2.9. Conversely, for every hermitian solution P of 2.9 there exists a unique A_--invariant subspace $\mathcal{V}$ of $\mathcal{X}_+(A_-)$ such that 3.12 holds. This subspace $\mathcal{V}$ is equal to $\mathcal{V} = \mathcal{X}_+(A_P)$, where A_P is defined by

$$A_P = A - B(R + B^* P B)^{-1}(C + B^* P A).$$

In addition, $\mathcal{X}_0(A_P) = \ker \Delta$ and $\mathcal{X}_-(A_P) = (\Delta \mathcal{V})^\perp \cap \mathcal{X}_-(A_+)$.

It will become clear in the proof that this result is actually little more than a reformulation of the last part of Theorem 2.1

Proof: First we show that it suffices to prove the theorem for equation 3.6. Note that $X_- = 0$ is the minimal solution of 3.6 and $X_+ = P_+ - P_-$ is the maximal solution. Moreover, by straightforward computation,

$$
\begin{aligned}
(A_-)_X &:= A_- - B(S_- + B^* X B)^{-1} B^* X A_- \\
&= A_{X+P_-}.
\end{aligned}
\tag{3.13}
$$

In particular this means that $(A_-)_{X_-} = A_-$ and $(A_-)_{X_+} = A_+$, i.e., the A_- and A_+ matrices remain the same. Because of Lemma 3.2 P is a solution of 2.9 if and only if $P = X + P_-$ for some solution of 3.6. Now assume the theorem is true for equation 3.6. Let $\mathcal{V}$ be an A_--invariant subspace of $\mathcal{X}_+(A_-)$. As the A_- matrix remains the same, we conclude that 3.11 holds. Furthermore, the matrix $X := X_+(I - \pi_\mathcal{V})$ is a hermitian solution of 3.6. This implies that $P := P_- + X = P_- \pi_\mathcal{V} + P_+(I - \pi_\mathcal{V})$ is a hermitian solution of 2.9. Conversely, let P is a solution of 2.9. Define $X := P - P_-$. There exists an A_--invariant subspace of $\mathcal{X}_+(A_-)$ such that $X = X_+(I - \pi_\mathcal{V})$. This however yields $P = P_- \pi_\mathcal{V} - P_+(I - \pi_\mathcal{V})$. Finally, since $(A_-)_X = A_P$ and $(A_-)_{X_+} = A_+$, we also find $\mathcal{V} = \mathcal{X}_+(A_P)$, $\ker \Delta = \mathcal{X}_0(A_P)$ and $\mathcal{X}_-(A_+) \cap (\Delta \mathcal{V})^\perp = \mathcal{X}_-(A_P)$.

It remains to prove the statements of the theorem for equation 3.6. The matrix T given by 3.1 looks particularly simple in the case of equation 3.6:

$$
T = \begin{pmatrix} A_- & -B S_-^{-1} B^* A_-^{*-1} \\ 0 & A_-^{*-1} \end{pmatrix},
$$

with $S_- > 0$. Note that A_- has all its eigenvalues on or outside the unit circle. Hence we have $\mathcal{X}_+(T) \subseteq \mathrm{im}\ \begin{pmatrix} I \\ 0 \end{pmatrix}$, more precisely,

$$\mathcal{X}_+(T) = \mathcal{X}_+(A_-) \times \{0\}.$$

So, T-invariant subspaces $\mathcal{N}$ with $\sigma(T \mid \mathcal{N}) \subset \mathcal{D}^e$ are precisely the subspaces of the form

$$\mathcal{N} = \mathcal{V} \times \{0\},$$

where $\mathcal{V}$ is A_--invariant and $\mathcal{V} \subseteq \mathcal{X}_+(A_-)$.

Now, let $\mathcal{V}$ be an A_--invariant subspace of $\mathcal{X}_+(A_-)$. We will prove 3.11 and that $X_+(I - \pi_\mathcal{V})$ is a hermitian solution of 3.6. According to Theorem 3.1, there is a unique solution X of 3.6 such that

$$\mathrm{im}\ \begin{pmatrix} I \\ X \end{pmatrix} \cap (\mathcal{X}_+(A_-) \times \{0\}) = \mathcal{V} \times \{0\}.$$

From [3], Sections 2 and 7 and Theorem 3.1 we have

$$\mathrm{im}\ \begin{pmatrix} I \\ X \end{pmatrix} = (\mathcal{V} \times \{0\}) \oplus (\mathrm{im}\ \begin{pmatrix} I \\ X \end{pmatrix} \cap \mathcal{X}_0(T)) \oplus (\mathcal{X}_-(T) \cap (\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{N})^\perp),$$

and moreover $\mathrm{im}\ \begin{pmatrix} I \\ X \end{pmatrix} \cap \mathcal{X}_0(T)$ is the same subspace for any hermitian solution X (here we also use the fact that the signs in the sign characteristic of $(T, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix})$ are all the same, see Theorem 1.2 in [4]). Then

$$
\begin{aligned}
\mathrm{im}\ \begin{pmatrix} I \\ X \end{pmatrix} \cap \mathcal{X}_0(T) &= \mathrm{im}\ \begin{pmatrix} I \\ X_+ \end{pmatrix} \cap \mathcal{X}_0(T) \\
&= \mathrm{im}\ \begin{pmatrix} I \\ X_- \end{pmatrix} \cap \mathcal{X}_0(T) \\
&= \mathrm{im}\ \begin{pmatrix} I \\ 0 \end{pmatrix} \cap \mathcal{X}_0(T) \\
&\subseteq \ker X_+ \times \{0\}.
\end{aligned}
$$

