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Linear r -matrix algebra for classical separable systems

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Abstract. We consider a hierarchy of the natural-type Hamiltonian systems of n degrees of freedom with polynomial potentials separable in general ellipsoidal and general paraboloidal coordinates. We give a Lax representation in terms of 2×2 matrices for the whole hierarchy and construct the associated linear r -matrix algebra with the r -matrix dependent on the dynamical variables. A Yang–Baxter equation of dynamical type is proposed. Using the method of variable separation, we provide the integration of the systems in classical mechanics constructing the separation equations and, hence, the explicit form of action variables. The quantization problem is discussed with the help of the separation variables.

1. Introduction

The method of separation of variables in the Hamilton–Jacobi equation,

$$H(p_1, \dots, p_n, x_1, \dots, x_n) = E \quad p_i = \frac{\partial W}{\partial x_i} \quad i = 1, \dots, n \quad (1.1)$$

is one of the most powerful methods for the construction of action for the Liouville integrable systems of classical mechanics [3]. We consider below systems of the *natural* form described by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 + V(x_1, \dots, x_n) \quad p_i, x_i \in \mathbf{R}. \quad (1.2)$$

The separation of variables means the solution of partial differential equation (1.1) for the action function W in the following additive form:

$$W = \sum_{i=1}^n W_i(\mu_i; H_1, \dots, H_n) \quad H_n = H$$

where μ_i will be called *separation variables*. Note that the partial functions W_i depend only on their separation variables μ_i , which define a new set of variables instead of the

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old ones $\{x_k\}$, and on the set of constants of motion, or *integrals of motion*, $\{H_k\}$. In the following we shall speak about *coordinate separation* where the separation variables $\{\mu_i\}$ are functions of the coordinates $\{x_k\}$ only. (The general change of variables may also include the corresponding momenta $\{p_k\}$.)

For a free particle ($V = 0$), the complete classification of all orthogonal coordinate systems in which the Hamilton–Jacobi equation (1.1) admits the separation of variables is known: these are *generalized n -dimensional ellipsoidal and paraboloidal coordinates* [8, 9] (see also the references therein). It is also known that the Hamiltonian systems (1.2) admitting an orthogonal coordinate separation with $V \neq 0$ are separated only in the *same coordinate systems*.

The modern approach to finite-dimensional integrable systems uses the language of the representations of r -matrix algebras [10, 15, 16, 17]. The classical method of separation of variables can be formulated within this language dealing with the representations of linear and quadratic r -matrix algebras [11, 12, 16, 18]. For the 2×2 L -operators, the recipe is to consider the zeros of one of the off-diagonal elements as the separation variables (see also a generalization of this approach to higher dimensions of L -matrix [18]). For $V = 0$ in [11, 12], 2×2 L -operators were given, satisfying the standard linear r -matrix algebra [10, 15],

$$\{L_1(u), L_2(v)\} = [r(u-v), L_1(u) + L_2(v)] \quad r(u) = \frac{1}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.3)$$

and the link with the separation of variables method was elucidated. In (1.3) we use the familiar notations for the tensor products of $L(u)$ and 2×2 identity matrix I , $L_1(u) = L(u) \otimes I$, $L_2(v) = I \otimes L(v)$.

In the present paper we construct 2×2 L -operators for systems (1.2) being separated in the generalized ellipsoidal and paraboloidal coordinates. In the case when the degree N of the potential V is equal to 1 or 2, the associated linear r -matrix algebra appears to be the standard one (1.3). In the case $N > 2$, the algebra is of the form

$$\{L_1(u), L_2(v)\} = [r(u-v), L_1(u) + L_2(v)] + [s(u, v), L_1(u) - L_2(v)] \quad (1.4)$$

with $s(u, v) = \alpha_N(u, v) \sigma_- \otimes \sigma_-$, where $\sigma_- = \sigma_1 - i\sigma_2$ and σ_i are the Pauli matrices, and $\alpha_N(u, v)$ is the function which equals 1 for $N = 3$ and depends on the dynamical variables $\{x_k, p_k\}_{k=1}^n$ for $N > 3$.

The study of completely integrable systems admitting a classical r -matrix Poisson structure with the r -matrix dependent on dynamical variables has attracted some attention [5, 6, 13]. It is remarkable that the celebrated Calogero–Moser system, whose complete integrability was demonstrated a number of years ago (cf [14]), has been found only recently to possess a classical r -matrix of dynamical type [4].

