Linear Stability Analysis of Runge-Kutta Methods for Singular Lane-Emden Equations

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Abstract

Runge-Kutta methods are efficient methods of computations in differential equations, the classical Runge-Kutta method of order 4 happens to be the most popular of these methods, and most times it is attached to the mind when Runge-Kutta methods are mentioned. However, there are numerous forms of them existing in lower and higher orders of the classical method. This work investigates the linear stabilities and abilities of some selected explicit members of these Runge-Kutta methods in integrating the singular Lane-Emden differential equations. The results obtained established the ability of the classical Runge-Kutta method and why is mostly used in computations.

Keywords: Classical, Stability, Explicit, Lane-Emden equations, Ordinary Differential Equations

1. Introduction

The most popular of the methods of Runge-Kutta is the classical Runge-Kutta method of order 4, ‘classical’ as related to the method obtained in the pre-computer era. However, many other forms of Runge-Kutta methods have been derived by numerous authors in the past. Existing methods of Runge-Kutta are of orders 2, 3, and orders greater than 4. Methods of orders greater than 4 are regarded as higher-order methods of Runge-Kutta. The numerical solutions of singular Initial Value Problems (IVP) and Boundary Value Problems (BVP) of second-order ordinary differential equations (ODEs) have been studied in this work. Existing methods for the singular problems are series, analytical methods and of recent non-series numerical methods as obtained by various authors.

The general form of second-order singular differential equations is denoted as:

\[ y''(x) + \frac{P(x)}{Q(x)}y'(x) + f(x,y) = g(x,y); \quad a \leq x \leq b. \]  \hspace{1cm} (1)

To solve equation (1), any of the conditions stated below need to be imposed.

\[ y(a) = \alpha, \ y'(a) = \beta \] \hspace{1cm} (2)
\[ y(a) = \alpha_1, \ y'(b) = \beta_1 \] \hspace{1cm} (3)

Equation (1) together with (2) are called Initial Value Problems (IVPs) while equations (1) and (3) are called Boundary Value Problems (BVPs).

Equation (1) is singular at \( Q(x) = 0 \), and \( f(x,y), \ g(x,y) \) are non-linear continuous functions. It is well known that some of these

2. Method

The general m-stage Runge-kutta method is defined by

$$y_{r+1} - y_r = h\theta(x_r, y_r, h),$$

$$\theta(x, y, h) = \sum_{i=1}^{m} q_i \mu_i$$

$$\mu_1 = f(x, y)$$

$$\mu_r = f(x + h\mu_{r-1}, y + h \sum_{i=1}^{r-1} b_{i,r}; i = 2, 3, \ldots, m)$$

$$p_i = \sum_{s=1}^{i-1} b_{is}, \; i = 2, 3, \ldots, m; h = x_i - x_{i-1}$$

It can be observed that an m-stage Runge-kutta method involves m-function evaluations per step. Each of the functions $\mu_i(x, y, h)$, $i = 1, 2, 3, \ldots, m$ may be interpreted as an approximation of the derivative $y'(x)$ and the function $\theta(x, y, h)$ as a weighted mean of these approximations.

There is a great deal of tedious manipulation involved in deriving Runge-Kutta methods of the higher order. However, the forms of Runge-Kutta methods in the scope of this work shall be given in the following section.

(i) 2-stage Runge-kutta method of Order 2 (RK2)

$$y_{i+1} - y_i = h\mu_2$$

$$\mu_1 = f(x_i, y_i)$$

$$\mu_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}h\mu_1)$$

(ii) 3-stage Runge-kutta method of Order 3 (RK3)

$$y_{i+1} - y_i = \frac{1}{6}h(\mu_1 + 4\mu_2 + \mu_3)$$

$$\mu_1 = f(x_i, y_i)$$

$$\mu_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}h\mu_1)$$

$$\mu_3 = f(x_i + h, y_i - h(\mu_1 - 2\mu_2))$$

(iii) 4-stage Runge-kutta method of Order 4 (RK4)

$$y_{i+1} - y_i = \frac{1}{6}h(\mu_1 + 2\mu_2 + 2\mu_3 + \mu_4)$$

$$\mu_1 = f(x_i, y_i)$$

$$\mu_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}h\mu_1)$$

$$\mu_3 = f(x_i + h, y_i + \frac{1}{2}h\mu_2)$$

$$\mu_4 = f(x_i + h, y_i + \frac{1}{2}h\mu_3)$$
(iv) 6-stage kutta-Nyström method of Order 5 (RK Nyström)

