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Linear Spin-Zero Quantum Fields in External Gravitational and Scalar Fields

I. A One Particle Structure for the Stationary Case

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Abstract. We give mathematically rigorous results on the quantization of the covariant Klein Gordon field with an external stationary scalar interaction in a stationary curved space-time.

We show how, following Segal, Weinless etc., the problem reduces to finding a "one particle structure" for the corresponding classical system.

Our main result is an existence theorem for such a one-particle structure for a precisely specified class of stationary space-times. Byproducts of our approach are:

1) A discussion of when a given "equal-time" surface in a given stationary space-time is Cauchy.

2) A modification and extension of the methods of Chernoff [3] for proving the essential self-adjointness of certain partial differential operators.

§0. Introduction

In this series of papers, we discuss the quantization of the equation

$$(g^{\mu\nu}\nabla_\mu \partial_\nu + m^2 + V)\varphi = 0, \tag{0.1}$$

— the covariant Klein Gordon equation in a fixed curved space-time $(\mathcal{M}, g^{\mu\nu})$ and interacting with a fixed external scalar field V. (We shall always take $(\mathcal{M}, g)$, V to be C^∞.)

Notwithstanding much recent work on "quantum field theory in curved space-times" [6, 12], it is perhaps fair to say that there has been, so far, no satisfactory statement of what it would mean to quantize our Equation (0.1) on a generic space-time. For the case of stationary space-times however, there is a well established procedure — at least at the heuristic level. We ought not to rest content with this procedure while the generic case remains unsolved (most space-time are not stationary, and even for stationary space-times, why should we single out

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particular coordinates for which the metric is stationary?). However, we shall show (in subsequent papers) how results on the stationary case play a role in a possible formulation of a quantization procedure suitable for generic case.

The present paper is concerned with giving rigorous mathematical results on the existence and uniqueness of quantization for the stationary case.

Sections 1–4 outline our general approach and state the principal results. Sections 5–7 contain details of proofs and further results.

§1. Stationary Space-Times

Note. 1) We use Hawking and Ellis [10] (H.E.) especially Chapters 1 and 6 as a reference throughout—except that we choose signature $(+ - - -)$. 2) All space-times are assumed to be space and time orientable.

By a stationary space-time, we mean a space and time orientable space-time $(\mathcal{M}, {}^4g)$ together with a global time-like Killing vector field X.

We shall find it convenient to have a representation for such space-times which picks out a preferred time-coordinate (equivalently a preferred family of equal-time surfaces):

Given an (orientable) Riemannian 3-manifold $(\mathscr{C}, {}^3g)$ and given $(\alpha, \beta) - \alpha$ a scalar field, β a vector field on $\mathscr{C}$ satisfying

$$\alpha > 0 \qquad \alpha^2 - {}^3g(\beta, \beta) > 0 \tag{1.1}$$

we define $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$ to be the space-time $(\mathbb{R} \times \mathscr{C}, {}^4g)$ where ${}^4g$ is given (in local coordinates $(t, x^i)) - x^i$ local coordinates on $\mathscr{C}$) by

$${}^4g_{\text{lower}} = \begin{pmatrix} \alpha^2 - \beta^i \beta_i & -\beta_j \\ -\beta_i & -{}^3g_{ij} \end{pmatrix} \tag{1.2}$$

together with the Killing vector field

$$X \left(\equiv \frac{\partial}{\partial t} \right) = \alpha N(\mathscr{C}) + \beta \tag{1.3}$$

[$N(\mathscr{C})$: unit future-pointing normal vector field to $\mathscr{C}$] α and β (sometimes "N" and "$N^i \partial_i$") are usually known [17] as lapse and shift fields.

Clearly, $\mathscr{C}$ (strictly, $\{s\} \times \mathscr{C}$ for any $s \in \mathbb{R}$) is always a smooth spacelike partial Cauchy surface in $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$.

In the sequel, we restrict ourselves to *globally hyperbolic* stationary space-times. By a theorem due to Geroch (H.E. Propositions 6.6.3, 6.6.8) global hyperbolicity is equivalent to the existence of a (smooth, spacelike) *global* Cauchy surface.

Thus, a globally-hyperbolic stationary space-time can always be realized as some $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$ in which $\mathscr{C}$ is globally Cauchy: Simply choose some Cauchy surface $\mathscr{C}$, set ${}^3g$ equal to the induced Riemannian metric and define α and β in terms of X by Equation (1.3).

In §5, we use results on the causal structure of space-times to find some simple conditions guaranteeing that $\mathscr{C}$ is Cauchy in $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$. For instance, it is always true when $\mathscr{C}$ is compact. The results obtained are interesting in their own right, and in any case are needed in §7.

§2. The Classical Theory as a "Linear Dynamical System" (Stationary Case)

Given some Cauchy surface, $\mathscr{C}$, let us denote by $D(\mathscr{C})$ the space of real smooth Cauchy data of compact support:

$$D(\mathscr{C})=C_0^\infty(\mathscr{C})+C_0^\infty(\mathscr{C}). \tag{2.1}$$

Then we have

2.1. Leray's Theorem. *Let $(\mathscr{M},g)$ be an oriented globally hyperbolic space-time; $\mathscr{C}$ some Cauchy surface—unit future pointing normal $N(\mathscr{C})$.*

Then, the Cauchy data $\Phi\in D(\mathscr{C})$ given by

$$\Phi=\begin{pmatrix} f\\ p\end{pmatrix}\qquad f=\varphi|_{\mathscr{C}}\qquad p=N(\mathscr{C})\varphi|_{\mathscr{C}}$$

define a unique solution in $C^\infty(\mathscr{M})$ having compact support on every other Cauchy surface. In fact, the solution has support in $J^+(\operatorname{supp}\Phi)\cup J^-(\operatorname{supp}\Phi)$—the union of the causal future and the causal past of the Cauchy data.

For the proof, see Leray [15], Choquet-Bruhat [4], and Lichnerowicz [16]. Thus, representing our space-time as $\operatorname{Stat}(\mathscr{C},{}^3g,\alpha,\beta)$ for some Cauchy surface $\mathscr{C}$, we may view time-evolution as a one-parameter group:

$$\mathscr{T}(t):D(\mathscr{C})\to D(\mathscr{C}).$$

Moreover, the existence of the conserved current $j_\mu=\varphi_1\overleftrightarrow{\partial}_\mu\varphi_2$ for any pair of solutions φ_1, φ_2 of (0.1) guarantees that $\mathscr{T}(t)$ preserves the symplectic form

$$\sigma(\Phi_1,\Phi_2)=\int_{\mathscr{C}}(f_1p_2-p_1f_2)d\eta(\mathscr{C}), \tag{2.2}$$

where

$$\Phi_1=\begin{pmatrix} f_1\\ p_1\end{pmatrix}\in D(\mathscr{C}),\qquad \Phi_2=\begin{pmatrix} f_2\\ p_2\end{pmatrix}\in D(\mathscr{C}),$$

$\eta(\mathscr{C})$ denotes the Riemannian volume element on $(\mathscr{C},{}^3g)$. Equivalently, we shall sometimes write

$$\sigma(\Phi_1,\Phi_2)=\langle\Phi_1|\boldsymbol{g}\Phi_2\rangle_{L^2(\mathscr{C},\eta)+L^2(\mathscr{C},\eta)}, \tag{2.3}$$

where

$$\boldsymbol{g}=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

We shall use the term *linear dynamical system* to denote any real symplectic space (D,σ) together with a one-parameter symplectic group $\mathscr{T}(t)$.