Below we briefly recap how to get the 2×2 L -operators for the separable systems (1.2) without the potential V [11, 12]. Let us consider a direct sum of the Lie algebras, each of rank 1: $\mathcal{A} = \bigoplus_{k=1}^n \mathfrak{so}_k(2, 1)$. Generators $s_k \in \mathbb{R}^3$, $k = 1, \dots, n$ of the \mathcal{A} algebra satisfy the Poisson brackets

$$\{s_k^i, s_m^j\} = \delta_{km} \varepsilon_{ijl} g_{ll} s_k^l \quad g = \text{diag}(1, -1, -1). \quad (1.5)$$

Throughout the paper, we imply that the g metric calculates the norm and scalar product of the vectors s_i :

$$\begin{aligned} s_i^2 &\equiv (s_i, s_i) = (s_i^1)^2 - (s_i^2)^2 - (s_i^3)^2 \\ (s_i, s_k) &= s_i^1 s_k^1 - s_i^2 s_k^2 - s_i^3 s_k^3. \end{aligned}$$

Let us fix the values of the Casimir elements of the \mathcal{A} algebra: $s_i^2 = c_i^2$, then variables s_i lie on the direct product of n hyperboloids in \mathbb{R}^3 . Let $c_i \in \mathbb{R}$ and choose the upper sheets of these double-sheeted hyperboloids. Denote the obtained manifold as K_n^+ . We will denote by *hyperbolic Gaudin magnet* [7] integrable Hamiltonian system on K_n^+ given by n integrals of motion H_i which are in involution with respect to the bracket (1.5),

$$H_i = 2 \sum_{k=1}^n \frac{(s_i, s_k)}{e_i - e_k} \quad \{H_i, H_k\} = 0 \quad e_i \neq e_k \quad \text{if } i \neq k. \tag{1.6}$$

To be more exact, one has to call this model an n -site $so(2,1)$ -XXX Gaudin magnet. Note that all the H_i are quadratic functions on generators of the \mathcal{A} algebra and the following equalities are valid:

$$\sum_{i=1}^n H_i = 0 \quad \sum_{i=1}^n e_i H_i = J^2 - \sum_{i=1}^n c_i^2.$$

Here a new variable $J = \sum_{i=1}^n s_i$ is introduced which is the total sum of the hyperbolic momenta s_i . The components of the vector J obey $so(2,1)$ Lie algebra with respect to the bracket (1.5) and are in involution with all the H_i . The complete set of involutive integrals of motion is provided by the following choice: H_i, J^2 and, for example, $(J^3)^2$. Integrals (1.6) are generated by the 2×2 L -operator (as well as the additional integrals J)

$$L(u) = \sum_{j=1}^n \frac{1}{u - e_j} \begin{pmatrix} is_j^3 & -(s_j^1 - s_j^2) \\ -(s_j^1 + s_j^2) & -is_j^3 \end{pmatrix}$$

$$\det L(u) = - \sum_{j=1}^n \left(\frac{H_j}{u - e_j} + \frac{c_j^2}{(u - e_j)^2} \right) \tag{1.7}$$

satisfying the standard linear r -matrix algebra (1.3). Let $c_i = 0, i = 1, \dots, n$, which turns the hyperboloids $s_i^2 = c_i^2$ into cones. The manifold K_n^+ admits in this case the following parameterization ($p_i, x_i \in \mathbb{R}$):

$$s_i^1 = \frac{p_i^2 + x_i^2}{4} \quad s_i^2 = \frac{p_i^2 - x_i^2}{4} \quad s_i^3 = \frac{p_i x_i}{2} \tag{1.8}$$

where the variables p_i and x_i are canonically conjugated. Using the isomorphism (1.8), the complete classification of the separable orthogonal coordinate systems was provided in [11, 12] by means of the corresponding L -operators satisfying the standard linear r -matrix algebra (1.3). The starting point for our investigation are these L -operators written for the cases of free motion on a sphere and in the Euclidean space.

The paper is organized as follows. In section 2 we describe the classical Poisson structure associated with the hierarchy of natural-type Hamiltonians separable in the three coordinate systems: the spherical (for motion on a sphere), and general ellipsoidal and paraboloidal (for n -dimensional Euclidean motion) coordinates. This structure is given in terms of the linear r -matrix formalism, providing a new example of the dynamical dependence of the r -matrices. We also introduce an analogue of the Yang–Baxter equation for our dynamical r -matrices. In section 3 we derive the Lax representation for all the hierarchy, as a consequence of the r -matrix representation given in section 2. Section 3 deals also with the aspect of variable separation. The question of quantization of the considered systems is briefly discussed.

