LINEAR QUADRATIC REGULATOR PROBLEM WITH POSITIVE CONTROLS

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Abstract

In this paper, the Linear Quadratic Regulator Problem with a positivity constraint on the admissible control set is addressed. Necessary and sufficient conditions for optimality are presented in terms of inner products, projections on closed convex sets, Pontryagin's maximum principle and dynamic programming. Sufficient and sometimes necessary conditions for the existence of positive stabilizing controls are incorporated. Convergence properties between the finite and infinite horizon case are presented. Besides these analytical methods, we describe briefly a method for the approximation of the optimal controls for the finite and infinite horizon problem.

1 Introduction

In literature (e.g. [1]), the Linear Quadratic Regulator Problem has been solved by using Riccati equations. In this article, the same problem will be treated with an additional positivity constraint: we require all components of the control to stay positive.

In many real-life problems, the influence we have on the system can be used only in one direction. One example is the controlling of the temperature in a room: the heating element can only put energy into the room, but it cannot extract energy. In process industry, the flow of a certain fluid or gas into a reactor is regulated by closing or opening a valve, resulting in one-way streams. Other examples of positive inputs can be found in economical systems, where quantities like investments and taxes are always positive.

Mainly, there are two classical approaches in optimal control theory. The first approach is the maximum principle, initiated by Pontryagin et al. ([13]). This original maximum principle has been used and extended by many others. See for instance, [10], [4], [12], [8], to mention a few. We will use the convergence results between the finite and infinite horizon problem to derive a maximum principle on infinite horizon for the "positive Linear Quadratic Regulator problem."

The second approach, Dynamic Programming, was originally conceived by R. Bellman ([2]) as a fruitful numerical method to compute optimal controls in discrete time processes. Later, people realized that the same ideas can be used for optimal control problems in continuous time. For continuous time problems, Dynamic Programming leads to a partial differential equation, the so-called Hamilton-Jacobi-Bellman equation, which has the value function among its solutions ([14]). In a lot of problems, the value function does not behave smoothly, which causes analytical and numerical problems. In case of nonsmoothness, the value function does not satisfy the HJB-equation in a classical sense, but in a more general and more complicated way: it are so-called "viscosity solutions" (see [5]). In this paper, it will be shown that the value function in our problem is continuously differentiable and satisfies the HJB-equation in a classical way. This simplifies Dynamic Programming in both analytical and numerical aspects.

In the formulation of the infinite horizon problem, we have to impose some boundedness requirements on the state trajectory: we allow only positive controls that result in square integrable state trajectories. This leads to the introduction of the concept of positive stabilizability. In [7] and [3], the problem of controllability of linear systems with positive controls is addressed. The result of these papers is used to derive sufficient and sometimes also necessary conditions for the existence of positive stabilizing controls.

2 Problem Formulation

We consider a linear system with input or control $u : [t,T] \to \mathbb{R}^m$, state $x : [t,T] \to \mathbb{R}^n$ and output $z : [t,T] \to \mathbb{R}^p$, given by

$$\dot{x} = Ax + Bu \tag{1}$$
$$z = Cx + Du, \tag{2}$$

where A, B, C and D are matrices of appropriate dimensions and T is a fixed end time. For every input function $u \in L_2[t,T]^m$ (Lebesgue space of square integrable functions) and initial condition (t, x_0), i.e. $x(t) = x_0$, the solution of (1) is an absolutely continuous state trajectory, denoted by $x_{t,x_0,u}$. The corresponding output can be written as

$$z_{t,x_0,u}(s) = \underbrace{Ce^{A(s-t)}x_0}_{(\mathcal{M}_{t,T}x_0)(s)} + \underbrace{+\int_t^s Ce^{A(s-\tau)}Bu(\tau)d\tau + Du(s)}_{(\mathcal{L}_{t,T}u)(s)}, \tag{3}$$

where $s \in [t,T]$, with $\mathcal{M}_{t,T}$ a bounded linear operator from $\mathbb{R}^n$ to $L_2[t,T]^p$ and $\mathcal{L}_{t,T}$ is a bounded linear operator from $L_2[t,T]^m$ to $L_2[t,T]^p$. Both operators are defined in (3).