\[
y_{i+1} - y_i = \frac{1}{192} h (23\mu_1 + 125\mu_3 - 81\mu_5 + 125\mu_6)
\]
\[
\mu_1 = f(x_i, y_i)
\]
\[
\mu_2 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h\mu_1)
\]
\[
\mu_3 = f(x_i + \frac{3}{5} h, y_i + \frac{3}{30} h(4\mu_1 + 6\mu_2))
\]
\[
\mu_4 = f(x_i + h, y_i + \frac{1}{3} h(\mu_1 - 12\mu_2 + 15\mu_3))
\]
\[
\mu_5 = f(x_i + \frac{2}{3} h, y_i + \frac{1}{6} h(6\mu_1 + 90\mu_2 - 50\mu_3 + 8\mu_4))
\]
\[
\mu_6 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{3} h(6\mu_1 + 36\mu_2 + 10\mu_3 + 8\mu_4))
\]

(v) 8-stage Huta method of Order 6 (RK 8)

\[
y_{i+1} - y_i = \frac{1}{840} h (84q_1 + 216q_2 + 272q_3 + 270q_4 + 216q_5 + 42q_6)
\]
\[
\mu_1 = f(x_i, y_i)
\]
\[
\mu_2 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h\mu_1)
\]
\[
\mu_3 = f(x_i + \frac{1}{3} h, y_i + \frac{3}{4} h\mu_2)
\]
\[
\mu_4 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h\mu_3)
\]
\[
\mu_5 = f(x_i + \frac{2}{3} h, y_i + \frac{3}{4} h\mu_4)
\]
\[
\mu_6 = f(x_i + \frac{3}{4} h, y_i + \frac{3}{4} h\mu_5)
\]
\[
\mu_7 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h\mu_4)
\]
\[
\mu_8 = f(x_i + h, y_i + \frac{1}{6} h\mu_7)
\]

For purpose of completion and without loss of generality, special Runge-kutta methods which were developed to exhibit certain properties were also considered.

(vi) 5-stage Merson’s Method of Order 4 (RK Merson)

\[
y_{i+1} - y_i = \frac{1}{6} h (\mu_1 + 4\mu_4 + \mu_5)
\]
\[
\mu_1 = f(x_i, y_i)
\]
\[
\mu_2 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h\mu_1)
\]
\[
\mu_3 = f(x_i + \frac{3}{5} h, y_i + \frac{3}{10} h(\mu_1 + \mu_2))
\]
\[
\mu_4 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h(\mu_1 + \mu_3))
\]
\[
\mu_5 = f(x_i + h, y_i + \frac{1}{2} h(\mu_1 - 3\mu_3 + 4\mu_4))
\]

(vii) 5-stage Scraton’s Method of Order 4 (RK Scraton)

\[
y_{i+1} - y_i = h \left(\frac{17}{1080} \mu_1 + \frac{81}{1080} \mu_3 + \frac{32}{1080} \mu_4 + \frac{250}{1080} \mu_5\right)
\]
\[
\mu_1 = f(x_i, y_i)
\]
\[
\mu_2 = f(x_i + \frac{1}{5} h, y_i + \frac{1}{5} h\mu_1)
\]
\[
\mu_3 = f(x_i + \frac{2}{5} h, y_i + \frac{4}{5} h(\mu_1 + \mu_2))
\]
\[
\mu_4 = f(x_i + \frac{3}{5} h, y_i + \frac{3}{5} h(\mu_1 + \mu_3))
\]
\[
\mu_5 = f(x_i + h, y_i + \frac{1}{5} h(\mu_1 + 3\mu_3))
\]
\[
\mu_6 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h(\mu_1 - 3\mu_3 + 4\mu_4))
\]

(viii) 6-stage England’s Method of Order 4 (RK England)

\[
y_{i+1} - y_i = \frac{1}{6} h (\mu_1 + 4\mu_3 + \mu_4)
\]
\[
\mu_1 = f(x_i, y_i)
\]
\[
\mu_2 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h\mu_1)
\]
\[
\mu_3 = f(x_i + \frac{1}{3} h, y_i + \frac{1}{3} h(\mu_1 + \mu_2))
\]
\[
\mu_4 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h(\mu_1 + \mu_3))
\]
\[
\mu_5 = f(x_i + \frac{2}{3} h, y_i + \frac{2}{3} h(\mu_1 + \mu_4))
\]
\[
\mu_6 = f(x_i + h, y_i + \frac{1}{2} h(\mu_1 + \mu_5))
\]
\[
\mu_7 = f(x_i + \frac{1}{2} h, y_i + \frac{1}{2} h(\mu_1 + \mu_6))
\]
\[
\mu_8 = f(x_i + h, y_i + \frac{1}{2} h(\mu_1 + \mu_7))
\]