2. Classical Poisson structure

Let us consider the following ansatz for the 2×2 L -operator

$$L_N(u) = \begin{pmatrix} A(u) & B(u) \\ C_N(u) & -A(u) \end{pmatrix} \tag{2.1}$$

where

$$B(u) = \varepsilon - \sum_{i=1}^n \frac{x_i^2}{u - e_i} \quad \varepsilon = 0, 1, \text{ or } 4(u - x_{n+1} + B) \tag{2.2}$$

$$A(u) = \dot{\varepsilon} + \frac{1}{2} \sum_{i=1}^n \frac{x_i p_i}{u - e_i} \tag{2.3}$$

$$C_N(u) = \sum_{i=1}^n \frac{p_i^2}{u - e_i} - V_N(u) \quad V_N(u) = \sum_{k=0}^N \mathcal{V}_k u^{N-k}. \tag{2.4}$$

Here the x_i, p_j are canonically conjugated variables ($\{p_i, x_j\} = \delta_{ij}$), \mathcal{V}_k are indeterminate functions of the x -variables; B and e_i are non-coincident real constants. Note that dot over ε means differentiation by time, and for natural Hamiltonian (1.2) one has $\dot{x}_{n+1} = p_{n+1}$.

Theorem 1. Let the curve $\det(L(u) - \lambda I) = 0$ for the L -operator (2.1) have the form

$$\lambda^2 - A(u)^2 - B(u)C_N(u) = \lambda^2 + \varepsilon u^N - \sum_{i=1}^n \frac{H_i}{u - e_i} = 0 \quad \text{for } \varepsilon = 0, 1 \tag{2.5}$$

$$\lambda^2 - A(u)^2 - B(u)C_N(u) = \lambda^2 + 16u^{N-2}(u + B)^2 + 8H - \sum_{i=1}^n \frac{H_i}{u - e_i} = 0$$

for $\varepsilon = 4(u - x_{n+1} + B)$ (2.6)

with some integrals of motion H_i and H , H in the case of (2.6). Then the following recurrence relations for \mathcal{V}_k are valid:

$$\mathcal{V}_k = \sum_{i=1}^n x_i^2 \sum_{j=0}^{k-1} \mathcal{V}_{k-1-j} e_i^j \quad \mathcal{V}_0 = 1 \quad \text{for } \varepsilon = 0, 1 \tag{2.7}$$

$$\mathcal{V}_k = (x_{n+1} + B)\mathcal{V}_{k-1} + \frac{1}{2} \sum_{i=1}^n x_i^2 \sum_{j=1}^{k-1} \mathcal{V}_{k-1-j} e_i^j \quad \mathcal{V}_0 = 0$$

for $\varepsilon = 4(u - x_{n+1} + B)$. (2.8)

The explicit formulae for the integrals H_i have the form

$$H_i = -\sum_{j=1}^{n'} \frac{M_{ij}^2}{e_i - e_j} + \varepsilon \cdot p_i^2 + x_i^2 \sum_{k=0}^N \mathcal{V}_k e_i^{N-k} \quad \text{for } \varepsilon = 0, 1 \tag{2.9}$$

$$H_i = 2x_i^2 \sum_{j=1}^{N-1} (-1)^{j-1} e_i^j \mathcal{V}_{N-j} + 4p_{n+1} p_i x_i - p_i^2(e_i + 4x_{n+1} - 4B) + \sum_{j \neq i} \frac{M_{ij}^2}{e_i - e_j}$$

for $\varepsilon = 4(u - x_{n+1} + B)$ (2.10)

where $M_{ij} = x_i p_j - x_j p_i$. The Hamiltonians H are given by

$$H \equiv \sum_{i=1}^n H_i = \varepsilon \cdot \sum_{i=1}^n p_i^2 + \mathcal{V}_{N+1} \quad \text{for } \varepsilon = 0, 1 \tag{2.11}$$

$$H = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + \mathcal{V}_N \quad \text{for } \varepsilon = 4(u - x_{n+1} + B). \tag{2.12}$$

The proof is straightforward and based on direct computations.

We remark that the above recurrence formulae for the potentials can be written in differential form. In particular, for the paraboloidal coordinates we have

$$\frac{\partial \mathcal{V}_N}{\partial x_i} = \frac{1}{2} \frac{\partial \mathcal{V}_{N-1}}{\partial x_{n+1}} x_i - A_i \frac{\partial \mathcal{V}_{N-1}}{\partial x_i} \quad i = 1, \dots, n \tag{2.13}$$

$$\frac{\partial \mathcal{V}_N}{\partial x_{n+1}} = \mathcal{V}_{N-1} + \frac{1}{2} \sum_{i=1}^n x_i \frac{\partial \mathcal{V}_{N-1}}{\partial x_i} + (x_{n+1} - B) \frac{\partial \mathcal{V}_{N-1}}{\partial x_{n+1}}. \tag{2.14}$$

Note that the case of $\varepsilon = 0$ is connected with the ellipsoidal coordinates on a sphere and two other cases $\varepsilon = 1$ and $4(u - x_{n+1} + B)$ describe the ellipsoidal and paraboloidal coordinates in the Euclidean space, respectively (see section 3.2 and [11, 12] for more detail). Recall that we study now the motion of a particle on these manifolds under the external field with the potential V that could be any linear combination of the homogeneous ones \mathcal{V}_k .