The closed convex cone of positive functions in $L_2[t,T]^m$ is defined by

$$P[t,T] := \{u \in L_2[t,T]^m \mid u(s) \in \Omega \text{ a.e. on } [t,T]\},$$

where $\Omega := \{\mu \in \mathbb{R}^m \mid \mu_i \geq 0\}$.

First, we consider the finite horizon case ($T < \infty$). The infinite horizon version ($T = \infty$) will be postponed to section 6. Our objective is to determine for every initial condition $(t, x_0) \in [0,T] \times \mathbb{R}^n$ a control input $u \in P[t,T]$, an optimal control, such that

$$J(t, x_0, u) := \|z_{t,x_0,u}\|_2^2 = \|\mathcal{M}_{t,T}x_0 + \mathcal{L}_{t,T}u\|_2^2 \tag{4}$$

is minimal. $\|\cdot\|_2^2$ denotes the standard L_2-norm, i.e. $\|z\|_2^2 = \int_t^T z^\top(s)z(s)ds$. In fact, this is a minimum-norm problem for a closed convex cone. The optimal value of J for all considered initial conditions is described by the value function.

Definition 1 (Value function) *The value function V is a function from $[0,T] \times \mathbb{R}^n$ to $\mathbb{R}$ and is defined for every $(t, x_0) \in [0,T] \times \mathbb{R}^n$ by*

$$V(t, x_0) := \inf_{u \in P[t,T]} J(t, x_0, u) \tag{5}$$

3 Example

To illustrate that there is no simple connection between the optimal unconstrained LQ-feedback ($u = Fx$) and the optimal control with positivity constraint, we consider a pendulum given (after linearizing) by the dynamics

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - u$$

with $u \geq 0$: it is allowed only to push from one side. Our objective is to minimize $\int_0^\infty \{x_1^2(s) + u^2(s)\}ds$. We computed $J(0, x_0), x_0 = (1,0)^\top$ for controls of the form $u = g\max(0, Fx), g \geq 0$. The results are plotted in figure 1. We observe that the cost function is not optimal for $g = 1$, because $u_{1.13}$ performs better. However, we can compute a nonnegative control achieving even 5.108, the value indicated by the dashed line in figure 1. We conclude that there is no simple connection between constrained and unconstrained optimal controls.

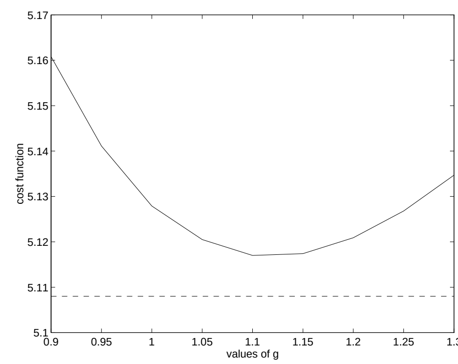

Figure 1: Cost function for various gains g.

4 Mathematical Preliminaries

A starting point is the fundamental theorem in Hilbert space theory concerning the minimum distance to a convex set. For a proof, see chapter 3 in [11].

Theorem 2 *Let x be a vector in a Hilbert space H with inner product $(\cdot \mid \cdot)$ and let K be a closed convex subset of H. Then there is exactly one $k_0 \in K$ such that $\|x - k_0\| \leq \|x - k\|$ for all $k \in K$. Furthermore, a necessary and sufficient condition that k_0 is the unique minimizing vector is that $(x - k_0 \mid k - k_0) \leq 0$ for all $k \in K$.*

Theorem 2 can be used to generalize orthogonal projections on closed subspaces in Hilbert spaces. Let K be a closed convex set of the Hilbert space H. We introduce the projection P_K onto K as

$$P_K x = k_0 \in K \iff \|x - k_0\| \leq \|x - k\| \ \forall k \in K \tag{6}$$

for $x \in H$. Theorem 2 justifies this definition and gives an equivalent characterisation of P_K:

$$P_K x = k_0 \in K \iff (x - k_0 \mid k - k_0) \leq 0 \ \forall k \in K. \tag{7}$$

We now state some simple properties of projections on closed convex sets.