2.1. Linear Stability

This is a behaviourial property related to $h > 0$. As in most literature, the linear stability will be analyzed using the Dalquist’s test

\[
y'(t) = \gamma y(t), \quad \Re(\gamma) < 0.
\]

Applying methods (9)-(16) on (17), we have obtained a recurrence equation

\[
y_i = M(z)y_{i-1}
\]

where $M(z) = hy$ for each of the method is given in Table 1 below. The contour for regions of absolute stability as obtained from their characteristics equations are given in Figure 1 - 8.

3. Numerical Experiment

This section contains the numerical example considered and their results presented in tables of absolute errors.

**Example 1: Variable Coefficient Non-homogeneous Singular IVP [3]**

\[
\begin{cases}
y''(x) = -\frac{2}{x} y'(x) - n^2 \cos(nx) \\ -\frac{2}{x} \sin(nx) \\ y(0) = 2, y'(0) = 0
\end{cases}
\]

Theoretical Solution: $y(x) = 1 + \cos(nx)$

*Absolute Error = |Exact Solution - Numerical Solution|, Numerical solution is obtained using the Runge-kutta methods.*

**Example 2: Variable Coefficient Homogeneous Singular Initial BVP [6]**

\[
\begin{cases}
(x^2 y')' = \beta(x y' + y(\alpha + \beta - 1))x^{n\alpha + \beta - 2}, & 0 \leq x \leq 1 \\
y(0) = 1, y(1) = \exp(1)
\end{cases}
\]

Theoretical Solution = $\exp(x^4)$
Table 1. Table of Characteristics Equations for Runge-kutta Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>$M(z)$, $z = h\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9)</td>
<td>$\frac{1}{4}z^4 + \frac{1}{2}z^2 + 1$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{1}{5}z^5 + \frac{1}{2}z^2 + z + 1$</td>
</tr>
<tr>
<td>(11)</td>
<td>$\frac{1}{6}z^6 + \frac{1}{2}z^2 + z + 1$</td>
</tr>
<tr>
<td>(12)</td>
<td>$\frac{1}{24}z^4 + \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + 1$</td>
</tr>
<tr>
<td>(13)</td>
<td>$\frac{1}{24}z^4 + \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + 1$</td>
</tr>
<tr>
<td>(14)</td>
<td>$\frac{1}{36}z^4 + \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + 1$</td>
</tr>
<tr>
<td>(15)</td>
<td>$\frac{1}{96}z^4 + \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + 1$</td>
</tr>
<tr>
<td>(16)</td>
<td>$\frac{1}{24}z^4 + \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + \frac{1}{12}$</td>
</tr>
</tbody>
</table>

Table 2. Table of Absolute Errors for Example 1, $n = 3$ at different $h$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$x$</th>
<th>$y(x)$</th>
<th>RK2</th>
<th>RK3</th>
<th>RK4</th>
<th>RK Nyström</th>
<th>RK Merson</th>
<th>RK Scratchon</th>
<th>RK England</th>
<th>RK8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>0.2</td>
<td>9.1982 × 10^{-2}</td>
<td>9.0000 × 10^{-6}</td>
<td>5.0392 × 10^{-2}</td>
<td>4.5167 × 10^{-2}</td>
<td>1.3355 × 10^{-1}</td>
<td>7.61289 × 10^{-1}</td>
<td>5.0392 × 10^{-2}</td>
<td>1.1722 × 10^{0}</td>
<td></td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>0.04</td>
<td>7.0148 × 10^{-4}</td>
<td>9.0000 × 10^{-6}</td>
<td>2.5148 × 10^{-4}</td>
<td>7.0046 × 10^{-4}</td>
<td>1.5808 × 10^{-3}</td>
<td>9.7960 × 10^{-4}</td>
<td>2.5148 × 10^{-4}</td>
<td>1.2152 × 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>0.09</td>
<td>4.7178 × 10^{-6}</td>
<td>9.0000 × 10^{-6}</td>
<td>1.1179 × 10^{-6}</td>
<td>9.4074 × 10^{-6}</td>
<td>1.8213 × 10^{-5}</td>
<td>1.1179 × 10^{-6}</td>
<td>1.6996 × 10^{-5}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Cmpt Time (s) | 0.0025 | 0.0022 | 0.0035 | 0.0059 | 0.0039 | 0.0077 | 0.0065 | 0.021 |