Note that, in our case, $\mathscr{T}(t)$ is generated in the sense of classical mechanics by the Hamiltonian:

$$H(f,p)=\tfrac{1}{2}\int_{\mathscr{C}} d\eta(\mathscr{C})\alpha(p^2+{}^3g^{ij}\partial_i f\,\partial_j f+(m^2+V)f^2)$$

$$+\int_{\mathscr{C}} d\eta(\mathscr{C})p\beta^i\partial_i f$$

$$=\tfrac{1}{2}\langle\Phi|A\Phi\rangle_{L^2(\mathscr{C},\eta)+L^2(\mathscr{C},\eta)}, \tag{2.4}$$

where

$$A=\begin{pmatrix} -(\partial^i\alpha)\partial_i+\alpha(m^2-\Delta(\mathscr{C})+V) & -(V_i\beta^i+\beta^i\partial_i) \\ \beta^i\partial_i & \alpha \end{pmatrix},$$

where $\Delta(\mathscr{C})$ is the Laplace-Beltrami operator for $(\mathscr{C},{}^3g)$; V_i denotes the covariant derivative for $(\mathscr{C},{}^3g)$.

The first order form equations can be written

$$\frac{d}{dt}\mathscr{T}(t)\Phi\bigg|_{t=0}=-\boldsymbol{h}\Phi, \tag{2.5}$$

where $\boldsymbol{h}=-\boldsymbol{g}\boldsymbol{A}$.

Note. Although it is not made manifest by our formalism, it is clear that the linear dynamical system we obtain depends only on the stationary space-time in question [i.e. on $(\mathscr{M},{}^4g,X)$], and not on the choice of Cauchy surface $\mathscr{C}$. More precisely, if we chose a different Cauchy surface $\mathscr{C}'$ [realizing our space-time as $\mathrm{Stat}(\mathscr{C}',{}^3g',\alpha',\beta')$ the resulting $(D(\mathscr{C}'),\sigma',\mathscr{T}'(t))$ would be equivalent to $(D(\mathscr{C}),\sigma,\mathscr{T}(t))$].

§3. Quantization of Linear Dynamical Systems [2, 5, 8, 14, 21–23, 25]

To quantize a linear dynamical system $(D,\sigma,\mathscr{T}(t))$, one seeks a "*quantization*" $(\mathscr{H},W,\Omega,V(t))$ [25] where $\mathscr{H}$ is a (separable) Hilbert space, W a map from D into unitaries on $\mathscr{H}$, Ω a vector in $\mathscr{H}$ and $V(t)$ a strongly continuous unitary group on $\mathscr{H}$ satisfying:

$$\left.\begin{aligned}
&1)\ W(\Phi+\Psi)=\exp(-i\sigma(\Phi,\Psi)/2)W(\Phi+\Psi),\\
&2)\ W(\mathscr{T}(t)\Phi)=V(t)W(\Phi)V(-t),\\
&3)\ V(t)\Omega=\Omega,\\
&4)\ V(t)\text{ has positive energy}^1,\\
&5)\ W(s\Phi)\text{ is weakly continuous in } s\in\mathbb{R},\\
&6)\ \Omega\text{ is cyclic for the } W(\Phi)\text{'s} \qquad \forall\Phi,\Psi\in D.
\end{aligned}\right\} \tag{3.1}$$

[1] I.e. its generator is positive

Such a quantization is determined (up to equivalence) by its generating functional:

$$\langle \Omega | W(\Phi)\Omega \rangle .$$

We now briefly explain the reason for this definition of quantization: For the generators $R(\Phi)$ of $W(t\Phi)$ and H of $V(t)$, (3.1) above is a rigorous version of:

$$\begin{aligned} &1)\ [R(\Phi), R(\Psi)] = i\sigma(\Phi, \Psi), \\ &2)\ [H, R(\Phi)] = -iR(\boldsymbol{h}\Phi), \\ &3)\ H\Omega = 0 \qquad \left(\boldsymbol{h} := -\frac{d}{dt}\mathcal{T}(t)\Big|_{t=0}\right). \end{aligned} \tag{3.2}$$

Here, $R(\Phi)$ may be thought of as $\sigma(\hat{\varphi}, \hat{\pi}; f, p)$ where $\hat{\varphi}$, $\hat{\pi}$ are the equal-time quantum fields corresponding to f, p. (The usual canonical momentum is $\sqrt{{}^3g}\,\hat{\pi}$.)

In (3.2), 1) expresses

"$[\hat{\varphi}(x), \hat{\varphi}(y)] = 0, \ [\hat{\pi}(x), \hat{\pi}(y)] = 0, \ [\hat{\varphi}(x), \hat{\pi}(y)] = i\delta^3(x, y)$",

2) are the quantum Hamiltonian equations of motion; and 3) gives the interpretation of Ω as the vacuum vector—or lowest energy state of the positive Hamiltonian H.

Roughly speaking then, seeking a quantization in our sense amounts to seeking a representation of the CCR in which a positive Hamiltonian exists generating the dynamics (with a vacuum vector Ω cyclic for the CCR).