Now we are ready to describe the linear algebra for the L -operator (2.1).

Theorem 2. Let the L -matrix be of the form (2.1) and satisfy the conditions of theorem 1. The following algebra is then valid for its entries:

$$\{B(u), B(v)\} = \{A(u), A(v)\} = 0 \tag{2.15}$$

$$\{C_N(u), C_N(v)\} = -4 \alpha_N(u, v) (A(u) - A(v)) \tag{2.16}$$

$$\{B(u), A(v)\} = \frac{2}{u - v} (B(u) - B(v)) \tag{2.17}$$

$$\{C_N(u), A(v)\} = -\frac{2}{u - v} (C_N(u) - C_N(v)) - 2 \alpha_N(u, v) B(v) \tag{2.18}$$

$$\{B(u), C_N(v)\} = -\frac{4}{u - v} (A(u) - A(v)) \tag{2.19}$$

where the function $\alpha_N(u, v)$ has the form

$$\alpha_N(u, v) = \frac{Q_N(u) - Q_N(v)}{u - v} = \sum_{k=1}^N Q_k \frac{u^k - v^k}{u - v} \tag{2.20}$$

$$Q_N(u) = \sum_{k=0}^N Q_k u^k \quad Q_k = \sum_{m=0}^k \mathcal{V}_m \mathcal{V}_{k-m}.$$

The proof is based on the recurrence relations (2.7), (2.8).

We remark that for the paraboloidal coordinates the following formula is valid:

$$Q(u) = u^{N-2} - \frac{1}{2} \sum_{k=0}^{N-3} \frac{\partial \mathcal{V}_{N-k-1}}{\partial x_{n+1}} \tag{2.21}$$

therefore in this case we have

$$Q(u) = \frac{1}{4} \frac{\partial C(u)}{\partial x_{n+1}}. \tag{2.22}$$

The algebra (2.15)–(2.19) can be rewritten in the matrix form as linear r -matrix algebra

$$\{L_1(u), L_2(v)\} = [r(u - v), L_1(u) + L_2(v)] + [s_N(u, v), L_1(u) - L_2(v)] \tag{2.23}$$

using 4×4 notations $L_1(u) = L(u) \otimes I$, $L_2(v) = I \otimes L(v)$; the matrices $r(u - v)$ and $s_N(u, v)$ are given by

$$r(u - v) = \frac{2}{u - v} P \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.24}$$

$$s_N(u, v) = 2\alpha_N(u, v) \sigma_- \otimes \sigma_- \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The algebra (2.15)–(2.19) or (2.23)–(2.24) contains all the information about the system under consideration. From it there follows the involutivity of the integrals of motion. Indeed, the determinant $d(u) \equiv \det L(u)$ is the generating function for the integrals of motion and it is simply to show that

$$\{d(u), d(v)\} = 0. \tag{2.25}$$

In particular, the integrals H_i (2.9), (2.10) are the residues of the function $d(u)$:

$$H_i = \text{res}|_{u=e_i} d(u) \quad i = 1, \dots, n.$$

The Hamiltonians H (2.11), (2.12) appear to be a residue at infinity. Let us rewrite the relation (2.23) in the form

$$\{L_1(u), L_2(v)\} = [d_{12}(u, v), L_1(u)] - [d_{21}(u, v)L_2(v)] \tag{2.26}$$

with $d_{ij} = r_{ij} + s_{ij}$, $d_{ji} = s_{ij} - r_{ij}$ at $i < j$.

Theorem 3. The following equations (the dynamical Yang–Baxter equations) are valid for the algebra (2.26)

$$\begin{aligned} & [d_{12}(u, v), d_{13}(u, w)] + [d_{12}(u, v), d_{23}(v, w)] + [d_{32}(w, v), d_{13}(u, w)] \\ & + \{L_2(v), d_{13}(u, w)\} - \{L_3(w), d_{12}(u, v)\} \\ & + [c(u, v, w), L_2(v) - L_3(w)] = 0 \end{aligned} \tag{2.27}$$

where $c(u, v, w)$ is some matrix dependent on dynamical variables. The other two equations are obtained from (2.27) by cyclic permutations.