Lemma 3
- *$P_K^2 = P_K$.*
- *$\|P_K x - P_K y\| \leq \|x - y\|$ for all $x, y \in H$.*

- *Besides K being closed and convex, assume it is a cone. Then $P_K(\alpha x) = \alpha P_K x$ for all $x \in H$ and $\alpha \geq 0$.*

5 Optimal Controls

5.1 Existence and Uniqueness

A standing assumption in the remainder of the paper will be the injectivity of D. In this case, $\mathcal{L}_{t,T}$ has a bounded left inverse, as we will see.

Consider a z in the range of $\mathcal{L}_{t,T}$, i.e. $z = \mathcal{L}_{t,T} u$ for some $u \in L_2[t,T]^m$. In that case z satisfies

$$\begin{aligned} \dot{x}(s) &= Ax(s) + Bu(s), \quad x(t) = 0 \\ z(s) &= Cx(s) + Du(s) \end{aligned}$$

for $s \in [t,T]$ and we can solve u by putting $u(s) = (D^\top D)^{-1} D^\top \{z(s) - Cx(s)\}$. Substituting this in the differential equation gives

$$\begin{aligned} \dot{x}(s) &= (A - B(D^\top D)^{-1} D^\top C)x(s) + \\ &\quad + B(D^\top D)^{-1} D^\top z(s), \qquad x(t) = 0 \\ u(s) &= (D^\top D)^{-1} D^\top \{z(s) - Cx(s)\}. \end{aligned}$$

This is a prescription in state space representation of a bounded left inverse of $\mathcal{L}_{t,T}$. We denote this particular left inverse by $\tilde{\mathcal{L}}_{t,T}$.

Theorem 4 *Let t, T be finite times with $t < T$ and $x_0 \in I\!R^n$. There is a unique control $u_{t,T,x_0} \in P[t,T]$ such that*

$$\|\mathcal{M}_{t,T} x_0 + \mathcal{L}_{t,T} u_{t,T,x_0}\|_2 \leq \|\mathcal{M}_{t,T} x_0 + \mathcal{L}_{t,T} u\|_2$$

for all $u \in P[t,T]$. A necessary and sufficient condition for $u^\star \in P[t,T]$ to be the unique minimizing control is that

$$(\mathcal{M}_{t,T} x_0 + \mathcal{L}_{t,T} u^\star \mid \mathcal{L}_{t,T} u - \mathcal{L}_{t,T} u^\star) \geq 0 \tag{8}$$

for all $u \in P[t,T]$.

Proof. If we realize that our problem consists of minimizing $\|v - \mathcal{M}_{t,T} x_0\|$ over $v \in \mathcal{L}_{t,T}(P[t,T])$, the result is an easy translation of Theorem 2, where we use $K = \mathcal{L}_{t,T}(P[t,T])$ and $k_0 = \mathcal{L}_{t,T}(u^\star)$. □

The optimal control, the optimal trajectory and the optimal output with initial conditions (t, x_0) and final time T will be denoted by u_{t,T,x_0}, x_{t,T,x_0} and z_{t,T,x_0}, respectively.

Considering the proof above, it is not hard to see that in terms of projections, we can write

$$u_{t,T,x_0} = \tilde{\mathcal{L}}_{t,T}\, P\,(-\mathcal{M}_{t,T} x_0), \tag{9}$$

where P is the projection on the closed convex cone $\mathcal{L}_{t,T}(P[t,T])$.

From the last expression and Lemma 3, it is clear that for $\alpha \geq 0$

$$u_{t,T,\alpha x_0} = \alpha u_{t,T,x_0} \tag{10}$$

and hence,

$$V(t, \alpha x_0) = \alpha^2 V(t, x_0). \tag{11}$$

5.2 Maximum Principle

The results in this subsection are classical (see e.g. [13]). They are stated here, because in section 6 this theory is used to derive convergence results between finite and infinite horizon optimal controls.