Conversely, for $x \in \ker X_+$ we have from 3.6

$$(A^*_- X_+ A_- x, x) = ((X_+ - A^*_- X_+ A_-)x, x) \leq 0.$$

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On the other hand $X_+ \geq 0$, by Lemma 3.4 so $(A^*_- X_+ A_- x, x) = 0$, i.e., $X_+ A_- x = 0$. It follows that ker X_+ is A_--invariant. We claim that

$$\ker X_+ \subseteq \mathcal{X}_0(A_-). \tag{3.14}$$

Indeed, assume $x \in \ker X_+$, $x \neq 0$, and $A_- x = \lambda x$. Then $T \begin{pmatrix} x \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} x \\ 0 \end{pmatrix} \in$ im $\begin{pmatrix} I \\ X_+ \end{pmatrix}$, by Lemma 3.3 we have $\lambda \in \bar{\mathcal{D}}$. On the other hand, since $\lambda \in \sigma(A_-)$, $\lambda \in \bar{\mathcal{D}}^e$. Thus $\lambda \in \mathcal{T}$, which proves the claim. It follows from 3.14 that

$$\text{im } \begin{pmatrix} I \\ X \end{pmatrix} \cap \mathcal{X}_0(T) = \ker X_+ \times \{0\}.$$

Next,

$$\mathcal{X}_-(T) = \mathcal{X}_-(T) \cap \text{im } \begin{pmatrix} I \\ X_+ \end{pmatrix},$$

so

$$\mathcal{X}_-(T) \cap (\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{N})^\perp \subseteq \text{im } \begin{pmatrix} I \\ X_+ \end{pmatrix} \cap (\mathcal{C}^n \times \mathcal{V}^\perp).$$

Now $\begin{pmatrix} x \\ X_+ x \end{pmatrix} \in \mathcal{C}^n \times \mathcal{V}^\perp$ implies $X_+ x \in \mathcal{V}^\perp$, i.e., $x \in (X_+ \mathcal{V})^\perp$. So

$$\mathcal{X}_-(T) \cap (\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{N})^\perp \subseteq \{ \begin{pmatrix} x \\ X_+ x \end{pmatrix} \mid x \in (X_+ \mathcal{V})^\perp \}.$$

Since obviously also ker $X_+ \subseteq (X_+ \mathcal{V})^\perp$, we find

$$\text{im } \begin{pmatrix} I \\ X \end{pmatrix} \subseteq (\mathcal{V} \times \{0\}) + \{ \begin{pmatrix} x \\ X_+ x \end{pmatrix} \mid x \in (X_+ \mathcal{V})^\perp \}. \tag{3.15}$$

Using the previous inclusion it is easy to see that $\mathcal{C}^n = \mathcal{V} + (X_+ \mathcal{V})^\perp$. We claim that the latter is, in fact, a direct sum. Indeed, $x \in \mathcal{V} \cap (X_+ \mathcal{V})^\perp$ implies $(x, X_+ x) = 0$ whence $X_+ x = 0$. Thus $\mathcal{V} \cap (X_+ \mathcal{V})^\perp \subseteq \ker X_+ \cap \mathcal{V} = 0$ (recall that $\mathcal{V} \subseteq \mathcal{X}_+(A_-)$ while ker $X_+ \subseteq \mathcal{X}^0(A_-)$). Now, if $\pi_\mathcal{V}$ is the projection onto $\mathcal{V}$ along $(X_+ \mathcal{V})^\perp$, it can be seen that 3.15 implies

$$X = X_+(I - \pi_\mathcal{V}),$$

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Next, conversely, let X be a hermitian solution of 3.6. Define

$$\mathcal{N} := \text{im } \begin{pmatrix} I \\ X \end{pmatrix} \cap \mathcal{X}_+(T).$$

Then $\mathcal{N}$ is a T-invariant subspace and $\sigma(T \mid \mathcal{N}) \subset \mathcal{D}^e$. Thus $\mathcal{N}$ has the form $\mathcal{N} = \mathcal{V} \times \{0\}$ for some A_--invariant subspace $\mathcal{V}$ of $\mathcal{X}_+(A_-)$. By repeating the argument in the first part of this proof it is then shown that we must have $X = X_+(I - \pi_\mathcal{V})$.

Finally, we will show that if $\mathcal{V} \subset \mathcal{X}_+(A_-)$ is A_--invariant and $X = X_+(I - \pi_\mathcal{V})$, then $\mathcal{V} = \mathcal{X}_+((A_-)_X)$, ker $X_+ = \mathcal{X}_0((A_-)_X)$ and $(\mathcal{X}_+\mathcal{V})^\perp \cap \mathcal{X}_-(A_+) = \mathcal{X}_-((A_-)_X)$.