To construct a quantization for a given linear dynamical system $(D, \sigma, \mathcal{T}(t))$, one seeks first a "*one particle structure*" $(K, \mathcal{H}, U(t))$ for $(D, \sigma, \mathcal{T}(t))$ where $\mathcal{H}$ is a separable Hilbert space, and $U(t)$ a unitary group with strictly positive energy[2] and K a real linear map from D to $\mathcal{H}$ satisfying:

$$\begin{aligned} &1)\ \operatorname{ran} K \text{ is dense in } \mathcal{H}, \\ &2)\ 2\operatorname{Im}\langle K\Phi | K\Psi \rangle = \sigma(\Phi, \Psi) \quad (\textit{i.e. } K \text{ is symplectic}), \\ &3)\ K(\mathcal{T}(t)\Phi) = U(t)(K(\Phi)) \quad [\text{i.e. } K \text{ intertwines } \mathcal{T}(t) \text{ and } U(t)]. \end{aligned} \tag{3.3}$$

We may then quantize using the Fock representation over $\mathcal{H}$ (Segal's "free Weyl process"). E.g. we can choose for $\mathcal{K}$ Fock space over $\mathcal{H}$:

$$\mathcal{K} = \mathcal{F}(\mathcal{H}) = \mathbb{C} + \mathcal{H} + (\mathcal{H} \otimes \mathcal{H})_s + (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})_s + \ldots$$

whereupon we have:

$$\Omega = 1 + 0 + 0 + \ldots$$

$$V(t) = \text{"}d\Gamma(U(t))\text{"} = 1 + U(t) + (U(t) \otimes U(t))_s + \ldots$$

$$W(\Phi) = \exp(-iR^F(K\Phi)),$$

where

$$R^F(K\Phi) = -i\overline{(a^*(K\Phi) - a^*(K\Phi)^*)}$$

[$a^*(\cdot)$ the usual creation operator on $\mathcal{F}(\mathcal{H})$].

[2] I.e. it has positive self-adjoint generator h with dense range

This quantization is characterized by the generating functional:

$$\langle \Omega | W(\Phi)\Omega \rangle = \exp(-\tfrac{1}{2}\|K\Phi\|^2_{\mathscr{H}}).$$

It is known that when a one-particle structure exists, it is unique up to unitary equivalence: The proof, though not explicitly contained, may easily be extracted from Weinless [25] (see [13]).

For theorems on the existence and uniqueness of a quantization as such, see again Weinless [25]—where again the concept of a "one-particle structure" plays an important role.

§4. Existence of a One-Particle Structure for Equation (0.1) on a Stationary Space-Time

The main result of this paper is on the existence of a one-particle structure for the linear dynamical system we constructed in §2 for our equation:

4.1. Theorem. *Given Equation* (0.1) *on a (globally hyperbolic) stationary space-time* $(\mathscr{M}, {}^4g, X)$.

Suppose i) *V is stationary and satisfies $V(x) > -m^2 + \varepsilon$ everywhere for some $\varepsilon > 0$.*

Suppose also that there exists a Cauchy surface $\mathscr{C}$ in $(\mathscr{M}, {}^4g)$ (thus realizing $(\mathscr{M}, {}^4g)$ as $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$*) such that*

ii) *α is bounded below away from zero: $\alpha > \varepsilon_1 > 0$ everywhere on $\mathscr{C}$ for some ε_1.*

iii) *$\alpha - \beta^i\beta_i/\alpha$ is bounded below away from zero: $\alpha - \beta^i\beta_i/\alpha > \varepsilon_2 > 0$ everywhere on $\mathscr{C}$ for some ε_2. (α and β being defined in terms of $(\mathscr{M}, g, X)$ and $\mathscr{C}$ by Equation* (1.3).*)*

Then, on the corresponding linear dynamical system (as in §2) there exists a (unique) one-particle structure $(K, \mathscr{H}, U(t))$.

Moreover, letting h be the self-adjoint generator of $U(t): U(t) = e^{-ith}$

1) *h has bounded inverse (i.e. there is a "mass gap"),*

2) *$KD + iKD$ is an invariant domain of essential self-adjointness for h.*

Note. a) that condition i) may be interpreted as avoiding "Klein paradox" situations; condition ii) prevents the Killing vector from becoming arbitrarily small; condition iii) prevents it from approaching a light-like vector.

b) In the case of compact $\mathscr{C}$, ii), iii), are automatically satisfied.

In §6 we prove this theorem, and in §7, we give a more detailed discussion of the static case when it is possible to construct $(K, \mathscr{H}, U(t))$ in a more concrete way.

§5. When is $\mathscr{C}$ Cauchy in Stat$(\mathscr{C}, {}^3g, \alpha, \beta)$?

We use the

5.1. Lemma. *Let $\mathscr{S}$ be a partial Cauchy surface in some space-time $(\mathscr{M}, g)$; then $\mathscr{S}$ is a global Cauchy surface if and only if every inextendible null geodesic in $\mathscr{M}$ intersects $\mathscr{S}$.*

Note. With slightly different definitions for the technical terms, this would be a special case of "Property 6" of Geroch [9]. A *proof* in our case (i.e. following the definitions of H.E.) can easily be constructed by combining the Corollary to H.E. Proposition 6.5.3 with the easily proved fact that a partial Cauchy surface is a global Cauchy surface if and only if its Horizon is empty. For details see [13].

Firstly, in the "special static case of lapse 1 and shift 0", we have:

5.2. Proposition. *$\mathscr{C}$ is Cauchy in* $\mathrm{Stat}(\mathscr{C}, {}^3g, 1, 0)$ *if and only if* $(\mathscr{C}, {}^3g)$ *is complete.*

(Recall: [11], p. 56: for Riemannian manifolds, geodesic completeness is equivalent to metric space completeness in the metric of geodesic length.)

Proof. It is easy to see that the null geodesics in $\begin{pmatrix} 1 & 0 \\ 0 & -{}^3g \end{pmatrix}$ can all be parametrized:

$$t \mapsto (t, \gamma(t)), \tag{A}$$

where $t \mapsto \gamma(t)$ is a geodesic in $(\mathscr{C}, {}^3g)$ parametrized by its Riemannian length. We have, (Lemma 5.1) $\mathscr{C}$ is Cauchy $\Leftrightarrow$ the image of every inextendible null geodesic cuts every $\{t\} \times \mathscr{C} \Leftrightarrow$[by (A) above] every inextendible geodesic $t \mapsto \gamma(t)$ is defined for all $t \in (-\infty, \infty) \Leftrightarrow (\mathscr{C}, {}^3g)$ is complete. □

For more general lapse and shift, we have partial results:

5.3. Proposition. *A sufficient condition for $\mathscr{C}$ Cauchy in* $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$ *is if* $(\mathscr{C}, {}^3g)$ *complete and α bounded on $\mathscr{C}$.*

Proof. We first prove the

Lemma. *Suppose $\mathscr{C}$ is complete in some Riemannian metric ${}^3h$ (perhaps, but not necessarily ${}^3g$) and $\exists \varepsilon > 0$ s.t. ${}^3h_{ij} n^i n^j / n^{02} < \varepsilon$ for every null-vector (n^0, n^i) in the tangent bundle. Then, $\mathscr{C}$ is Cauchy in* $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$.