Table 3. Table of Absolute Errors for Example 2 Using $\alpha = 2$, $\beta = 4$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$x$</th>
<th>$y(x)$</th>
<th>RK2</th>
<th>RK3</th>
<th>RK4</th>
<th>RK Nyström</th>
<th>RK Merson</th>
<th>RK Scratchon</th>
<th>RK England</th>
<th>RK8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{10}$</td>
<td>0.2</td>
<td>1.5913 × 10^{-1}</td>
<td>1.5954 × 10^{-2}</td>
<td>1.5745 × 10^{-3}</td>
<td>1.5610 × 10^{-3}</td>
<td>1.5706 × 10^{-3}</td>
<td>7.0036 × 10^{-4}</td>
<td>1.5705 × 10^{-3}</td>
<td>1.5705 × 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>0.05</td>
<td>5.8226 × 10^{-2}</td>
<td>5.8697 × 10^{-2}</td>
<td>5.7050 × 10^{-2}</td>
<td>5.7009 × 10^{-2}</td>
<td>5.7014 × 10^{-2}</td>
<td>2.5489 × 10^{-2}</td>
<td>5.7050 × 10^{-2}</td>
<td>5.7175 × 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>0.8</td>
<td>3.8263 × 10^{-1}</td>
<td>3.8559 × 10^{-1}</td>
<td>3.7177 × 10^{-1}</td>
<td>3.7160 × 10^{-1}</td>
<td>3.7166 × 10^{-1}</td>
<td>5.7962 × 10^{-1}</td>
<td>3.7177 × 10^{-1}</td>
<td>3.7554 × 10^{-1}</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>0.125</td>
<td>2.4357 × 10^{-1}</td>
<td>2.4382 × 10^{-1}</td>
<td>2.4258 × 10^{-2}</td>
<td>2.4231 × 10^{-2}</td>
<td>2.4234 × 10^{-2}</td>
<td>2.4234 × 10^{-2}</td>
<td>2.4258 × 10^{-2}</td>
<td>2.4233 × 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>0.5</td>
<td>5.7534 × 10^{-2}</td>
<td>5.7757 × 10^{-2}</td>
<td>5.7026 × 10^{-2}</td>
<td>5.7019 × 10^{-2}</td>
<td>5.7020 × 10^{-2}</td>
<td>2.5563 × 10^{-2}</td>
<td>5.7062 × 10^{-2}</td>
<td>5.7273 × 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>0.75</td>
<td>2.6863 × 10^{-1}</td>
<td>2.8731 × 10^{-1}</td>
<td>2.8304 × 10^{-1}</td>
<td>2.8302 × 10^{-1}</td>
<td>2.8303 × 10^{-1}</td>
<td>3.7761 × 10^{-1}</td>
<td>2.8304 × 10^{-1}</td>
<td>2.8600 × 10^{-1}</td>
<td></td>
</tr>
</tbody>
</table>

Cmpt Time (s) | 0.0023 | 0.0026 | 0.0042 | 0.0055 | 0.0033 | 0.0061 | 0.0055 | 0.011 |
**Example 3:** Variable Coefficient Non-linear Homogeneous Singular IVP [3]

\[
y''(x) = -\frac{3}{2}y'(x) - y^5(x); \quad x \in [0, 1] \quad y(0) = 1, y'(0) = 0
\]

Theoretical Solution: \( y(x) = \frac{1}{\sqrt{1 + \frac{3}{2}x}} \)

\[
(21)
\]

**Example 4:** Van der Pol Singular Problem [11]

\[
y'' = \frac{y'(1-y^2)y}{y} ;
\]

\[
y(0) = 2, \quad y'(0) = -\frac{\mu}{2} + \frac{10}{87}\mu - \frac{292}{2187}\mu^2 - \frac{1814}{19683}\mu^3 ; \quad \mu = 10^{-1}.
\]

\[
(22)
\]

The methods considered in this work were used to approximate the problem over the interval \([0, 0.55139]\) for \(h = 10^{-3}\).
Table 4. Table of Absolute Errors for Example 3 Using different h