As in the proof of Theorem 3.4 one shows that $\mathcal{X}_0(A_-) \subseteq$ ker X_+. Combined with 3.14 this yields ker $X_+ = \mathcal{X}_0(A_-)$. Since $0 \leq X \leq X_+$, ker $X_+ \subseteq$ ker X. Hence $(A_-)_X \mid$ ker $X_+ = A_- \mid$ ker X_+. This yields

$$\text{ker } X_+ = \mathcal{X}_0((A_-)_X),$$

as desired. We will now show that

$$\mathcal{V} \subseteq \mathcal{X}_+((A_-)_X) \tag{3.16}$$

Indeed, note that $\mathcal{V} \subseteq$ ker X. Hence $(A_-)_X \mid \mathcal{V} = A_- \mid \mathcal{V}$, which yields 3.16. Next, we show that

$$(X_+\mathcal{V})^\perp \subseteq \mathcal{X}_0((A_-)_X) \oplus \mathcal{X}_-((A_-)_X). \tag{3.17}$$

In order to prove this, first note that $A_-\mathcal{V} = \mathcal{V}$, since $\mathcal{V}$ is A_--invariant and since A_- is invertible. Now, the Riccati equation 3.6 with $X = X_+$ can be written as $X_+ = A_-^* X_+ A_+$. We claim that $(X_+\mathcal{V})^\perp$ is A_+-invariant. Let $w \in (X_+\mathcal{V})^\perp$. Then for all $v \in \mathcal{V}$

$$0 = v^* X_+ w = v^* A_-^* X_+ A_+ w.$$

Hence $A_+ w \in (X_+ A_-\mathcal{V})^\perp = (X_+\mathcal{V})^\perp$. Next, by straightforward caculation, one shows that

$$(A_-)_X = A_+ - B(R + B^*(X + P_-)B)^{-1}B^*(X - X_+)A_+. \tag{3.18}$$

Since $X \mid (X_+\mathcal{V})^\perp = X_+ \mid (X_+\mathcal{V})^\perp$, 3.18 yields $(A_-)_X \mid (X_+\mathcal{V})^\perp = A_+ \mid (X_+\mathcal{V})^\perp$. Since $\sigma(A_+) \subset \bar{\mathcal{D}}$, this implies $\sigma((A_-)_X \mid (X_+\mathcal{V})^\perp) \subset \bar{\mathcal{D}}$, which yields 3.17.

By combining 3.16, 3.17 and the fact that $\mathcal{V} \oplus (X_+\mathcal{V})^\perp = \mathcal{C}^n$ we find that the inclusions 3.16 and 3.17 are, in fact, equalities. Finally, since $(A_-)_X \mid (X_+\mathcal{V})^\perp = A_+ \mid (X_+\mathcal{V})^\perp$, we find

$$(X_+ \mathcal{V})^\perp \cap \mathcal{X}_-(A_+) = \mathcal{X}_-((A_-)_X).$$ □

4 A proof of Theorem 2.1

The proof is split up into several lemmas, which are all discrete time counterparts of results in [6]. In this section, let $\mathcal{L}$ be an arbitrary but fixed subspace of C^n. We will first study the finiteness of the optimal cost $V_{\mathcal{L}}(x_0)$. Note that our assumption that (A, B) is controllable is sufficient to guarantee that $V_{\mathcal{L}}(x_0) < +\infty$ for all x_0. In the sequel we shall establish a sufficient condition to guarantee $V_{\mathcal{L}}(x_0) > -\infty$ for all x_0. From section 2 recall the definition of negative semi-definiteness on $\mathcal{L}$ of a given hermitian matrix P. It turns out that if the smallest solution P_- of the Riccati equation 2.9 is negative semi-definite on $\mathcal{L}$, then the optimal cost is finite:

Lemma 4.1 *Assume that (A, B) is controllable, $A - BR^{-1}C$ is non-singular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Furthermore, assume that 2.9 has at least one hermitian solution. Then we have: if P_- is negative semi-definite on $\mathcal{L}$ then $V_{\mathcal{L}}(x_0) \in \mathcal{R}$ for all $x_0 \in C^n$*

Our proof of Lemma 4.1 uses the following two lemmas:

Lemma 4.2 *Let $\mathcal{L}$ be a subspace of C^n and let H be a matrix such that $\mathcal{L} = \ker H$. Let $P \in C^{n \times n}$ be hermitian. Then P is negative semi-definite on $\mathcal{L}$ if and only if there exists $\lambda \in \mathcal{R}$ such that $P - \lambda H^* H$ is negative semi-definite.* □

For a proof of the above lemma we refer to [6].

Lemma 4.3 *For any $x_0 \in C^n$, any sequence u and any hermitian solution P of 2.9 we have*

$$\begin{aligned} J_T(x_0, u) &= x_0^* P x_0 - x_{T+1}^* P x_{T+1} \\ &\quad + \Sigma_{k=0}^T \|u_k + (R + B^* P B)^{-1}(C + B^* P A)x_k\|_{R+B^*PB}^2. \end{aligned}$$

Here, $\|v\|_S^2 := v^* S v$. □

Giving a proof of this lemma is just a matter of standard computation.