Proof. Parametrize null geodesics in $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, \beta)$ by t [thus $t \mapsto (t, \gamma(t))$]. Suppose an inextendible such geodesic has l.u.b. $\{t\} = T$. Let $t_i \to T$, then $\{\gamma(t_i)\}$ is Cauchy since

$$d(\gamma(t_i), \gamma(t_j)) \leqq \int_{t_i}^{t_j} \left({}^3h_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right)^{1/2} dt < \varepsilon^{1/2}(t_j - t_i).$$

Therefore $\gamma(t_i) \to \gamma(T)$ (say) [since $t \mapsto \gamma(t)$ continuous and $(\mathscr{C}, {}^3h)$ complete]. Also, [using ${}^3h(\dot\gamma(t_i), \dot\gamma(t_j)) < \varepsilon$ again] we easily have $(1, \dot\gamma(t_i)) \to (1, \dot\gamma(T))$ [$\dot\gamma(T)$ "left tangent at T"] in the tangent bundle to our space-time whereupon, we can extend the geodesic beyond $(T, \gamma(T))$ contra. hypoth.—so the geodesic is defined for all $t > T$. Similarly, it is extendible into the past.

Therefore every inextendible null geodesic takes on all values of t.

Therefore $\mathscr{C}$ is Cauchy.

Proof of Proposition. It remains to verify the conditions of our Lemma. Now, given a null vector (n^0, n^i), we have [using Eq. (1.2)]

$$(\alpha^2 - \beta^i \beta_i) n^{02} - 2n^0 \beta^i n_i - {}^3g_{ij} n^i n^j = 0$$

i.e.

$$g_{ij} \left(\frac{n^i}{n^0} + \beta^i \right) \left(\frac{n^j}{n^0} + \beta^j \right) = \alpha^2 .$$

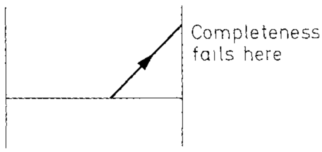

Fig. 1

Fig. 2

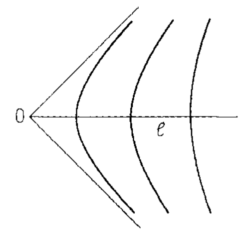

Fig. 3

In other words: calling n^i/n^0 "N" and writing $\| \; \|$ for the Euclidean norm in our tangent space:

$\|N+\beta\| = \alpha$

also, $\alpha^2 - \beta^i\beta_i > 0 \Rightarrow \|\beta\| < \alpha$ so $\|N\| \leqq \|N+\beta\| + \|\beta\| \leqq 2\alpha$ i.e. ${}^3g^{ij}n^in^j/n^{0^2} \leqq 4\alpha^2$. □

5.4. Corollary. *$\mathscr{C}$ compact implies $\mathscr{C}$ Cauchy in* Stat$(\mathscr{C}, {}^3g, \alpha, \beta)$ *for any* ${}^3g, \alpha, \beta$ *on* $\mathscr{C}$.

Proof. $(\mathscr{C}, {}^3g)$ complete, since in metric spaces compact $\Rightarrow$ complete.
Finally, $\mathscr{C}$ compact and α continuous $\Rightarrow$ α bounded. □

5.5. *Examples.* Intuitively, $\mathscr{C}$ fails to be Cauchy in Stat$(\mathscr{C}, {}^3g, \alpha, \beta)$ if signals (null or timelike curves) can hit the "edge" in a finite time. We give a series of examples for 2 dimensional space-times.

1) Choose $\mathscr{C} = (0, 1)$ with usual metric, $\alpha = 1, \beta = 0$. $\mathscr{C}$ is *not* Cauchy (see Fig. 1).

2) We could also envisage the lapse function growing too fast at "infinity" on a complete manifold:

Take the previous example under a conformal transformation which sends $(0, 1)$ (usual metric) to $(-\infty, \infty)$ (usual metric) i.e. stretches it out lengthways thus making $(\mathscr{C}, g)$ complete but preserving the causal structure. (Note: conformal transformations on a space-time always preserve the causal structure.) See Figure 2.

3) Finally, we show that if the lapse function *shrinks* fast enough, $(\mathscr{C}, g)$ need not be geodesically complete for $\mathscr{C}$ to be Cauchy. Take the "Rindler Wedge" i.e. one of the connected pieces of 2 dimensional Minkowski space-time which is spacelike separated from some origin 0. Define α and β by taking for $\frac{\partial}{\partial t}$ the Killing vector tangent to Lorentz boosts.

$\mathscr{C} = (0, \infty)$ is clearly Cauchy. See Figure 3.

§6. Proof of the Existence Theorem

In this section, we prove Theorem 4.1. At a heuristic level, the strategy is well known (introduction of "complex structure" etc. [1, 2, 6, 12, 22, 23]). The main technical steps involve proving the essential self-adjointness of certain Hilbert space operators related to our differential equation. For these essential self-adjointness proofs, it turns out that Leray's theorem combines nicely with a well known Lemma of Nelson [18]. In fact, in §7, we shall need to use a generalization of Nelson's Lemma due to Chernoff which we now state:

6.1. Chernoff's Lemma [3]. *Let T be a symmetric operator with dense domain $\mathscr{D} \subset \mathscr{H}$ a (complex) Hilbert space. Suppose T maps $\mathscr{D}$ into itself. Suppose in addition that there is a one-parameter group $V(t)$ of unitary operators on $\mathscr{H}$ such that $V(t)\mathscr{D} \subset \mathscr{D}$, $V(t)T = TV(t)$ on $\mathscr{D}$ and*

$$\frac{d}{dt} V(t)u = iTV(t)u\,; \qquad u \in \mathscr{D}\,.$$

Then T^n is essentially self-adjoint for all n. (We shall refer to "Nelson's Lemma" when we only use the case $n=1$.)

Proof of Theorem 4.1. We proceed in 5 stages:

1) $\langle \Phi | A\Phi \rangle_{L^2+L^2} > \varepsilon \|\Phi\|^2_{L^2+L^2}$ for some $\varepsilon > 0$.