Proof. Let us write the Jacobi identity as

$$\{L_1(u), \{L_2(v), L_3(w)\}\} + \{L_3(w), \{L_1(u), L_2(v)\}\} + \{L_2(v), \{L_3(w), L_1(u)\}\} = 0 \tag{2.28}$$

with $L_1(u) = L(u) \otimes I \otimes I$, $L_2(v) = I \otimes L(v) \otimes I$, $L_3(w) = I \otimes I \otimes L(w)$. The extended form of (2.28) reads [13]

$$\begin{aligned} & [L_1(u), [d_{12}(u, v), d_{13}(u, w)]] + [d_{12}(u, v), d_{23}(v, w)] + [d_{32}(w, v), d_{13}(u, w)] \\ & + [L_1(u), \{L_2(v), d_{13}(u, w)\}] - \{L_3(w), d_{12}(u, v)\} \\ & + \text{cyclic permutations} = 0. \end{aligned} \tag{2.29}$$

Further on we restrict ourselves to proving (2.27) only in the paraboloidal case (other cases can be handled in a similar way). To complete the derivation of (2.27), we shall prove the following equality for all the members of the hierarchy

$$\begin{aligned} & \{L_2(v), s_{13}(u, w)\} - \{L_3(w), s_{12}(u, v)\} \\ & = 2\beta_N(u, v, w)[P_{23}, s_{13} + s_{12}] - \frac{\partial \beta_N(u, v, w)}{\partial x_{n+1}} [s, L_2(v) - L_3(w)] \end{aligned} \tag{2.30}$$

(with cyclic permutations). In (2.30) the matrix $s = \sigma_- \otimes \sigma_- \otimes \sigma_-$ and

$$\beta_N(u, v, w) = \frac{Q_N(u)(v-w) + Q_N(v)(w-u) + Q_N(w)(u-v)}{(u-v)(v-w)(w-u)}. \tag{2.31}$$

In the extended form (2.30) can be rewritten as

$$\{Q(u), Q(v)\} = \{B(u), Q(v)\} = 0 \tag{2.32}$$

$$\{A(u), Q(v)\} = 4\alpha_N(u, v) - \frac{1}{2} \frac{\partial \alpha_N(u, v)}{\partial x_{n+1}} B(u) \tag{2.33}$$

$$\begin{aligned} \{A(w), \alpha_N(u, v)\} \\ = \frac{4}{u-v} (\alpha_N(w, u) - \alpha_N(w, v)) - \frac{1}{2} \frac{B(w)}{u-v} \left(\frac{\partial \alpha_N(w, v)}{\partial x_{n+1}} - \frac{\partial \alpha_N(w, u)}{\partial x_{n+1}} \right) \end{aligned} \tag{2.34}$$

$$\{Q(u), C(v)\} + \{C(u), Q(v)\} = \frac{\partial \alpha_N(u, v)}{\partial x_{n+1}} (A(u) - A(v)). \tag{2.35}$$

The equality (2.32) is trivial and equation (2.33) is derived by differentiating (2.18). Equation (2.34) follows from the definition of $Q(u)$ and (2.33). To prove (2.35) we write it using the explicit form of $C_N(u)$ and $A(u)$ as

$$\begin{aligned} \sum_{i=1}^n \frac{p_i}{(v-e_i)(u-e_i)} \left((u-e_i) \frac{\partial \alpha_N(w, u)}{\partial x_i} - (v-e_i) \frac{\partial \alpha_N(w, v)}{\partial x_i} \right) \\ = \frac{1}{2} \sum_{i=1}^n \frac{p_i x_i}{(u-e_i)(v-e_i)} \left(\frac{\partial \alpha_N(w, u)}{\partial x_{n+1}} - \frac{\partial \alpha_N(w, v)}{\partial x_{n+1}} \right). \end{aligned} \tag{2.36}$$

Using the identity

$$u \frac{w^k - u^k}{w-u} - v \frac{w^k - v^k}{w-v} = \frac{w^{k+1} - u^{k+1}}{w-u} - \frac{w^{k+1} - v^{k+1}}{w-v}$$

and the recurrence relation (2.13), we find that the equality (2.36) is valid. Therefore the equations (2.27) follow with the matrix $c(u, v, w) = \partial \beta(u, v, w) / \partial x_{n+1} \sigma_- \otimes \sigma_- \otimes \sigma_-$. The proof is completed.

We remark that the validity of equations (2.27) with an arbitrary matrix $c(u, v, w)$ is sufficient for the validity of (2.28) and, therefore, (2.27) can be interpreted as some dynamical classical Yang–Baxter equation, i.e. the associativity condition for the linear r -matrix algebra. These equations have an extra term $[c, L_i - L_j]$ in comparison with the extended Yang–Baxter equations in [13].

We would like to emphasize that all statements of this section can be generalized to the following form of the potential term $V_N(u)$ in (2.4):

$$V_{MN}(u) = \sum_{k=-M}^N f_k \mathcal{V}_k u^{N-k} \quad f_k \in \mathbb{C}.$$

This form corresponds to the linear combinations of homogeneous terms \mathcal{V}_k as potential V and also includes the negative degrees to separable potential (see the end of section 3.1 for more detail).