Proof of Lemma 4.1 : Let $x_0 \in C^n$. Since (A, B) is controllable there is an input sequence $u \in U_{\mathcal{L}}(x_0)$ such that $J(x_0, u) < +\infty$ (in fact, one can steer from x_0 to the origin in finite time). Thus $V_{\mathcal{L}}(x_0) \in \mathcal{R} \cup \{-\infty\}$. Now let $u \in U_{\mathcal{L}}(x_0)$ be arbitrary. Let H be such that $\mathcal{L} = \ker H$ and let $\lambda \in \mathcal{R}$ be such that $P_- - \lambda H^* H$ is negative semi-definite. According to Lemma 4.3, for all T we have

$$J_T(x_0, u) = x_0^* P_- x_0 - x_{T+1}^*(P_- - \lambda H^* H)x_{T+1} - \lambda \|H x_{T+1}\|^2$$

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$$+ \Sigma_{k=0}^{T}\|u_k + (R + B^*PB)^{-1}(C + B^*PA)x_k\|_{R+B^*PB}^2.$$

Thus, for all $T \geq 0$,

$$J_T(x_0, u) \geq x_0^* P_- x_0 - \lambda \|Hx_{T+1}\|^2.$$

Since x_T converges to $\mathcal{L}$ as $T \to \infty$, we have $Hx_{T+1} \to 0$. It follows that

$$J(x_0, u) = \lim_{T \to \infty} J_T(x_0, u) \geq x_0^* P_- x_0.$$

Since the latter holds for all $u \in U_{\mathcal{L}}(x_0)$ this proves our claim. $\square$

The next few lemmas give some general properties of linear systems.

Lemma 4.4 *Consider the system*

$$x_{k+1} = Ax_k + v_k, \quad y_k = Cx_k \,,$$

and suppose that (C, A) is observable. Then if $\{v_k\}_{k=0}^{\infty} \in \ell_2$ and $\{y_k\}_{k=0}^{\infty} \in \ell_{\infty}$ necessarily $\{x_k\}_{k=0}^{\infty} \in \ell_{\infty}$.

Proof: Since (C, A) is observable there exists a matrix L such that $\sigma(A + LC) \subset \mathcal{D}$. Obviously, $\{x_k\}_{k=0}^{\infty}$ satisfies the difference equation

$$x_{k+1} = (A + LC)x_k - Ly_k + v_k.$$

Using some straightforward estimates we see that $v \in \ell_{\infty}$ and $y \in \ell_{\infty}$ imply $x \in \ell_{\infty}$. $\square$

In the following, if $\mathcal{C}_g$ is a subset of $\mathcal{C}$, then $\mathcal{X}_g(A)$ will denote the spectral subspace of A associated with its eigenvalues in $\mathcal{C}_g$, i.e., the largest A-invariant subspace $\mathcal{V}$ with the property that $\sigma(A \mid \mathcal{V}) \subset \mathcal{C}_g$. Using the previous lemma we can now prove the following:

Lemma 4.5 *Consider the system*

$$x_{k+1} = Ax_k + v_k, \quad y_k = Cx_k.$$

Assume that (C, A) is detectable (relative to $\mathcal{C}_g$). Let the state space $\mathcal{C}^n$ be decomposed into $\mathcal{C}^n = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1$ is A-invariant. In this decomposition, let

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$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}.$$

Assume that $\sigma(A \mid \mathcal{X}_1) \subset \mathcal{C}_g$ and $\sigma(A_2) \subset \mathcal{C}/\mathcal{C}_g$. Then for every initial condition x_0 we have: if $\{v_k\}_{k=0}^{\infty} \in \ell_2$ and $\{y_k\}_{k=0}^{\infty} \in \ell_{\infty}$ then $\{x_{2,k}\}_{k=0}^{\infty} \in \ell_{\infty}$.

Proof: Clearly $\mathcal{X}_1 = \mathcal{X}_g(A)$. By the fact that (C, A) is detectable with respect to $\mathcal{C}_g$ we may now conclude that $< \ker C \mid A > \subseteq \mathcal{X}_1$. Decompose $\mathcal{X}_1 = \mathcal{X}_{11} \oplus \mathcal{X}_{12}$, with $\mathcal{X}_{11} := < \ker C \mid A >$ and $\mathcal{X}_{12}$ arbitrary. Accordingly, partition $x = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$. We then have $\mathcal{C}^n = \mathcal{X}_{11} \oplus \mathcal{X}_{12} \oplus \mathcal{X}_2$ with $x = (x_{11}^T, x_{12}^T, x_2^T)^T$. In this decomposition let

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & C_2 & C_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_{11} \\ v_{12} \\ v_2 \end{pmatrix}.$$

Obviously, the system

$$\left(\begin{pmatrix} C_2 & C_3 \end{pmatrix}, \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \right)$$

is observable. Moreover,

$$\begin{pmatrix} x_{12,k+1} \\ x_{2,k+1} \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \begin{pmatrix} x_{12,k} \\ x_{2,k} \end{pmatrix} + \begin{pmatrix} v_{12,k} \\ v_{2,k} \end{pmatrix}, \quad y_k = \begin{pmatrix} C_2 & C_3 \end{pmatrix} \begin{pmatrix} v_{12,k} \\ v_{2,k} \end{pmatrix}.$$

It now follows from the previous lemma that $\begin{pmatrix} x_{12} \\ x_2 \end{pmatrix} \in \ell_{\infty}$, which implies that $x_2 \in \ell_{\infty}$. $\quad \square$

The next lemma tells us that a semi-stable controllable system has the property that all initial states can be steered to the origin with arbitrary small controls (in ℓ_2-sense).

Lemma 4.6 *Consider the controllable system*

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 \text{ given.}$$

Assume that $\sigma(A) \subset \bar{\mathcal{D}}$. Then for all $\epsilon > 0$ there exists a $u \in \ell_2$ such that $\|u\|_2 < \epsilon$ and $x(x_0, u)_k \to 0$ as $k \to \infty$.