<table>
<thead>
<tr>
<th>h</th>
<th>x</th>
<th>y(x)</th>
<th>RK2</th>
<th>RK3</th>
<th>RK4</th>
<th>RK Nystrom</th>
<th>RK Merson</th>
<th>RK Scarton</th>
<th>RK England</th>
<th>RK8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/7</td>
<td>0.2857</td>
<td>2.4565 × 10⁻²</td>
<td>9.7182 × 10⁻⁶</td>
<td>2.7524 × 10⁻²</td>
<td>3.4496 × 10⁻²</td>
<td>4.1169 × 10⁻²</td>
<td>3.5739 × 10⁻²</td>
<td>2.7525 × 10⁻²</td>
<td>1.1722 × 10⁻¹</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5714</td>
<td>2.8265 × 10⁻²</td>
<td>9.7182 × 10⁻⁶</td>
<td>3.1850 × 10⁻²</td>
<td>3.8382 × 10⁻²</td>
<td>4.54985 × 10⁻²</td>
<td>3.9917 × 10⁻²</td>
<td>3.1851 × 10⁻²</td>
<td>1.2281 × 10⁻¹</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.8571</td>
<td>1.0144 × 10⁻²</td>
<td>9.7182 × 10⁻⁶</td>
<td>1.4411 × 10⁻²</td>
<td>2.1093 × 10⁻²</td>
<td>2.7780 × 10⁻²</td>
<td>4.0945 × 10⁻²</td>
<td>1.4112 × 10⁻²</td>
<td>1.0723 × 10⁻¹</td>
<td></td>
</tr>
<tr>
<td>1/11</td>
<td>0.2857</td>
<td>2.9836 × 10⁻²</td>
<td>1.2148 × 10⁻⁶</td>
<td>3.0700 × 10⁻²</td>
<td>3.2486 × 10⁻²</td>
<td>3.4150 × 10⁻²</td>
<td>2.1129 × 10⁻²</td>
<td>3.0701 × 10⁻²</td>
<td>5.4061 × 10⁻²</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4286</td>
<td>3.4167 × 10⁻²</td>
<td>1.2148 × 10⁻⁶</td>
<td>3.0686 × 10⁻²</td>
<td>3.8524 × 10⁻²</td>
<td>3.2139 × 10⁻²</td>
<td>3.5075 × 10⁻²</td>
<td>5.9065 × 10⁻²</td>
<td>5.5316 × 10⁻²</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.6429</td>
<td>2.8969 × 10⁻²</td>
<td>1.2148 × 10⁻⁶</td>
<td>2.9944 × 10⁻²</td>
<td>3.1729 × 10⁻²</td>
<td>2.4973 × 10⁻²</td>
<td>2.9944 × 10⁻²</td>
<td>2.4823 × 10⁻²</td>
<td>2.4823 × 10⁻²</td>
<td></td>
</tr>
</tbody>
</table>

| Cmpt Time (s) | 0.0025 | 0.0021 | 0.0035 | 0.0059 | 0.0045 | 0.0077 | 0.0051 | 0.021 |

Figure 8. Stability Region for England’s Method

Table 5. Numerical Results for Example 4 at x = 0.55139

<table>
<thead>
<tr>
<th>h</th>
<th>Method</th>
<th>y(x)</th>
<th>Cmpt. Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10⁻³</td>
<td>RK2</td>
<td>0.625821430245524</td>
<td>0.0025</td>
</tr>
<tr>
<td></td>
<td>RK3</td>
<td>0.625821606669667</td>
<td>0.0019</td>
</tr>
<tr>
<td></td>
<td>RK4</td>
<td>0.625821602543592</td>
<td>0.0065</td>
</tr>
<tr>
<td></td>
<td>RK Nystrom</td>
<td>0.625821602551168</td>
<td>0.018</td>
</tr>
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</tbody>
</table>

4. Discussion of Results and Conclusion

It is worthy to note that numerical methods were programmed via MATLAB 9.2 version on a personal computer with the following specifications:

- System name- HP Pavilion x360 Convertible
- Processor- Intel(R) Core(TM) i3-7100U CPU @ 2.40GHz
- Installed memory (RAM)- 8.00GB
- System Type- 64-bits Operating System, x64-based processor

- Operating system- Windows 10

Cmpt time is the computer computation time measured in seconds(s). This paper presents the performances of different orders of Runge-Kutta methods in the integration of singular Lane-Emden equations. From Tables 2-4, It could be observed that the Runge-Kutta method of order 4 performs the best in terms of accuracy. The methods of order 3, Scraton’s, and order 8 were found to have failed in the solution of singular problems in ordinary differential equations. The England’s method performs more satisfactorily as it shows some competitive strength against the order 4 method. It could be concluded that the Runge-Kutta method of order 4 outperforms all other methods under consideration and this property makes it the most stable and accurate of the Runge-Kutta methods.

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References