3. Consequences of the r -matrix representation

3.1. Lax representation

Following the article [5] we can consider the Poisson structure (2.26) for the powers of the L -operator

$$\{(L_1(u))^k, (L_2(v))^l\} = [d_{12}^{(k,l)}(u, v), L_1(u)] - [d_{21}^{(k,l)}(u, v), L_2(v)] \quad (3.1)$$

with

$$d_{ij}^{(k,l)}(u, v) = \sum_{p=0}^{k-1} \sum_{q=0}^{l-1} (L_1(u))^{k-p-1} (L_2(v))^{l-q-1} d_{ij}(u, v) (L_1(u))^p (L_2(v))^q. \quad (3.2)$$

As an immediate consequence of (3.1), (3.2) we obtain that the conserved quantities H and H_i are in involution. Indeed, we have

$$\{\text{Tr}(L_1(u))^2, \text{Tr}(L_2(v))^2\} = \text{Tr}\{(L_1(u))^2, (L_2(v))^2\} \quad (3.3)$$

and after applying the equality (3.1) at $k = l = 2$ to this equation and taking the trace, we obtain the desired involutivity. Further, let us define differentiation by time as

$$\dot{L}(u) = \frac{d}{dt} L(u) = \text{Tr}_2\{L_1(u), (L_2(u))^2\} \quad (3.4)$$

where the trace is taken over the second space. Applying the equation (3.1) at $k = 1, l = 2$ to (3.4), we obtain the Lax representation in the form $\dot{L}(u) = [M(u), L(u)]$ with the matrix $M(u)$ given by

$$M(u) = 2 \lim_{v \rightarrow u} \text{Tr}_2 L_1(v) (r(u-v) - s(u, v)). \quad (3.5)$$

After the calculation in which we take into account the asymptotic behaviour of the L -operator (2.1), we obtain the following explicit Lax representation:

$$\begin{aligned} \dot{L}(u) &= [M(u), L(u)] \\ L(u) &= \begin{pmatrix} A(u) & B(u) \\ C_N(u) & -A(u) \end{pmatrix} \quad M(u) = \begin{pmatrix} 0 & 1 \\ Q_N(u) & 0 \end{pmatrix} \end{aligned} \quad (3.6)$$

where $Q_N(u)$ was defined by equations (2.20). Lax representations for the higher flows can be obtained in a similar way.

As follows from (3.6),

$$A(u) = -\frac{1}{2} \dot{B}(u) \quad C_N(u) = -\frac{1}{2} \ddot{B}(u) - B(u) Q_N(u)$$

so our L -matrix can be given in the form

$$L(u) = \begin{pmatrix} -\frac{1}{2} \dot{B}(u) & B(u) \\ -\frac{1}{2} \ddot{B}(u) - B(u) Q_N(u) & \frac{1}{2} \dot{B}(u) \end{pmatrix}. \quad (3.7)$$

The equations of motion, which follow from (3.6) with the L -matrix from (3.7), have the form

$$\mathcal{B}_1[Q_N] \cdot B(u) = 0 \quad (3.8)$$

where

$$\mathcal{B}_1[Q_N] \equiv \frac{1}{4} \partial^3 + \frac{1}{2} \{\partial, Q_N\} \quad \partial \equiv \frac{d}{dt}$$

with curly brackets standing for the anticommutator. Operator B_1 is the Hamiltonian operator of the first Hamiltonian structure for the coupled KdV equation [1, 2]. Equation (3.8), considered as one for the unknown function $B(u)$, was solved in the three cases (2.2),

$$B(u) = \varepsilon - \sum_{i=1}^n \frac{x_i^2}{u - e_i} \quad \varepsilon = 0, 1, 4(u - x_{n+1} + B)$$

in [1] and [2]. General solution of this equation as one for the $Q(u)$ has the form

$$Q_{MN}(u) = \sum_{k=-M}^N f_k Q_k u^k \quad f_k \in \mathbb{C} \tag{3.9}$$

where the coefficients Q_k are defined from the generating function $\tilde{Q}(u)$

$$\tilde{Q}(u) \equiv B^{-2}(u) = \sum_{k=-\infty}^{+\infty} Q_k u^k. \tag{3.10}$$

Recall that we can write the element $C_N(u)$ of the L -matrix (2.1) in two different forms (using the Q or V functions)

$$C_N(u) = -\frac{1}{2} \tilde{B}(u) - B(u) Q_N(u) = -\sum_{i=1}^n \frac{p_i^2}{u - e_i} - V(u)$$

where function $V(u) = \sum_{k=0}^N \mathcal{V}_k u^{N-k}$ was defined in (2.4). The general form of the function $V(u)$ is

$$V_{MN}(u) = \sum_{k=-M}^N f_k \mathcal{V}_k u^k \quad f_k \in \mathbb{C} \tag{3.11}$$

where coefficients \mathcal{V}_k are defined by the generating function $\tilde{V}(u)$

$$\tilde{V}(u) \equiv B^{-1}(u) = \sum_{k=-\infty}^{+\infty} \mathcal{V}_k u^k. \tag{3.12}$$