Proof: We will first show that it suffices to prove the statement of the lemma with $\bar{\mathcal{D}}$ replaced by $\mathcal{T}$. Indeed, we can always choose a basis in $\mathcal{C}^n$ such that A and B have the form

$$
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
$$

with $\sigma(A_1) \subset \mathcal{D}$ and $\sigma(A_2) \subset \mathcal{T}$. Consider the subsystem $x_{1,k+1} = A_1 x_{1,k} + B u_k$, with $x_{1,0}$ given. Obviously, for any input sequence $u \in \ell_2$ we automatically have that $x_1 \in \ell_2$ and hence $x_{1,k} \to 0$ as $k \to \infty$.

Assume therefore that $\sigma(A) \subset \mathcal{T}$. For any $\delta > 0$, any initial state x_0 and any input sequence u define the quadratic cost

$$
J_\delta(x_0, u) = \sum_{k=0}^{\infty} \|u_k\|^2 + \delta^2 \|x(x_0, u)_k\|^2 \tag{4.1}
$$

and consider the optimization problem

$$
\inf_u J_\delta(x_0, u). \tag{4.2}
$$

Note that 4.2 can be considered as a 'standard' discrete time linear quadratic problem of minimizing $\sum_{k=0}^{\infty} u_k^* R u_k + x_k^* D^* D x_k$, with $R = I$ and $D = \delta I$ (see for example [7]). Since (A, B) is controllable and $(\delta I, A)$ is observable, the infimum 4.2 is equal to $x_0^* P(\delta) x_0$, with $P(\delta)$ the unique positive semi-definite solution of the Riccati equation

$$
P = A^* P A + \delta^2 I - A^* P B (I + B^* P B)^{-1} B^* P A.
$$

(In fact, $P(\delta) > 0$). Furthermore, for any x_0 there exists a unique optimal input u^+ which is given by the state feedback control law

$$
u^+ = -(I + B^* P(\delta) B)^{-1} B^* P(\delta) A x^+,
$$

and the corresponding optimal closed loop matrix

$$
A - B(I + B^* P(\delta) B)^{-1} B^* P(\delta) A
$$

is stable, i.e., has all its eigenvalues in $\mathcal{D}$. Now, we shall analyse what happens if $\delta \downarrow 0$. We claim that $P(\delta) \downarrow 0$. Indeed, $P(\delta) \geq 0$ and $P(\delta)$ is monotonically decreasing as $\delta \downarrow 0$. Hence there exists a hermitian matrix $\bar{P} \geq 0$ such that $P(\delta) \downarrow \bar{P}$. Clearly, $\bar{P}$ satisfies the Riccati

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equation.

$$P = A^* P A - A^* P B (I + B^* P B)^{-1} B^* P A \tag{4.3}$$

Now, by applying Theorem 3.4 we find that the latter Riccati equation has a largest hermitian solution, say, P_+. We contend that $P_+ = 0$. Indeed, $P = 0$ is a solution of 4.3 and it has the property that $\sigma(A_P) \subset \bar{\mathcal{D}}$ (since $A_P = A$ and $\sigma(A) \subset \mathcal{T}$). Thus, we must have $\bar{P} \leq 0$. Our conclusion is that $P = 0$.

In order to complete the proof, let $\epsilon > 0$. Choose $\delta > 0$ such that $x_0^* P(\delta) x_0 < \epsilon$. Choose the input sequence given by $u^+ = -(I + B^* P(\delta) B) B^* P(\delta) A x^+$. Then we have

$$\|u^+\|^2 \leq J_\delta(x_0, u^+) = x_0^* P(\delta) x_0 < \epsilon.$$

Since the corresponding closed loop matrix is stable, the corresponding state trajectory converges to zero as $k \to \infty$. $\qquad\square$

We proceed by decomposing $\mathcal{C}^n$ as follows. Let

$$\mathcal{X}_1 := <\mathcal{L} \cap \ker P_- \mid A_- > \cap \mathcal{X}_+(A_-) = \mathcal{V}(\mathcal{L}),$$

and let $P_{\mathcal{L}}$ be the solution corresponding to $\mathcal{V}(\mathcal{L})$ according to Theorem 3.5. Put

$$A_{\mathcal{L}} := A - B(R + B^* P_{\mathcal{L}} B)^{-1} (C + B^* P_{\mathcal{L}} A).$$

According to Theorem 3.5 we have $\mathcal{X}_+(A_{\mathcal{L}}) = \mathcal{X}_1$. In addition, we define

$$\mathcal{X}_2 := \mathcal{X}_0(A_{\mathcal{L}}) = \mathcal{X}_0(A_-) = \ker \Delta,$$

$$\mathcal{X}_3 := \mathcal{X}_-(A_{\mathcal{L}}) = (\Delta \mathcal{V}(\mathcal{L}))^\perp \cap \mathcal{X}_-(A_+).$$

With respect to the decomposition $\mathcal{C}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ we have

$$A_- = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \quad A_{\mathcal{L}} = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A'_{33} \end{pmatrix},$$

Here, we have used that $A_- \mid \mathcal{X}_1 = A_{\mathcal{L}} \mid \mathcal{X}_1$ and that $A_- \mid \mathcal{X}_2 = A_{\mathcal{L}} \mid \mathcal{X}_2$. This follows most easily by combining 3.13 and the facts that $P_{\mathcal{L}} \mid \mathcal{V}(\mathcal{L}) = P_- \mid \mathcal{V}(\mathcal{L})$ and $P_{\mathcal{L}} \mid (\Delta \mathcal{V}(\mathcal{L}))^\perp =$