Consider the Hamiltonian

$$\begin{aligned} H(x,\mu,\phi) &= x^\top A^\top \phi + \mu^\top B^\top \phi + x^\top C^\top C x + \\ &\quad + 2x^\top C^\top D\mu + \mu^\top D^\top D\mu \end{aligned} \tag{12}$$

for $(x,\mu,\phi) \in I\!R^n \times I\!R^m \times I\!R^n$. The adjoint or costate equation reads

$$\begin{aligned} \dot{\phi} &= -\frac{\partial H(x,u,\phi)}{\partial x} = \\ &A^\top \phi - 2C^\top Cx - 2C^\top Du = -A^\top \phi - 2C^\top z, \end{aligned} \tag{13}$$

with terminal condition

$$\phi(T) = 0 \tag{14}$$

and $z = Cx + Du$. The optimal control u_{0,T,x_0}, for shortness denoted by u_{opt}, satisfies for all $s \in [0,T]$

$$u_{opt}(s) \in \arg\min_{\mu \in \Omega} H(x_{opt}(s), \mu, \phi_{opt}(s)), \tag{15}$$

where x_{opt} is the optimal trajectory and ϕ_{opt} the solution to the adjoint equation (13) with z equal to the optimal output $z_{opt} = z_{0,T,x_0}$. $\Omega = \{\mu \in I\!R^m \mid \mu_i \geq 0\}$.

We introduce $\|\cdot\|_{D^\top D}$ as the norm induced by the inner product $(x \mid y)_{D^\top D} = x^\top D^\top D y$ for $x, y \in I\!R^n$ in the Hilbert space $I\!R^n$. Furthermore, in this Hilbert space, P_Ω denotes the projection on Ω, which is continuous according to Lemma 3.

Lemma 5 *Let the final time $T > 0$ and initial state $x_0 \in I\!R^n$ be fixed. The optimal control u_{opt} satisfies*

$$u_{opt}(s) = P_\Omega(-\frac{1}{2}(D^\top D)^{-1}\{B^\top \phi(s) + 2D^\top Cx(s)\}) \tag{16}$$

for all $s \in [0,T]$, where the functions x and ϕ are given by

$$\begin{aligned} \dot{x} &= Ax + Bu_{opt} & x(0) &= x_0 \\ \dot{\phi} &= -A^\top \phi - 2C^\top Cx - 2C^\top Du_{opt} & \phi(T) &= 0 \end{aligned} \tag{17}$$

Hence, the optimal control is continuous in time.

Proof. This lemma is a straightforward application of the theory above. By (15), $u_{opt}(s)$ is the unique pointwise minimizer of the costs

$$\begin{aligned} &\mu^\top D^\top D\mu + \mu^\top g(s) = \\ &\|\mu + \frac{1}{2}(D^\top D)^{-1} g(s)\|^2_{D^\top D} - \frac{1}{4} g^\top(s)(D^\top D)^{-1} g(s), \end{aligned} \tag{18}$$

taken over all $\mu \in \Omega$, where $g(s) := B^\top \phi_{opt}(s) + 2D^\top C x_{opt}(s)$. □

Substituting (16) in (17) gives a two-point boundary value problem. Its solutions provide us with a set of candidates containing the optimal control, because the maximum principle is a necessary condition for optimality.

In case $D^\top D = \frac{1}{2}I$, the expression $P_\Omega(-\frac{1}{2}(D^\top D)^{-1}\{B^\top \phi(s) + 2D^\top Cx(s)\})$ simplifies to

$$\max\{0, -B^\top \phi(s) - 2D^\top Cx(s)\},$$

where "max" means taking the maximum componentwise.

5.3 Dynamic Programming

In the introduction, we discussed the problems with nonsmooth value functions and announced to show that in the positive LQ-problem, the value function is continuously differentiable and satisfies the Hamilton-Jacobi-Bellman equation in classical sense. The proof of this fact, being rather technical, is not included, but can be found in [6].

We start with a short overview of the technique of dynamic programming. To do so, we introduce the function L for $(x, \mu) \in \mathbb{R}^n \times \Omega$ by

$$L(x,\mu) = z^\top z = \\ = x^\top C^\top Cx + 2x^\top C^\top D\mu + \mu^\top D^\top D\mu. \quad (19)$$

The HJB-equation is given by

$$V_t(t,x) + \tilde{H}(x, V_x(t,x)) = 0, \quad (20)$$

where $\tilde{H}$ is given by

$$\tilde{H}(x,p) = \inf_{\mu\in\Omega}\{p^\top Ax + p^\top B\mu + L(t,x,\mu)\} \quad (21)$$

for $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$. V_t means the time-derivative of the value function and V_x the gradient of V.