Proof. By Equation (2.4):

$$\begin{aligned}\langle \Phi | A\Phi \rangle_{L^2+L^2} &= \int_{\mathscr{C}} d\eta\alpha(p^2 + {}^3g^{ij}\partial_i f \partial_j f + (m^2+V)f^2 + 2\int_{\mathscr{C}} d\eta p\beta^i\partial_i f \\ &= \int_{\mathscr{C}} d\eta\alpha \left\{\left(1 - \frac{\beta_i\beta^i}{\alpha^2}\right)p^2 + {}^3g^{ij}\left(\partial_i f + \frac{\beta_i}{\alpha}p\right)\left(\partial_j f + \frac{\beta_j}{\alpha}p\right) + (m^2+V)f^2\right\} \\ &\geqq \int_{\mathscr{C}} d\eta\{\varepsilon_2 p^2 + \varepsilon_1\varepsilon f^2\} (\varepsilon, \varepsilon_1, \varepsilon_2 \text{ as in } \S 4) \\ &\geqq \min(\varepsilon_2, \varepsilon_1\varepsilon)\int d\eta(p^2+f^2)\,.\end{aligned}$$

2) Define $\mathscr{A}$: the completion of $D(\mathscr{C})$ in $\boldsymbol{A}$ norm
(A norm: $\|\Phi\|^2_{\mathscr{A}} := \langle \Phi | \boldsymbol{A}\Phi \rangle_{L^2+L^2}$) then

a) $\boldsymbol{h}: D \to D$ and is skew symmetric on $D \subset \mathscr{A}$,

b) σ is continuous in $\boldsymbol{A}$ norm i.e.

$$\sigma(\Phi, \Psi) \leqq C \|\Phi\|_{\mathscr{A}} \|\Psi\|_{\mathscr{A}}\,,$$

c) $\mathscr{T}(t): D \to D$ extends to a strongly continuous unitary group $\mathscr{T}'(t)$ with $\left.\frac{d\mathscr{T}'(t)}{dt}\right|_{t=0} \overset{s}{=} -\boldsymbol{h}$ on D and $\mathscr{T}'(t)\boldsymbol{h} = \boldsymbol{h}\mathscr{T}'(t)$ on D.

Proof. a) Clearly, $\boldsymbol{h}: D \to D$ since $\boldsymbol{h} = -\boldsymbol{g}A$ and both $\boldsymbol{g}$ and A are matrices of C^∞ differential operators. Let $\Phi, \Psi \in D$ then $\langle \Phi | \boldsymbol{h}\Psi \rangle_{\mathscr{A}} = \langle \Phi | A(-\boldsymbol{g}A)\Psi \rangle_{L^2+L^2}$ $= \langle \boldsymbol{g}A\Phi | A\Psi \rangle_{L^2+L^2}$ (since $\boldsymbol{g}^+ = -\boldsymbol{g}$ on L^2+L^2) $= -\langle \boldsymbol{h}\Phi | \Psi \rangle_{\mathscr{A}}$.

b) $\sigma(\Phi, \Psi) = \langle \Phi | \boldsymbol{g}\Psi \rangle_{L^2+L^2} \leqq \|\Phi\|_{L^2+L^2} \|\Psi\|_{L^2+L^2}$ (since $\boldsymbol{g}$ bounded in L^2+L^2) $\leqq \frac{1}{\varepsilon^2} \|\Phi\|_{\mathscr{A}} \|\Psi\|_{\mathscr{A}}$ by 1) above.

c) Follows easily after checking that $\mathscr{T}(t)$ is:

i) isometric on D;

ii) strongly continuous on D;

iii) $\left.\frac{d}{dt}\mathscr{T}(t)\right|_{t=0} \overset{s}{=} -\boldsymbol{h}$ on D.

i) Follows from

$$\frac{d}{dt}\langle \mathcal{T}(t)\Phi|\mathcal{T}(t)\Phi\rangle_{\mathcal{A}} = \frac{d}{dt}\int \overline{\mathcal{T}(t)\Phi} A \mathcal{T}(t)\Phi d\eta = 0\ ;$$

ii) from $\lim_{t\to 0}\|(\mathcal{T}(t)-1)\Phi\|^2 = \lim_{t\to 0}\int \overline{(\mathcal{T}(t)-1)\Phi} A(\mathcal{T}(t)-1)\Phi d\eta = 0$ and

iii) from

$$\lim_{t\to 0}\left\|\left(\frac{\mathcal{T}(t)-1}{t}+\boldsymbol{h}\right)\Phi\right\|^2 = \lim_{t\to 0}\int \overline{\left(\frac{\mathcal{T}(t)-1}{t}+\boldsymbol{h}\right)\Phi} A\left(\frac{\mathcal{T}(t)-1}{t}+\boldsymbol{h}\right)\Phi d\eta = 0.$$

In each case, the last equality following by interchanging the derivative in i) or limit in ii) and iii) with the integral sign and applying the pointwise results $\left[\lim(\mathcal{T}(t)-1)=0,\ \frac{d\mathcal{T}(t)}{dt}\Big|_{t=0} = -\boldsymbol{h}\mathcal{T}(t)\right]$.

In each case the conditions justifying the interchange being that the integrand [and its t derivative for i)] is a jointly continuous function of t and x and that the integral over x is over a compact set with finite measure (cf. Dieudonné [7], I, p. 176, §VII.11). These conditions holding thanks to Leray's theorem.

3) a) $\boldsymbol{h}$ is essentially skew-adjoint on D.

b) $\bar{\boldsymbol{h}}^{-1}$ exists and is bounded.

c) σ extends uniquely to σ' on $\mathcal{A}$. Moreover

$$\sigma'(\Phi,\Psi) = \langle \Phi|\bar{\boldsymbol{h}}^{-1}\Psi\rangle_{\mathcal{A}}.$$

Proof. 3a) follows by Nelson's lemma[3] using 2a and 2c. 2b and the Riesz lemma imply ∃ a bounded skew adjoint operator T s.t.

$$\sigma(\Phi,\Psi) = \langle \Phi|T\Psi\rangle_{\mathcal{A}} \qquad \Phi,\Psi\in D.$$

This last equation then extends σ uniquely by continuity to $\mathcal{A}$ call the extension σ'. Now note:

$$\Phi,\Psi\in D \Rightarrow \langle \Phi|T\boldsymbol{h}\Psi\rangle_{\mathcal{A}} = \sigma(\Phi, -\boldsymbol{g}A\Psi)$$

$$= \langle \Phi|\boldsymbol{g}(-\boldsymbol{g}A)\Psi\rangle_{L^2+L^2} = \langle \Phi,\Psi\rangle_{\mathcal{A}} \quad \text{i.e.} \quad T\boldsymbol{h}=1 \quad \text{on} \quad D. \qquad (*)$$

Therefore $\operatorname{ran} T$ dense and T^{-1} is an unbounded skew-adjoint operator with $D(T^{-1}) = \operatorname{Ran} T$ (see e.g. Rudin [20], Theorem 13.11b)) $(*)\Rightarrow T^{-1}=\boldsymbol{h}$ on D.

Therefore $T^{-1}=\bar{\boldsymbol{h}}$ since $\boldsymbol{h}$ essentially skew-adjoint on D. We have thus proved 3b), 3c).