$P_+ \mid (\Delta \mathcal{V}(\mathcal{L}))^{\perp}$. We also have

$$P_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{22}^- & P_{23}^- \\ 0 & P_{23}^{-*} & P_{33}^- \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix}.$$

and

$$P_+ = P_- + \Delta = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & P_{22}^- & P_{23}^- \\ 0 & P_{23}^{-*} & P_{33}^+ \end{pmatrix}, \quad P_{\mathcal{L}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{22}^- & P_{23}^- \\ 0 & P_{23}^{-*} & P_{33}^+ \end{pmatrix},$$

where $P_{33}^+ := P_{33}^- + \Delta_{33}$.

The next lemma states that $P_{\mathcal{L}}$ yields a lower bound for the linear quadratic problem 2.8 under consideration.

Lemma 4.7 *Suppose (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Assume further 2.9 has at least one hermitian solution and assume P_- is negative semi-definite on $\mathcal{L}$. Then for all x_0 and for all $u \in U_{\mathcal{L}}(x_0)$ we have:*

$$J(x_0, u) \geq x_0^* P_{\mathcal{L}} x_0 + \Sigma_{k=0}^{\infty} \|u_k + (R + B^* P_{\mathcal{L}} B)^{-1}(C + B^* P_{\mathcal{L}} A)x_k\|_{R+B^* P_{\mathcal{L}} B}^2.$$

Proof: Let H be a matrix such that $L = \ker H$. Let $\lambda \in \mathcal{R}$ be such that $P_- - \lambda H^* H \leq 0$ (see Lemma 4.2). Take an arbitrary $u \in U_{\mathcal{L}}(x_0)$. It follows from Lemma 4.1 that $J(x_0, u) \in \mathcal{R} \cup \{+\infty\}$. If it is equal to $+\infty$ then the inequality trivially holds. Assume therefore that $J(x_0, u)$ is finite. Put

$$v_k = u_k + (R + B^* P_- B)^{-1}(C + B^* P_- A)x_k.$$

From Lemma 4.3 we have

$$
\begin{aligned}
\Sigma_{k=0}^T &\|v_k\|_{R+B^* P_- B}^2 \\
&= J_T(x_0, u) - x_0^* P_- x_0 + x_{T+1}^*(P_- - \lambda H^* H)x_{T+1} + \lambda \|H x_{T+1}\|^2 \\
&\leq J_T(x_0, u) - x_0^* P_- x_0 + \lambda \|H x_{T+1}\|^2.
\end{aligned}
\tag{4.4}
$$

Since $J_T(x_0, u) \to J(x_0, u)$ and $H x_T \to 0$ we find that $\{v_k\} \in \ell_2$. Again using 4.4 this implies that

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$$\lim_{T \to \infty} x_T^*(P_- - \lambda H^*H)x_T$$

exists and is finite. Thus $\lim_{T \to \infty} x_T^* P_- x_T$ exists and is finite. Also. since $P_- - \lambda H^*H$ is semi-definite, $(P_- - \lambda H^*H)x_k$ and hence $P_- x_k$ are bounded functions of k. Denote

$$y_k = \begin{pmatrix} P_- \\ H \end{pmatrix} x_k.$$

Then $\{y_k\} \in \ell_\infty$. Since $x_{k+1} = Ax_k + Bu_k$, we have that $\{x_k\}$, $\{y_k\}$ and $\{v_k\}$ are related by

$$x_{k+1} = A_- x_k + Bv_k, \quad y_k = \begin{pmatrix} P_- \\ H \end{pmatrix} x_k.$$

Now decompose C^n as above: $C^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$. Since

$$\mathcal{X}_1 \subseteq \ker \begin{pmatrix} P_- \\ H \end{pmatrix},$$

we have

$$\begin{pmatrix} P_- \\ H \end{pmatrix} = \begin{pmatrix} 0 & D_2 & D_3 \end{pmatrix} \tag{4.5}$$

for given matrices D_2 and D_3. With respect to the given decomposition, let

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

Since $\mathcal{X}_1$ is the undetectable subspace (relative to $\bar{\mathcal{D}}$) of the system

$$(\begin{pmatrix} P_- \\ H \end{pmatrix}, \ A_-),$$

it is easily verified that the pair

$$\left(\begin{pmatrix} D_2 & D_3 \end{pmatrix}, \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix}\right)$$

is detectable (relative to $\bar{\mathcal{D}}$). Since $\sigma(A_-) \subset \bar{\mathcal{D}}^e$ and $\mathcal{X}_2 = \mathcal{X}_0(A_-)$, we have $\sigma(A_{22}) \subset \mathcal{T}$ and

$$\sigma\left(\begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix}\right) \subset \mathcal{D}^e.$$

Hence $\sigma(A_{33}) \subset \mathcal{D}^e$. Also we have

$$\begin{pmatrix} x_{2,k+1} \\ x_{3,k+1} \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \begin{pmatrix} x_{2,k} \\ x_{3,k} \end{pmatrix} + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} v_k, \quad y_k = \begin{pmatrix} D_2 & D_3 \end{pmatrix} \begin{pmatrix} x_{2,k} \\ x_{3,k} \end{pmatrix}.$$