By manipulating (21), we get

$$\tilde{H}(x,p) = x^\top C^\top Cx + \\ + p^\top Ax - \frac{1}{4}g^\top(x,p)(D^\top D)^{-1}g(x,p) + \\ + \inf_{\mu\in\Omega}\|\mu + \frac{1}{2}(D^\top D)^{-1}g(x,p)\|^2_{D^\top D} \quad (22)$$

with $g(x,p) = B^\top p + 2D^\top Cx$ and $\|\cdot\|_{D^\top D}$ as in the previous subsection.

The minimizing μ equals $P_\Omega(-\frac{1}{2}(D^\top D)^{-1}g(x,p))$, where P_Ω denotes the projection on Ω in the Hilbert space $\mathbb{R}^n$ with inner product $(\cdot \mid \cdot)_{D^\top D}$. If V is continuously differentiable and the optimal controls are continuous, the value function satisfies the HJB-equation (see [5]). The Maximum Principle shows that the optimal controls in our problem are continuous.

Theorem 6 (Differentiability of V w.r.t. x) *The value function V is continuously differentiable with respect to x and its directional derivative in the point $(t,x_0) \in [t,T] \times \mathbb{R}^n$ with increment $h \in \mathbb{R}^n$ is given by*

$$(V_x(t,x_0) \mid h) = 2\,(z_{t,T,x_0} \mid \mathcal{M}_{t,T}h)_2 \quad (23)$$

or, explicitly,

$$V_x(t,x_0) = 2\int_t^T e^{A^\top(s-t)}C^\top z_{t,T,x_0}(s)ds. \quad (24)$$

z_{t,T,x_0} is the output corresponding to initial condition (t,x_0) with optimal control u_{t,T,x_0}.

Notice that in (23) the left inner product is the Euclidean inner product in $\mathbb{R}^n$ and the right inner product is the L_2-inner product.

Theorem 7 (Differentiability of V w.r.t. t) *The value function V is differentiable with respect to t and the partial derivative in the point $(t,x_0) \in [t,T] \times \mathbb{R}^n$ equals*

$$V_t(t,x_0) = -(V_x(t,x_0) \mid Ax + Bu_{t,T,x_0}(t)) + \\ - z^\top_{t,T,x_0}(t)z_{t,T,x_0}(t) \quad (25)$$

In [6], it is shown also that both partial derivatives are continuous in both arguments. We can conclude now that V is a classical solution to the HJB-equation with the boundary condition $V(T,x) = 0$ for all $x \in \mathbb{R}^n$. The so-called *verification theorem* in [5] gives now that the optimal control u_{t,T,x_0} is the pointwise minimizer of (21) along the optimal trajectory, i.e.

$$u_{t,T,x_0}(s) = \\ P_\Omega(-\frac{1}{2}(D^\top D)^{-1}g(x_{t,T,x_0}(s), V_x(s, x_{t,T,x_0}(s)))), \quad (26)$$

This defines a time-varying optimal feedback. In fact, by using (26), we see that (25) is the HJB-equation.

Comparing (24) with the adjoint equation of the Maximum Principle yields a connection between the adjoint variable and the gradient of V:

$$\phi_{t,T,x_0}(s) = V_x(s, x_{t,T,x_0}(s)), \quad (27)$$

where ϕ_{t,T,x_0} is the solution to the adjoint equation corresponding to u_{t,T,x_0}.

Moreover, we can even conclude that both dynamic programming and the maximum principle are necessary and sufficient conditions for optimality. This follows from the link (27) between the two methods. Necessity of the maximum principle in terms of Lemma 5 translates via (27) into necessity of dynamic programming in terms of (26). Vice versa, the sufficiency of dynamic programming leads to sufficiency of the maximum principle along the same lines.

6 Infinite Horizon

6.1 Problem Formulation

In this section we consider the infinite horizon case ($T = \infty$). The time-invariance of the problem guarantees that we can take $t = 0$ without loss of generality.

The problem is to minimize

$$J_\infty(x_0, u) := \int_0^\infty z_{x_0,u}^\top(s) z_{x_0,u}(s) ds \tag{28}$$

over $u \in P[0,\infty)$ subject to the relations (1), (2), $x(0) = x_0$ and the additional constraint that the state trajectory x should be contained in $L_2[0,\infty)^n$. Notice that $x \in L_2[0,\infty)^n$ implies $x(s) \longrightarrow 0 (s \longrightarrow \infty)$, because x is necessarily absolutely continuous.