Note. By this stage, we have extended dynamics to the linear dynamical system $(\mathcal{A},\sigma',\mathcal{T}'(t))-(\mathcal{A},\sigma')$ represents physically the phase space of all finite-energy classical solutions. Mathematically, the fact that our enlarged symplectic space is a

[3] Applied to the "natural complexification" $\mathcal{A}_{\mathbb{C}}$ of $\mathcal{A}$: i.e. $\mathcal{A}+\mathcal{A}$ with $i(x,y)=(-y,x)$.
$\langle (x_1,y_1)|(x_2,y_2)\rangle_{\mathcal{A}_{\mathbb{C}}} = \langle x_1|x_2\rangle_{\mathcal{A}} + \langle y_1|y_2\rangle_{\mathcal{A}} + i(\langle x_1|y_2\rangle_{\mathcal{A}} - \langle y_1|x_2\rangle_{\mathcal{A}})$

(real) *Hilbert space* (and not just a prehilbert space) allows us to complete the construction of K and $\mathscr{H}$:

4) Construction of $(K, \mathscr{H}, U(t))$. Firstly, we seek a "complex structure" J on $\mathscr{A}$ i.e. a real unitary operator satisfying

i) $J^2 = -1$,

ii) $J\bar{\boldsymbol{h}} = \bar{\boldsymbol{h}}J$,

iii) $\sigma'(\Phi, J\Psi) > 0 \, (\Leftrightarrow \bar{\boldsymbol{h}}^{-1}J \geqq 0)$.

The following constructions for J are easily seen to be equivalent and to satisfy i), ii), iii) above:

Either define J via the polar decomposition

$$J = |\bar{\boldsymbol{h}}|^{-1}\bar{\boldsymbol{h}} \, (= \bar{\boldsymbol{h}}|\bar{\boldsymbol{h}}|^{-1}),$$

Or, equivalently, note that $i(\bar{\boldsymbol{h}} + \bar{\boldsymbol{h}})$ on $\mathscr{A}_{\mathbb{C}}$ (see Footnote 3) is self-adjoint. Defining P^+, P^- the projections onto the positive and negative parts of its spectrum, we have

$$J = i(P^+ - P^-)$$

restricted to $\mathscr{A} \oplus 0$.

(This is the appropriate generalization of the familiar "positive/negative frequency decomposition".)

Finally, we define $(K, \mathscr{H}, U(t))$, $K: \mathscr{A} \to \mathscr{H}$ using J:

a) $(a+ib)K(\Phi) := K(a\Phi) + K(bJ\Phi)$ $a, b \in \mathbb{R}$;

b) $$\langle K(\Phi)|K(\Psi)\rangle_{\mathscr{H}} := \tfrac{1}{2}\sigma'(\Phi, J\Psi) + \tfrac{1}{2} i\sigma(\Phi, \Psi)$$
$$\equiv \tfrac{1}{2}\langle \Phi \| \bar{\boldsymbol{h}}|^{-1}\Psi\rangle + \tfrac{1}{2} i\langle \Phi|\bar{\boldsymbol{h}}^{-1}\Psi\rangle_{\mathscr{A}};$$

c) $\mathscr{H}$ is the completion of $K(\mathscr{A})$ in $\mathscr{H}$ norm given by b) [it being easily checked that $K(\mathscr{A})$ defined by a), b) is a complex prehilbert space];

d) $U(t)K(\Phi) := K(e^{-\bar{\boldsymbol{h}}t}\Phi)$.

We need to check that KD is dense in $\mathscr{H}$: we have

$$\|K(\Phi)\|^2_{\mathscr{H}} = \tfrac{1}{2}\sigma'(\Phi, J\Phi) = \tfrac{1}{2}\langle \Phi \| \bar{\boldsymbol{h}}|^{-1}\Phi\rangle_{\mathscr{A}} \leqq \| |\bar{\boldsymbol{h}}|^{-1}\|_{\mathscr{A}} \|\Phi\|^2_{\mathscr{A}}$$

verifying K continuous from $D \subset \mathscr{A}$ to $\mathscr{H}$.

The result follows since in topological spaces, D dense in S_1 and $K: D \to S_2$ continuous $\Rightarrow KD$ dense in S_2. That K is symplectic follows from b).

$U(t)$ as defined in d) is clearly an isometric group on $K(\mathscr{A})$ in $\mathscr{H}$; in fact it is easy to check it preserves both real and imaginary parts of the inner product since $\bar{\boldsymbol{h}}$, $|\bar{\boldsymbol{h}}|$, $e^{-\bar{\boldsymbol{h}}t}$ all commute. We extend it by continuity to $\mathscr{H}$ calling the extension $U(t)$ also.

The strictly positive energy is verified in:

5) Verification of Last Paragraph of Theorem. Using $K: \mathscr{A} \supset D \to \mathscr{H}$ continuous [proved in 4) above] and $e^{-\bar{\boldsymbol{h}}t}: D \to D$ [$\mathscr{T}'(t) = e^{-\bar{\boldsymbol{h}}t}$ by 2c), 3a) and Stone's theorem] we have by 4d) that $U(t)$ is strongly differentiable on KD. Writing

$U(t)=e^{-i\hbar t}$ we have from 4d)

$$-i\hbar K(\Phi)=K(-\bar{\boldsymbol{h}}\Phi)=-K(\boldsymbol{h}\Phi).$$

Clearly, all the hypotheses of Nelson's lemma then hold for $U(t)$, $\hbar$ by taking as our invariant (complex linear) domain $K(D)+iK(D)$ so $\hbar$ is essentially self-adjoint on $K(D)+iK(D)$ $(=K(D+JD))$.

To show that $\hbar$ has bounded inverse we proceed as in the proof of 3b) above. Firstly, we show that $F:=iK\bar{\boldsymbol{h}}^{-1}K^{-1}\equiv K|\bar{\boldsymbol{h}}|^{-1}K^{-1}$ is bounded and positive on $K(\mathscr{A})\subset\mathscr{H}$. Note: $K(\mathscr{A})$ is a complex linear domain containing $K(D)+iK(D)$ and F is easily seen to be complex linear:

F bounded:

$$\begin{aligned}\|FK\Phi\|^2_{\mathscr{H}}&=\|K|\bar{\boldsymbol{h}}|^{-1}\Phi\|^2_{\mathscr{H}}=\tfrac{1}{2}\langle|\bar{\boldsymbol{h}}|^{-1}\Phi\,||\bar{\boldsymbol{h}}|^{-1}\bar{\boldsymbol{h}}^{-1}\Phi\rangle_{\mathscr{A}}\\&=\tfrac{1}{2}\langle(|\bar{\boldsymbol{h}}|^{-1})^2|\bar{\boldsymbol{h}}|^{-1/2}\Phi\,||\bar{\boldsymbol{h}}|^{-1/2}\Phi\rangle_{\mathscr{A}}\leqq\tfrac{1}{2}\|\bar{\boldsymbol{h}}^{-1}\|^2_{\mathscr{A}}\langle\Phi\,||\bar{\boldsymbol{h}}|^{-1}\Phi\rangle_{\mathscr{A}}\\&=\tfrac{1}{2}\|\bar{\boldsymbol{h}}^{-1}\|^2_{\mathscr{A}}\|\Phi\|^2_{\mathscr{H}}\qquad\Phi\in D.\end{aligned}$$