Since $\{v_k\} \in \ell_2$ and $\{y_k\} \in \ell_\infty$, by Lemma 4.5 (applied with $C_g = \bar{\mathcal{D}}$) we have $\{x_{3,k}\} \in \ell_\infty$. Now consider Lemma 4.3. We have

$$J_T(x_0, u) = x_0^* P_{\mathcal{L}} x_0 - x_{T+1}^* P_{\mathcal{L}} x_{T+1} + \Sigma_{k=0}^T \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2,$$

where

$$w_k = u_k + (R + B^* P_{\mathcal{L}} B)^{-1} (C + B^* P_{\mathcal{L}} A) x_k$$

Then

$$\begin{aligned} J_T(x_0, u) &= x_0^* P_{\mathcal{L}} x_0 + \Sigma_{k=0}^T \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 \\ &\quad - \begin{pmatrix} x_{2,T+1} \\ x_{3,T+1} \end{pmatrix}^* \begin{pmatrix} P_{22}^- & P_{23}^- \\ P_{23}^{-*} & P_{33}^+ \end{pmatrix} \begin{pmatrix} x_{2,k} \\ x_{3,k} \end{pmatrix} \\ &= x_0^* P_{\mathcal{L}} x_0 + \Sigma_{k=0}^T \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 \\ &\quad - x_{3,T+1}^* \Delta_{33} x_{3,T+1} - x_{T+1}^* P_- x_{T+1}. \end{aligned} \tag{4.6}$$

Now $x_{T+1}^* P_- x_{T+1}$, $J_T(x_0, u)$ are bounded as $T \to \infty$, and likewise $x_{3,T+1}$ is bounded as $T \to \infty$. It follows that

$$\Sigma_{k=0}^\infty \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 < \infty,$$

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and as $R + B^* P_{\mathcal{L}} B \geq R + B^* P_- B > 0$ we have $\{w_k\} \in \ell_2$. Now considering

$$x_{k+1} = A x_k + B u_k = A_{\mathcal{L}} x_k + B w_k,$$

one sees $x_{3,k+1} = A_{33}' x_{3,k} + B_3 w_k$. As $\sigma(A_{33}') \subset \mathcal{D}$ (because of the fact that $\mathcal{X}_3 = \mathcal{X}_-(A_{\mathcal{L}})$) we obtain $\{x_{3,k}\} \in \ell_2$. Hence $\lim_{k \to \infty} x_{3,k} = 0$. Now from 4.6 we have

$$
\begin{aligned}
J_T(x_0, u) &= x_0^* P_{\mathcal{L}} x_0 - x_{T+1}^*(P_- - \lambda H^* H) x_{T+1} \\
&\quad - \lambda \|H x_{T+1}\|^2 + \Sigma_{k=0}^T \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 - x_{3,T+1}^* \triangle_{33} x_{3,T+1} \\
&\geq x_0^* P_{\mathcal{L}} x_0 + \lambda \|H x_{T+1}\|^2 + \Sigma_{k=0}^T \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 - x_{3,T+1}^* \triangle_{33} x_{3,T+1}.
\end{aligned}
$$

The desired result then follows by taking the limit as $T \to \infty$ in the above inequality. $\quad\square$

Our next lemma states that $V_{\mathcal{L}}(x_0) = x_0^* P_{\mathcal{L}} x_0$, taking into account the previous lemma.

Lemma 4.8 *Suppose (A, B) is controllable, $A - BR^{-1}C$ is nonsingular and $\Psi(\eta) > 0$ for some $\eta \in \mathcal{T}$. Assume further 2.9 has at least one hermitian solution and assume P_- is negative semi-definite on $\mathcal{L}$. Then for all x_0 and for all $\epsilon > 0$ there is a $u \in U_{\mathcal{L}}(x_0)$ such that $J(x_0, u) \leq x_0^* P_{\mathcal{L}} x_0 + \epsilon$.*

Proof: Put $w_k = u_k + (R + B^* P_{\mathcal{L}} B)^{-1}(C + B^* P_{\mathcal{L}} A) x_k$. From 4.6 we have for all $u \in U_{\mathcal{L}}(x_0)$:

$$
\begin{aligned}
J_T(x_0, u) &= x_0^* P_{\mathcal{L}} x_0 + \Sigma_{k=0}^T \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 \\
&\quad - \begin{pmatrix} x_{2,T+1} \\ x_{3,T+1} \end{pmatrix}^* \begin{pmatrix} P_{22}^- & P_{23}^- \\ P_{23}^{-*} & P_{33}^+ \end{pmatrix} \begin{pmatrix} x_{2,k} \\ x_{3,k} \end{pmatrix}.
\end{aligned}
$$

Moreover $x_{k+1} = A_{\mathcal{L}} x_k + B w_k$, so

$$
\begin{pmatrix} x_{2,k+1} \\ x_{3,k+1} \end{pmatrix} = \begin{pmatrix} A_{22} & 0 \\ 0 & A_{33}' \end{pmatrix} \begin{pmatrix} x_{2,k} \\ x_{3,k} \end{pmatrix} + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} w_k.
$$

Now $\sigma(A_{22}) \subset \mathcal{T}$, $\sigma(A_{33}') \subset \mathcal{D}$. By Lemma 4.8 there is $w \in \ell_2$ such that

$$\Sigma_{k=0}^\infty \|w_k\|_{R+B^* P_{\mathcal{L}} B}^2 < \epsilon$$

and $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} (x_0, w)_k \to 0$ as $k \to \infty$. Introduce

$$u_k = w_k - (R + B^* P_{\mathcal{L}} B)^{-1} (C + B^* P_{\mathcal{L}} A) x_k.$$

Then

$$
\begin{aligned}
J(x_0, u) &= \lim_{T \to \infty} J_T(x_0, u) \\
&= x_0^* P_{\mathcal{L}} x_0 + \Sigma_{k=0}^{\infty} \| w_k \|_{R + B^* P_{\mathcal{L}} B}^2 \\
&\leq \epsilon + x_0^* P_{\mathcal{L}} x_0.
\end{aligned}
$$

$\square$

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1: (i) is proved in Lemmas 4.1, 4.7 and 4.4. It remains to prove (ii).