Definition 8 *A control is said to be stabilizing for initial state x_0, if the corresponding state trajectory x satisfies $x \in L_2[0,\infty)^n$. (A, B) is said to be positive stabilizable, if for every x_0 there exists a stabilizing control $u \in P[0,\infty)$.*

Notice that $u \in L_2[0,\infty)^m$ and $x \in L_2[0,\infty)^n$ imply $z \in L_2[0,\infty)^p$ and thus the finiteness of J_∞. So the positive stabilizability of (A, B) guarantees the existence of positive controls, which keep the cost function finite. This is necessary to formulate a descent infinite horizon problem.

6.2 Positive Stabilizability

In this subsection we present sufficient conditions for the positive stabilizability of (A, B). In the single input case ($m = 1$), they turn out to be sufficient and necessary. The main results are stated without proofs. The proofs can be found in [6].

Theorem 9 *Consider the system given by (A, B, C, D) with control restraint set*

$$\Omega = \{(\mu_1, \ldots, \mu_m)^\top \in I\!R^m \mid \mu_i \geq 0 \ , \forall_{i=1,\ldots,m}\}.$$

If (A, B) is stabilizable and all real eigenvectors v of $A^\top$ corresponding to a nonnegative eigenvalue of $A^\top$ have the property that $B^\top v$ has positive components, then (A, B) is positively stabilizable.

Notice that a real eigenvector v of $A^\top$ corresponding to a nonnegative eigenvalue $B^\top v$ should also have negative components, because $-v$ is an eigenvector too. For $m = 1$ the conditions are also necessary.

Corollary 1 *Consider the system (A, B, C, D) with $m = 1$ and control restraint set*

$$\Omega = \{\mu \in I\!R \mid \mu \geq 0\} = [0, \infty).$$

(A, B) is positively stabilizable if and only if (A, B) stabilizable and $\sigma(A) \cap [0,\infty) = \emptyset$.

6.3 Connection between Finite and Infinite Horizon

In this subsection, we present only the results. The proofs are rather technical and therefore, omitted. However, they can be found in [6]. In the remainder of this paper, we concentrate on systems with the following properties.

1. (A, B) is positively stabilizable;
2. D is injective;
3. (A, B, C, D) is minimum phase.

Items 2. and 3. can be replaced by item 3. and $(C + DF_2, A + BF_2)$ is detectable, where F_2 is given by $(C + DF_2)^\top D = 0$.

Under these assumptions, the optimal controls with infinite horizon exist and are unique.

We define the operators $\Pi_T : L_2[0,\infty)^m \to L_2[0,T]^m$ for $s \in [0,T]$ by $(\Pi_T u)(s) = u(s)$ and $\Pi_T^* : L_2[0,T]^m \to L_2[0,\infty)^m$ for all $u \in L_2^m[0,T]$ by

$$(\Pi_T^* u)(s) = \begin{cases} u(s) & s \leq T \\ 0 & s > T \end{cases}$$

In the sequel, we denote by u_T, x_T and z_T the optimal control,the optimal state trajectory and the optimal output on $[0,T]$ with fixed initial condition $(0, x_0)$ for $T > 0$ or $T = \infty$.

Theorem 10 *For $T \longrightarrow \infty$, we have the convergences $\Pi_T^* u_T \longrightarrow u_\infty, \Pi_T^* z_T \longrightarrow z_\infty$ and $\Pi_T^* x_T \longrightarrow x_\infty$ in the L_2-norm.*

Also the optimal costs of the finite horizon converge to the optimal costs of the infinite horizon, if the horizon tends to infinity.

Pontryagin's Maximum Principle for the finite horizon converges to a maximum principle for the infinite horizon, which implies the continuity of the optimal control on infinite horizon.

Theorem 11 *Fix initial state x_0. The corresponding optimal control, denoted by u_∞, satisfies*

$$u_\infty(s) = P_\Omega((D^\top D)^{-1}\{\frac{1}{2}B^\top \phi_\infty(s) - D^\top C x_\infty(s)\}), \tag{29}$$

where the continuous function $\phi_\infty \in L_2[0,\infty)^n$ is initially given by $\phi_\infty(0)$ and the differential equation

$$\dot{\phi}_\infty = -A^\top \phi_\infty - 2C^\top z_\infty. \tag{30}$$

$\phi_\infty(0)$ is the limit of a converging sequence $\{\phi_{T_i}(0)\}_{i \in I\!N}$, where $\{T_i\}_{i \in I\!N}$ is a sequence of positive numbers, which tend to infinity.