F positive on KD:

$$\begin{aligned}\langle K\Phi|FK\Phi\rangle_{\mathscr{H}}&=\langle K\Phi|\,K|\bar{\boldsymbol{h}}|^{-1}\Phi\rangle_{\mathscr{H}}=\tfrac{1}{2}\langle\Phi\,||\bar{\boldsymbol{h}}|^{-2}\Phi\rangle_{\mathscr{A}}\\&=\tfrac{1}{2}\langle|\bar{\boldsymbol{h}}|^{-1}\Phi\,||\bar{\boldsymbol{h}}|^{-1}\Phi\rangle_{\mathscr{A}}>0.\end{aligned}$$

Calling the continuous extension of F, F also—which is then a bounded positive operator; we easily have $F\hbar=1$ on $KD+iKD$ whereupon $F^{-1}\supset(\hbar$ on $KD+iKD)$ and hence (since $\hbar$ is essentially self-adjoint on $KD+iKD$)

$$F^{-1}=\bar{\hbar}.\quad\Box$$

§7. The Special Static Cases

When our space-time is not just stationary but *static*, we can find a time coordinate such that the equal-time surfaces $\mathscr{C}$ are orthogonal to the Killing trajectories. In other words, we can realize our space-time as $\mathrm{Stat}(\mathscr{C},{}^3g,\alpha,0)$—with lapse but no shift. It is known [26] that if such a space-time possesses a Cauchy surface at all (i.e. if it is globally hyperbolic) then these particular surfaces are Cauchy. Now the matrix A [Eq. (2.4)] becomes diagonal:

$$A=\begin{pmatrix}-(\partial^i\alpha)\partial_i+\alpha(m^2-\Delta(\mathscr{C})+V) & 0\\ 0 & \alpha\end{pmatrix}\tag{7.1}$$

and we can construct K and $\mathscr{H}$ in a more concrete way. The simplest case to deal with is when the lapse function is actually 1 i.e. when the unit normals coincide with the time-like Killing vectors:

$$A=\begin{pmatrix}m^2-\Delta(\mathscr{C})+V & 0\\ 0 & 1\end{pmatrix}.\tag{7.2}$$

7.1. Theorem (cf. Chernoff [3] last paragraph). *Let V satisfy* i) *of Theorem 4.1 and $(\mathscr{C},{}^3g)$ be complete. Then $m^2-\Delta(\mathscr{C})+V$ is essentially self-adjoint on $C_0^\infty(\mathscr{C})\subset L^2(\mathscr{C},d\eta)$—its closure being positive, invertible.*

Proof. Completeness of $(\mathscr{C}, {}^3g)$ gives $\mathscr{C}$ Cauchy in $\mathrm{Stat}(\mathscr{C}, {}^3g, 1, 0)$ by Proposition 5.2. Calling $m^2 - \Delta(\mathscr{C}) + V$ "A", *we have*

$$\langle x|Ax\rangle_{L^2} \geqq \varepsilon \langle x|x\rangle_{L^2} \qquad x \in C_0^\infty(\mathscr{C}). \tag{7.3}$$

ε as in i) of Theorem 4.1.

Now, 2a) and 2c) of the proof of Theorem 4.1 allow us to conclude by Chernoff's lemma that $\bar{\boldsymbol{h}}^2$ is essentially self-adjoint on $C_0^\infty(\mathscr{C}) + C_0^\infty(\mathscr{C})$ in $\mathscr{A}$ $(\boldsymbol{h} = -\boldsymbol{g}A)$.

Here: $\langle f_1, p_1 | f_2, p_2\rangle_{\mathscr{A}} = \langle f_1 | A f_2\rangle_{L^2} + \langle p_1 | p_2\rangle_{L^2}$ i.e. $\mathscr{A} = \mathscr{A}_1 + L^2$, where $\mathscr{A}_1$ is the completion of C_0^∞ in $\langle \cdot | A \cdot\rangle_{L^2}$ and L^2 is $L^2(\mathscr{C}, \eta)$.

Also

$$\boldsymbol{h} = -\boldsymbol{g}A = \begin{pmatrix} 0 & -1 \\ A & 0 \end{pmatrix} \therefore \boldsymbol{h}^2 = \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}$$

whereupon (by projection) A essentially self adjoint in $C_0^\infty(\mathscr{C}) \subset L^2(\mathscr{C}, \eta)$. Furthermore, $\bar{A}$ being a unique s.a. extension must coincide with the Friedrichs extension (see Reed and Simon, II, [19], §X.3) which, having the same lower bound as A, has by Equation (7.3) lower bound to its spectrum ε. □

We then obtain a concrete form for $(K, \mathscr{H}, U(t))$ (easily checked to satisfy the necessary properties and therefore equivalent to the unique one particle structure constructed in Theorem 4.1).

Let $\mathscr{H} = \mathrm{L}^2_{\mathbb{C}}(\mathscr{C}, \eta)$ and define k (replacing K): $C_0^\infty(\mathscr{C}) + C_0^\infty(\mathscr{C}) \to L^2_{\mathbb{C}}(\mathscr{C}, \eta)$

$$(f, p) \mapsto \frac{\bar{A}^{1/4} f + i\bar{A}^{-1/4} p}{\sqrt{2}}.$$

$k(f, p)$ generalizes the familiar Newton-Wigner wave-function—*note* we have a preferred complex conjugation on $\mathscr{H}$. [The natural one on $L^2_{\mathbb{C}}(\mathscr{C}, \eta)$!] representing time-reversal. Finally, $U(t)$ is represented by $\exp(-i\bar{A}^{1/2}t)$ i.e. h becomes $\bar{A}^{1/2}$.

We now deal with the slightly more complicated general case [Eq. (7.1)]. Abbreviating $-(\partial^i \alpha)\partial_i + \alpha(m^2 - \Delta(\mathscr{C}) + V)$ to A also, we have

$$\boldsymbol{A} = \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}.$$

7.2. Theorem. *Let V satisfy* i) *of Theorem 4.1; $(\mathscr{C}, {}^3g)$ be complete and α be bounded above and below away from zero*

$$0 < \varepsilon_1 < \alpha < C.$$

Then $A(= -(\partial^i \alpha)\partial_i + \alpha(m^2 - \Delta(\mathscr{C}) + V)$ and α are essentially self-adjoint on $C_0^\infty(\mathscr{C}) \subset L^2(\mathscr{C}, \eta)$ -their closure being positive, invertible.