Potentials \mathcal{V}_k are connected with coefficients Q_k . Indeed, using generating functions (3.10) and (3.12), we have

$$\tilde{Q} = B^{-2}(u) = \tilde{V}(u) \cdot \tilde{V}(u) = \left(\sum_{k=-\infty}^{+\infty} \mathcal{V}_k u^k \right) \left(\sum_{k=-\infty}^{+\infty} \mathcal{V}_k u^k \right) = \sum_{k=-\infty}^{+\infty} u^k \sum_{j=0}^k \mathcal{V}_k \mathcal{V}_{k-j}$$

and, therefore, $Q_k = \sum_{j=0}^k \mathcal{V}_k \mathcal{V}_{k-j}$. Thus we have recovered the formula (2.20) for the s -matrix.

3.2. Separation of variables

Let K denote the number of degrees of freedom: $K = n - 1$ for ellipsoidal coordinates on a sphere, $K = n$ for ellipsoidal coordinates in the Euclidean space, and $K = n + 1$ for paraboloidal coordinates in the Euclidean space. The separation of variables (cf [12, 17]) is understood in the context of the given hierarchy of Hamiltonian systems as the construction of K pairs of canonical variables $\pi_i, \mu_i, i = 1, \dots, K$,

$$\{\mu_i, \mu_k\} = \{\pi_i, \pi_k\} = 0 \quad \{\pi_i, \mu_k\} = \delta_{ik} \tag{3.13}$$

and K functions Φ_j such that

$$\Phi_j \left(\mu_j, \pi_j, H_N^{(1)}, \dots, H_N^{(K)} \right) = 0 \quad j = 1, 2, \dots, K \tag{3.14}$$

where $H_N^{(i)}$ are the integrals of motion in involution. Equations (3.14) are the *separation equations*. The integrable systems considered admit the Lax representation in the form of 2×2 matrices (3.6) and we will introduce the separation variables π_i, μ_i as

$$B(\mu_i) = 0 \quad \pi_i = A(\mu_i) \quad i = 1, \dots, K. \quad (3.15)$$

Below we write explicitly these formulae for our systems. The set of zeros $\mu_j, j = 1, \dots, K$ of the function $B(u)$ defines the spherical ($\varepsilon = 0$), general ellipsoidal ($\varepsilon = 1$) and general paraboloidal ($\varepsilon = 4(u - x_{n+1} + B)$) coordinates given by the formulae [8, 9, 12]

$$x_m^2 = c \frac{\prod_{j=1}^{n-1} (\mu_j - e_m)}{\prod_{k \neq m} (e_m - e_k)} \quad m = 1, \dots, n \quad \text{where } c = \sum_{k=1}^n x_k^2 \quad \text{for } \varepsilon = 0 \quad (3.16)$$

$$x_m^2 = -4 \frac{\prod_{j=1}^n (\mu_j - e_m)}{\prod_{k \neq m} (e_m - e_k)} \quad m = 1, \dots, n \quad \text{for } \varepsilon = 1 \quad (3.17)$$

$$x_{n+1} = - \sum_{i=1}^n e_i + B + \sum_{i=1}^{n+1} \mu_i$$

$$x_m^2 = -4 \frac{\prod_{j=1}^{n+1} (\mu_j - e_m)}{\prod_{k \neq m} (e_m - e_k)} \quad m = 1, \dots, n \quad \text{for } \varepsilon = 4(u - x_{n+1} + B). \quad (3.18)$$

Theorem 4. The coordinates μ_i, π_i given by (3.15) are canonically conjugated.

Proof. Let us list the commutation relations between $B(v)$ and $A(u)$,

$$\{B(u), B(v)\} = \{A(u), A(v)\} = 0 \quad (3.19)$$

$$\{A(u), B(v)\} = \frac{2}{v-u} (B(u) - B(v)). \quad (3.20)$$

The equalities $\{\mu_i, \mu_j\} = 0$ follow from (3.19). To derive the equality $\{\mu_i, \pi_j\} = -\delta_{ij}$ we substitute $u = \mu_j$ in (3.20) and obtain

$$\{\pi_j, B(v)\} = -\frac{2}{v - \mu_j} B(v)$$

which together with the equation

$$0 = \{\pi_j, B(\mu_i)\} = \{\pi_j, B(v)\} |_{v=\mu_i} + B'(\mu_i) \{\pi_j, \mu_i\}$$

gives

$$\{\pi_j, \mu_i\} = -\frac{1}{B'(\mu_i)} \{\pi_j, B(v)\} |_{v=\mu_i} = \delta_{ij}.$$