By using the maximum principle in finite and infinite horizon, we can extend the convergence results of Theorem 10 to pointwise convergence.

Theorem 12

$$u_T(s) \longrightarrow u_\infty(s) \quad (T \longrightarrow \infty),$$

for all $s \in [0, \infty)$.

6.4 Approximation

It is obvious that for all $s \geq 0$ $u_{t,x_0}(s+t) = u_{x_0}(s)$. From this it follows that that there exists a time-invariant optimal feedback u_{fdb} defined by

$$u_{fdb}(x_0) := u_{t,x_0}(t) = u_{0,x_0}(0) = \lim_{T \longrightarrow \infty} u_{0,T,x_0}(0). \quad (31)$$

The limit in this equation provides us with a method to approximate the optimal feedback. Discretization in time and state is a method to approximate the finite horizon optimal controller. This technique is based on discrete Dynamic Programming, as initiated by R. Bellman ([2]). Such techniques can be found in e.g. [9] and [8]. If T is large enough we can use u_{0,T,x_0} as an approximation for $u_{fdb}(x_0)$. In [6], some numerical result are presented.

7 Conclusions

In this paper, we addressed three approaches to Linear Quadratic Regulator problem with a positivity constraint on the admissible controls. As main contributions, we consider the necessary and sufficient conditions in terms of inner products, projections, the maximum principle and dynamic programming. The maximum principle results in a two-point boundary value problem, dynamic programming in a partial differential equation. If one of these equations are solved, their solutions lead analytically to the optimal control. The conditions for positive stability are verified easily and can be used to check the well-posedness of the infinite horizon problem. Another essential result is the maximum principle for the infinite horizon case and the convergence results between finite horizon optimal controls and the infinite horizon control which result in a method to approximate the optimal positive feedback in the considered problem. The results hold for LQ-problems with arbitrary closed convex cones as control restraint set as well.

References

[1] Anderson B.D.O., Moore J.B., "Optimal Control - Linear Quadratic Methods", *Prentice Hall*, (1990).

[2] Bellman R., "Introduction to the Mathematical Theory of Control Processes", *Academic Press*, (1967).

[3] Brammer R.F., "Controllability in Linear Autonomous Systems with Positive Controllers", *Siam J. of Control*, **10-2**, (1972).

[4] Feichtinger G., Hartl R.F., "Ökonomischer Prozesse: Anwendung des Maximumprinzips in den Wirtschaftswissenschaften," *De Gruyter*, (1986).

[5] Fleming W.H., Soner H.M., "Controlled Markov Processes and Viscosity Solutions", *Springer-Verlag*, (1993).

[6] Heemels W.P.M.H., "Optimal Positive Control & Optimal Control with Positive State Entries", *Master's thesis Technical University of Eindhoven*, (1995).

[7] Heymann M., Stern R.J., "Controllability of Linear Systems with Positive Controls: Geometric Considerations", *Journal of Mathematical Analysis and Applications*, **52**, pages 36-41, (1975).

[8] Kirk D.E., "Optimal Control Theory, An Introduction", *Prentice-Hall Inc.*, (1970).

[9] Kushner H.J., and Dupuis P.G., "Numerical Methods for Stochastic Control Problems in Continuous Time", *Springer-Verlag*, (1992).

[10] Lee E.B., Markus L., "Foundations of Optimal Control Theory", *John Wiley & Sons, Inc.*, (1967).

[11] Luenberger D.G., "Optimization by Vector Space Methods." *John Wiley & Sons, Inc.*, (1969).

[12] Macki J., Strauss A., "Introduction to Optimal Control Theory", *Springer-Verlag Inc.*, (1982).

[13] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F., "The Mathematical Theory of Optimal Processes." *Interscience Publishers, John Wiley & Sons*, (1962).

[14] Fleming W.H., Rishel R.W., "Deterministic and Stochastic Optimal Control", *Springer-Verlag*, (1975).