Proof. Completeness of $(\mathscr{C}, {}^3g)$ together with α bounded above gives by Proposition 5.3 that $\mathscr{C}$ is Cauchy in $\mathrm{Stat}(\mathscr{C}, {}^3g, \alpha, 0)$. V satisfying i) of Theorem 4.1 and $\alpha > \varepsilon_1 > 0$ gives

$$\langle x|Ax\rangle_{L^2} \geqq \varepsilon' \langle x|x\rangle_{L^2} \quad \text{for some} \quad \varepsilon' > 0. \tag{7.4}$$

Now, note α bounded above and below away from zero gives immediately that—as a multiplication operator on $L^2(\mathscr{C},\eta)$, $\bar{\alpha}$ is bounded, invertible, positive self-adjoint. We can deal with A by an extension of Chernoff's method (see previous theorem) exploiting the "very nice properties of α":

Write

$$\mathscr{A}=\mathscr{A}_1+\mathscr{A}_2,$$

where $\mathscr{A}_1$ is the completion of C_0^∞ in $\langle\cdot|A\cdot\rangle$ norm and $\mathscr{A}_2$ is the completion of C_0^∞ in $\langle\cdot|\alpha\cdot\rangle$ norm. Also

$$\boldsymbol{h}=-\boldsymbol{g}A=\begin{pmatrix}0 & -\alpha\\ A & 0\end{pmatrix}$$

therefore

$$\boldsymbol{h}^2=\begin{pmatrix}-\alpha A & 0\\ 0 & -A\alpha\end{pmatrix}.$$

By restriction, $A\alpha$ is ess. s.a. on $C_0^\infty\subset\mathscr{A}_2$.

Now, $U:\mathscr{A}_2\to L^2(\mathscr{C},\eta)$ defined by $C_0^\infty(\mathscr{C},\eta)\ni p\mapsto\alpha^{1/2}p$ is unitary. Moreover $U:C_0^\infty\to C_0^\infty$.

Hence, $\alpha^{1/2}A\alpha^{1/2}$ is ess. s.a. on $C_0^\infty\subset L^2$ (since it is unitarily equivalent to A on $C_0^\infty\subset\mathscr{A}_2$). Now, using Equation (8.4) and the fact that is bounded below away from zero: we get (Friedrichs extension = s.a. extension since unique) $\overline{\alpha^{1/2}A\alpha^{1/2}}$ positive s.a. invertible.

Now, $\bar{\alpha}^{1/2}$ is bounded invertible and maps C_0^∞ onto C_0^∞. We therefore have $\overline{\alpha^{1/2}A\alpha^{1/2}}=\overline{\bar{\alpha}^{1/2}A\bar{\alpha}^{1/2}}=\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2}$ (proof that $\overline{\bar{\alpha}^{1/2}A\bar{\alpha}^{1/2}}=\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2}$:

Note that $\bar{\alpha}^{1/2}:L^2\to L^2$ homeomorphism in norm topology. Therefore $\bar{\alpha}^{-1/2}\oplus\bar{\alpha}^{1/2}:L^2+L^2\to L^2+L^2$

$$\{p,Ap\}\mapsto\{\bar{\alpha}^{-1/2}p,(\bar{\alpha}^{1/2}A\bar{\alpha}^{1/2})\bar{\alpha}^{1/2}p\}$$

is a homeomorphism sending the graph of A to the graph of $\bar{\alpha}^{1/2}A\bar{\alpha}^{1/2}$. Therefore it sends the graph of $\bar{A}$ to the graph of $\overline{\bar{\alpha}^{1/2}A\bar{\alpha}^{1/2}}$ which must therefore equal $\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2}$.)

Finally, then, $\bar{A}$ must be self-adjoint (and positive with bounded inverse) since

$$\begin{aligned}\bar{A}^{-1}&=\bar{\alpha}^{1/2}\bar{\alpha}^{-1/2}\bar{A}^{-1}\bar{\alpha}^{-1/2}\bar{\alpha}^{1/2}\\ &=\bar{\alpha}^{1/2}(\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2})^{-1}\bar{\alpha}^{1/2}=\bar{\alpha}^{1/2}(\overline{\bar{\alpha}^{1/2}A\bar{\alpha}^{1/2}})^{-1}\bar{\alpha}^{1/2}\end{aligned}$$

which is bounded, self-adjoint, positive. □

We sketch a "concrete" description of $(K,\mathscr{H},U(t))$.

1) $\bar{A}$, $\bar{\alpha}$ positive, self-adjoint, invertible allow us to identify $\mathscr{A}$ as $L^2(\mathscr{C},\eta)+L^2(\mathscr{C},\eta)$ [$\mathscr{A}_1\supset C_0^\infty(\mathscr{C})\ni f\mapsto\bar{A}^{1/2}f\in L^2(\mathscr{C},\eta)$, and $\mathscr{A}_2\supset C_0^\infty(\mathscr{C})\ni p\mapsto\bar{\alpha}^{1/2}p\in L^2(\mathscr{C},\eta)$ define unitaries]. With this identification, it is easy to follow through the construction in §6 to get

$$J=\begin{pmatrix}0 & -j\\ j^+ & 0\end{pmatrix}, \tag{7.5}$$

where the real-unitary operator $j=\bar{A}^{1/2}\bar{\alpha}^{1/2}(\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2})^{-1/2}$. Ashtekar and Magnon [1] have previously given an equivalent (non-rigorous) formula.

2) From J above, there are several (unitarily equivalent) analogues to Newton-Wigner fields in which $\mathscr{H}$ appears as $L^2_{\mathbb{C}}(\mathscr{C},\eta)$. Take

$$\begin{aligned} &k: C_0^\infty(\mathscr{C})+C_0^\infty(\mathscr{C})\to L^2_{\mathbb{C}}(\mathscr{C},\eta)\\ &(f,p)\mapsto \frac{Xf+iX^{+-1}p}{\sqrt{2}}, \end{aligned} \tag{7.6}$$

where X is closed and satisfies:

$$X^+X=\bar{\alpha}^{-1/2}j^+\bar{A}^{1/2}=\bar{\alpha}^{-1/2}(\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2})^{1/2}\bar{\alpha}^{-1/2}.$$

E.g. let

$$X=(\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2})^{1/4}\bar{\alpha}^{-1/2}.$$

We then have the one particle Hamiltonian

$$h=(\bar{\alpha}^{1/2}\bar{A}\bar{\alpha}^{1/2})^{1/2}$$

(cf. Klein [14], Theorem 2: which gives a unitarily equivalent construction in a special case, see [13]).

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