Equalities $\{\pi_i, \pi_j\} = 0$ can be verified in the similar way:

$$\begin{aligned} \{\pi_i, \pi_j\} &= \{A(\mu_i), A(\mu_j)\} \\ &= \{A(\mu_j), A(v)\} |_{v=\mu_i} + A'(\mu_i) \{\mu_i, A(\mu_j)\} \\ &= A'(\mu_j) \{A(\mu_i), \mu_j\} + A'(\mu_i) \{\mu_i, A(\mu_j)\} = 0. \end{aligned}$$

The separation equations have the form

$$\pi_i^2 = d(\mu_i) \quad (3.21)$$

where the function $d(u)$ is the determinant of the L -operator (2.25).

3.3. Quantization

The separation of variables has a direct quantum counterpart [11, 19]. To pass to quantum mechanics we change the variables π_i, μ_i to operators and the Poisson brackets (3.13) to the commutators

$$[\mu_j, \mu_k] = [\pi_j, \pi_k] = 0 \quad [\pi_j, \mu_k] = -i\delta_{jk}. \tag{3.22}$$

Suppose that the common spectrum of μ_i is simple and the momenta π_i are realized as the derivatives $\pi_j = -i\partial/\partial\mu_j$. The separation equations (3.21) become the operator equations, where the non-commuting operators are assumed to be ordered precisely in the order as those listed in (3.14), that is, $\pi_i, \mu_i, H_N^{(1)}, \dots, H_N^{(K)}$. Let $\Psi(\mu_1, \dots, \mu_K)$ be a common eigenfunction of the quantum integrals of motion:

$$H_N^{(i)}\Psi = \lambda_i\Psi, \quad i = 1, \dots, K. \tag{3.23}$$

Then the operator separation equations lead to the set of differential equations

$$\Phi_j\left(-i\frac{\partial}{\partial\mu_j}, \mu_j, H_N^{(1)}, \dots, H_N^{(K)}\right)\Psi(\mu_1, \dots, \mu_K) = 0 \quad j = 1, \dots, K \tag{3.24}$$

which allows the separation of variables

$$\Psi(\mu_1, \dots, \mu_K) = \prod_{j=1}^K \psi_j(\mu_j). \tag{3.25}$$

The original multidimensional spectral problem is therefore reduced to the set of one-dimensional multiparametric spectral problems which have the following form in the context of the problems under consideration:

$$\left(\frac{d^2}{du^2} + \varepsilon u^N + \sum_{i=1}^n \frac{\lambda_i}{u - e_i}\right) \psi_j(u; \lambda_1, \dots, \lambda_n) = 0 \quad \text{for } \varepsilon = 0, 1 \tag{3.26}$$

$$\left(\frac{d^2}{du^2} + 16u^{N-2}(u+B)^2 + 8\lambda_{n+1} + \sum_{i=1}^n \frac{\lambda_i}{u - e_i}\right) \psi_j(u; \lambda_1, \dots, \lambda_{n+1}) = 0$$

for $\varepsilon = 4(u - x_{n+1} + B)$ (3.27)

with the spectral parameters $\lambda_1, \dots, \lambda_{n+1}$. The problems (3.26), (3.27) must be solved on the different intervals ('permitted zones') for the variable u .

4. Conclusion

We should emphasise that a large family of integrable systems has been studied in the present paper. As partial cases it includes, for example, the classical Coulomb problem, the oscillator and many others that can be separated in general orthogonal coordinate systems. In other terms we can claim that every coordinately separable Hamiltonian of natural type, with the separation variables lying on the hyperelliptic curve, is in our family. We have to mention also a recent preprint [20] where it was shown that the elliptic Calogero–Moser problem provides one more example of the integrable system of natural type described through L -operator satisfying the algebra (1.4) with a slightly more general dynamical Yang–Baxter equation (2.27). But it is not coordinately separable any more.

We remark also that all systems considered in the paper yield an algebra which has general properties that are independent of the type of the system. Therefore, it would be

interesting to consider its Lie-algebraic origin within the general approach to the classical r -matrices [16].

There exists an interesting link of the algebra studied here with the restricted flow formalism for the stationary flows of the coupled KdV (cKdV) equations [1]. The Lax pairs which have been derived in the paper from the algebraic point of view were recently found in [2] by considering the bi-Hamiltonian structure of cKdV.

It seems to be interesting to examine the same questions for the generalized hierarchy of Gelfand–Dickey differential operators for which the corresponding L -operators have to be the $n \times n$ matrices.

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