(ii) First assume that for all x_0 there is an optimal control, i.e., $u^+ \in U_{\mathcal{L}}(x_0)$ for which $V_{\mathcal{L}}(x_0) = J(x_0, u^+)$. Choose x_0 and let u^+ be the corresponding optimal control. Put $x_k^+ = x(x_0, u^+)_k$. By Lemma 4.7

$$x_0^* P_{\mathcal{L}} x_0 = J(x_0, u^+) \geq x_0^* P_{\mathcal{L}} x_0 + \Sigma_{k=0}^{\infty} \| w_k \|_{R + B^* P_{\mathcal{L}} B}^2,$$

where $w_k = u_k^+ + (R + B^* P_{\mathcal{L}} B)^{-1} (C + B^* P_{\mathcal{L}} A) x_k^+$. Hence $w_k = 0$, i.e.,

$$u_k^+ = -(R + B^* P_{\mathcal{L}} B)^{-1} (C + B^* P_{\mathcal{L}} A) x_k^+. \tag{4.7}$$

Since $x_{k+1} = A_{\mathcal{L}} x_k + B w_k$ this yields $x_{k+1}^+ = A_{\mathcal{L}} x_k^+$, in particular

$$
\begin{pmatrix} x_{2,k+1}^+ \\ x_{3,k+1}^+ \end{pmatrix} = \begin{pmatrix} A_{22} & 0 \\ 0 & A_{33}' \end{pmatrix} \begin{pmatrix} x_{2,k}^+ \\ x_{3,k}^+ \end{pmatrix}
$$

As $\sigma(A_{33}') \subset \mathcal{D}$ we have $x_{3,k}^+ \to 0$ as $k \to \infty$. From 4.6 we see that

$$J_T(x_0, u^+) = x_0^* P_{\mathcal{L}} x_0 - x_{3,T+1}^{+*} \Delta_{33} x_{3,T+1}^+ - x_{T+1}^{+*} (P_- - \lambda H^* H) x_{T+1}^+ - \| H x_{T+1} \|^2.$$

As $J_T(x_0, u^+) - x_0^* P_{\mathcal{L}} x_0 \to 0$, $H x_{T+1} \to 0$ and $x_{3,T+1}^{+*} \Delta_{33} x_{3,T+1}^+ \to 0$ as $T \to \infty$, we get $x_{T+1}^{+*} (P_- - \lambda H^* H) x_{T+1}^+ \to 0$. As $P_- - \lambda H^* H \leq 0$ this gives $(P_- - \lambda H^* H) x_{T+1}^+ \to 0$ and hence $P_- x_T \to 0$. In turn this implies that $D_2 x_{2,k}^+ + D_3 x_{3,k}^+ \to 0$ (see 4.5). As $x_{3,k} \to 0$ we

obtain $D_2 x_{2,k}^+ \to 0$. Using $x_{2,k+1}^+ = A_{22} x_{2,k}^+$ together with the fact that $\sigma(A_{22}) \subset \mathcal{T}$, this yields $D_2 = 0$ (note that x_0, so $x_{2,0}$, is arbitrary). We conclude that

$$\ker \Delta = \mathcal{X}_2 \subseteq \ker \begin{pmatrix} P_- \\ H \end{pmatrix} = \mathcal{L} \cap \ker P_-.$$

Conversely, suppose that $\ker \Delta \subseteq \mathcal{L} \cap \ker P_-$. Then we have $P_{22}^- = 0$ and $P_{23}^- = 0$. Also $D_2 = 0$. Put $u = \{u_k\}$, where u_k is given by

$$u_k = -(R + B^* P_{\mathcal{L}} B)^{-1}(C + B^* P_{\mathcal{L}} A) x_k.$$

Then by 4.6

$$J_T(x_0, u) = x_0^* P_{\mathcal{L}} x_0 - x_{3,T+1}^* P_{33}^+ x_{3,T+1}.$$

Since $x_{3,k+1} = A_{33}' x_{3,k}$ and $\sigma(A_{33})' \subset \mathcal{D}$, we have $x_{3,T+1} \to 0$. Hence $J_T(x_0, u) \to x_0^* P_{\mathcal{L}} x_0$ so $J(x_0, u) = x_0^* P_{\mathcal{L}} x_0$. Also note that $\begin{pmatrix} P_- \\ H \end{pmatrix} x_k = D_3 x_{3,k} \to 0$ and hence $H x_{3,k} \to 0$. Thus $u \in U_{\mathcal{L}}(x_0)$ and we can conclude that u is optimal.

The second part of (ii) was already proved (c.f. 4.7). $\qquad\square$

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